A BRIEF GUIDE TO "ABSOLUTE VALUE"

For High-School Students

This material is based on:

THINKING MATHEMATICS!
Volume 4: **Functions and Their Graphs**
Chapter 1 and *Chapter 3*

CONTENTS

ABSOLUTE VALUE AS DISTANCE

Let's start with a series of warm-up questions.

EXERCISE 1: An ant stands on a very long ruler. Every inch mark is marked with a whole number and the ant is currently standing at the position marked "8 inches."

a) How many inches is it from the ant to the 12 inch mark?

b) How many inches is it from the ant to the 4 inch mark?

c) How many inches is it from the ant to the -6 inch mark? (I said it was a very long ruler!)

d) How many inches it from the ant to the 243 inch mark?

e) How many inches is it between the 16 inch mark and the 89 inch mark?

f) How many inches are between the marks -12 and 30?

g) How many inches are between the marks 0 and 20?

h) How many inches are between the marks 0 and -20?

i) How many inches are between the marks -677 and 402? And how many inches are between the marks 402 and -677?

Is it possible to write a formula for the number of inches between the a inch mark and the b inch mark? What do you think? (Think carefully!)

EXERCISE 2:

a) What is the value of $-x$ if x is 16?

b) What is the value of $-x$ if x is -16?

c) What is the value of $-x$ if x is 0?

EXERCISE 3: The *radix* symbol $\sqrt{\ }$ is a symbol from geometry in which all quantities are considered positive: positive length, positive area, and so forth. (There are no negative quantities in geometry.) Thus, the radix refers only to the positive square root of a number. For example:

$$\sqrt{9}=3 \qquad \sqrt{100}=10 \qquad \sqrt{289}=17$$

(Even though -3 is also a square root of 9, for instance, writing $\sqrt{9}=-3$ is technically incorrect. Even writing $\sqrt{9}=\pm 3$ is still technically incorrect!)

a) What is $\sqrt{169}$?

b) What is $\sqrt{400}$?

c) What is the square root of zero? Is zero considered positive or negative, both or neither? Could, or should, geometry allow quantities of measure zero?

d) Solve $x^2=169$. (Is there one or two solutions? Is this a geometry question or an algebra question?)

Now let's get a little strange!

e) What is the value of $\sqrt{7^2}$?

f) What is the value of $\sqrt{836^2}$?

g) What is the value of $\sqrt{(-7)^2}$?

h) What is the value of $\sqrt{(-78)^2}$?

i) What is the value of $\sqrt{x^2}$ if x is 95?

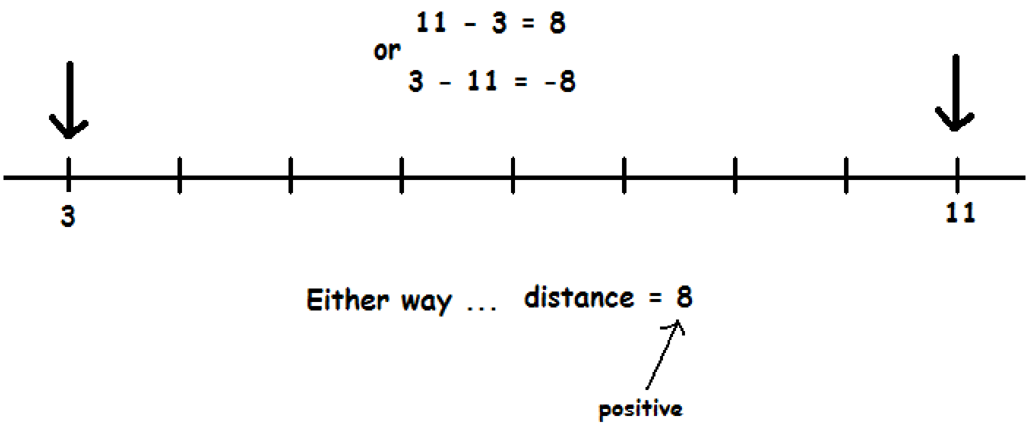
j) What is the value of $\sqrt{x^2}$ if x is -95?

k) What is the value of $\sqrt{x^2}$ if x is -38074?

l) What is the value of $\sqrt{x^2}$ if x is -0.0086?

In general, what can you say about the quantity $\sqrt{x^2}$ if x is a positive number? What can you say about the value of $\sqrt{x^2}$ if x is a negative number?

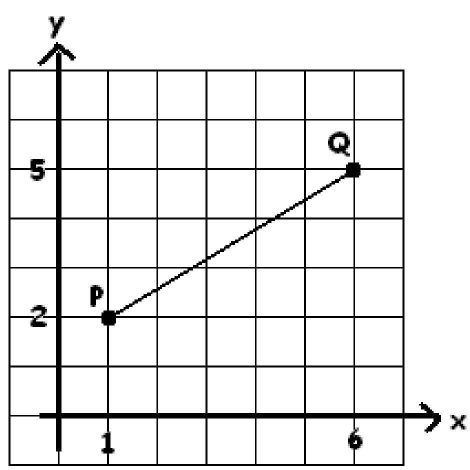
Exercise 1 shows that if we wish to compute the distance between any two points on a number line, we need to compute the positive difference of those two numbers.



Exercises 2 and 3 show that there are at least two ways to make sure answers are positive. We've already seen the second way in our geometry course. Recall …

THE DISTANCE FORMULA

Consider the points P and Q on the coordinate plane shown. What is the distance between them?



