Chapter 4A - Introduction to Functions

Definition of Function

In almost every aspect of our lives, we find examples of situations where one quantity depends on another. For example, the wind chill depends on the speed of the wind; the area of a circle depends on its radius; and the amount earned by your investment depends on the interest rate.

Definitions:

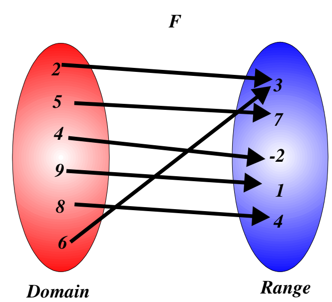
- A **function** is a relation in which no two different ordered pairs have the same first element.

$$F = \{(2,3),\ (5,7),\ (4,-2),\ (9,1),\ (8,4),\ (6,3)\}$$

- A **function** can also be defined as a rule that assigns exactly one element in set B to each element in set A.
- Set A, the set of all first coordinates, is called the domain of the function.
- Set B, the set of all second coordinates, is called the range.

In the function F above, the domain is the set of first elements: $\{2,5,4,9,8,6\}$ and the range is the set of second elements: $\{3,7,-2,1,4\}$.

The function F is depicted pictorially below:



The characteristic that distinguishes a function from any other relation is that there is only one output (y-value) for each input (x-value).

Example 1: Which of the following relations are functions?

a) $\{(2,3),(-1,3),(5,3),(8,3),(0,3)\}$

b)

x	y
-3	4
-2	3
-1	2
0	1
1	0

c) $\{(1,5),(2,7)(3,9),(1,4)\}$

Solution:

a) Function. Although each output is the same, 3, there is only one output for each input so the relation is a function.

b) Function. Each x is paired with only one y.

c) Not a function. 1 is paired with two outputs: 5 and 4. We have two ordered pairs with the same first element.

Example 2: Many functions are programmed into your calculator. Use the $\sqrt{\ }$ function to find the outputs for the following inputs.

Input	$\sqrt{\ }$	Output
25		
9		
0		
$\frac{1}{4}$		
-4		

a) What output did you get for -4?

b) List 4 numbers that are in the domain of $y = \sqrt{x}$ and 4 numbers in its range.

Solution:

Input	$\sqrt{\ }$	Output
25	$\sqrt{25}$	5
9	$\sqrt{9}$	3
0	$\sqrt{0}$	0
$\frac{1}{4}$	$\sqrt{\frac{1}{4}}$	$\frac{1}{2}$
-4	$\sqrt{-4}$	*ERROR*

a) ERROR on your calculator means that $\sqrt{-4}$ is not a real number. Therefore, -4 is not in the domain of the function.

b) Answers will vary. Domain will be numbers you key into your calculator (input). Range will be the output numbers that the calculator provides on your screen.

Example 3: Which of the following represent functions? Determine the domain and range of each set.

a) $\{(2,5), (-1,6), (5,7), (8,5), (2,4)\}$

b) $\{(95,A), (82,B), (92,A), (62,D), (74,C), (85,B), (58,F), (77,C)\}$

c)

x	y
-1	5
0	3
1	-1
2	-3
3	-5

d)

Input	6	2	1	3	6
Output	7	9	4	7	5

Solution:

a) Not a function. 2 is paired with both 5 and 4. Domain: $\{2,-1,5,8\}$ Range: $\{5,6,7,4\}$

b) Function. Domain: $\{95,82,92,62,74,85,58,77\}$ Range: $\{A,B,C,D,F\}$

c) Function. Domain: $\{-1,0,1,2,3\}$ Range: $\{5,3,-1,-3,-5\}$

d) Not a function. 6 is paired with both 7 and 5. Domain: $\{6,2,1,3\}$ Range: $\{7,9,4,5\}$

Function Notation

To communicate functional information more effectively, we often use the symbol $f(x)$ read "f of x" instead of y to give the output for the input x. That is, $y = f(x)$. The use of $f(x)$ instead of y allows to state that f is a function and it gives us the input value x as well.

Input x	Function Rule: $f(x) = 3x - 1$	Output $f(x)$	Ordered Pair $(x, f(x))$
5	$f(5) = 3(5) - 1$	14	$(5, 14)$
3	$f(3) = 3(3) - 1$	8	$(3, 8)$
0	$f(0) = 3(0) - 1$	-1	$(0, -1)$

$$\begin{array}{l} \quad\; \textit{input} = 5 \\ \qquad \downarrow \\ \text{We write that } \underbrace{f(5)} = 14, \text{ meaning that the output } (y) \text{ is 14 when the input } (x) \text{ is 5 .} \\ \qquad \uparrow \\ \quad\; \textit{output} = 14 \end{array}$$

Question: Which of the following are true about the statement $f(3) = 8$?

a) $f \cdot 3 = 8 \;\Rightarrow\; f = \dfrac{8}{3}$

b) 3 is the input.

c) $f(3)$ is the output.

d) 8 is the output.

e) $(3, 8)$ is an ordered pair of the function f.

Answer:

a) False. The symbol $f(x)$ does NOT mean multiply f *times* x.

b) True.

c) True.

d) True.

e)True.

Evaluating Functions

To evaluate the function f, we find the output for a given input. Consider the function $f(x) = 2x^2 - x + 1$. To evaluate $f(2)$, we plug $x = 2$ into f to find the functional value (y-value).

$$f(\ x\ ) = 2 \cdot x^2 - x + 1$$
$$\updownarrow \qquad \updownarrow \quad \updownarrow$$
$$f(\ 2\ ) = 2 \cdot 2^2 - 2 + 1 = 2(4) - 2 + 1 = 7 \quad \Rightarrow \quad f(2) = 7$$

Example 1: For $f(x) = 2x^2 - x + 1$, find a) $f(-3)$, b) $f(a+h)$, c) $f(-x)$.

Solution:

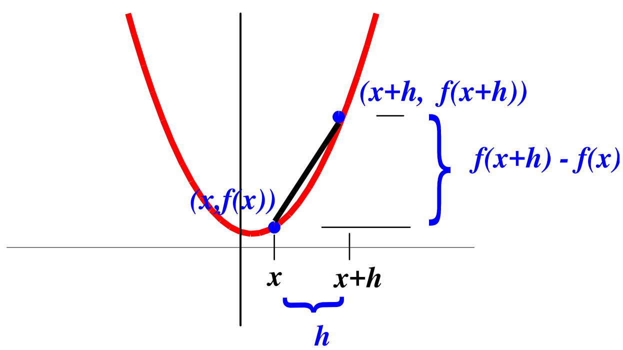
a) Replace x with -3: $f(-3) = 2(-3)^2 - (-3) + 1 = 22$

b) Replace x with the quantity $(a+h)$:

$$f(a+h) = 2(a+h)^2 - (a+h) + 1 = 2(a^2 + 2ah + h^2) - (a+h) + 1$$
$$= 2a^2 + 4ah + 2h^2 - a - h + 1$$

c) $f(-x) = 2(-x)^2 - (-x) + 1 = 2x^2 + x + 1$

An important expression in mathematics is the **difference quotient**. Graphically, the difference quotient is the slope of the line that goes through two particular points on the graph of a function. For example, the slope of the black line that connects the points $(x, f(x))$ and $(x+h, f(x+h))$ for the graph of $f(x) = 2x^2 - x + 1$ is shown below. Note that the **slope** of the line $m = \dfrac{f(x+h) - f(x)}{h}$ is the **difference quotient**.



Algebraically, the difference quotient is the average rate of change of the y-values with respect to their x-values. : $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$.

Although the mathematical expression may seem a bit complicated at first, it is just a set of instructions written in mathematical symbols. To evaluate the difference quotient for a given function just follow the directions given in the formula:

1) Evaluate $f(x+h)$ for the function. You did this easily above.
2) Subtract the original function $f(x)$ from your result. Remember to distribute the "$-$" sign.
3) Divide the result by h and simplify.

Example 2: Evaluate $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$ for the above function $f(x) = 2x^2 - x + 1$.

Solution:

$$\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^2 - (x+h) + 1] - [2x^2 - x + 1]}{h}$$
$$= \frac{[2x^2 + 4xh + 2h^2 - x - h + 1] - [2x^2 - x + 1]}{h}$$
$$= \frac{2x^2 + 4xh + 2h^2 - x - h + 1 - 2x^2 + x - 1}{h}$$
$$= \frac{4xh + 2h^2 - h}{h} = 4x + 2h - 1$$

IMPORTANT NOTE: $f(x+h) \neq 2x^2 - x + 1 + h$.

To avoid this mistake, write the function with ( ) in place of x. Then write the input expression inside the ( ).

For example, to find $f(x+h)$ for the function $f(x) = 3x^4 - 5x^2 + 6$:write

$f(x+h) = 3(\quad)^4 - 5(\quad)^2 + 6$.

Fill in the parentheses with the input $(x+h)$ to get $f(x+h) = 3(x+h)^4 - 5(x+h)^2 + 6$.

Example 3: Evaluate $f(x) = 2x + 5$ for each of the following:

a) $f(-5)$ b) $f(-c)$ c) $f(a+h)$ d) $\frac{f(a+h)-f(a)}{h}$, $h \neq 0$

Solution:

a) $f(-5) = 2(-5) + 5 = -5$

b) $f(-c) = 2(-c) + 5 = -2c + 5$

c) $f(a+h) = 2(a+h) + 5 = 2a + 2h + 5$

d) $\frac{f(a+h)-f(a)}{h} = \frac{(2(a+h)+5) - (2a+5)}{h} = \frac{2a + 2h + 5 - 2a - 5}{h} = \frac{2h}{h} = 2$

Example 4: Evaluate the difference quotient $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$ for each of the following:

a) $f(x) = -x + 3$ b) $f(x) = 2x + 5$ c) $f(x) = x^2 + x - 4$ d) $f(x) = -x^2 - x - 1$

Solution:

a) $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$

$$\frac{[-(x+h)+3] - [-x+3]}{h} = \frac{-x - h + 3 + x - 3}{h} = \frac{-h}{h} = -1$$

b) $$\frac{[2(x+h)+5] - [2x+5]}{h} = \frac{[2x + 2h + 5] - [2x+5]}{h} = \frac{2h}{h} = 2$$

c) $$\frac{\left[(x+h)^2 + (x+h) - 4\right] - [x^2 + x - 4]}{h} = \frac{x^2 + 2xh + h^2 + x + h - 4 - x^2 - x + 4}{h} =$$
$$\frac{2xh + h^2 + h}{h} = 2x + h + 1$$

d) $$\frac{\left[-(x+h)^2 - (x+h) - 1\right] - [-x^2 - x - 1]}{h} = \frac{-(x^2 + 2xh + h^2) - x - h + 1 + x^2 + x + 1}{h} =$$
$$\frac{-x^2 - 2xh - h^2 - x - h + 1 + x^2 + x + 1}{h} = \frac{-2xh - h^2 - h}{h} = -2x - h - 1$$

Domain of a Function

Consider the function $f(x) = \sqrt{x}$. The table below gives some pairs (*input*, *output*). Notice that there is no real valued output for input $x = -4$. This is because there is no real number that can be squared to get -4. $(2)^2 = 4$ and $(-2)^2 = 4$

Input x	Output $f(x) = \sqrt{x}$
25	$\sqrt{25} = 5$
9	$\sqrt{9} = 3$
0	$\sqrt{0} = 0$
$\frac{1}{4}$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$
-4	$\sqrt{-4}$ = not a real number

Question: In general, what values of x will not yield a real number for $\sqrt{x}$?

Answer: x cannot be a negative number. Therefore, the domain of f consists of all real numbers that are greater than or equal to 0. We write the domain in interval notation as $[0, +\infty)$.

Example 1: Evaluate $f(x) = \dfrac{3}{x-1}$ for each of the following and determine which of the values of x are in the domain of f:

a) $f(-1)$ b) $f(0)$ c) $f(1)$ d) $f(2)$ e) $f(3)$

Solution:

a) $f(-1) = \dfrac{3}{-1-1} = -\dfrac{3}{2}$ -1 is in the domain of f.

b) $f(0) = \dfrac{3}{0-1} = -3$ 0 is in the domain of f.

c) $f(1) = \dfrac{3}{1-1} = \dfrac{3}{0}$ = *undefined*. 1 is NOT in the domain of f because $f(1)$ is not a real number.

d) $f(2) = \dfrac{3}{2-1} = 3$ 2 is in the domain of f.

e) $f(3) = \dfrac{3}{3-1} = \dfrac{3}{2}$ 3 is in the domain of f.

Questions:

- What value of x gave an output for f that was undefined and why?
- Are there any other values for x that would not be in the domain of f?
- What is the domain of f?

Answers:

- f was undefined when $x = 1$ because it caused the denominator of the fraction to be $1 - 1 = 0$.
- No. f is defined for all other values for x.
- The domain is all real numbers except 1. We can write the domain in interval notation as $(-\infty, 1) \cup (1, +\infty)$. That is, the domain is the union $\cup$ of two different intervals $(-\infty, 1)$ and $(1, +\infty)$, neither of which include the value 1. Notice the ( ) which denote that 1 is not included in the set. Be sure you can write domains and ranges using interval notation.

The domain of most algebraic functions is the set of all real numbers EXCEPT:

1. any real numbers that cause the denominator of a fraction to be 0.
2. any real numbers that cause us to take the square root (or any radical with an even index) of a negative number.

There are other exceptions too, but these two are the ones we will encounter most often.

Example 2: Find the domain of each function:

a) $f(x) = \dfrac{2}{x^2 - 4}$ b) $g(x) = \sqrt{2x - 3}$

Solution:

a) The function f is a fraction so it is not defined when its denominator is 0. Since $f(x) = \dfrac{2}{x^2 - 4} = \dfrac{2}{(x+2)(x-2)}$, f is undefined when $x = 2$ *or* -2.

Therefore, Domain_f = all real numbers except 2 and –2.

In interval notation we write: $Domain_f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

b) Since g is defined using a square root, we must eliminate all values of x that cause the radicand (the number or expression under the radical) to be a negative number. Therefore, $2x + 3$ must be greater than or equal to 0.

Solving for x: $2x + 3 \geq 0 \Rightarrow 2x \geq -3 \Rightarrow x \geq \dfrac{-3}{2}$

$Domain_g = \left[\dfrac{-3}{2}, \infty\right)$

Example 3: Find the domain of each function:

a) $f(x) = x^2 - 5x + 6$ b) $g(x) = \sqrt{x^2 - 5x + 6}$

Solution:

a) Since f is not a fraction and does not contain a radical, we do not have to eliminate any values for x in the domain. Therefore, the domain is all real numbers.

$Domain_f = (-\infty, +\infty)$

b) Since $x^2 - 5x + 6$ is under the radical, we must make sure that $x^2 - 5x + 6 \geq 0$. Factoring, we get $(x-2)(x-3) \geq 0 \Rightarrow (x-2)(x-3) = 0$ when $x = 2$ or $x = 3$.

We must determine the sign of g on the intervals on either side of the zeros of g.

Intervals	$x < 2$	$2 < x < 3$	$x > 3$
Test number	0	2.5	4
$x^2 - 5x + 6$	$(0-2)(0-3) = (-)(-) = +$	$(2.5-2)(2.5-3) = (+)(-) = -$	$(4-2)(4-3) = (+)(+) = +$

From the chart we see that $g(x)$ is negative when x is between 2 and 3; therefore,
$Domain_g = (-\infty, 2] \cup [3, +\infty)$

Example 4: Find the domain of $f(x) = \frac{2x-1}{3x+2}$.

Solution: Since f is a fraction we must eliminate from the domain all values of x that cause the denominator to be 0.

That is, $3x+2 \neq 0 \Rightarrow x \neq -\frac{2}{3}$. Therefore, the domain is all real numbers except $-\frac{2}{3}$.

$Domain_f = \left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$

Example 5: Find the domain of $f(x) = \frac{\sqrt{x+2}}{x-1}$.

Solution: Since f is a fraction we must eliminate from the domain all values of x that cause the denominator to be 0. Since f contains a radical, we must eliminate all values of x that cause the radicand to be negative. That is, $x-1 \neq 0 \Rightarrow x \neq 1$ AND $x+2 \geq 0 \Rightarrow x \geq -2$. Therefore, the domain is all real numbers greater or equal to -2 except 1. $Domain_f = [-2, 1) \cup (1, +\infty)$

Applying Functions

Algebra is a tool that can help us in many circumstances in our personal lives as well as our careers. Any time you enter a formula into a spreadsheet you are using algebra. In fact, many of the times you use algebraic thinking you are not aware that you are using it. In either instance, the ability to formulate quantitative problems mathematically is critical to use algebra as a problem solving tool. In this section we will begin the process by translating some situations into functions.

Example 1: The Student Government is considering ways to raise funds for a gift they wish to purchase for the school. One suggestion is to sponsor a Back to School Dance at the Brazos Center at the beginning of the semester. Rent for the Brazos Center is \$150, and the social chairman knows a deejay who will only charge \$100 for the event. A local caterer has agreed to provide the refreshments for \$2/ticket sold. Suppose the SG charges \$5 per ticket.

a) How many tickets must be sold to break even?

b) How much will they clear if they sell a ticket to 10% of the 7600 students enrolled at Blinn-Bryan?

c) How many tickets must be sold if they are to fully fund the gift which costs \$300?

d) Use your grapher to draw a plot of the function. Label the axes and include the number scale for each axis.

e) If you were on the executive committee of SG, how would you vote? Why?

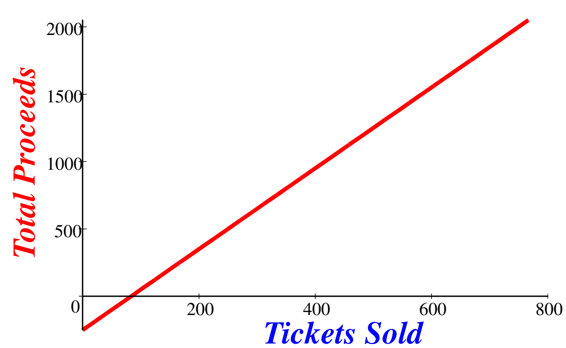
Solution: We will write a function to model the information we have: Let x = the number of tickets sold. Since the proceeds depend on the number of tickets sold, $p(x)$ = the proceeds if x tickets are sold. Since proceeds = revenue - costs,

$$p(x) = 5x - (2x + 250) = 3x - 250$$

a) To break even, $p(x) = 0 \Rightarrow 3x - 250 = 0 \Rightarrow 3x = 250 \Rightarrow x = \frac{250}{3} = 83\frac{1}{3} \Rightarrow$ 84 tickets must be sold.

b) If 10% of 7600 $=$ 760 students buy tickets, $x = 760 \Rightarrow p(760) = 3(760) - 250 = 2030$
Proceeds will be \$2030 if 10% of students buy tickets.

c) For proceeds of \$300, $300 = 3x - 250 \Rightarrow 3x = 550 \Rightarrow x = \frac{550}{3} = 183\frac{1}{3} \Rightarrow$ 184 tickets must be sold to buy gift.

d) 

e) Your opinion here...

Example 2: A. J. Ramirez, Inc. purchased a strip shopping center for $1.5 million. If the depreciation is $50,000 per year for 30 years, find the following:

a) Write a function for the current value of the shopping center.

b) Determine the domain.

c) Use your grapher to draw a plot of the function. Label the axes and include the x-scale and y-scale.

d) Find the value of the center in 20 years.

e) When will the center be worth $1 million?

Solution:

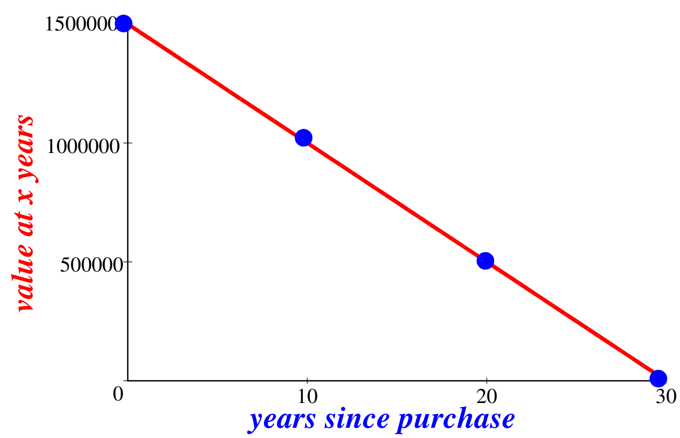
a) Let x = the number of years since the shopping center was purchased.

v = the current value of the center.

Then $v(x) = 1500000 - 50000x$.

b) We must deal with a non-negative number of years, so $x \geq 0$. But the building is completely depreciated in 30 years. Therefore, the domain is $[0, 30]$.

c)



d) $v(20) = 1500000 - 50000(20) = 500000 \Rightarrow$ The building is worth $500,000 in 20 years.

e) We are given $v(x) = 1,000,000$ and we must find the number of years x.

Solve for x: $\Rightarrow 1500000 - 50000x = 1000000 \Rightarrow -50000x = -500000 \Rightarrow x = 10$

The building will be worth $1 million in 10 years. Note that the graph supports the algebraic solution.

Remember the key to determining if an equation represents a function is whether there is one output for each input in the domain of the function. For example, equations of circles do not represent functions, but most lines do represent functions.

Question: What kind of line does **not** represent a function?

Answer: Vertical line.

In this section we'll look at some examples of geometric formulas which are functions.

Consider the formula for the area of a square $A = s^2$. The input is the length of the side of the square s and the output is the area of that particular square. Using function notation, we would write $A(s) = s^2$. Although the domain of function A is normally $(-\infty, +\infty)$, in this case we would restrict the domain to $[0, +\infty)$ because the function is an application of a geometric formula. The lengths of sides and areas of squares, as well as other geometric figures, are always non-negative.

Example 3: Write the perimeter of a square (the distance around the figure) in terms of the length of one of its sides. Find the domain of the resulting function.

Solution: $p(s) = 4s$ Domain: $[0, +\infty)$

Example 4: Use the above formulas involving squares to complete the table below:

p	20	64	100	84	68	10
A						

. Explain what you did to accomplish the task.

Solution:

p	20	64	100	84	68	10
A	25	256	625	441	289	6.25

You know that $p = 4s$, so you probably divided the perimeter by 4 to get the length of one side of the square. To find the area, you squared the length of the side, because $A = s^2$.

Example 5: We can even write a function to describe the process you used to complete the above table. That is, write A as a function of p.

Solution: Since A and p are both functions of s, we can eliminate the s by first solving $p = 4s$ for s: $s = \frac{p}{4}$. Substituting $\frac{p}{4}$ for s into $A = s^2$, we get $A = \left(\frac{p}{4}\right)^2$. In functional notation we write $A(p) = \frac{p^2}{16}$. Check this function out for each input in the table above. For example, $A(20) = \frac{(20)^2}{16} = 25$. Isn't this the result you obtained in the above example?

Example 6: Write the perimeter of a square p as a function of A. $[p(A)]$

Solution: That means, we want p to be the output and A to be the input. Start with the function you have for the output.

- Start with the function for p: $p = 4s$
- Solve $A = s^2$ for s: $s = \sqrt{A}$
- Replace the input s with its equivalent $\sqrt{A}$ into $p = 4s$: $p = 4\left(\sqrt{A}\right) \Rightarrow p(A) = 4\sqrt{A}$.

Example 7: Use the formulas for the circumference ($C = 2\pi r$) and area ($A = \pi r^2$) of a circle to write each of the following functions. Restrict their domain as necessary.

a) write the radius as a function of the circumference. $[r(C)]$

b) write the radius as a function of the area. $[r(A)]$

c) write the circumference as a function of the area. $[C(A)]$

d) write the area as a function of the circumference. $[A(C)]$

Solution:

a) Solve $C = 2\pi r$ for r: $r = \frac{C}{2\pi}$ $\Rightarrow$ in function notation: $r(C) = \frac{C}{2\pi}$ Domain: $[0, +\infty)$

b) Solve $A = \pi r^2$ for r: $r^2 = \frac{A}{\pi} \Rightarrow r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{A \cdot \pi}{\pi \cdot \pi}} = \frac{\sqrt{A\pi}}{\pi}$ (We only want the postive root.) Domain: $[0, +\infty)$

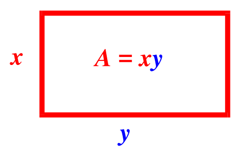
c) Start with $C = 2\pi r$. Solve $A = \pi r^2$ for r: $r = \frac{\sqrt{A\pi}}{\pi}$ (See results of b)

Substitute for r: $C = 2\pi\left(\frac{\sqrt{A\pi}}{\pi}\right) = 2\sqrt{A\pi} \Rightarrow C(A) = 2\sqrt{A\pi}$ Domain: $[0, +\infty)$

d) Start with $A = \pi r^2$. Solve $C = 2\pi r$ for r: $r = \frac{C}{2\pi}$ Substitute for r:

$A = \pi\left(\frac{C}{2\pi}\right)^2 = \pi\left(\frac{C^2}{4\pi^2}\right) = \frac{C^2}{4\pi} \Rightarrow A(C) = \frac{C^2}{4\pi}$ Domain: $[0, +\infty)$

Suppose we wanted to write the area of a rectangle as a function of the length of one of its sides.



Let $x =$ the number of units in the length of one side of the rectangle

and $\mathbf{y} =$ the number of units in the length of the other side of the rectangle

and $A =$ the number of square units in the area of the rectangle

We know that the relationship between the area and the length and width of the rectangle is $A = xy$ and the perimeter is $p = 2x + 2\mathbf{y}$.

We start with the formula for the area of the rectangle: $A = x\mathbf{y}$

We can then use the perimeter to write $\mathbf{y}$ in terms of x: $p = 2x + 2\mathbf{y} \Rightarrow \mathbf{y} = \frac{p - 2x}{2} = \frac{p}{2} - x$

Substitute into $A = x\mathbf{y}$: $A = x\left(\frac{p}{2} - x\right) \Rightarrow A(x) = \frac{p}{2}x - x^2$

In the following, we chose $p = 200$ so that $\mathbf{y} = \frac{200}{2} - x = 100 - x$ and $A = x(100 - x)$ or $A = 100x - x^2$. Consider A in factored form: $A(x) = x(100 - x)$. Fill in the table below:

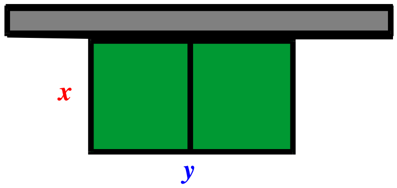
x	0	10	40	50	60	75	90	100
$y = 100 - x$	100	90	60	50	40	25	10	0
$A(x)$	0	900	2400	2500	2400	1875	900	0

Note that with a limited perimeter, if x is increasing, then y must be decreasing. The area will be 0 when $x = 0$ or $y = 0$.

Question: What is the domain of $A(x) = 100x - x^2$?

Answer: Since x represents the length of a side of a rectangle, x cannot be a negative number, so $x \geq 0$. But if $x > 100$, then y will be a negative number. Therefore, the domain is $[0, 100]$.

Example 8: A farmer wishes to fence a rectangular area adjacent to a straight brick wall so he doesn't need to fence along the wall. He also wants to divide the fenced area into two smaller areas as shown below. If he has a total of 4000 feet of fencing material, write a function for the total fenced area in terms of the length of one of the sides.



Solution: Let x = the number of feet of one side of the rectangle, y = the number of feet of the other side, and A = the number of square feet in the area. Then $A = xy$. Since 4000 feet of fencing $\Rightarrow 4000 = y + 3x \Rightarrow y = 4000 - 3x$. We can substitute $y = 4000 - 3x$ for y:
$A = x(4000 - 3x) \Rightarrow A(x) = 4000x - 3x^2$.

Question: What is the domain of $A(x) = x(4000 - 3x)$?

Solution: If $x = 0$, $A(0) = 0$. If $y = 0$, then $4000 - 3x = 0 \Rightarrow x = \frac{4000}{3} = 1333\frac{1}{3}$

Domain: $[0, 1333\frac{1}{3}]$

Exercises for Chapter 4A - Introduction to Functions

1. Determine whether each of the following represent functions?

a) $\{(-2,7),(-1,6),(0,5),(1,5),(2,3),(3,2),(4,1)\}$

b) $\{(3,6),(-1,5),(5,1),(-1,2),(6,5)\}$

c)

x	y
1	–2
3	5
5	5
7	6
9	–2
0	8

d)

x	y
1	–9
2	–8
4	–6
7	–6
7	–5
9	–3

e)

x	y
–2	–4
–1	–2
0	0
1	2
2	4
3	6

f)

x	y
–2	3
–1	3
0	3
1	3
2	3
3	3

g)

x	y
–2	–2
–2	–1
–2	0
–2	1
–2	2
–2	3

2. Write the domain and range for each set in #1.

3. Does there appear to be a rule that defines any of the relations listed in exercise #1?

4. a) Is the following a function?

x	y
Joe	Schnauser
Ali	Collie
Lisa	Doberman Pinscher
Mary	German Shephard
Selena	Lhasa Apso
Juan	Labrador Retriever

b) Write a rule that describes the function.

c) What is the domain and range?

5. If $f(2) = 5$, then the input is _____ and the output is _____.

6. If $(4,9)$ is an ordered pair in function f, then $f($___$) = $_____.

7. If the output is -2 for input 8 in function g, then $g($____$) = $______.

8. Use function notation to describe each pair in the following functions:

a) function f: $\{(-2,7),(-1,6),(0,5),(1,5),(2,3),(3,2),(4,1)\}$

b) function g :

x	y
–2	–4
–1	–2
0	0
1	2
2	4
3	6

c) function: *dog*:

x	y
Joe	Schnauser
Ali	Collie
Lisa	Doberman Pinscher
Mary	German Shephard
Selena	Lhasa Apso
Juan	Labrador Retriever

9. Complete the table for each function below:

input	output
−3	
−2	
−1	
0	
1	
2	

a) $f(x) = x^2 - x + 1$
b) $g(x) = x^3 + 2$
c) $h(x) = -x + 3$

10. For $f(x) = 4x - 3$, evaluate each of the following:
a) $f(2)$
b) $f(-4)$
c) $f(a)$
d) $f(2a)$
e) $f(a + h)$
f) $f(a + h) - f(a)$
g) $\dfrac{f(a + h) - f(a)}{h}$, $h \neq 0$

11. For $f(x) = -x^2 - x$
a) $f(2)$
b) $f(-4)$
c) $f(a)$
d) $f(2a)$
e) $f(x + h)$
f) $f(x + h) - f(x)$
g) $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$

12. For $f(x) = 3x^2 - 2x - 1$
a) $f(2)$
b) $f(-4)$
c) $f(a)$
d) $f(2a)$
e) $f(a + h)$
f) $f(a + h) - f(a)$
g) $\dfrac{f(a + h) - f(a)}{h}$, $h \neq 0$

13. Determine which of the following functions have a domain $(-\infty, +\infty)$.
a) $f(x) = x^4 - 3x + 1$
b) $g(x) = \dfrac{1}{x - 3}$
c) $p(x) = \sqrt{2x - 3}$
d) $h(x) = \sqrt[3]{x + 1}$

14. Find the domain of each of the following functions.
a) $f(x) = 3x - 1$
b) $f(x) = \dfrac{x}{x^2 - 1}$

c) $f(x) = \sqrt{x-2}$

d) $g(x) = \dfrac{2x-3}{x^2+1}$

e) $h(x) = \sqrt[5]{x-5}$

f) $g(x) = \dfrac{\sqrt{x}}{x-1}$

g) $f(x) = \sqrt{x^2-9}$

15. Ajax Car Rental charges a daily rate of \$10 plus \$0.10 per mile.

a. Write the daily charge as a function of the number of miles.

b. Find the daily charge if 200 miles are driven.

c. If Comet Car Rental charges a flat rate of \$35 per day, which car company would be the better choice if you were driving 300 miles?

d. Use your grapher to draw a plot of both functions on the same graph. Does the graph confirm your findings?

e. Interpret the graph.What information do you get from the point $(250, 35)$?

16. Write the surface area of a cube as a function of its edge. $[A(e)]$

17. Write the surface area of a cube as a function of its volume. $[A(V)]$

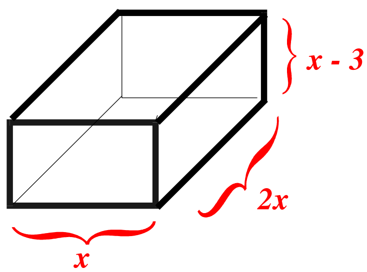
18. If $G = 3x$ and $H = x^3$, write a) $G(H)$ and b)$H(G)$.

19. If x is the side of an equilateral triangle, write the area as a function of x.

20. If x and y are the lengths of the adjacent sides of a rectangle whose perimeter is 40, write y as a function of x.

21. If x and y are the lengths of the adjacent sides of a rectangle whose perimeter is p, write y as a function of x.

22. a) Write the surface area of the box below as a function of x:



b) What is an algebraic expression for the volume of the box?

23. An open box is constructed from a 12-inch square piece of cardboard by cutting squares of equal length from each corner and turning up the sides. Write a function for the volume of the cardboard box in terms of the length of the cutout square. Restrict the domain appropriately. Use your grapher to plot the graph. Record the results. Label each axis and include the viewing window and scales. Note that the $v(x)$ is positive when $x \geq 6$. Explain why these values are not in the domain of $v(x)$.

Answers to Exercises for Chapter 4A - Introduction to Functions

1. a) function. no ordered pairs have same first coordinate
b) relation only. -1 is paired with both 5 and 2.
c) function. No x is paired with more than 1 y.
d) Relation only. -7 is paired with -6 and -5.
e) Function. No two outputs have the same input.
f) functions. No input has two outputs.
g) relation only. -2 is paired with many different outputs.

2. a) domain: $\{-2,-1,0,1,2,3,4\}$ range: $\{1,2,3,5,6,7\}$
b) domain: $\{-1,3,5,6\}$ range: $\{1,2,5,6\}$
c) domain: $\{1,3,5,7,9,0\}$ range: $\{-2,5,6,8\}$
d) domain: $\{1,2,4,7,9\}$ range: $\{-9,-8,-6,-5,-3\}$
e) domain: $\{-2,-1,0,1,2,3\}$ range: $\{-4,-2,0,2,4,6\}$
f) domain: $\{-2,-1,0,1,2,3\}$ range: $\{3\}$
g) domain: $\{-2\}$ range: $\{-2,-1,0,1,2,3\}$

3. a) no particular rule
b) no particular rule
c) no particular rule
d) no particular rule
e) In each pair, the output is twice the inpur: $y = 2x$
f) In each case, the output is 3: $y = 3$
g) In each case, the input is -2 : $x = -2$

4. a) yes
b) rule: the breed of dog each has as a pet or the person's favorite breed of dog, etc. We could write $y = dog(x)$. That is, y is the type of dog x has.
c) domain: {Joe, Ali, Lisa, Mary, Selena, Juan}, range: {Schnauser, Collie, Doberman Pinscher, German Shephard, Lhasa Apso, Labrador Retreiver}

5. input 2; output 5

6. $f(4) = 9$

7. $g(8) = -2$

8. a) $f(-2) = 7,\ f(-1) = 6,\ f(0) = 5,\ f(1) = 5,\ f(2) = 3,\ f(3) = 2,\ f(4) = 1$
b) $g(-2) = 4,\ g(-1) = -2,\ g(0) = 0,\ g(1) = 2,\ g(2) = 4,\ g(3) = 6$
c) $dog(Joe) = Schnauser,\ dog(Ali) = Collie,\ dog(Lisa) = Doberman\ Pinscher,$
$dog(Mary) = German\ Shepherd,\ dog(Selena) = Lhasa Apso,$
$dog(Juan) = Labrador\ Retriever$

9. a)

input	output
–3	13
–2	7
–1	3
0	1
1	1
2	3

b)

input	output
–3	–25
–2	–6
–1	1
0	2
1	3
2	10

c)

input	output
–3	6
–2	7
–1	4
0	3
1	2
2	1

10. a) $f(2) = 4(2) - 3 = 5$

b) $f(-4) = 4(-4) - 3 = -19$

c) $f(a) = 4(a) - 3 = 4a - 3$

d) $f(2a) = 4(2a) - 3 = 8a - 3$

e) $f(a+h) = 4(a+h) - 3 = 4a + 4h - 3$

f) $[4(a+h) - 3] - [4a - 3] = 4a + 4h - 3 - 4a + 3 = 4h$

g) $\dfrac{[4(a+h) - 3] - [4a - 3]}{h} = \dfrac{4a + 4h - 3 - 4a + 3}{h} = \dfrac{4h}{h} = 4$

11. For $f(x) = -x^2 - x$

a) $f(2) = -(2)^2 - 2 = -6$

b) $f(-4) = -(-4)^2 - (-4) = -12$

c) $f(a) = -(a)^2 - a = -a^2 - a$

d) $f(2a) = -(2a)^2 - (2a) = -(2a)^2 - 2a = -4a^2 - 2a$

e) $f(x+h) = -(x+h)^2 - (x+h) = -(x^2 + 2xh + h^2) - (x+h) = -x^2 - 2xh - h^2 - x - h$

f) $[-(x+h)^2 - (x+h)] - [-x^2 - x] = -x^2 - 2xh - h^2 - x - h + x^2 + x = -2xh - h^2 - h$

g) $\dfrac{\left[-(x+h)^2 - (x+h)\right] - [-x^2 - x]}{h} = \dfrac{-x^2 - 2xh - h^2 - x - h + x^2 + x}{h}$

$= \dfrac{-2xh - h^2 - h}{h} = -2x - h - 1$

12. For $f(x) = 3x^2 - 2x - 1$

a) $f(2) = 3(2)^2 - 2(2) - 1 = 7$

b) $f(-4) = 3(-4)^2 - 2(-4) - 1 = 55$

c) $f(a) = 3(a)^2 - 2(a) - 1 = 3a^2 - 2a - 1$

d) $f(2a) = 3(2a)^2 - 2(2a) - 1 = 12a^2 - 4a - 1$

e) $f(a+h) = 3(a+h)^2 - 2(a+h) - 1 = 3(a+h)^2 - 2(a+h) - 1 = 3a^2 + 6ah + 3h^2 - 2a - 2h - 1$

f) $f(a+h) - f(a) = \left[3(a+h)^2 - 2(a+h) - 1\right] - [3a^2 - 2a - 1]$

$= [3a^2 + 6ah + 3h^2 - 2a - 2h - 1] - [3a^2 - 2a - 1]$

$= 3a^2 + 6ah + 3h^2 - 2a - 2h - 1 - 3a^2 + 2a + 1 = 6ah + 3h^2 - 2h$

g) $\dfrac{f(a+h) - f(a)}{h} = \dfrac{6ah + 3h^2 - 2h}{h} = 6a + 3h - 2$ (see results of part f if necessary)

13. a and d have neither fractions or even-indexed radicals.