Pythagoras's theorem shows that the distance between points P and Q is:

$$d(P,Q) = \sqrt{5^2 + 3^2} = \sqrt{34}$$

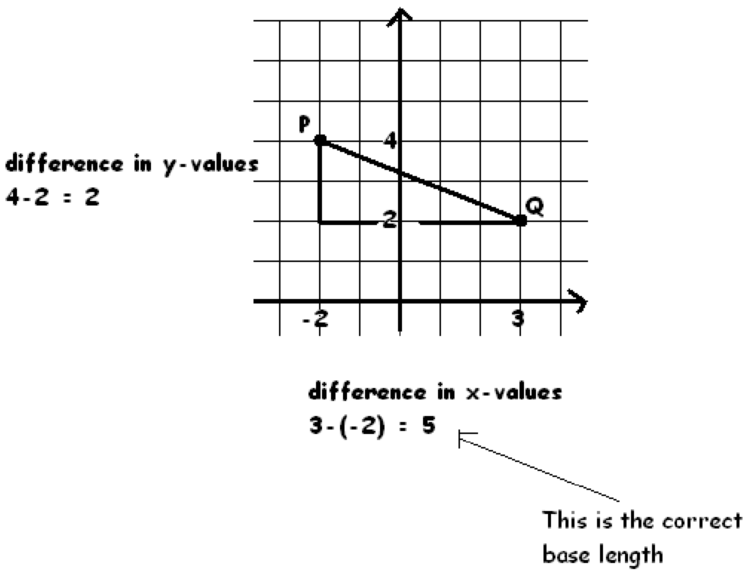
The radix here makes sure that the answer is positive – as it should be!

We have:

$$\text{distance } = \sqrt{\left(\text{difference in x-values}\right)^2 + \left(\text{difference in y-values}\right)^2}$$

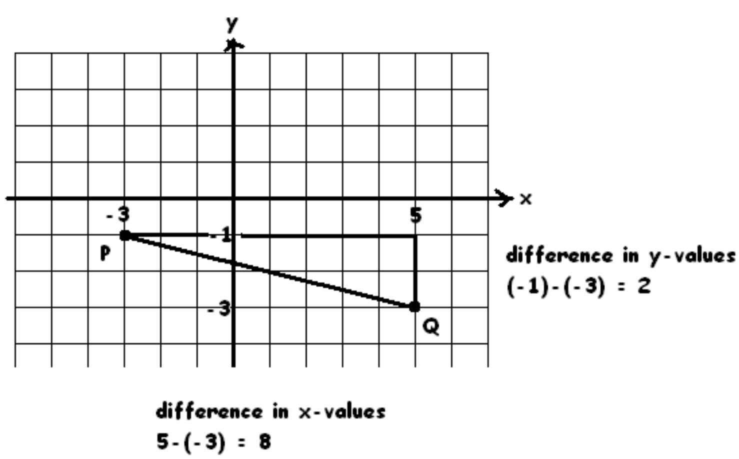
This is valid even if the points involved lie in alternative quadrants of the plane.

EXAMPLE:



$d(P,Q) = \sqrt{5^2 + 2^2} = \sqrt{29}$

EXAMPLE:



$d(P,Q) = \sqrt{8^2 + 2^2} = \sqrt{68}$

In general, for points in the plane:

DISTANCE FORMULA: The distance between two points $P=(x,y)$ and $Q=(a,b)$ is given by:

$$d(P,Q)=\sqrt{(x-a)^2+(y-b)^2}$$

In three dimensions points have three coordinates. It turns out (see page 14) that the distance formula in three dimensions is:

If $P=(x,y,z)$ and $Q=(a,b,c)$, then $d(P,Q)=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$.

And it seems irresistible to say that in ONE-DIMENSION the distance formula should read:

$$d(P,Q)=\sqrt{(x-a)^2}$$

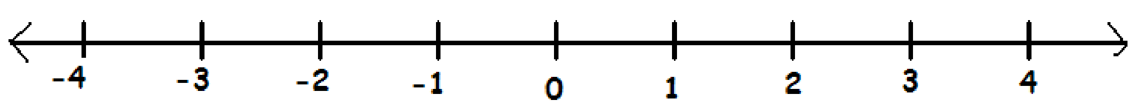
Can we make sense of this?

STEP 1: What is the "one-th" dimension?

Three dimensional space is the space we move in. We can move left and right, forward and back, and up and down.

A two-dimensional world is a plane. In it we can just move left and right, and forward and back (but not up and down).

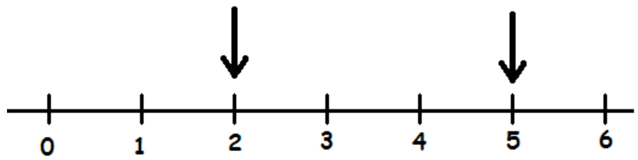
A one-dimensional world must therefore be a line. We can move only left and right.



one-dimensional world = number line

STEP 2: What is the distance formula saying for our number line?

Let's pick two points on the number line, say, 2 and 5.



Then the formula

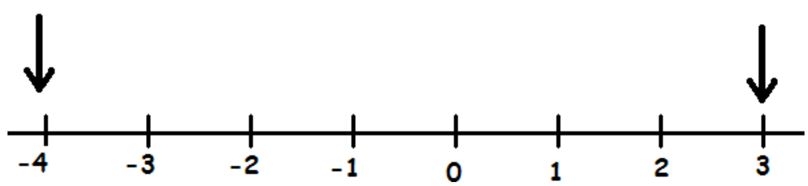
$$d(P,Q) = \sqrt{(x-a)^2}$$

reads

$$d(2,5) = \sqrt{(2-5)^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$$

This is indeed the distance between the points 2 and 5 on the number line!