14. a) $(-\infty, +\infty)$

b) $f(x) = \dfrac{x}{x^2 - 1} = \dfrac{x}{(x-1)(x+1)}$. Set denominator equal to 0. $(x-1)(x+1) = 0$ when $x = -1$ and $x = 1$. Therefore we must eliminate those values from the domain of f. $Dom_f = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

c) Since $f(x) = \sqrt{x-2}$ is defined by a square root, the radicand must be greater than or equal to 0. Set $x - 2 \geq 0 \Rightarrow x \geq 2$. Therefore, $Dom_f = [2, +\infty)$.

d) Since g is a fraction, we must check the denominator: $x^2 + 1 = 0 \Rightarrow x^2 = -1$ which is not possible. Therefore the denominator is never 0 so that $Dom_g = (-\infty, +\infty)$.

e) Since $h(x) = \sqrt[5]{x-5}$ is a radical with odd index, it doesn't matter if the radicand is negative $\Rightarrow$ $Dom_h = (-\infty, +\infty)$

f) Since g involves both a fraction and a radical, we must leave out all values for x for which the denominator is 0 and the radicand is negative. Radicand: $x \geq 0$ Denominator: $x - 1 = 0 \Rightarrow x = 1 \Rightarrow 1$ cannot be in the domain. The domain is all $x \geq 0$ except 1. $dom_g = [0,1) \cup (1, +\infty)$

g) Radical: $x^2 - 9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow |x| \geq 3 \Rightarrow x \leq -3$ or $x \geq 3$. $Dom_f = [-3, 3]$

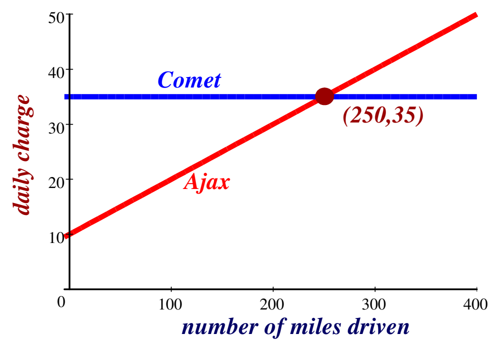
15. Ajax Car Rental charges a daily rate of $10 plus $0.10 per mile.

a. If x = number of miles driven in one day, c = daily charge $c(x) = 0.10x + 10$

b. If 200 miles are driven, $x = 200$, $c(200) = .10(200) + 10 = 30$. The charge for the day is $30.

c. Ajax: If x = number of miles driven in one day, c = daily charge $c(x) = 0.10x + 10$

Comet: Daily charge is $35. The charge for Ajax would be $40 so Comet would be the better choice.



d. The costs for each company are the same for 250 miles. For less than 250 miles, Ajax is the better choice; for more than 250 miles, Comet is the better rate.

16. Surface area of a cube: $A = 6e^2$. Solve for e: $\frac{A}{6} = e^2 \Rightarrow e = \sqrt{\frac{A}{6}} = \sqrt{\frac{A \cdot 6}{6 \cdot 6}} = \frac{\sqrt{6A}}{6}$.

Thus, $e(A) = \frac{\sqrt{6A}}{6}$

17. Start with surface area: $A = 6e^2$. Solve $V = e^3$ for e: $e = \sqrt[3]{V}$. Substitute into A: $A(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{\frac{2}{3}}$

18. a) Start with G: $G = 3x$. Solve $H = x^3$ for x: $x = \sqrt[3]{H}$. Substitute into $G = 3x$: $G = 3\left(\sqrt[3]{H}\right)$. Therfore, $G(H) = 3\sqrt[3]{H} = 3H^{\frac{1}{3}}$

b) Start with H: $H = x^3$. Solve G for x: $G = 3x \Rightarrow x = \frac{G}{3}$. Substitute into H:

$H = \left(\frac{G}{3}\right)^3$. Therefore, $H(G) = \left(\frac{G}{3}\right)^3 = \frac{G^3}{27}$

19. The area of a triangle is $A = \frac{1}{2}bh$, where b is the length of the base (one side of the triangle) and h is the perpendicular distance from the remaining vertex to the base.

Here, we have a equililateral triangle, so we know the base is x. To find the height, we draw the perpendicular distance and notice that it divides the triangle into two equal right triangles each have a base of $\frac{1}{2}x$, a height of h, and a hypotenuse of x. We can now use Pythagorean's Theorem to find h:

$(\frac{1}{2}x)^2 + h^2 = x^2 \Rightarrow h^2 = x^2 - \frac{1}{4}x^2 \Rightarrow h^2 = \frac{3}{4}x^2 \Rightarrow h = \frac{\sqrt{3}}{2}x$

Therefore, the area of the triangle is: $A(x) = \frac{1}{2}x(\frac{\sqrt{3}}{2}x) = \frac{\sqrt{3}}{4}x^2$.

20. If $p = 2x + 2y$ and $p = 40$, then $40 = 2x + 2y \Rightarrow 2y = 40 - 2x \Rightarrow y = 20 - x$ so one side is x and the other side is $20 - x$.

21. If $p = 2x + 2y$, then $2y = p - 2x \Rightarrow y = \frac{p - 2x}{2} = \frac{p}{2} - x$ so one side is x and the other side is $\frac{p}{2} - x$.

22. The surface area consists of the sum of the areas of the sides of the box:

Area of top and bottom are each: $x(2x) = 2x^2$

Area of front and back are each: $x(x-3)$

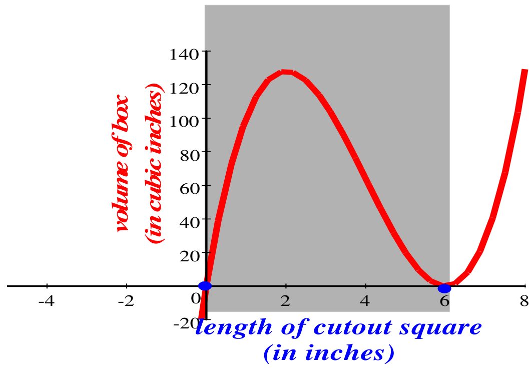
Area of left and right side are each: $2x(x-3)$

The total surface area A :

$A = 2(2x^2) + 2(x^2 - 3x) + 2(2x^2 - 6x) = 4x^2 + 2x^2 - 6x + 4x^2 - 12x = 10x^2 - 18xa$

b) The volume $V = lwh$. For the given box: $V(x) = (x)(2x)(x-3) \Rightarrow V(x) = 2x^3 - 6x^2$

23. Let x = number of inches in the length of side of cutout square, V = number of cubic inches in the volume of the box. $V(x) = x(12-2x)^2$, Domain: $[0,6]$. Only include the part of the



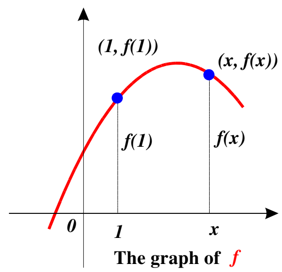
graph in the gray area.

Chapter 4B - Graphs of Functions

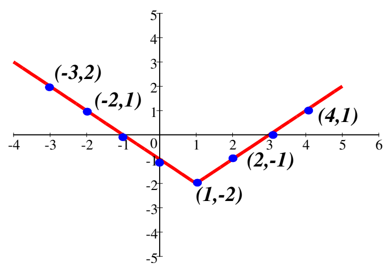
Graphs of Functions

We can represent functions graphically just as we represent any equation by plotting all pairs $(x, f(x))$ using the rectangular coordinate system. That is, we plot the pairs $(input, output)$.

Definition: If f is a function with domain A, the **graph of** f is the set of ordered pairs $(x, f(x))$, where $x \in A$. That is, $y = f(x)$.



Consider the graph of function f below.



Example 1: Use the graph of function f to evaluate each of the following:

a) $f(0)$ b) $f(-3)$ c) $f(2)$ d) $f(1)$ e) $f(-1)$

Solution: a) -1. The y-coordinate of the point where $x = 0$ is -1. Therefore, $f(0) = -1$.
b) 2. The y-coordinate of the point where $x = -3$ is 2. Therefore, $f(-3) = 2$.
c) -1. The y-coordinate of the point where $x = 2$ is -1. Therefore, $f(2) = -1$.
d) -2. The y-coordinate of the point where $x = 1$ is -2. Therefore, $f(1) = -2$.
e) 0. The y-coordinate of the point where $x = -1$ is 0. Therefore, $f(-1) = 0$.

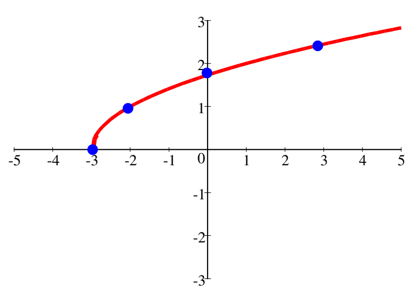
Example 2: Graph the function $f(x) = \sqrt{x+3}$.

Solution:

Make a table of values.

x	$f(x)$	$(x, f(x))$
-4	$\sqrt{-1}$ is not real	$-$
-3	$\sqrt{0} = 0$	$(-3, 0)$
-2	$\sqrt{1} = 1$	$(-2, 1)$
0	$\sqrt{3} \approx 1.7$	$(0, \sqrt{3})$
3	$\sqrt{6} \approx 2.4$	$(3, \sqrt{6})$

Plot points and graph:



Note that the domain of f is $[-3, +\infty)$; therefore, there are no points on the graph for values of x that are less than -3. From the graph, we can see that there are no points with y-coordinates less than 0. Therefore, the range is $[0, +\infty)$. Also take note that the y-intercept is $\sqrt{3}$ and the x-intercept is -3.

Find the y-intercept of a function:

For any function, we can calculate the y-intercept by finding the *output* value when $x = 0$, or $f(0)$. In the function above, $f(0) = \sqrt{0+3} = \sqrt{3}$ is the y-intercept.

Find the x-intercept of a function:

Similarly, we can calculate the x-intercept by finding the *input* when the *output* is 0. That is, set $f(x) = 0$ and solve for x.

$$f(x) = 0 \quad \Rightarrow \quad \sqrt{x+3} = 0 \quad \Rightarrow \quad \left(\sqrt{x+3}\right)^2 = (0)^2$$

Therefore, $x + 3 = 0 \quad \Rightarrow \quad x = -3$ is the x-intercept.

Example 3: Find the x- and y-intercepts of the graph of $f(x) = x^3 - 9x$.

Solution: y-intercept: $f(0) = (0)^3 - 9(0) = 0 \quad \Rightarrow \quad (0,0)$ is the y-intercept.

x-intercept(s): $f(x) = 0 \quad \Rightarrow \quad x^3 - 9x = 0$

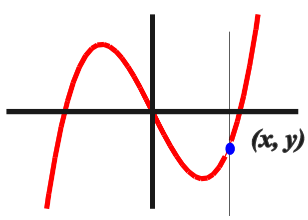
$$x^3 - 9x = x(x^2-9) = x(x+3)(x-3) = 0 \Rightarrow$$

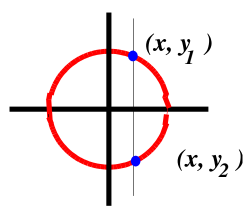
$$x = 0 \quad x = -3 \quad x = 3 \quad \Rightarrow$$

$(0,0), (-3,0), (3,0)$ are x-intercepts.

Recall that a set of ordered pairs of numbers is a function if and only if each x in the domain is paired with exactly one y in the range. If we apply this definition to points on a graph, a vertical line through any value of x would intersect the graph in no more than one point.

Notice the graphs below. The graph on the left is a function because each x is paired with one y. In the graph on the right, we see at least one x is paired with two values for y, so that the second graph does not represent a function.





FUNCTION **NOT A FUNCTION**

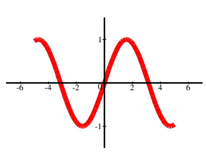
We formalize this process as the vertical line test.

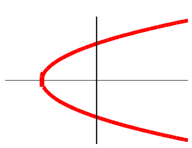
Vertical Line Test:

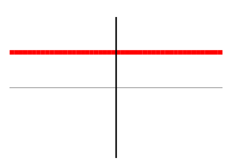
A set of points in the coordinate plane represents a function if and only if no vertical line intersects the graph in more that one point.

Example 4: Apply the vertical line test to each of the following graphs to determine whether the graph represents a function.

a) b) c)



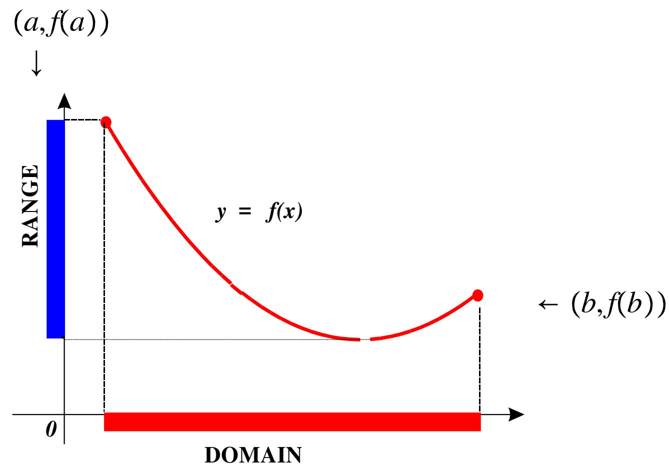




Solution: Graphs a and c represent functions. Graph b does not.

Domain and Range from Graphs

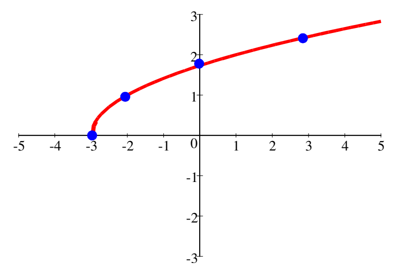
We can find the domain and range of a function by reading its graph. The domain is the reflection of the graph onto the x-axis, while the range is the reflection of the graph onto the y-axis as shown below.



Domain: $[a,b]$ Range: $[f(b),f(a)]$

Example 1: Find the domain and range of $f(x) = \sqrt{x+3}$.

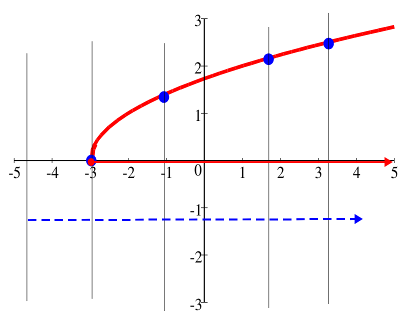
Solution: The graph of f is

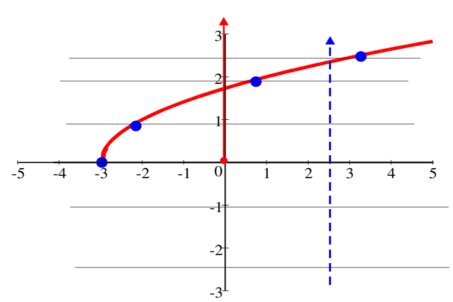


.

To find the domain of f imagine a vertical line scanning the x-axis from $-\infty$ to $+\infty$. If the line intersects the graph at point (x, y), $x \in Domain_f$.

To find the range of f imagine a horizontal line scanning the y-axis from $-\infty$ to $+\infty$. If the line intersects the graph at point (x, y), $y \in Range_f$.



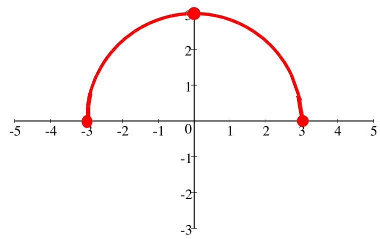


$Domain_f = [-3, +\infty)$
Vertical lines intersect the graph when $x \geq -3$.

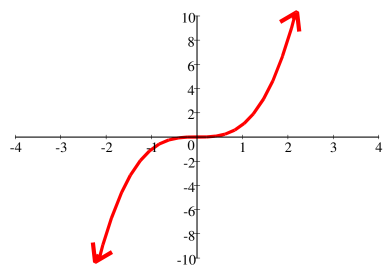
$Range_f = [0, +\infty)$
Horizontal lines intersect the graph when $y \geq 0$.

Example 2: Use the graphs to find the domain and range of each function:

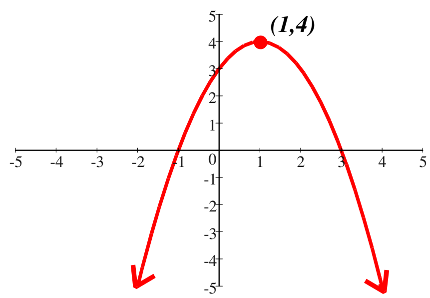
a.



b.



c.



Solution:

a. Domain: $[-3,3]$ Range: $[0,3]$

b. Domain: $(-\infty,+\infty)$ Range: $(-\infty,+\infty)$

c. Domain: $(-\infty,+\infty)$ Range: $(-\infty,4]$

Catalog of Basic Functions

Some functions that are commonly used in the study of algebra and calculus are listed in the following collection of basic functions.

Write a table of values and graph each of the functions on graph paper. Label the *x*- and *y*-intercepts and determine the domain and range of each. Study each function and commit its graph to memory. However, before commiting your graph to memory, be sure that it is correct.

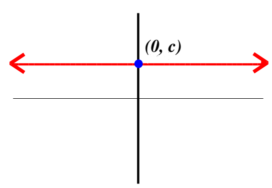
Don't peek at the answer until you have completed the exercise for yourself.