Let's do this again, this time with the points -4 and 3.



We have: $d(4,-3) = \sqrt{(4-(-3))^2} = \sqrt{(7)^2} = \sqrt{49} = 7$.

This is indeed the distance between 4 and -3 on the number line!

EXERCISE 4: Draw a number line and show on it the two numbers -16 and -5. Compute the distance $d(-16,-5)$ and verify that the answer 11, the distance between these two points, does indeed appear.

EXERCISE 5: Does $d(x,y) = d(y,x)$? Always?

EXERCISE 6: According to the distance formula, what is the value of $d(4,4)$? Does this make sense geometrically?

EXERCISE 7:

a) Find two values of x so that $d(x,6)=13$? Draw a number line and mark the locations of your two x-values.

b) Find two values of w so that $d(w,-2)=5$? Draw a number line and mark the locations of your two w-values.

c) Find all values k so that $d(\frac{116}{51},k)=0$.

A CHANGE OF NOTATION:

The notion of distance relies on all answers being positive. Mathematicians have given a name to a "value made positive."

Definition: The absolute value of a number x, denoted $|x|$ is that number "made positive." That is:

If x is already positive, then $|x|$ is just x. (No change.)
If x is negative, then $|x|$ is $-x$. (Change the sign.)

There is one case where the word "positive" or "negative" doesn't apply. People say:

If x is zero, then $|x|$ is just x. (No change).

For example:

$|18| = 18$ $\qquad |-56| = -(-56) = 56$ $\qquad |1.3| = 1.3$

$|-\frac{1}{2}| = \frac{1}{2}$ $\qquad |-\pi| = \pi$ $\qquad |0| = 0$

COMMENT: Recall from exercise 2 that $-x$ is a positive number if x is already a negative number. (Look at $|-56|$ in the above example.)

HISTORICAL COMMENT: The notion of "absolute value" as a handy concept to consider in its own right was developed somewhat late in the history of mathematics. Explicit mention of this notion didn't start occurring until the mid-1800s. In 1841, German mathematician Karl Weierstrass put forward the idea of using vertical bars as notation for it.

Our distance formula also makes quantities positive. For example,

$$d(5,2)=\sqrt{(5-2)^2}=\text{"}5-2\text{"}\,\textit{made postive}=3$$
$$d(7,18)=\sqrt{(7-18)^2}=\text{"}7-18\text{"}\,\textit{made postive}=11$$
$$d(-30,-4)=\sqrt{(-30-(-4))^2}=\text{"}-30-(-4)\text{"}\,\textit{made postive}=26$$

That is,

$$d(5,2)=|5-2|$$
$$d(7,18)=|7-18|$$
$$d(-30,-4)=|-30-(-4)|$$

OBSERVATION:

> **THE ABSOLUTE VALUE OF A DIFFERENCE $|x-a|$ IS PRECISELY THE DISTANCE BETWEEN x AND a ON THE NUMBER LINE. THAT IS:**
>
> $$|x-a| = d(x,a)$$
>
> **IT IS ALSO THE QUANTITY $(x-a)$ "MADE POSITIVE."**

EXAMPLE: $|5-7|$ is the distance between 5 and 7, and has answer 2. (And, of course, $|5-7|=|-2|=2$.)

EXAMPLE: $|1-10|$ is the distance between 1 and 10, and has answer 9. (And, of course, $|1-10|=|-9|=9$.)

EXAMPLE: $|2+30|$ is the distance between 2 and -30 and so has answer 32. (And, of course, $|2+30|=|32|=32$.)

EXAMPLE: $|-6-2|$ is the distance between -6 and 2 and so has answer 8. (And, of course, $|-6-2|=|-8|=8$.)

FROM NOW ON …

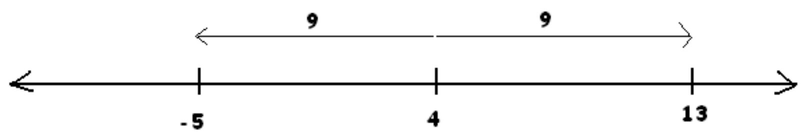
> **The distance between two numbers x and a shall be written: $|x-a|$.**

EXAMPLE: Find all values *x* that satisfy: $|x-4|=9$.

Approach 1: (GEOMETRY)

$|x-4|=9$ reads: "The distance between *x* and 4 is 9."

That is, if $|x-4|=9$, then *x* is a number whose distance from 4 is 9.That is, *x* is either 9 units up or 9 units down from 4.



Thus $x=13$ or $x=-5$. □

Approach 2: (ARITHMETIC)

If $|x-4|=9$, then $(x-4)$ is a quantity, which, when made positive, equals 9.

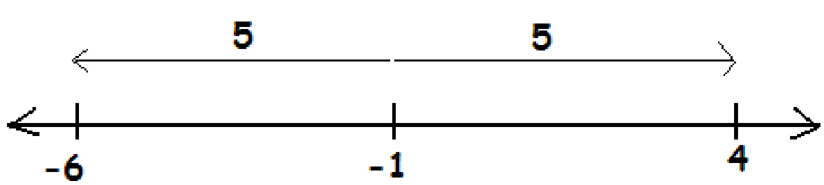
So either $x-4=9$ or $x-4=-9$

Now add 4 throughout to obtain: $x=13$ or $x=-5$. □

EXAMPLE: Find all values *w* that satisfy: $|w+1|=5$.

Approach 1: (GEOMETRY)

If $|w-(-1)|=5$ then the distance between *w* and -1 is 5.That is, *w* is either 5 units up or 5 units down from -1 .



Thus $w=-6$ or $w=4$. □

Approach 2: (ARITHMETIC)

If $|w+1|=5$, then $w+1$ is a quantity, which, when made positive, equals 5.

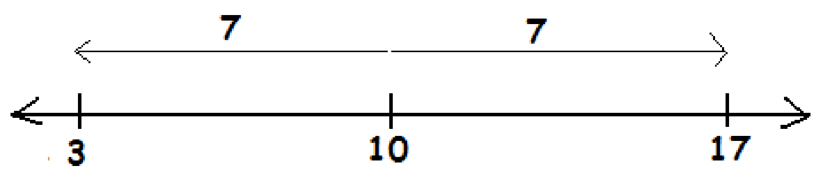
So either $w+1=5$ or $w+1=-5$

Now subtract 1 throughout to obtain: $w=4$ or $w=-6$. □

EXAMPLE: Find all values *x* that satisfy $|x-10|<7$.

Answer 1: Perhaps a **geometry** approach is best here.

The statement $|x-10|<7$ reads: "*x* is a number whose distance from 10 is less than 7."
A diagram now makes it clear that *x* must lie somewhere between 3 and 17.



Our final answer is:

$|x-10|<7$ has solution $3<x<17$.

□

EXERCISE 8: Do the necessary work to show that $|x+5|\le 9$ has solution $-14\le x\le 4$.

EXERCISE 9: Solve $|x-3|>2$. How should one present its solution?

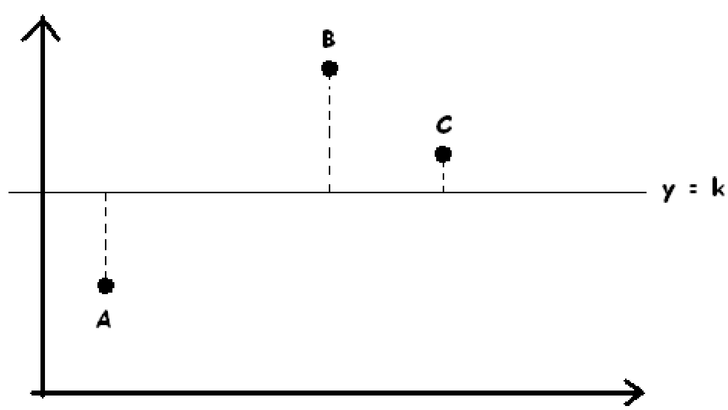
EXERCISE 10: For each equality or inequality, find all possible values of *x* that satisfy it.

a) $|x-700|=800$
b) $|x+3|=5$
c) $|x-1|=1$
d) $|x|=20$
e) $|x^2-25|=9$
f) $|x+8|<4$
g) $|x-10|\le 2$
h) $|x-6|\ge 23$
i) $|2x-14|<20$
j) $|x|=-7$

EXERCISE 11: Cuthbert says that $|a-b|$ will have the same value as $|b-a|$. What do you think?

EXERCISE 12: Angeline says that $|x|$ is "the distance of the point *x* from zero." What do you think?

EXERCISE 13: (OPTIONAL)
Three data points $A=(2,3)$, $B=(5,8)$ and $C=(7,5)$ are plotted on a graph.

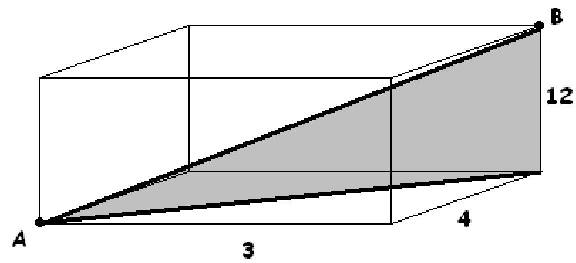


A horizontal line $y=k$ will be drawn for some value *k*.

a) If $k=4$ what is the sum of the three vertical distances shown?
b) If $k=6$, what is the sum of the three vertical distances? (WARNING: The horizontal line is now above the point C.)
c) **VERY TOUGH CHALLENGE:** What value for k gives the smallest value for the sum of the three distances?

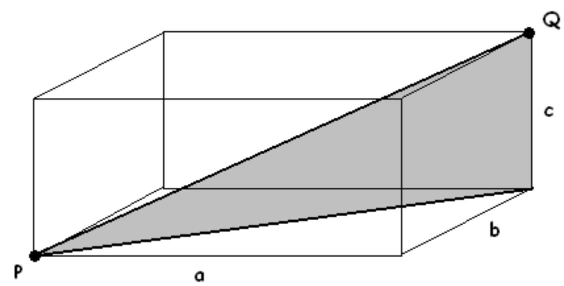
EXERCISE 14: On page 6 we said presented the three-dimensional version of the distance formula. Let's justify it here.

a) Find the length of the line segment $\overline{AB}$ given as the interior diagonal of the rectangular box shown:

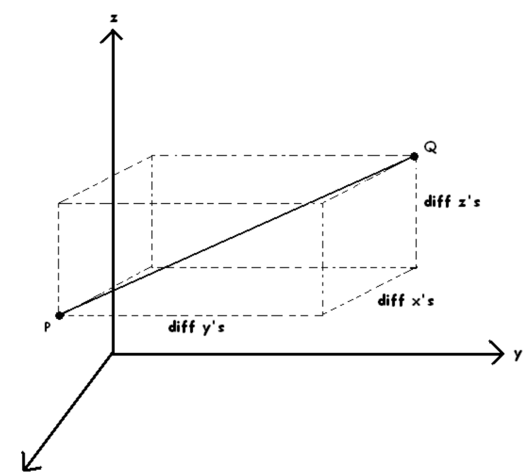


b) In general … A rectangular box has length *a* inches, width *b* inches, and height *c* inches. Show that the length of the longest diagonal in the box, $\overline{PQ}$, is given by:

$$d(P,Q)=\sqrt{a^2+b^2+c^2}$$



COMMENT: Does the following diagram now justify the distance formula given on page 6?



c) What is the distance between the four-dimensional points $A=(1,2,0,-3)$ and $B=(2,4,-1,2)$?

EXERCISE 15:

a) If $|x-5|\le 7$, what is the largest value x could be? What is the smallest value x could be?

b) Write the range of values $10 < x < 30$ as an "absolute value inequality."

c) Sketch the region of x-values given by $|x+2|\le 10$ on the number line.

d) Sketch the region of x-values given by $|x-3|>5$ on the number line.

EXERCISE 16:

a) Write the range of values $50 < x < 100$ as an absolute value inequality.

b) Write the range of values $20 \le x \le 80$ as an absolute value inequality.

c) Write the range of values $-10 \le x \le 40$ as an absolute value inequality.

d) Write the range of values $-50 < x < -30$ as an absolute value inequality.

EXERCISE 17: Suppose a and b are positive numbers.

Solve for x in each of the following. (That is, give the range of possible values for x.)

a) $|x-a|<b$

b) $|x-a|\le a$

c) $|x+a|<b$

d) $|x+a|\ge a$

HINT: Would it help to draw a picture of the number line for each of them?

ABSOLUTE VALUE GRAPHS

Let's now think of absolute value as function. That is, consider the function given by the formula:

$$y = |x|$$

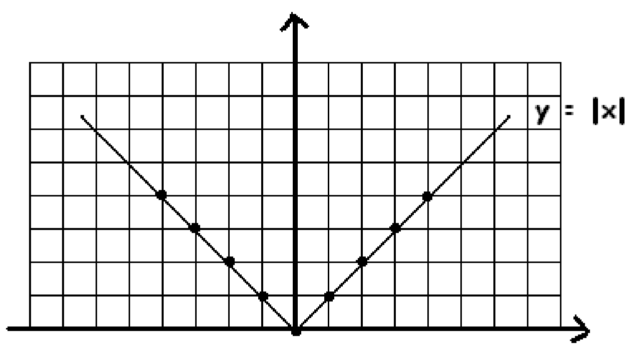
When $x = 3$, we have $y = 3$.
When $x = -67$, we have $y = 67$.
When $x = \pi^2 - 300$, we have $y = 300 - \pi^2$.
and so on.

What does the graph of the function $y = |x|$ look like?

NOTE: At any stage of sophistication in one's mathematical development there is absolutely no shame in drawing tables of values and plotting points. In fact, this is often the only thing one can do when presented with a new type of function for the first time.

Here's a table and the resultant graph:

x	-4	-3	-2	-1	0	1	2	3	4
y = |x|	4	3	2	1	0	1	2	3	4

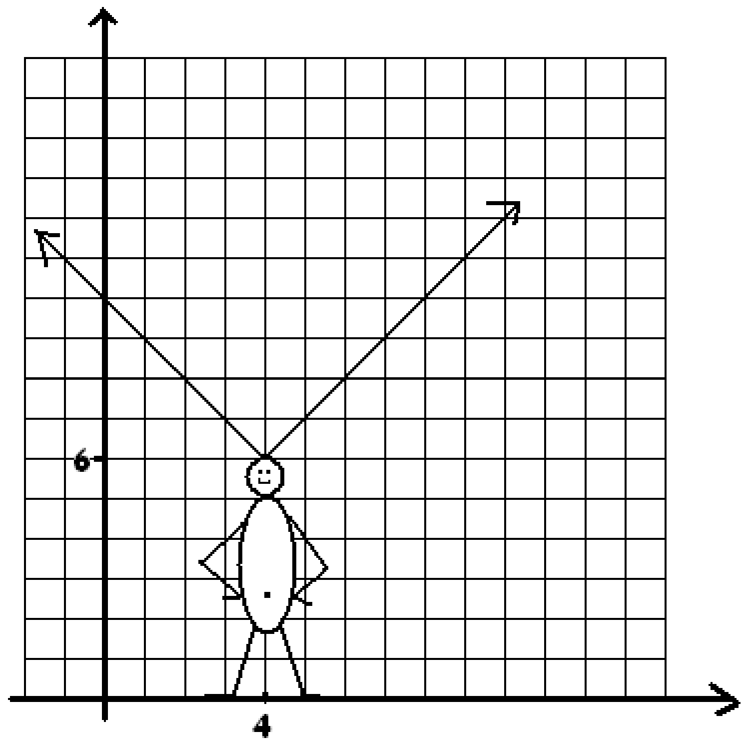


We see that the absolute value function produces a V-shaped graph with <u>vertex</u> at the origin.

Some people like to note that on the side of positive x-values, the graph is the same as the line $y = x$. On the side of the negative x-values, the graph is the line $y = -x$. (Does this make sense?)

Here's a challenge question.