Catalog of Basic Functions

1.	Constant Function:	$f(x) = c,\ c$ is a real number
2.	Identity Function:	$f(x) = x$
3.	Squaring Function:	$f(x) = x^2$
4.	Cubing Function:	$f(x) = x^3$
5.	Square Root Function:	$f(x) = \sqrt{x}$
6.	Absolute Value Function:	$f(x) = \lvert x\rvert$
7.	Reciprocal Function:	$f(x) = \frac{1}{x}$

Solutions:

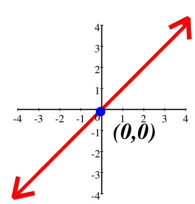
1. **Constant Function**: $f(x) = c$



y-intercept: $(0, c)$ Domain: $(-\infty, +\infty)$

x-intercept: none Range: $\{c\}$

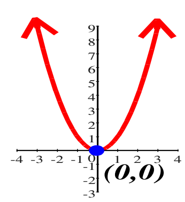
2. **Identity Function**: $f(x) = x$



y-intercept: $(0,0)$ Domain: $(-\infty, +\infty)$

x-intercept: $(0,0)$ Range: $(-\infty, +\infty)$

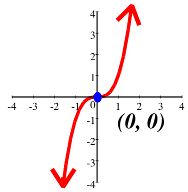
3. **Squaring Function**: $f(x) = x^2$



y-intercept: $(0,0)$ Domain: $(-\infty, +\infty)$

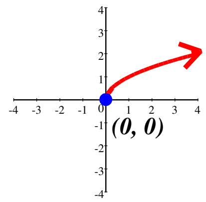
x-intercept: $(0,0)$ Range: $(-\infty, +\infty)$

4. **Cubing Function**: $f(x) = x^3$



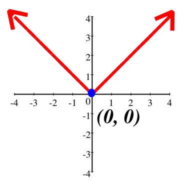
y-intercept: $(0,0)$ Domain: $(-\infty,+\infty)$
x-intercept: $(0,0)$ Range: $(-\infty,+\infty)$

5. **Square Root Function**: $f(x) = \sqrt{x}$



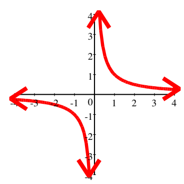
y-intercept: $(0,0)$ Domain: $[0,+\infty)$
x-intercept: $(0,0)$ Range: $[0,+\infty)$

6. **Absolute Value Function**: $f(x) = |x|$



y-intercept: $(0,0)$ Domain: $(-\infty,+\infty)$
x-intercept: $(0,0)$ Range: $[0,+\infty)$

7. **Reciprocal Function**: $f(x) = \frac{1}{x}$



y-intercept: none* Domain: $(-\infty,0) \cup (0,+\infty)$
x-intercept: none** Range: $(-\infty,0) \cup (0,+\infty)$

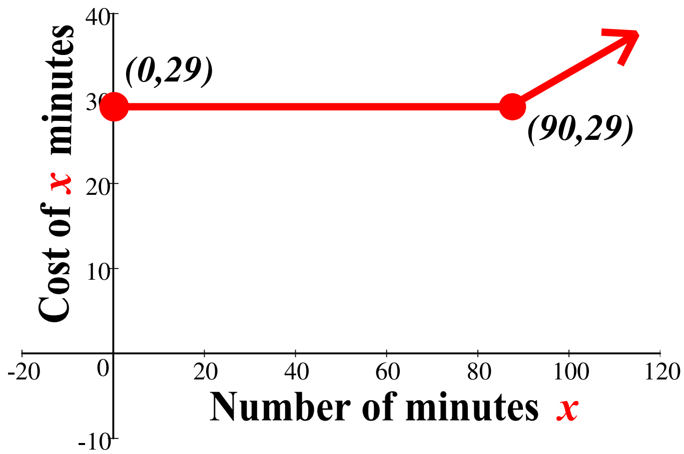
* $f(0) = \frac{1}{0} =$ undefined $\Rightarrow$ no y-intercepts

** $f(x) = 0 \;\Rightarrow\; \frac{1}{x} = 0 \;\Rightarrow\; x \cdot \frac{1}{x} = x \cdot 0 \;\Rightarrow\; 1 = 0 \;\Rightarrow$ no x-intercepts

Piecewise Functions

Sometimes one individual rule does not describe the function. For example, one company's cell phone plan charges \$29/month for 90 minutes and an additional \$0.35/minute for every minute over 90 minutes. The function $c(x)$ describing the monthly costs of operating the cell phone for x minutes would be

$$c(x) = \begin{cases} 29 & 0 \le x \le 90 \\ 29 + .35(x - 90) & x > 90 \end{cases}.$$

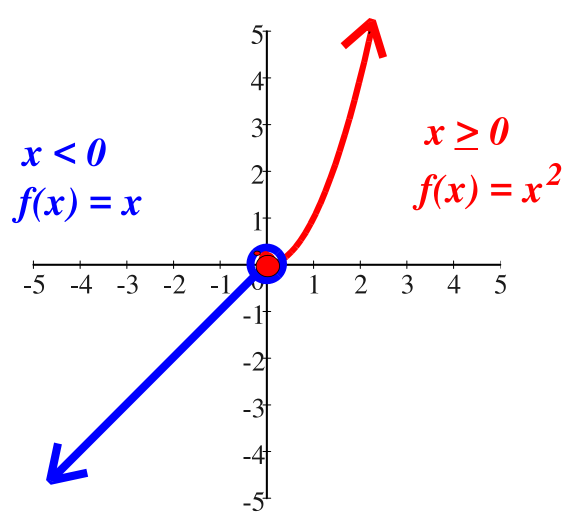


Functions of this nature are called piecewise functions because they are defined in pieces.

Example 1: Graph $f(x) = \begin{cases} x, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ by plotting points.

Solution: If $x < 0$, find $f(x)$ by substituting into $f(x) = x$.
If $x \ge 0$, find $f(x)$ by substituting into $f(x) = x^2$.

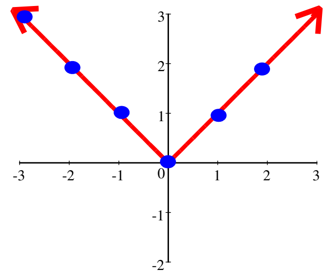
x	y
−2	−2
−1	−1
0	0
1	1
2	4
3	9
4	16



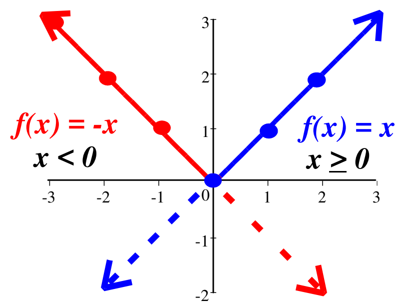
Example 2: Graph the piecewise function: $f(x) = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$.

Solution:

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2

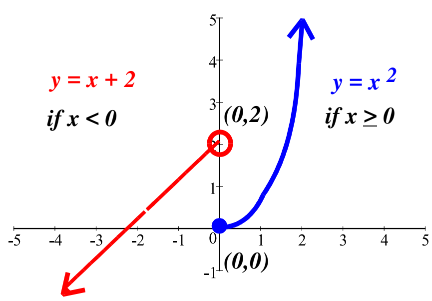


Question: Which basic function is graphed above? How do you know they are the same function?
Answer: The absolute values function $f(x) = |x|$. The algebraic definition of absolute value is $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. Therefore, $f(x) = |x|$ and $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ are the same functions.
Note further that f is the part of the line $y = x$ when $x \geq 0$ and part of the line $y = -x$ when $x < 0$.



Example 3: Graph the piecewise function: $f(x) = \begin{cases} x^2 & if \quad x \geq 0 \\ x+2 & if \quad x < 0 \end{cases}$.

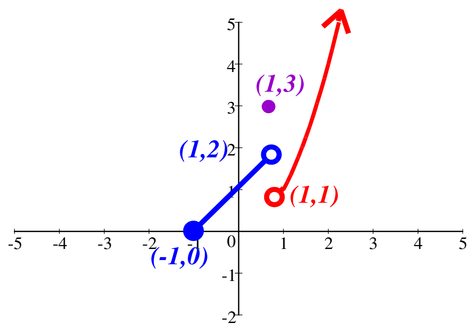
Solution: The graph of f consists of the part of $f(x) = x^2$ where $x \geq 0$, and the line $y = x+2$ where $x < 0$ as shown below.



To show that the point $(0,2)$ is **not** included in the graph, we use an open circle. To show that the point $(0,0)$ **is** included, we color in the point. It is important to indicate clearly that one of the points when $x = 0$ is NOT included. If both are included the graph does not pass the vertical line test at $x = 0$.

Example 4: Graph the piecewise function: $f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 3 & \text{if } x = 1 \\ -x+1 & \text{if } -1 \leq x < 1 \end{cases}$.

Solution:



Questions:

a) What is the domain of f?

b) What is the range?

c) Evaluate $f(1)$.

d) What are the intercepts?

Answers:

a) Domain: $[-1, +\infty)$

b) Range: $[0, +\infty)$

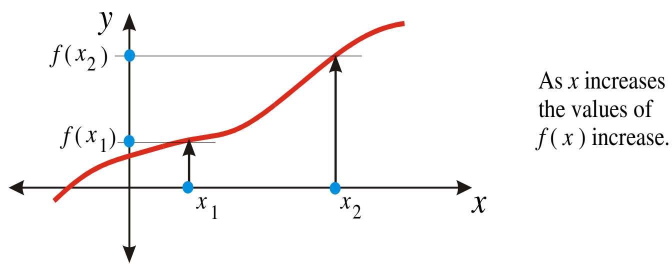
c) $f(1) = 3$.

d) y-intercept: 1 x-intercept: -1

Increasing, Decreasing and Constant

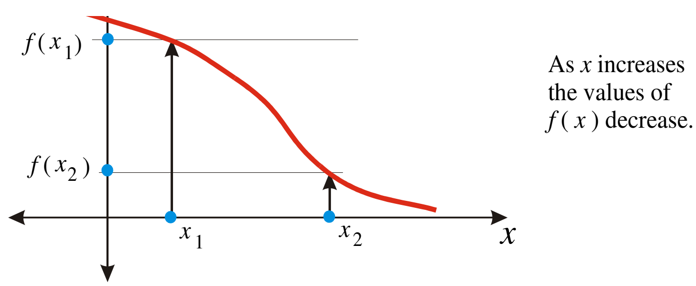
By inspecting graphs we can get information very quickly about how changes in *x* affect the functional values. In our earlier discussion of lines we saw that lines with positive slopes were increasing (uphill from left to right) while lines with negative slopes were decreasing. In this section we will apply the concept of increasing/decreasing to the graph of any function.

Definition: A function *f* is **increasing** on an interval *I* if and only if for every $x_1 < x_2 \in I,\ f(x_1) < f(x_2)$.



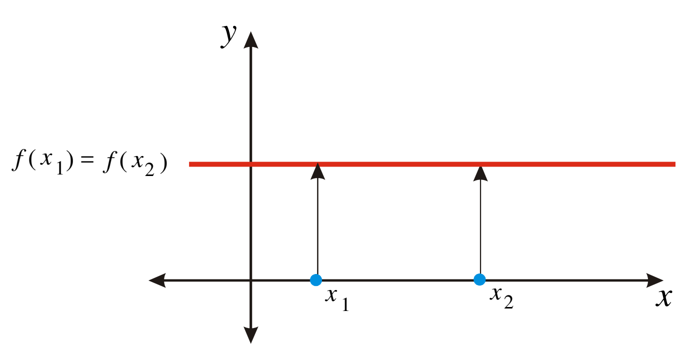
Graph: uphill from left to right.

Definition: A function *f* is **decreasing** on an interval *I* if and only if for every $x_1 < x_2 \in I,\ f(x_1) > f(x_2)$.



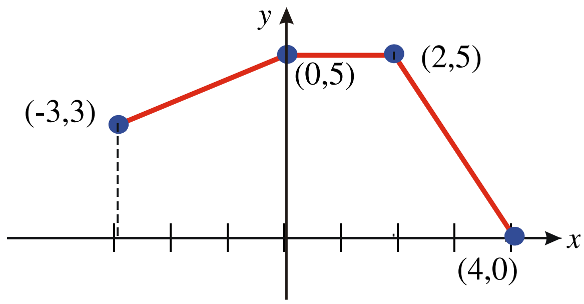
Graph: downhill from left to right.

Definition: A function *f* is **constant** on an interval *I* if and only if for every $x_{1,}\ x_2 \in I,\ f(x_1) = f(x_2)$.



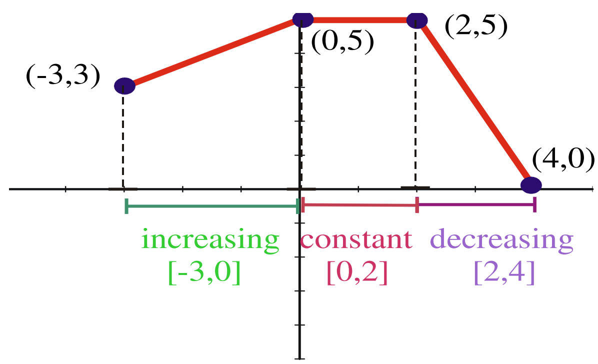
Graph: horizontal over interval *I*.

Example 1: Determine where function f is increasing, decreasing, and/or constant.



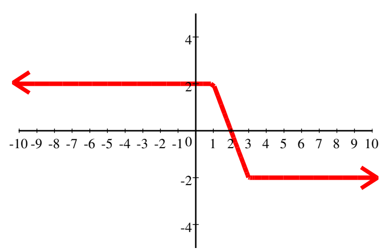
Solution: Inspect the graph for intervals where the graph is going uphill, downhill or remaining constant from left to right.

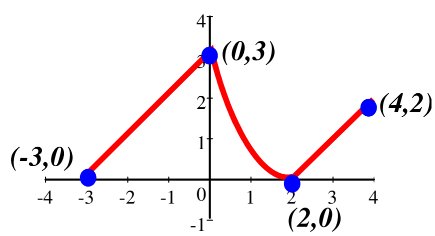
Read the intervals on the *x*-axis!!



f is increasing on the interval $[-3,0]$, constant on $[0,2]$, and decreasing on $[2,4]$.

Example 2: Determine intervals where f is increasing, decreasing, and constant for each of the following:

a) 

b) 

Solution: Read the intervals on the x-axis!!!!

a) Decreasing: $[1,3]$ (As we input values of x from 1 to 3, the y-values decrease.)

Constant: on $(-\infty,1]$ and constant on $[3,+\infty)$ (For $x \le 1$, $y = 2$, and or $x \ge 3$, $y = -2$)

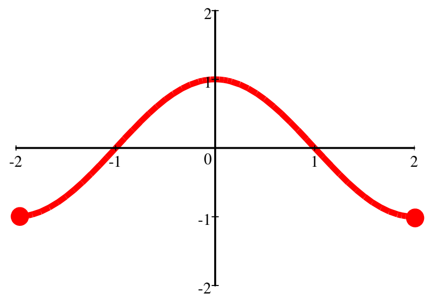
b) Increasing: on $[-3,0]$ and increasing on $[2,4]$ (As x increases from -3 to 0 y increases, and as x increases from 2 to 4, y increases.)

Decreasing: $[0,2]$ (As x increases from 0 to 2, y decreases.)

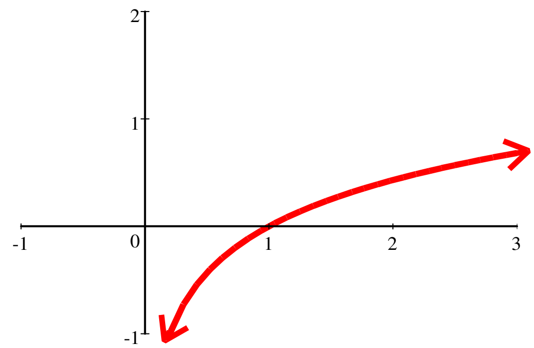
Example 3: Use the graph to find the following for each function:

- Domain
- Range
- Intercepts
- Intervals where *f* is increasing, decreasing and constant

a)



b)



Solution:

a) Domain: $[-2,2]$
Range: $[-1,1]$
y-intercept: $(0,1) \Rightarrow y = 1$
x-intercepts: $(-1,0),\ (1,0) \Rightarrow x = -1,1$
Increasing: $[-2,0]$
Decreasing: $[0,2]$

b) Domain: $(0,+\infty)$
Range: $(-\infty,+\infty)$
y-intercept: *none*
x-intercept: $(1,0) \Rightarrow x = 1$
Increasing: $(0,+\infty)$
Decreasing: *nowhere*

Exercises for Chapter 4B - Graphs of Functions

1. Find and label the x- and y-intercepts. Graph the function.

a. $f(x) = x^2 - 2x + 1$

b. $f(x) = \sqrt{x - 3}$

c. $f(x) = x^3 - 1$

d. $f(x) = 2x - 3$

e. $f(x) = x^2 + 3$

f. $f(x) = x^2 - 5x + 6$

g. $f(x) = x^3 - 4x$

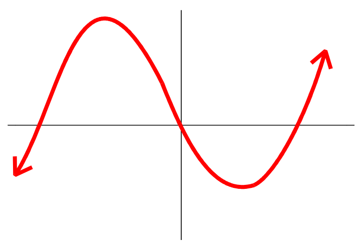
h. $f(x) = x^3 + 5x^2 - 9x - 45$

i. $f(x) = 6x - 5$

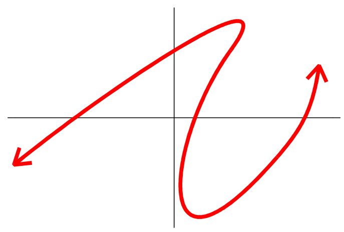
j. $f(x) = 3$

2. Use the vertical line test to determine which of the following graphs represent functions.

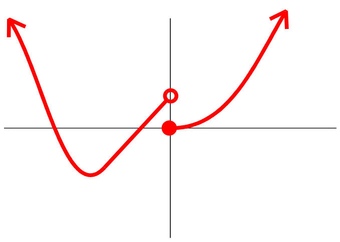
a)



b)

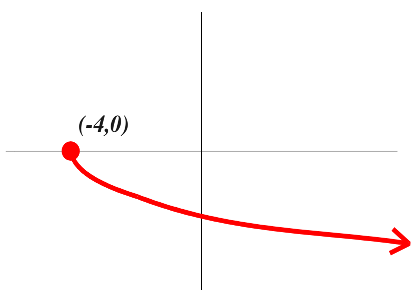


c)

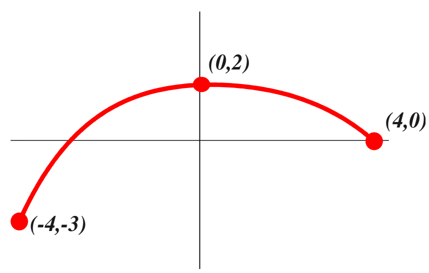


3. Find the domain and range of each of the following functions:

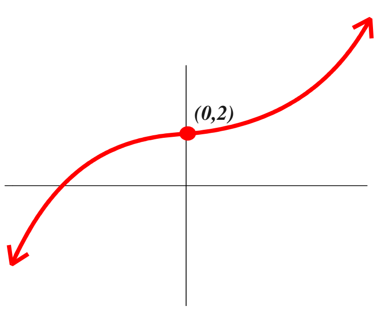
a)



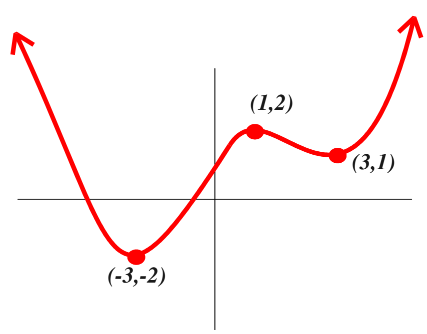
b)



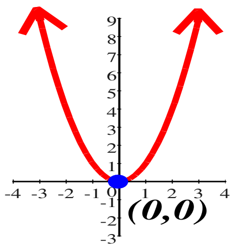
c)