I am six units tall and am standing at the position $x = 4$ on the horizontal axis. Is it possible to write down a formula for a function whose graph is the same V-shape but positioned so as to be balancing on my head?



This means we are looking for a function that gives a table of values that looks like:

x	0	1	2	3	4	5	6	7	8
f(x)	10	9	8	7	6	7	8	9	10

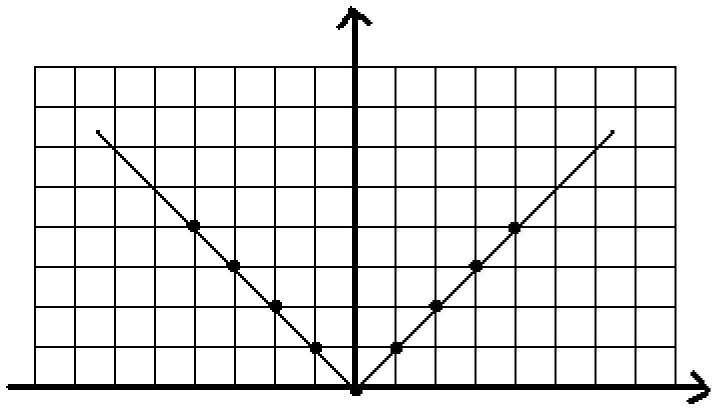
Play with this challenge for a while. What could you do to formula $y = |x|$ to make the V-shape appear with the vertex at $(4,6)$. Perhaps use pencil and paper to plot points for different formulas. Just try different things. If you don't have success, worry not! All will become clear a little later.

Move on to the next set of pages only after having given this challenge a good try.

BASIC GRAPH MANEUVERS:

THE GRAPH OF $y = |x - k|$

Here again if the graph of $y = |x|$:



It has table of values:

x	-4	-3	-2	-1	0	1	2	3	4
y = |x|	4	3	2	1	0	1	2	3	4

One thing to notice is that this graph has the "dip" of its V-shape at $x = 0$.

Now consider the function:

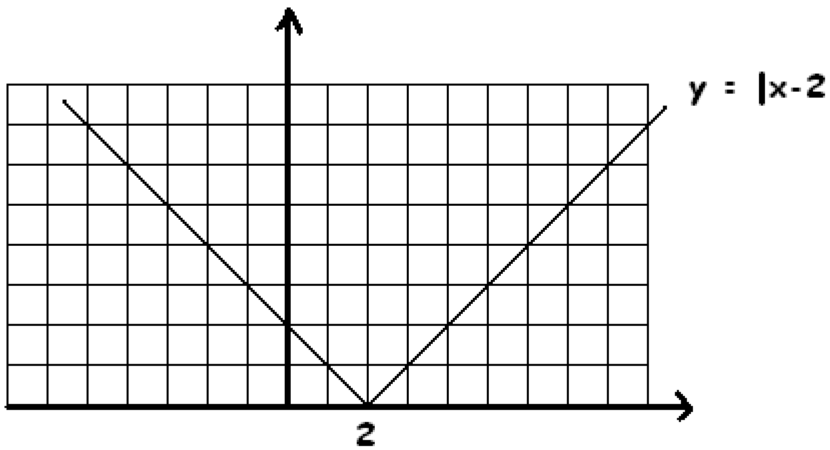
$$y = |x - 2|$$

Notice that when we put $x = 2$ into this formula we obtain the value $|0|$. That is, the number 2 is "behaving" just like $x = 0$ was for the original function.

In $|x - 2|$ we have that 2 is the "new zero" for the *x*-values.

So whatever the original function was doing at $x = 0$, it is now doing it at $x = 2$.

The original function "dips" at $x = 0$ so the graph of the function now dips at $x = 2$.



The entire graph has been shifted horizontally.

CHECK THIS: We should plot points to make sure we are correct:

x	-4	-3	-2	-1	0	1	2	3	4
y = |x-2|	6	5	4	3	2	1	0	1	2

Take to time to verify that the values in this table are correct.

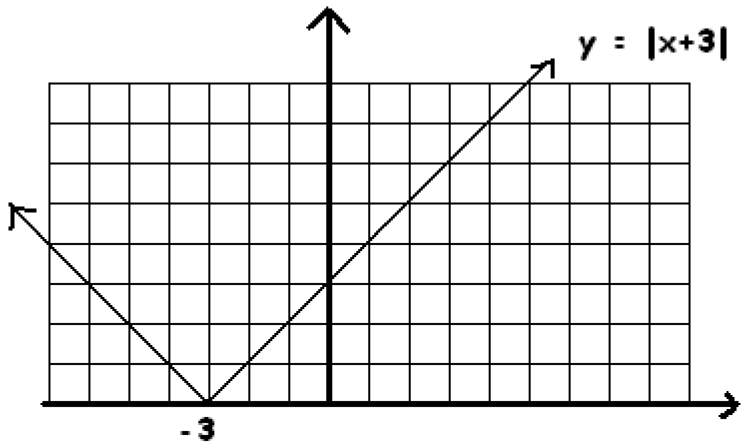
Now take the time to verify that the values in the table do indeed match the graph given at the top of this page.

This agrees with the picture at the top of this page.

EXAMPLE: Sketch a graph of $y = |x+3|$.

Answer: What value of *x* behaves like zero for $y = |x+3|$? Answer: $x = -3$ does!

So $y = |x+3|$ looks like $y = |x|$ but with $x = -3$ the new zero.