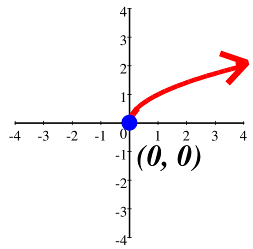


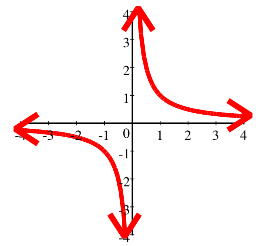
d)

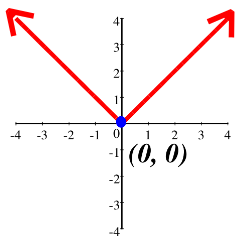


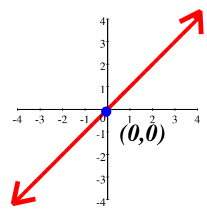
4. Write the function rule for each of the following:

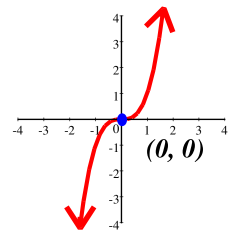
a) 

b) 

c) 

d) 

e) 

f) 

5. Graph the basic functions over the given domains. Label the endpoints and indicate whether they are included in the graph or not.

a. $f(x) = 5, \quad -1 \le x \le 6$

b. $f(x) = |x|, \quad x < 3$

c. $f(x) = x^3, \quad -2 \le x \le 2$

d. $f(x) = \sqrt{x}, \quad x > 1$

e. $f(x) = x^2, \quad x \ge 0$

6. i) Graph each of the following piecewise functions.

ii) Label the endpoints of segments and indicate whether they are included or excluded with an open or closed circle.

iii) Use the graph to determine the domain and range.

a. $f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x+2 & \text{if } x \ge 0 \end{cases}$

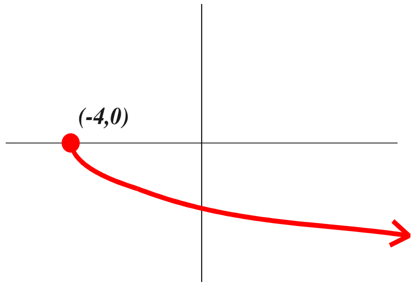
b. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

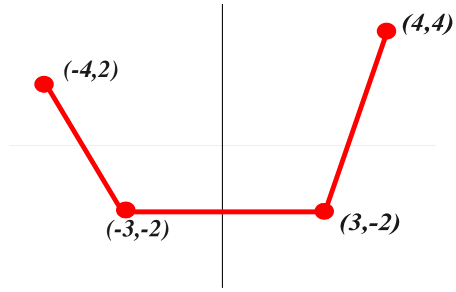
c. $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \ne 0 \\ 2 & \text{if } x = 0 \end{cases}$

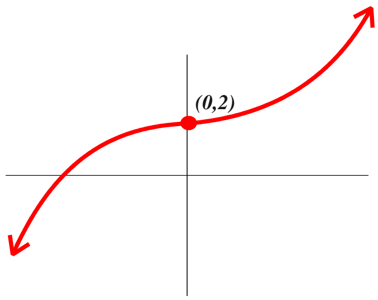
d. $f(x) = \begin{cases} -1, & x < -1 \\ x, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$

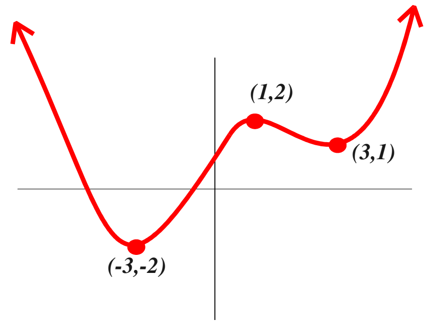
e. $f(x) = \begin{cases} 3, & -2 \le x < 5 \\ -x+5, & 5 \le x \le 8 \end{cases}$

7. Write the intervals where each function is increasing, decreasing or constant.

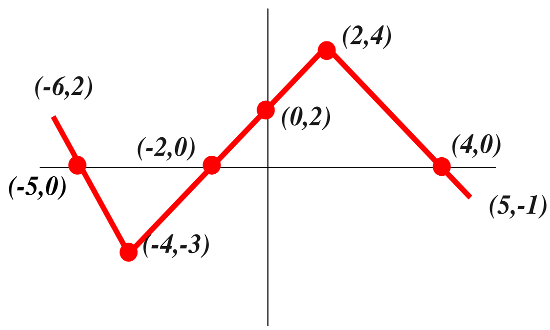
a) 

b) 

c) 

d) 

8. Use the graph to find the following:

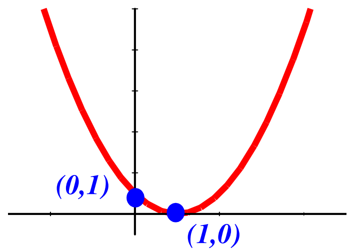


a. Domain
b. Range
c. $f(2)$
d. y-intercept
e. x-intercept
f. intervals where f is increasing
g. intervals where f is decreasing

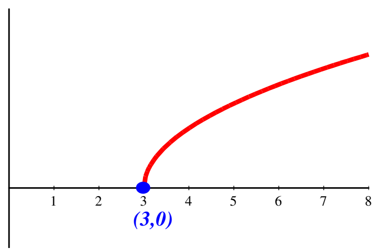
Answers to Exercises for Chapter 4B - Graphs of Functions

1. Find and label the *x*- and *y*-intercepts. Graph the function.

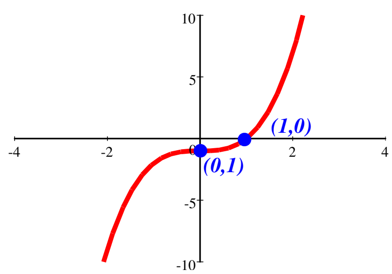
a. *y*-intercept: $(0,1)$ *x*-intercept: $(1,0)$



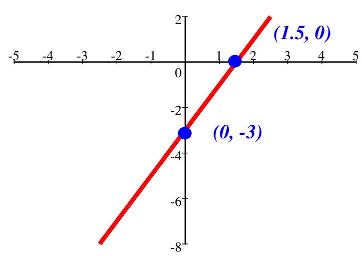
b. *y*-intercept: none *x*-intercept: $(3,0)$



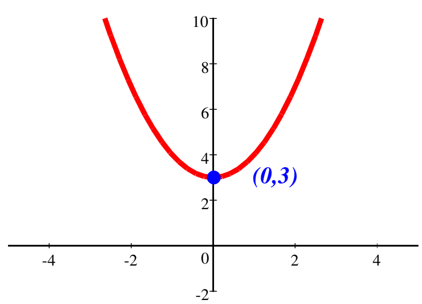
c. *y*-intercept: $(0,-1)$ *x*-intercept: $(1,0)$



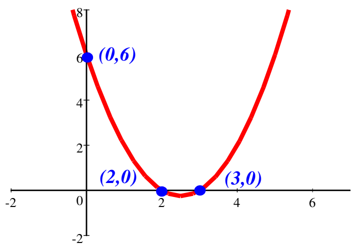
d. *y*-intercept: $(0,-3)$ *x*-intercept: $(\frac{3}{2},0)$



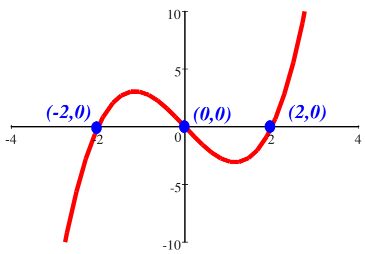
e. *y*-intercept: $(0,3)$ *x*-intercept: none



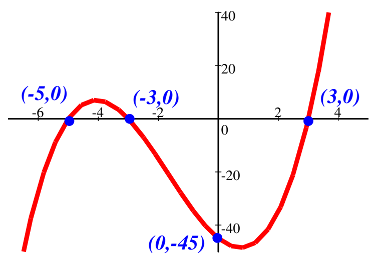
f. y-intercept: $(0,6)$ x-intercepts: $(2,0)$ and $(3,0)$



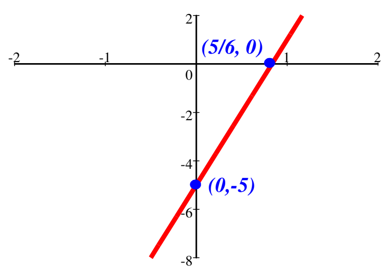
g. y-intercept: $(0,0)$ x-intercepts: $(-2,0)$, $(0,0)$, $(2,0)$



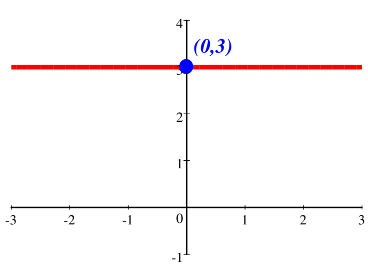
h. y-intercept: $(0,-45)$ x-intercepts: $(-5,0)$, $(-3,0)$, $(3,0)$



i. y-intercept: $(0,-5)$ x-intercept: $(5/6,0)$



j. y-intercept: $(0,3)$ x-intercept: none



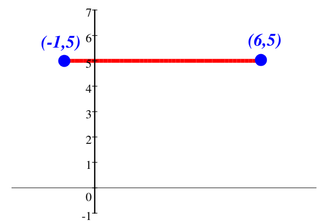
2. a and c represent functions. b does NOT represent a function.

3. a) Domain: $[-4,+\infty)$; Range: $(-\infty,0]$
b) Domain: $[-4,4]$; Range: $[-3,2]$
c) Domain: $(-\infty,+\infty)$; Range: $(-\infty,+\infty)$
d) Domain: $(-\infty,+\infty)$; Range: $[-2,+\infty)$

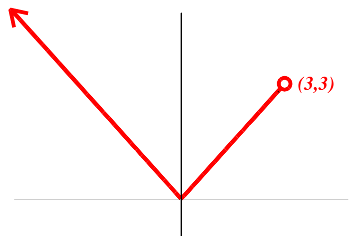
4. a) $f(x) = x^2$ b) $f(x) = \sqrt{x}$ c) $f(x) = \frac{1}{x}$ d) $f(x) = |x|$ e) $f(x) = x$ f) $f(x) = x^3$

5.

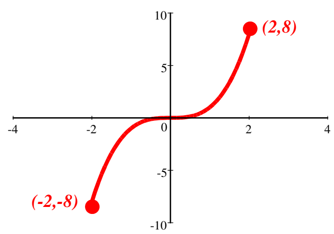
a.



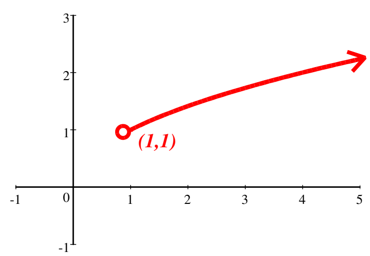
b.



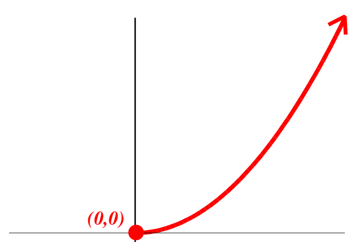
c.



d.

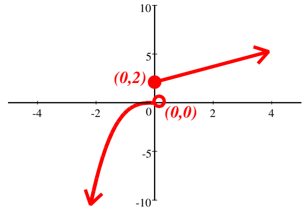


e.

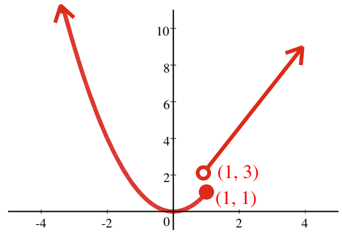


6.

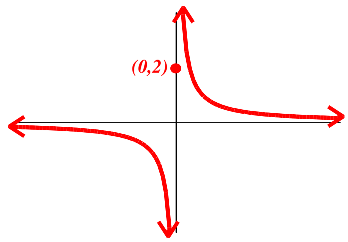
a. Domain: $(-\infty,+\infty)$ Range: $(-\infty,0)\cup[2,+\infty)$



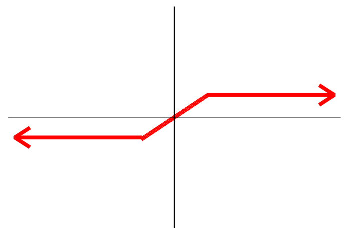
b. Domain: $(-\infty,+\infty)$ Range: $[0,+\infty)$



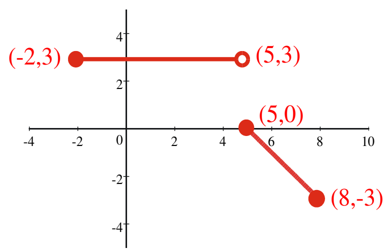
c. Domain: $(-\infty,+\infty)$ Range: $(-\infty,0)\cup(0,+\infty)$



d. Domain: $(-\infty.+\infty)$ Range: $[-1,1]$



e. Domain: $[-2,8]$ Range: $[-3,0]\cup\{3\}$



7. a) decreasing: $[-4,+\infty)$
b) decreasing: $[-4,-3]$ constant: $[-3,3]$ increasing: $[3,4]$
c) increasing: $(-\infty,+\infty)$
d) decreasing: $(-\infty,-3]$, $[1,3]$ increasing: $[-3,1]$, $[3,+\infty)$

8.

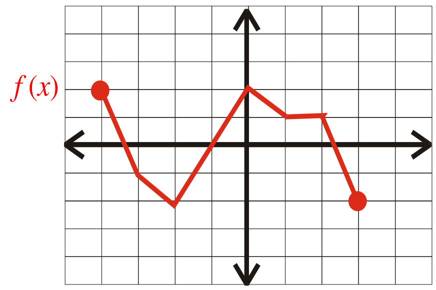
a. Domain: $[-6,5]$
b. Range: $[-4,4]$
c. $f(2)=4$
d. y-intercept: 2
e. x-intercepts: $-5,-2,4$
f. increasing: $[-4,2]$
g. decreasing: $[-6,-4]$, $[2,5]$

Chapter 4C - Transformations of Functions

Horizontal and Vertical Shifts

Knowledge of transformations of functions allows us to graph families of functions by simply observing the function rule and moving the graph of the appropriate basic function accordingly.

Example 1:



a) Copy the graph above on graph paper and use it to find $f(-4)$, $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$.

b) Use the above graph of f to graph $y = f(x) + 2$.

c) Use the above graph of f to graph $y = f(x - 1)$.

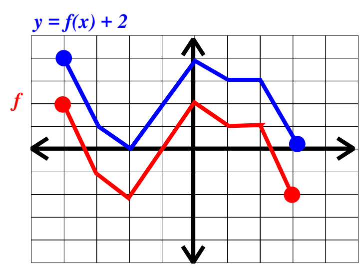
Solution:

a)

x	$f(x)$
–4	2
–3	–1
–2	–2
–1	0
0	2
1	1
2	1
3	–2

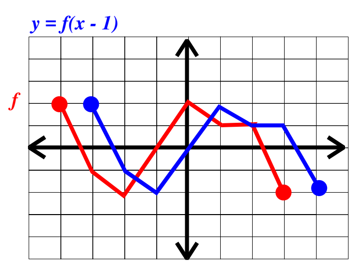
b)

x	$f(x)$	$f(x) + 2$
–4	2	4
–3	–1	1
–2	–2	0
–1	0	2
0	2	4
1	1	3
2	1	3
3	–2	0



c)

x	$f(x)$	$f(x - 1)$
–3	–1	2
–2	–2	–1
–1	0	–2
0	2	0
1	1	2
2	1	1
3	–2	1
4		–2



INVESTIGATION: Graph each set of functions on the same axes. Investigate the changes in the graph of the basic function (listed first) caused by adding or subtracting a constant.

a) $y_1 = x^2$
$y_2 = x^2 + 1$
$y_3 = x^2 - 2$

b) $y_1 = \sqrt{x}$
$y_2 = \sqrt{x} - 3$
$y_3 = \sqrt{x} + 4$

- **Question**: What effect did the constant terms have on the graphs of the basic functions?
  Answer: The graph is shifted vertically. If c is a positive number, $f(x) + c$ shifts the graph up c units. $f(x) - c$ shifts the graph down c units.

c) $y_1 = |x|$
$y_2 = |x + 2|$
$y_3 = |x - 1|$

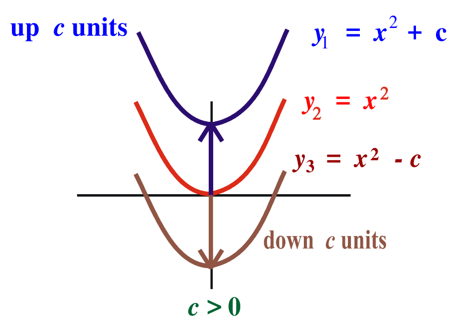
d) $y_1 = x^3$
$y_2 = (x - 2)^3$
$y_3 = (x + 4)^3$

- **Question**: What effect did the constants have on the graph of the original function?
  Answer: The graphs were shifted horizontally. If c is a positive number, $f(x + c)$ shifts the graph of f to the left c units. $f(x - c)$ shifts the graph of f to the right c units.

We can generalize the above findings:

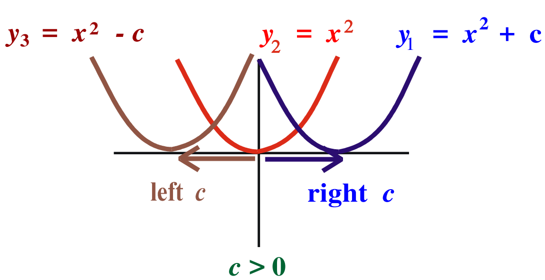
Vertical shift: **Add a constant c to the function**: $y = f(x) + c$

EQUATION	**TRANSFORMATION**
$y = f(x) + c,\ c > 0$	Shift up c units
$y = f(x) - c,\ c > 0$	Shift down c units



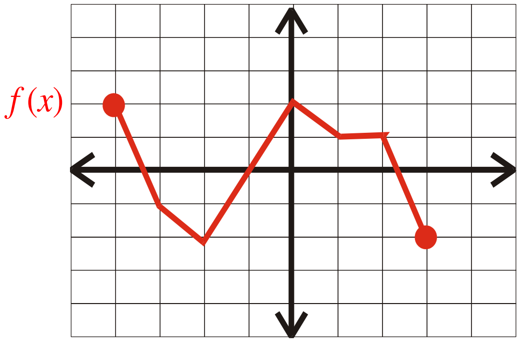
Horizontal shift: **Add constant c to x before applying function**: $y = f(x + c)$

EQUATION	**TRANSFORMATION**
$y = f(x + c),\ \ c > 0$	Shift left c units
$y = f(x - c),\ \ c > 0$	Shift right c units



Since the shape of the graph remains unchanged, both horizontal and vertical shifts are examples of rigid transformations.

Example 2: Transform the graph of f below to obtain the graph of each of the following functions:



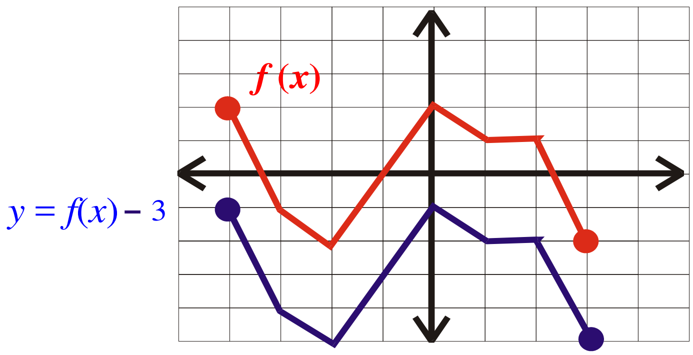
a) $y_1 = f(x) - 3$

b) $y_2 = f(x + 1)$

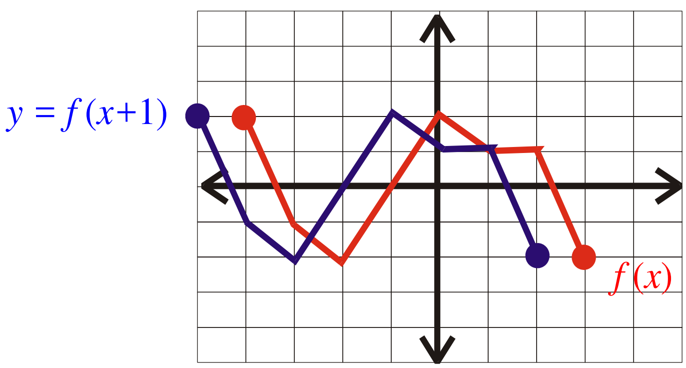
c) $y_3 = f(x - 2)$

Solution:

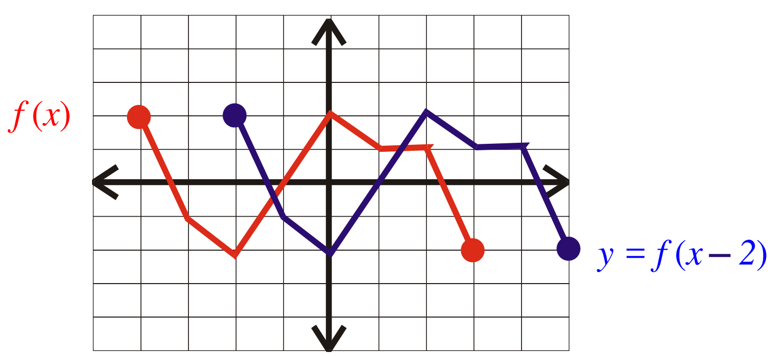
a. Vertical shift down 3:



b. Horizontal shift left 1:



c. Horizontal shift right 2:



Example 3: Apply vertical and horizontal shifts appropriately to the basic functions to obtain the graphs of the following. State the shift you applied. Determine the domain and range of each.

a) $f(x) = x^3 + 2$ b) $f(x) = (x+2)^3$

c) $f(x) = |x - 3|$ d) $f(x) = |x| - 3$

e) $f(x) = \dfrac{1}{x-2}$ f) $f(x) = \dfrac{1}{x} - 2$

Solution:

a) $f(x) = x^3 + 2$

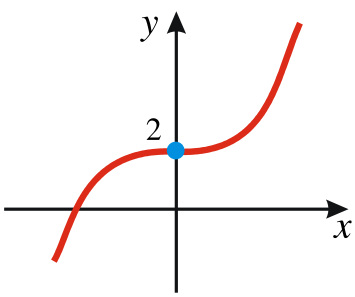
Cubing function: vertical shift up 2:

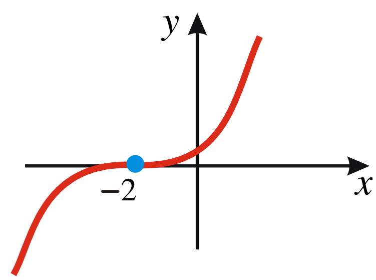
Domain: $(-\infty, +\infty)$ Range: $(-\infty, +\infty)$

b) $f(x) = (\underbrace{x+2})^3 = (x - \underbrace{(-2)})^3$

Horizontal shift: left 2

Domain: $(-\infty, +\infty)$ Range: $(-\infty, +\infty)$





c) $f(x) = |x - 3|$

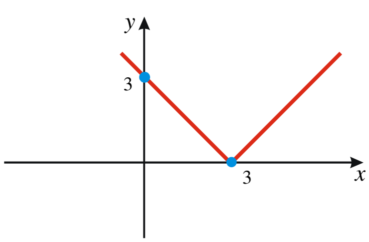
Absolute value function: Horizontal shift right 3.

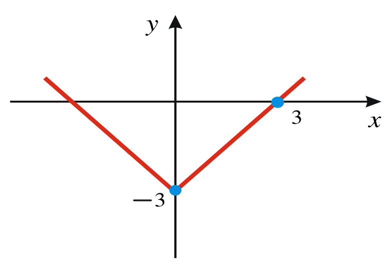
Domain: $(-\infty, +\infty)$ Range: $[0, +\infty)$

d) $f(x) = |x| - 3$

Vertical shift down 3

Domain: $(-\infty, +\infty)$ Range: $[-3, +\infty)$





e) $f(x) = \dfrac{1}{x-2}$

Reciprocal function: horizontal shift right 2

Domain: $(-\infty, 2) \cup (2, +\infty)$

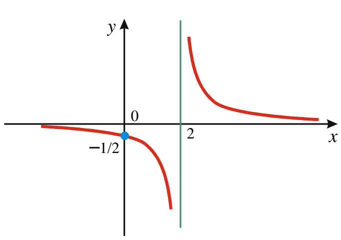
Range: $(-\infty, 0) \cup (0, +\infty)$

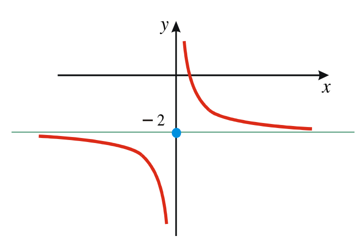
f) $f(x) = \dfrac{1}{x} - 2$

vertical shift down 2

Domain: $(-\infty, 0) \cup (0, +\infty)$

Range: $(-\infty, -2) \cup (-2, +\infty)$



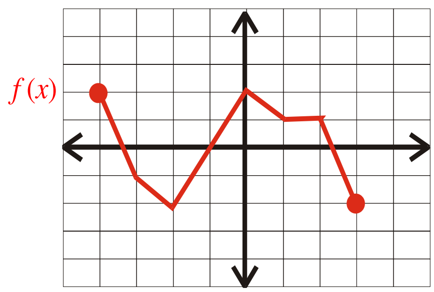


Reflections about the x- and y-axis

To reflect a graph about the x- or y-axis means much the same thing as your reflection in a mirror. Imagine picking the graph up by each end of one of the axes and rotating it 180°. The result would be the reflection of the graph about that axis. In this exercise, we will investigate how changing the sign of x and $f(x)$ affects the graph.

Example 1:

Recall the graph of f



and its table of values:

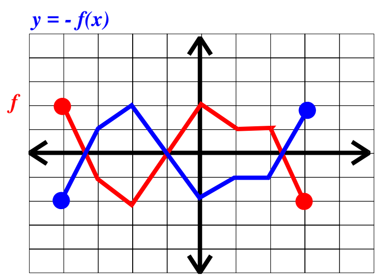
x	$f(x)$
–4	2
–3	–1
–2	–2
–1	0
0	2
1	1
2	1
3	–2

a) Find the graph of $y_1 = -f(x)$.
b) Find the graph of $y_2 = f(-x)$

Solution:

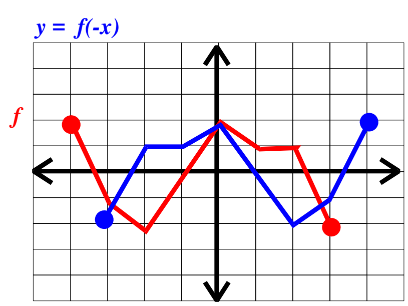
a)

x	$f(x)$	$-f(x)$
–4	2	–2
–3	–1	–1
–2	–2	2
–1	0	0
0	2	–2
1	1	–1
2	1	–1
3	–2	2



b)

x	$f(x)$	$f(-x)$
–4	2	$f(4) = -$
–3	–1	$f(3) = -2$
–2	–2	$f(2) = 1$
–1	0	$f(1) = 1$
0	2	$f(0) = 2$
1	1	$f(-1) = 0$
2	1	$f(-2) = -2$
3	–2	$f(-3) = 2$



Investigate the changes in the graph of each of the following sets of functions. Graph one function at a time in an appropriate viewing window of your grapher.

1. $y_1 = x^2$
   $y_2 = -x^2$
   $y_3 = (-x)^2$

2. $y_1 = \sqrt{x}$
   $y_2 = -\sqrt{x}$
   $y_3 = \sqrt{-x}$

3. $y_1 = |x|$
   $y_2 = -|x|$
   $y_3 = |-x|$

4. $y_1 = x^3$
   $y_2 = -x^3$
   $y_3 = (-x)^3$

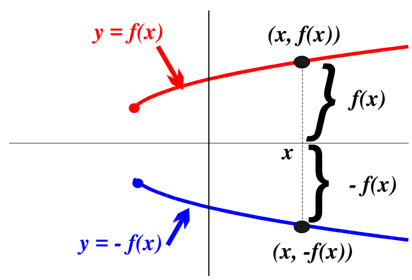
Question: Explain how the sign was applied differently in the groups of functions. What effect did each sign application have on the graph of the original function?

Answer: In the second function of each group, y_2, the negative sign was applied to the basic function. In the third function, y_3, the negative sign was applied to x before the basic function was applied. When the negative sign was applied after the function rule, as in the y_2 functions, the graph was reflected about the x-axis. When the negative sign was applied to x before the function was applied, y_3, the graph was reflected about the y-axis.

Your findings should agree with the following generalizations.

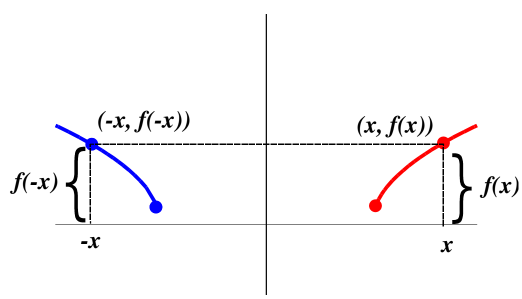
Reflection about the x-axis:

EQUATION	TRANSFORMATION
$y = -f(x)$	Reflect about the x-axis



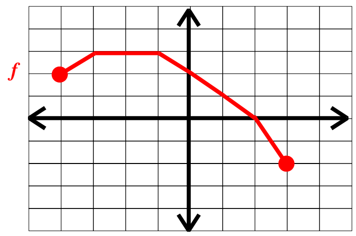
Reflection about the y-axis:

EQUATION	TRANSFORMATION
$y = f(-x)$	Reflect about the y-axis



Reflections about the x- and y-axes are also examples of rigid transformations.

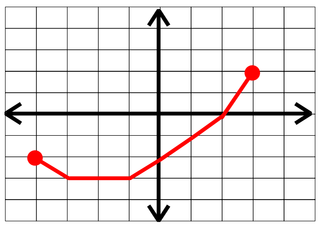
Example 2: Transform the graph of f below to obtain the graph of each of the following:



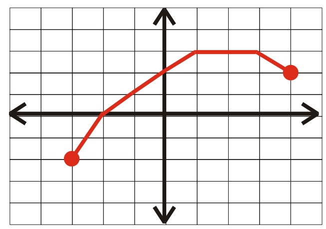
a. $y_1 = -f(x)$
b. $y_2 = f(-x)$

Solution:

a. Reflect about the x-axis:



b. Reflect about the y-axis:

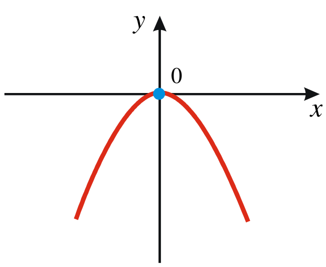


Example 3: Apply appropriate transformations to the graphs of basic functions to obtain the graphs of each of the following.

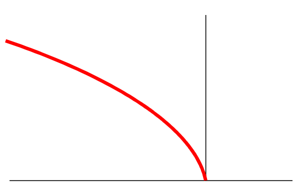
a) $y = -x^2$ b) $y = \sqrt{-x}$ c) $y = -|x|$

Solution:

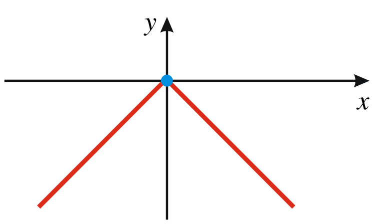
a) Basic function: Squaring function $y = x^2$ reflected about the x-axis: $y = -x^2$



b) Basic function: Square root function $y = \sqrt{x}$ reflected about the y-axis: $y = \sqrt{-x}$

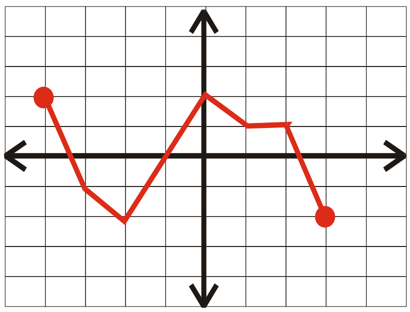


c) Basic function: Absolute value function: $y = |x|$ reflected about the x-axis: $y = -|x|$



Vertical Stretch and Shrink

Graphs can be "stretched" or "shrunk" by multiplying the function by a number. Examine the effect that multipliers have on the graph of function *f*.

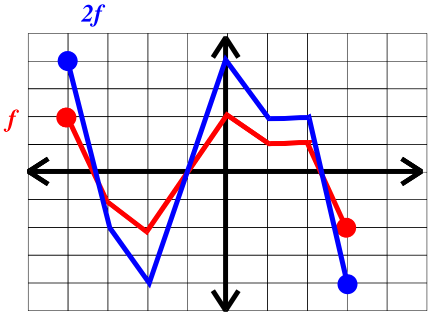


x	$f(x)$
–4	2
–3	–1
–2	–2
–1	0
0	2
1	1
2	1
3	–2

Example 1: Find the graph of $y = 2 \cdot f(x)$.

Solution:

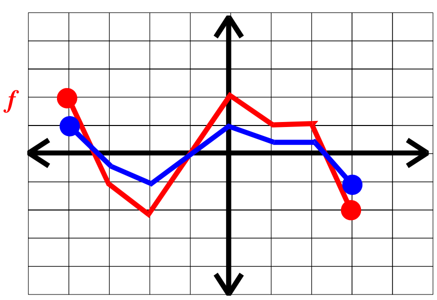
x	$f(x)$	$2f(x)$
–4	2	4
–3	–1	–2
–2	–2	–4
–1	0	0
0	2	4
1	1	2
2	1	2
3	–2	–4



Example 2: Find the graph of $y = \frac{1}{2} \cdot f(x)$.

Solution:

x	$f(x)$	$\frac{1}{2}f(x)$
–4	2	1
–3	–1	$-\frac{1}{2}$
–2	–2	–1
–1	0	0
0	2	1
1	1	$\frac{1}{2}$
2	1	$\frac{1}{2}$
3	–2	–1



Investigate the changes in the graph of each of the following sets of functions. Graph one function at a time in an appropriate viewing window of your grapher. Observe how the coefficient changes the graph of the basic function.

a) $y_1 = x^2$
$y_2 = 2x^2$
$y_3 = \frac{1}{2}x^2$

b) $y_1 = \sqrt{x}$
$y_2 = 3\sqrt{x}$
$y_3 = \frac{1}{3}\sqrt{x}$

c) $y_1 = |x|$
$y_2 = 4|x|$
$y_3 = \frac{1}{4}|x|$

d) $y_1 = x^3$
$y_2 = 2x^3$
$y_3 = .5(x)^3$

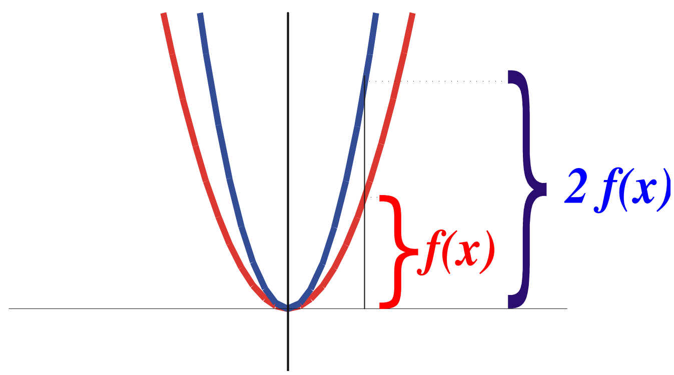
Question: How did multiplying a function f by a constant c effect the graph of f if

i) $c > 1$?
ii) $0 < c < 1$?

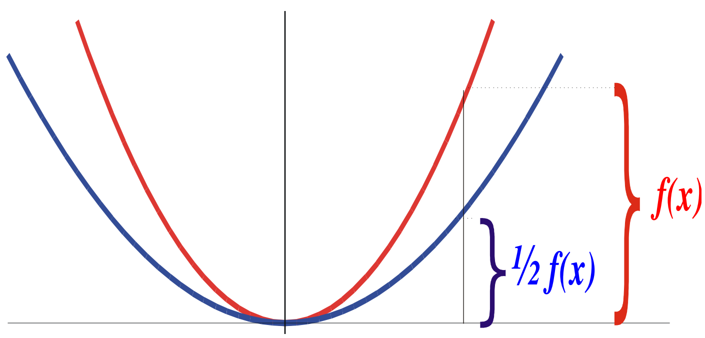
Answer: i) The graph is stretched vertically by a factor of c. ii) The graph is shrunk vertically by the factor $1/c$

Your findings should agree with the following generalizations.

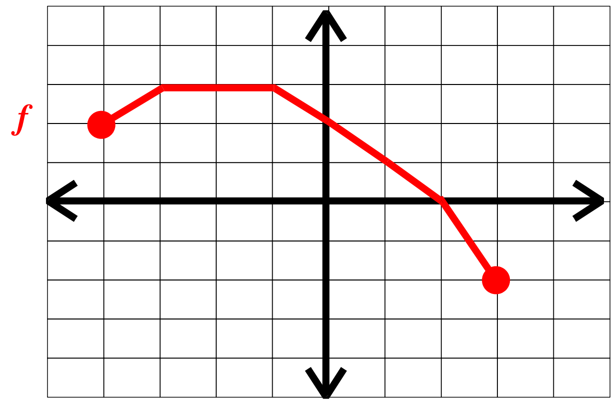
EQUATION	TRANSFORMATION
$y = cf(x),\ c > 1$	Stretch vertically by a factor of c.



EQUATION	TRANSFORMATION
$y = cf(x),\ 0 < c < 1$	Shrink vertically by a factor of $1/c$.