And we can check ourselves by computing points:

At $x = -4$ we have $|x+3| = |-1| = 1$.
At $x = -3$ we have $|x+3| = |0| = 0$.
At $x = -2$ we have $|x+3| = |1| = 1$.
At $x = 0$ we have $|x+3| = |3| = 3$.
At $x = 2$ we have $|x+3| = |5| = 5$.

EXERCISE 15:

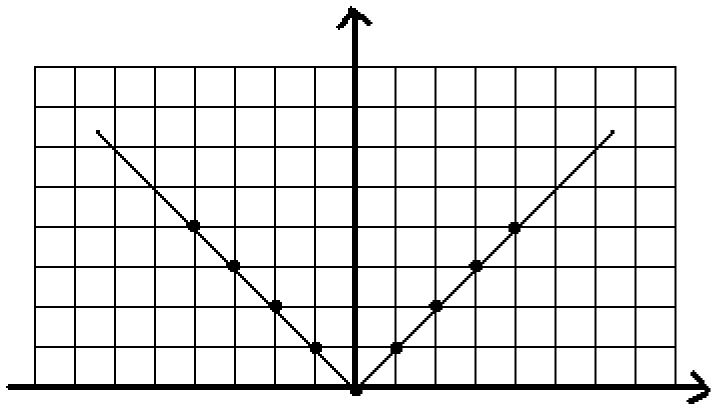
a) Sketch a graph of $y = |x-4|$

b) Sketch a graph of $y = |x + \frac{1}{2}|$

Ask yourself in each case "What value of *x* is the new zero of the x-values for the function?"

THE GRAPH OF $y = |x| \pm k$

Here again if the graph of $y = |x|$ and its table of values:



x	-4	-3	-2	-1	0	1	2	3	4
y = |x|	4	3	2	1	0	1	2	3	4

How would the graph of $y = |x| + 2$ appear?

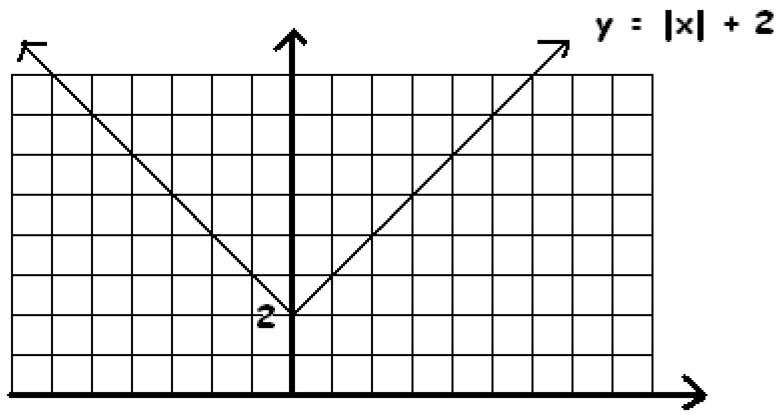
Notice that this new function is adding two units to each output of $y = |x|$.

At $x = 1$ $|1| = 1$ and $|1| + 2 = 1 + 2 = 3$
At $x = -3$ $|-3| = 3$ and $|-3| + 2 = 3 + 2 = 5$
At $x = 9$ $|9| = 9$ and $|9| + 2 = 9 + 2 = 11$

We have:

x	-4	-3	-2	-1	0	1	2	3	4
y = |x| + 2	6	5	4	3	2	3	4	5	6

This has the effect of raising the entire graphs two units in the vertical direction.



The graph of the function $y = |x| - 5$ would be the same graph shifted downwards 5 units, and the graph of $y = |x| + \sqrt{3}$ would the graph shifted upwards $\sqrt{3}$ units.

EXERCISE 16:

a) Sketch a graph of $y = |x| + 1$
b) Sketch a graph of $y = |x| - 1$
c) Sketch a graph of $y = |x| + \frac{5}{2}$
d) Sketch a graph of $y = |x| - 10$

e) Where does the graph $y = |x| - 202$ cross the *x*-axis?

HORIZONTAL AND VERTICAL SHIFTS TOGETHER

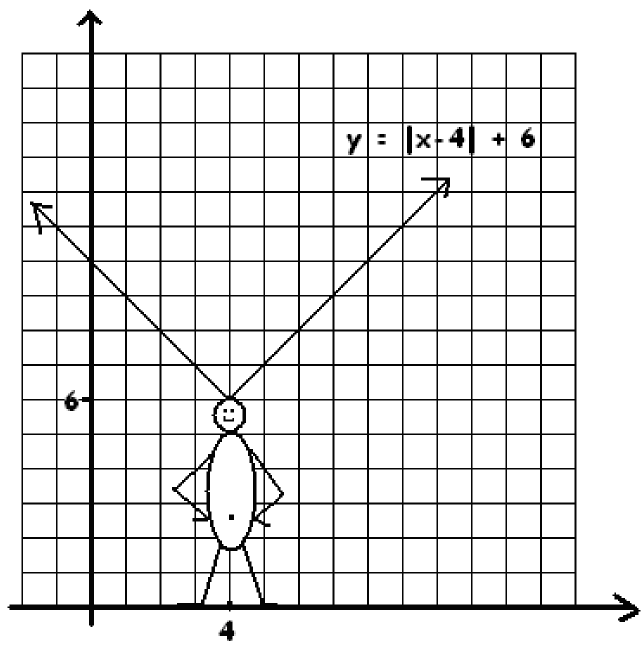
Let's return to the challenge of page 3. We need to take the graph of $y = |x|$ and shift it horizontally and vertically so that it sits balanced at the point $(4,6)$.