Example 3: Transform the graph of f below to obtain the graph of each of the following:

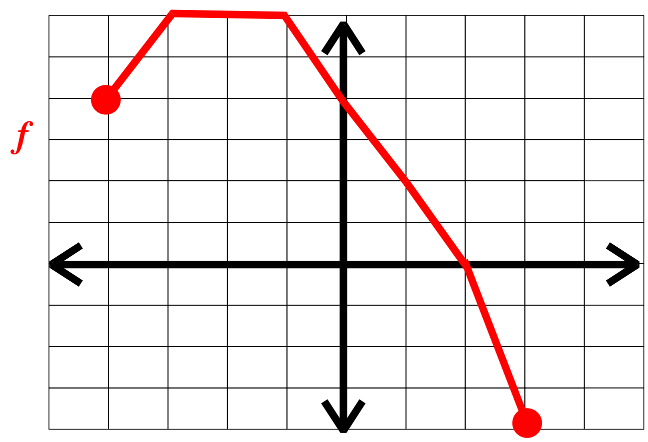


a. $y_1 = 2f(x)$

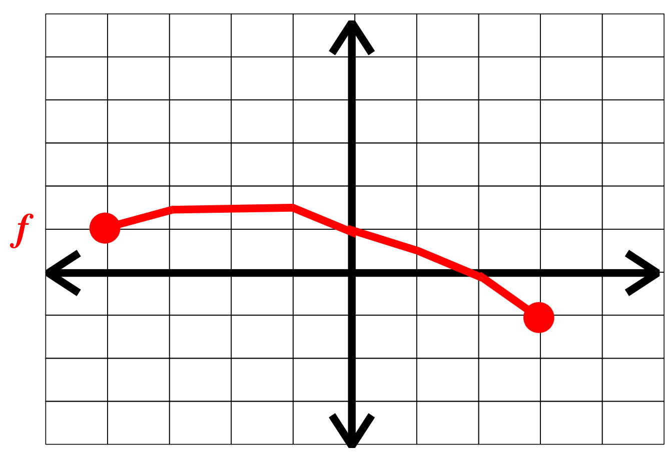
b. $y_2 = \frac{1}{2}f(x)$

Solution:

a. Stretch by a factor of 2:



b. Shrink by a factor of 2:



Example 4: Apply appropriate transformations to the graphs of basic functions to obtain the graphs of each of the following.

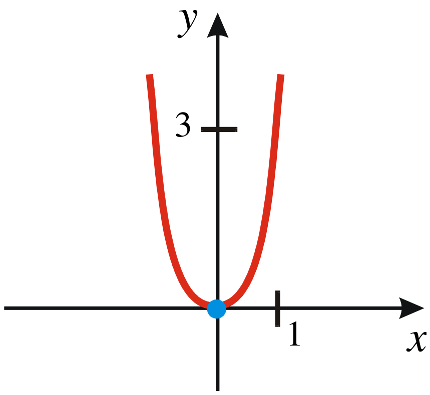
a. $y = 3x^2$

b. $y = 2\sqrt{x}$

c. $y = \frac{1}{2}|x|$

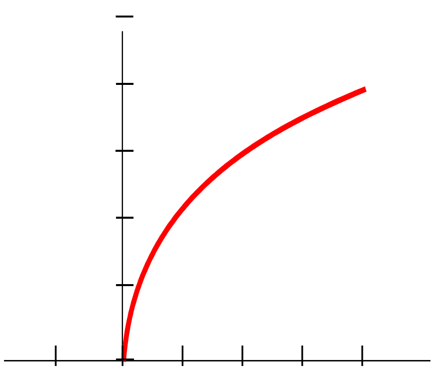
Solution:

a. Basic function: Squaring function $y = x^2$ stretched by a factor of 3: $y = 3x^2$



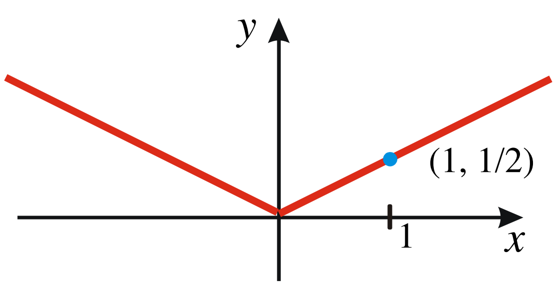
1.

b. Basic function: Square root function $y = \sqrt{x}$ stretched by a factor of 2: $y = 2\sqrt{x}$



1.

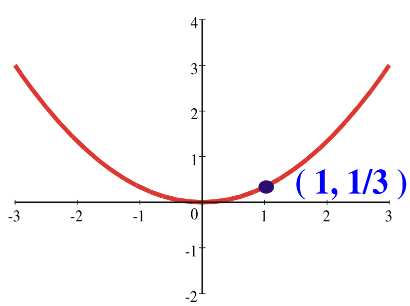
c. Basic function: Absolute value function: $y = |x|$ shrunk by a factor of 2: $y = \frac{1}{2}|x|$



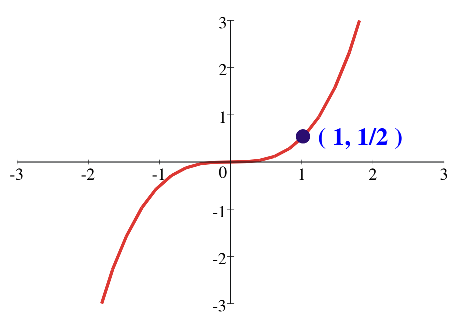
1.

Example 5: The following graphs consist of transformations of basic functions. Write the function rule for each:

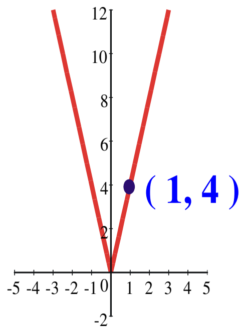
a)



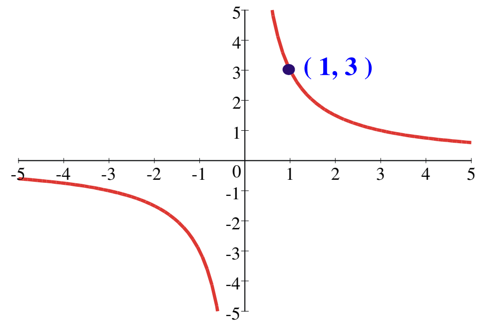
b)



c)



d)



Solution:

- a) Basic Shape: Squaring function. Since the point $\left(1, \frac{1}{3}\right)$ is on the graph and $(1, 1)$ is not, the graph has been shrunk by a factor of 3. Therefore, the function is $y = \frac{1}{3}x^2$.

  b) Basic shape: Cubing function. Shrunk by a factor of 2. Therefore, the function is $y = \frac{1}{2}x^3$.

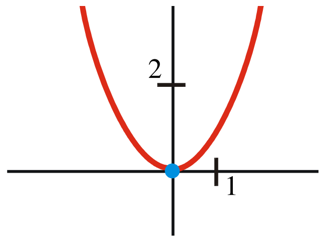
  c) Basic shape: Absolute value function. Stretched by a factor of 4. The function is $y = 4|x|$.

  d) Basic shape: Reciprocal function. Stretched by a factor of 3. The function is $y = 3 \cdot \frac{1}{x}$ or $y = \frac{3}{x}$.

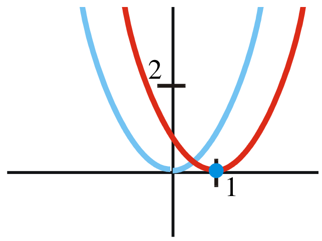
Combining Transformations

Consider the function $f(x) = (x-1)^2 + 2$

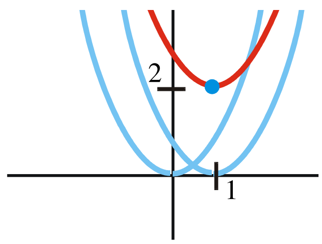
First we recognize the basic function is the squaring function: $f(x) = (x-1)^{\boxed{2}} + 2$



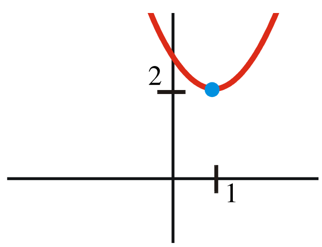
We see a horizontal shift of 1 to the right: $f(x) = \boxed{(x-1)}^2 + 2$



And a vertical shift of 2 up: $f(x) = (x-1)^2 \boxed{+2}$



Therefore the graph of $f(x) = (x-1)^2 + 2$ is:



Example 1:

a) Use what you know about the above graph to determine the domain and range of *f*.

b) Determine the intervals where *f* is increasing and decreasing.

Solution:

a) Domain: $(-\infty, +\infty)$ Range: $[1, +\infty)$

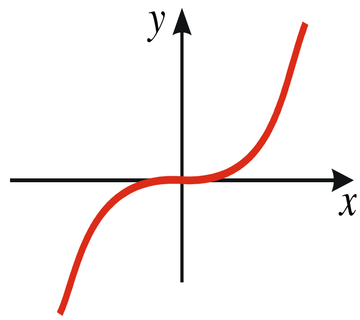
b) *f* is increasing on the interval $[1, +\infty)$ and decreasing on $(-\infty, 1]$.

Example 2: Use transformations of basic functions to graph $y = -x^3 - 2$. State the basic function, and each transformation applied. Use the graph to determine the following:

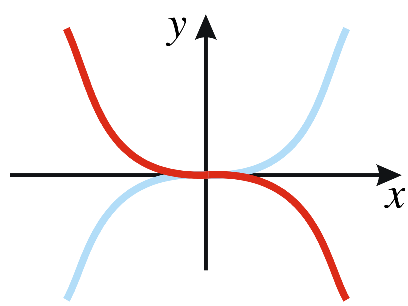
a) domain and range
b) intervals where y is increasing and decreasing.
c) symmetry

Solution: $y = -x^3 - 2$

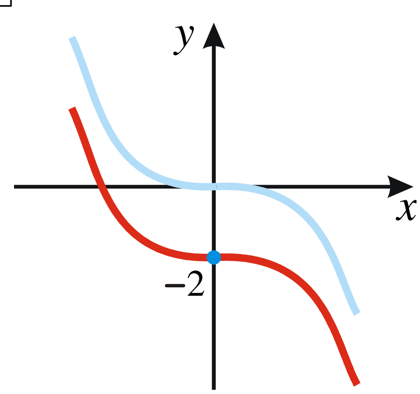
Basic function: cubing function: $y = -\boxed{x^3} - 2$



Reflect about the x-axis: $y = \boxed{-}x^3 - 2$



Vertical shift down 2: $y = -x^3\boxed{-2}$



a) Domain: $(-\infty, +\infty)$ Range: $(-\infty, +\infty)$

b) Decreasing: $(-\infty, +\infty)$

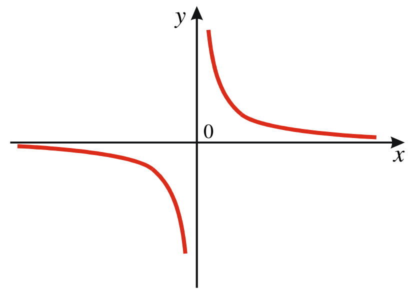
c) No symmetry about the axes or origin. There is symmetry about the point $(0, -2)$.

Example 3: Use transformations of basic functions to graph $y = \frac{1}{x-2} + 3$. State the basic function, and each transformation applied. Use the graph to determine the following:

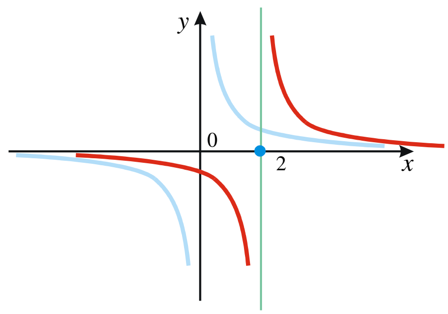
a) domain and range
b) intervals where y is increasing and decreasing.
c) symmetry

Solution:

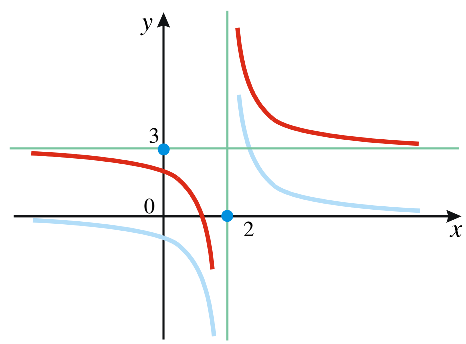
Basic function: reciprocal function: $y = \boxed{\frac{1}{x-2}} + 3$



Translate 2 units to the right: $y = y = \frac{1}{\boxed{x-2}} + 3$



Vertical shift 3 units up: $y = \frac{1}{x-2} + \boxed{3}$



a) Domain: $(-\infty, 2) \cup (2, \infty)$ Range: $(-\infty, 3) \cup (3, \infty)$

b) Decreasing on $(-\infty, 2)$ and decreasing on $(2, \infty)$

c) No symmetry about the axes or origin. There is symmetry about the point $(2, 3)$.

Example 4: Use transformations of basic functions to graph $y = 2|x+1|$. State the basic function, and each transformation applied. Use the graph to determine the following:

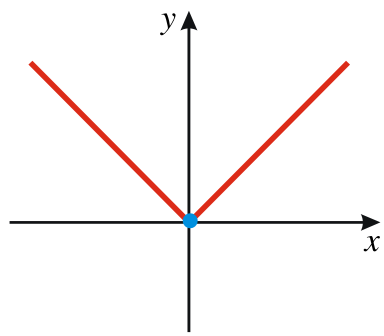
a) domain and range

b) intervals where y is increasing and decreasing.

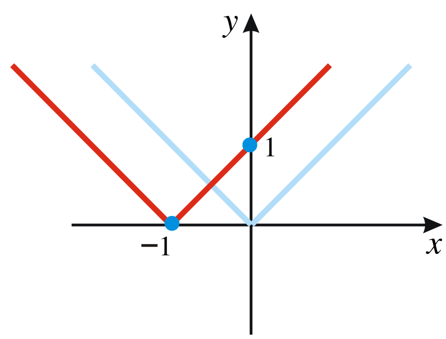
c) symmetry

Solution:

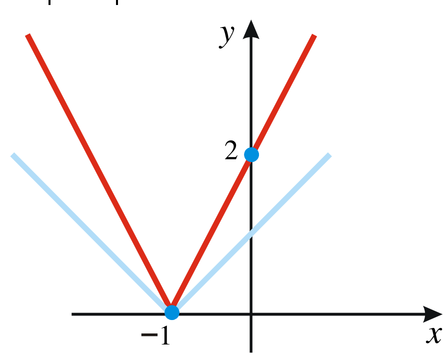
Basic function: absolute value: $y = 2\boxed{|x+1|}$



Horizontal shift 1 unit left: : $y = 2|x+1|$



Vertical stretch by factor of 2: $y = 2|x+1|$



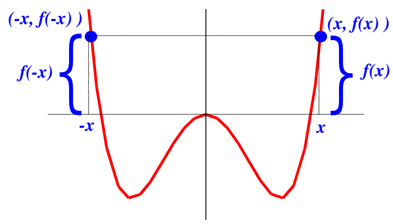
a) Domain: $(-\infty, +\infty)$ Range: $[0, +\infty)$

b) Decreasing: $(-\infty, -1]$ Increasing: $[-1, +\infty)$

c) No symmetry about the axes or origin. There is symmetry about the line $x = -1$.

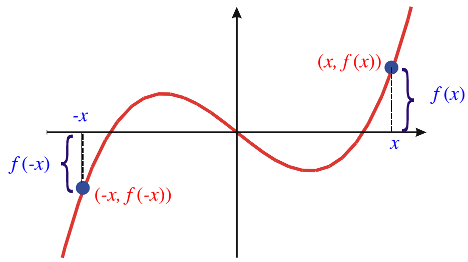
Even and Odd Functions

Definition: A function is **even** if and only if $f(-x) = f(x)$, for all x in the domain of f. The graph of an even function has symmetry about the y-axis.



For every pair x and its opposite $-x$, the functional values are equal: $f(-x) = f(x)$.

Definition: Function f is said to be **odd** if and only if $f(-x) = -f(x)$ for every x in the domain of f. The graph of an odd function has symmetry about the origin.



For each x and its opposite $-x$, the functional values are opposites: $f(-x) = -f(x)$.

To determine whether a function is even, odd or neither, evaluate $f(-x)$.

- If $f(-x) = f(x)$, the function is even.
- If $f(-x) = -f(x)$, the function is odd.
- If neither of the above is true, the function is neither even nor odd.

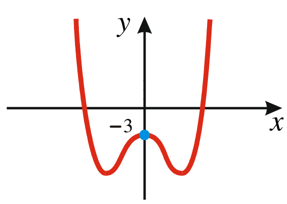
Example 1: Prove the following algebraically and confirm graphically.

a. $f(x) = x^4 - 4x^2 - 3$ is an even function.

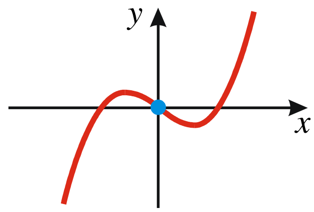
b. $f(x) = x^3 - 4x$ is an odd function.

Solution:

a. We must show that $f(-x) = f(x)$, so we evaluate $f(-x)$:
$f(-x) = (-x)^4 - 4(-x)^2 = x^4 - 4x^2 = f(x)$. Therefore, f is an even function. The graph appears to have symmetry about the y-axis.



b. We must show that $f(-x) = -f(x)$ so we evaluate $f(-x)$:
$f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x = -(x^3 - 4x) = -f(x)$. Therefore, f is an odd function. The graph appears to have symmetry about the origin.



Example 2: Determine whether the following functions are even, odd, or neither.

a. $f(x) = \dfrac{2x^4}{x^2 - 1}$

b. $f(x) = 3x^5 - 4x^4 + 3x - 1$

c. $f(x) = x|x^2 + 5|$

Solution: Evaluate $f(-x)$:

a. $f(-x) = \dfrac{2(-x)^4}{(-x)^2 - 1} = \dfrac{2x^4}{x^2 - 1} = f(x) \quad \Rightarrow \quad f$ is even.

b. $f(-x) = 3(-x)^5 - 4(-x)^4 + 3(-x) - 1 = -3x^5 + 4x^4 - 3x - 1$

$-f(x) = -(3x^5 - 4x^4 + 3x - 1) = -3x^5 + 4x^4 - 3x + 1$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x) \Rightarrow f$ is neither even nor odd.

c. $f(-x) = (-x)\left|(-x)^2 + 5\right| = -x|x^2 + 5| = -(x|x^2 + 5|) = -f(x) \Rightarrow f$ is an odd function.