This requires making $x = 4$ the new zero for the *x*-values and shifting the graph 6 units upwards.

Thus:

$$y = |x - 4| + 6$$

does the trick.



EXERCISE 17:

a) Complete the following table of values for the function $y = |x - 4| + 6$.

x	-4	-3	-2	-1	0	1	2	3	4
y = |x-4| + 6									

b) Take the time to check that your values in the table do indeed match the points on the graph balancing on my head.

EXERCISE 18: Make quick sketches of the graphs of the following functions:

a) $y = |x - 3| + 1$

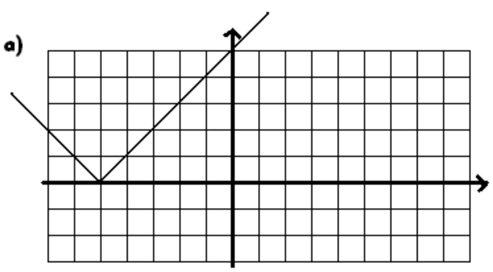
b) $y = |x + 2| + 4$

c) $y = |x + 5| - 3$

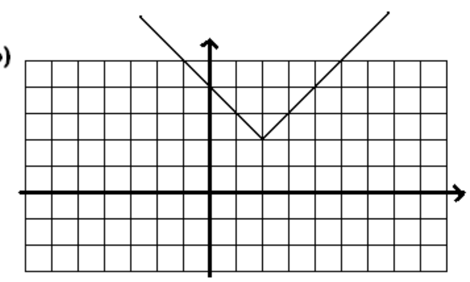
d) $y - 4 = |x + 6|$

EXERCISE 19: Write down an equation for each of these graphs:

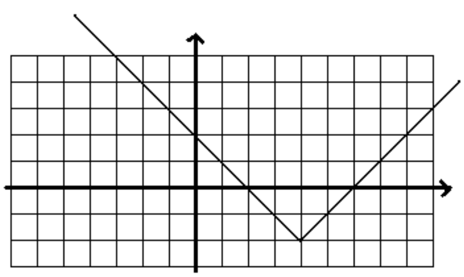
a)



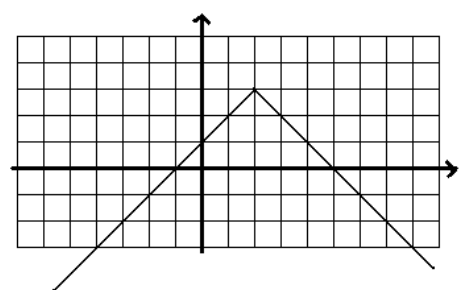
b)



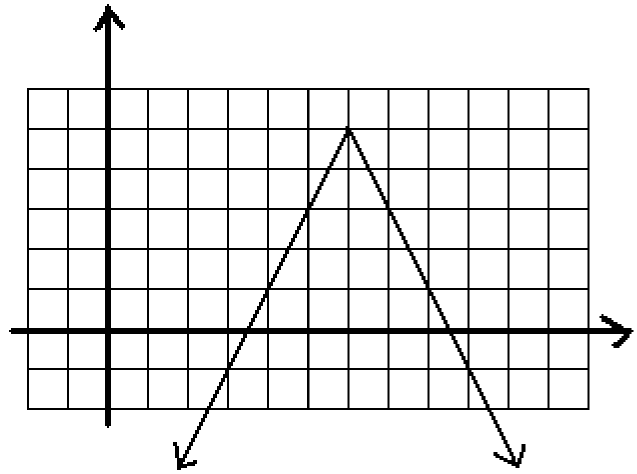
c)



d)

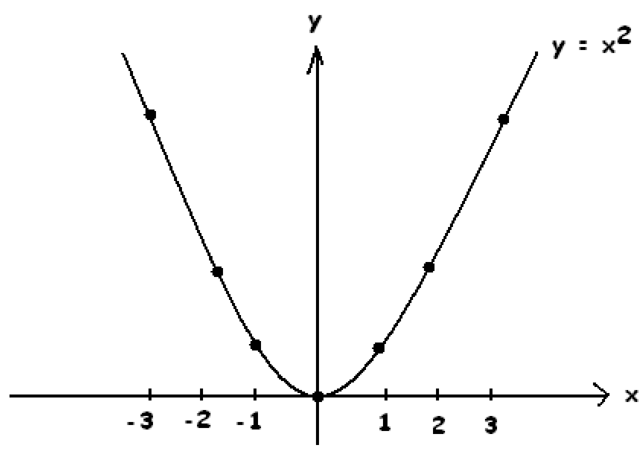


EXERCISE 20: CHALLENGE!
Write down an equation for this graph. Test whether or not you are correct by examining a table of values.



EXERCISE 21: CHALLENGE!
The graph of $y = x^2$ is an upward-facing U-shape with vertex at $(0,0)$ as follows:

x	0	1	2	3	-1	-2	-3
y	0	1	4	9	1	4	9



a) Write down the formula for the same parabola, but shifted so that its vertex is now at $(3,4)$.

b) Make a quick sketch of the graph of $y = (x-18)^2 - 40$.

COMMENT: See

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for easy ways to sketch more complicated absolute value equations (if this sort of thing floats your boat!) such as:

$$y = \frac{2}{3}|7x - 5| + 4 \quad \text{and} \quad y = \frac{5 - |17 + 2x|}{6}$$

One never has to work too hard if one has clarity of mind and a firm understanding of matters at hand.