Exercises for Chapter 4C - Transformations of Functions

1. Determine whether each of the following will involve a horizontal or vertical shift and state the number of units and direction.

a. $f(x) = \sqrt{x+3}$

b. $f(x) = 3 + \sqrt{x}$

c. $f(x) = x - 2$

d. $f(x) = (x+5)^2$

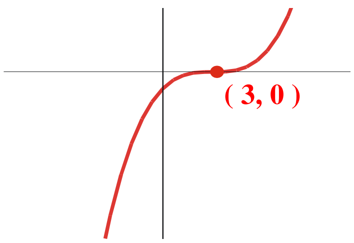
e. $f(x) = x^2 + 5$

f. $f(x) = \frac{1}{x} - 2$

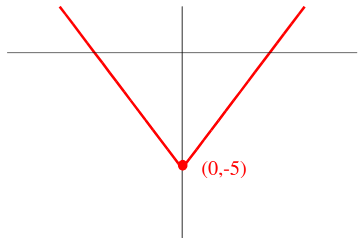
g. $f(x) = \frac{1}{x-2}$

2. Each of the following is a transformation of a basic function. Write the rule.

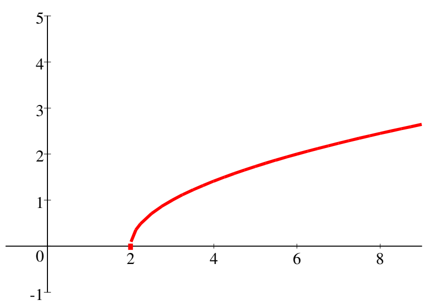
a)



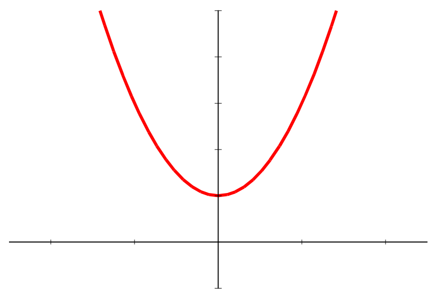
b)



c)



d)



3. Determine whether each of the following is symmetric about the x- or y-axis.

a. $f(x) = -x^2$

b. $f(x) = \sqrt{-x}$

c. $f(x) = |-x|$

d. $f(x) = (-x)^2$

e. $f(x) = -\sqrt{x}$

4. Describe the relationship between f and g in terms of basic functions and their reflections. Graph each pair of functions on the same axes.

a. $f(x) = x^2$ and $g(x) = (-x)^2$

b. $f(x) = |x|$ and $g(x) = |-x|$

c. What did you observe in each case? Explain why this is true in these cases.

5. Describe the relationship between f and g in terms of basic functions and their reflections. Graph each pair on the same axes.

a. $f(x) = -x^3$ and $g(x) = (-x)^3$.

b. $f(x) = \frac{1}{-x}$ and $g(x) = -\frac{1}{x}$.

c. What did you observe in each case? Explain why this is true in these cases.

6. Determine whether each of the following will be a vertical stretch or shrink of a basic function.

a. $f(x) = 2x$

b. $f(x) = \frac{1}{3}x^3$

c. $f(x) = 0.5x^2$

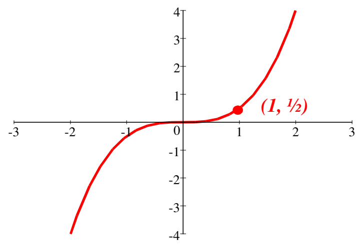
d. $f(x) = 2|x|$

e. $f(x) = \dfrac{3}{x}$

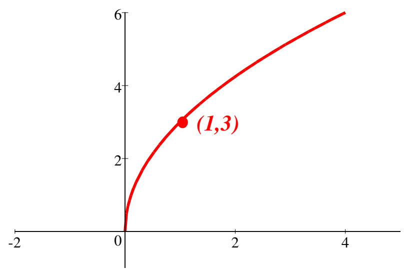
f. $f(x) = \dfrac{x^2}{2}$

7. Write the rule for each.

a.



b.



8. Identify the basic function and the transformations you used to graph the following:

a. $f(x) = (x-2)^2 + 3$

b. $f(x) = -\sqrt{x+1}$

c. $f(x) = -2x + 3$

d. $f(x) = -x^2 - 2$

e. $f(x) = 2|x| - 1$

f. $f(x) = \dfrac{1}{x+2} - 1$

9. Use the graph $f(x) = -2|x+1| - 3$ to determine the following:

a. Domain

b. Range

c. Intervals where *f* is increasing.

d. Intervals where *f* is decreasing.

e. Symmetry about any line or point?

10. Determine algebraically whether each of the following functions is even, odd, or neither.

a. $f(x) = 4x^4 - 3x^2 + 3$

b. $f(x) = x^3 - 5x - 1$

c. $f(x) = \dfrac{5x}{x^2+1}$

d. $f(x) = 5x^3 - 3x$

e. $f(x) = -|x| + 2$

f. $f(x) = \sqrt{x} + 2$

g. $f(x) = \dfrac{5x^2}{x+1}$

11. Use symmetry to determine whether each basic function is even, odd, or neither.

a. $y = 5$

b. $y = x$

c. $y = x^2$

d. $y = x^3$

e. $y = |x|$

f. $y = \frac{1}{x}$

g. $y = \sqrt{x}$

12. Discuss the relationship between symmetry, reflections about the axes, and even/odd functions.

Answers to Exercises for Chapter 4C - Transformations of Functions

1. **a.** horizontal shift 3 to the left.
b. vertical shift up 3.
c. vertical shift down 2.
d. horizontal shift 5 to the left.
e. vertical shift 5 up.
f. vertical shift 2 down.
g. horizontal shift 2 to the right.

2. a) $f(x) = (x-3)^3$ b) $f(x) = |x| - 5$ c) $f(x) = \sqrt{x-2}$ d) $f(x) = x^2 + 2$

3. Determine whether each of the following is symmetric about the x- or y-axis.
a. x-axis
b. y-axis
c. y-axis
d. y-axis
e. x-axis

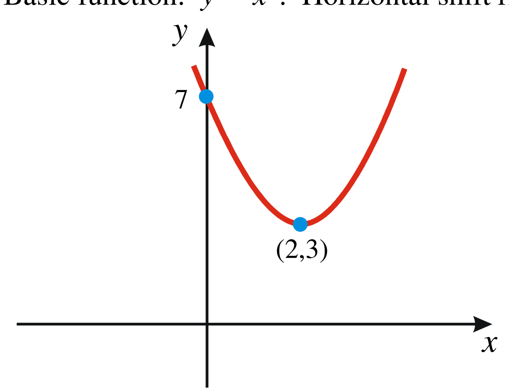
4. In each set, f is a basic function and g is the reflection of f about the y-axis. Graphs of f and g are the same in both cases.

5. In each set, one of the functions is the reflection of a basic function about the x-axis and the other is the reflection about the y-axis. Both graphs are the same.

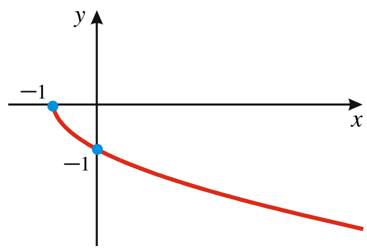
6. Determine whether each of the following will be a vertical stretch or shrink of a basic function.
a. stretch by a factor of 2.
b. shrink by a factor of 3.
c. shrink by a factor of 2. Why 2?
d. stretch by a factor of 2
e. stretch by a factor of 3.
f. shrink by a factor of 2.

7.
a. $f(x) = \frac{1}{2}x^3$
b. $f(x) = 3\sqrt{x}$

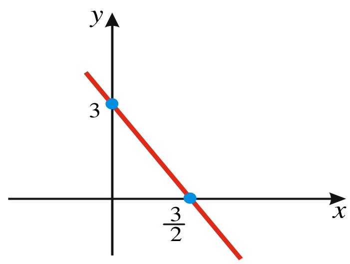
8. Identify the basic function and the transformations you used to graph the following:
a. Basic function: $y = x^2$. Horizontal shift right 2. Vertical shift up 3



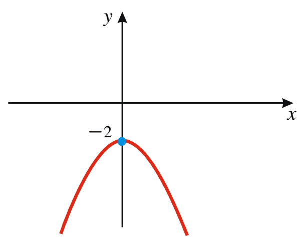
b. Basic function: $y = \sqrt{x}$. Horizontal shift left 1. Reflection about x-axis



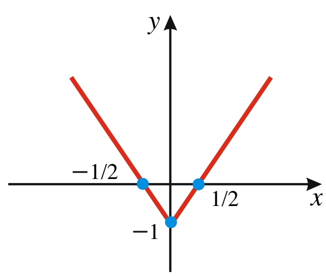
c. Basic function: $y = x$. Vertical shift up 3. Reflect about x-axis. Stretch by a factor of 2



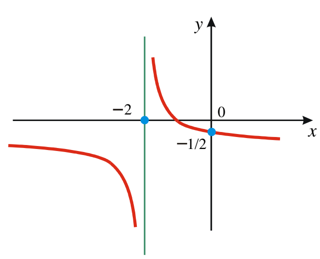
d. Basic function: $y = x^2$. Vertical shift down 2. Reflect about the x-axis



e. Basic function: $y = |x|$. Vertical shift down 1. Stretch by a factor of 2



f. Basic function: $y = \frac{1}{x}$. Horizontal shift left 2. Vertical shift down 1



9.

a. $(-\infty, +\infty)$
b. $(-\infty, -3]$
c. $(-\infty, -1]$
d. $[-1, +\infty)$
e. Symmetry about the line $x = -1$.

10.

- **a**. $f(-x) = 4(-x)^4 - 3(-x)^2 + 3 = 4x^4 - 3x^2 + 3 = f(x)$. Thus, $f(x)$ is an even function.
- **b**. neither
- **c**. odd
- **d**. odd
- **e**. even
- **f**. neither
- **g**. neither

11.

- **a**. even
- **b**. odd
- **c**. even
- **d**. odd
- **e**. even
- **f**. odd
- **g**. neither

12. An even function is symmetric with respect to the y-axis. An odd function is symmetric about the origin.

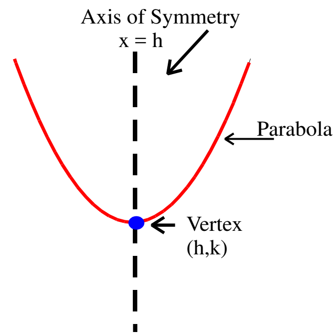
Chapter 4D - Maximum/Minimum Function Values

Quadratic Functions

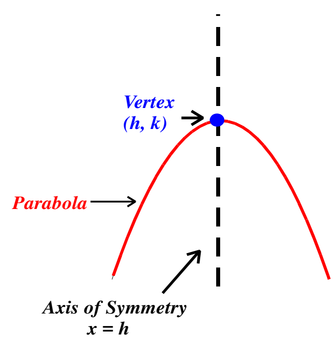
In the preceding section we graphed the function $f(x) = (x-1)^2 + 2$ by transforming the graph of the squaring function $y = x^2$. If we expand $(x-1)^2 + 2$ we get $f(x) = x^2 - 2x + 1 + 2 \Rightarrow f(x) = x^2 - 2x + 3$.

- **Definition**: A **quadratic function** is one that can be written in the form, $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. This form is called the **general form** of a quadratic function.

The graph of a quadratic function is a parabola.



- **Definition**: If the parabola opens up, the vertex (h, k) is the lowest point on the parabola. We say that k is the **minimum functional value of** f. It occurs when $x = h$. If the parabola opens down, k is the **maximum functional value of** f, and occurs when $x = h$.



- **Definition**: Maximum and minimum functional values are called **extreme functional values**.

The extreme values for a quadratic function are easy to find if the function is written in **standard form**.

- **Definition**: A quadratic function in the form $f(x) = a(x-h)^2 + k$, where $a \neq 0$, is said to be in **standard form**.

We can apply our knowledge of transformations to a quadratic function in standard form to easily graph the function and determine the extreme functional values.

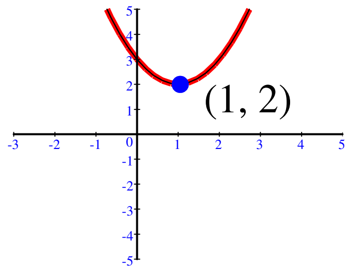
- If $a > 0$, the graph opens up; if $a < 0$, the graph opens down (reflection about the x-axis).
- Horizontal shift h units.
- Vertical shift k units.
- If $|a| > 1$, vertical stretch by a factor of $|a|$; if $0 < |a| < 1$, vertical shrink by a factor of $|a|$.

Example 1: Graph $f(x) = (x-1)^2 + 2$ by transforming $y = x^2$.

Solution:

- $a = 1 > 0 \Rightarrow$graph opens up
- $h = 1 \Rightarrow$horizontal shift right 1

- $k = 2 \Rightarrow$vertical shift up 2



We can inspect the graph to find that

- f has a vertex at the point $(1, 2)$
- the axis of symmetry is the line $x = 1$;
- domain = $(-\infty, +\infty)$ and range = $[2, +\infty)$.
- 2 is the minimum functional value of f, and occurs when $x = 1$.
- f is decreasing on $(-\infty, 1]$ and increasing on $[1, +\infty)$.

Converting quadratics from general form to standard form.

Example 2: Write $f(x) = x^2 + 10x + 16$ in standard form.

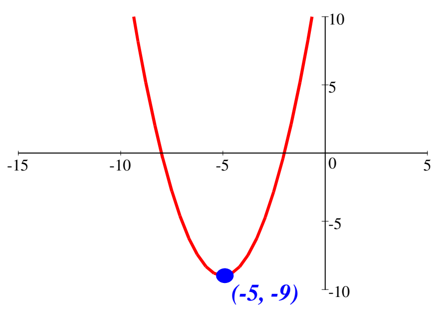
- Group the x-terms together: $f(x) = (x^2 + 10x \quad) + 16$
- Complete the square on the x-terms: Add $\left(\frac{1}{2} \cdot 10\right)^2 = 25$ to $x^2 + 10x$ to make it a perfect square trinomial.
- Add and subtract the computed number to complete the square:

$$\begin{aligned} f(x) &= \left(x^2 + 10x + \boxed{25}\right) + 16 - \boxed{25} \\ &= (x+5)^2 - 9 \end{aligned}$$

- $f(x)$ is now written in standard form.

Example 3: Determine the domain and range of $f(x) = x^2 + 10x + 16$ using its standard form derived above. Does f have a maximum or minimum functional value? What is the extreme functional value and for which value of x does it occur? Find the intervals where f is increasing and decreasing.

Solution: The graph of f is the parabola $y = x^2$ shifted 5 units to the left, and shifted down 9 units. $a > 0 \Rightarrow$the parabola opens up$\Rightarrow$ f has a minimum functional value of -9 when $x = -5$.



Domain: $(-\infty, +\infty)$

Range: $(-5, -9)$

f has a minimum functional value of -9

It occurs when $x = -5$.

Increasing: $(-5, +\infty)$

Decreasing: $(-\infty. -5)$

Example 4: Determine whether $f(x) = -2x^2 + 8x + 3$ has a maximum or minimum functional value. Write the function in standard form to determine its value.

Solution: Since $a = -2 < 0$, the parabola opens down $\Rightarrow f$ has a maximum functional value.

Write in standard form:

$$f(x) = \boxed{-2}(x^2 - 4x \quad) + 3 \qquad \text{Group } x\text{-terms together. Factor out } a = -2.$$

$$f(x) = \boxed{-2}\left(x^2 - 4x + \boxed{4}\right) + 3 + \boxed{8} \qquad 4(-2) = -8 \quad \text{was added and must be subtracted}$$

$$f(x) = -2(x-2)^2 + 11$$

Since we have written the quadratic function $f(x)$ in standard form it is easy to see that the vertex occurs at the point $(2, 11)$. Morover, since the coefficient of the squared term is negative, the parabola opens down. Thus, the function value $f(2) = 11$ is the maximum value of this function.

$$\boxed{\text{Maximum functional value is 11 when } x = 2.}$$

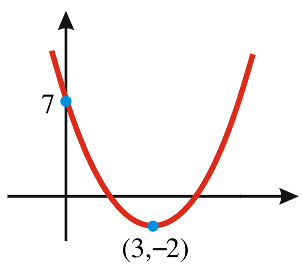
Example 5: Write $f(x) = x^2 - 6x + 7$ in standard form. Draw the graph and find a) vertex, b) axis of symmetry, c) domain, d) range, e) the minimum or maximum functional value, f) intervals where f is increasing and decreasing.

Solution: Group the x-terms together: $f(x) = (x^2 - 6x \quad) + 7$

Complete the square on $x^2 - 6x$: $\left[\text{Add } \left(\frac{1}{2} \cdot (-6)\right)^2 = (-3)^2 = 9\right]$

If we add 9, we must subtract it also to keep the equation equivalent:

$$f(x) = (x^2 - 6x + \mathbf{9}) + 7 - \mathbf{9}$$
$$= (x-3)^2 - 2$$



$a = 1$	Opens up
$h = 3$	Horizontal shift right 3
$k = -2$	Vertical shift down 2

a) vertex: $(3, -2)$ b) axis of symmetry: $x = 3$ c) domain: $(-\infty, +\infty)$ d) range: $[-2, +\infty)$.

e) Since the graph opens up f will have a minimum functional value -2 occurring when $x = 3$.

f) f is increasing on $[3, +\infty)$ and decreasing on $(-\infty, 3]$.

Example 6: Write $f(x) = -2x^2 + 10x - 7$ in standard form. Draw the graph and find a) vertex, b) axis of symmetry, c) domain, d) range, e) the minimum or maximum functional value, f) intervals where f is increasing and decreasing.

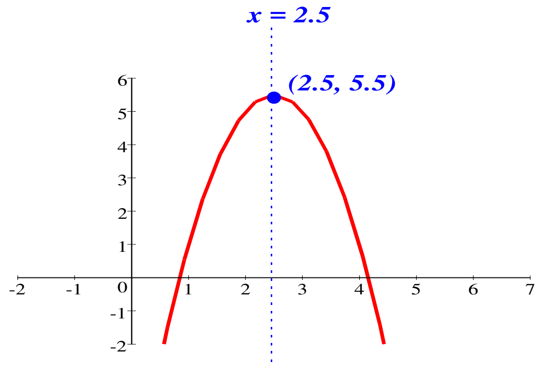
Solution: $f(x) = -2x^2 + 10x - 7$ Factor out -2 from the x terms.

$f(x) = -2(x^2 - 5x) - 7$ Complete the square: $\left[\frac{-5}{2}\right]^2 = \frac{25}{4}$

$f(x) = \boxed{-2}\left(x^2 - 5x + \boxed{\frac{25}{4}}\right) - 7 + \boxed{\frac{25}{2}}$ Note that $\frac{25}{4}$ was multiplied by -2 so that we

$\nwarrow \quad \nearrow \quad \uparrow$ really subtracted $\frac{25}{2}$. Therefore we add $\frac{25}{2}$.

$$\boxed{f(x) = -2\left(x - \frac{5}{2}\right)^2 + \frac{11}{2}} \qquad \text{Standard form}$$



$\Rightarrow$ $\begin{cases} a = -2 < 0 & \text{Opens down} \\ & \text{Vertical stech by factor of 2} \\ h = \frac{5}{2} = 2.5 & \text{Horizontal shift right } \frac{5}{2} \\ k = \frac{11}{2} = 5.5 & \text{Vertical shift up } \frac{11}{2} \end{cases}$

a) vertex: $(2.5, 5.5)$ b) axis of symmetry: $x = 2.5$ c) domain: $(-\infty, +\infty)$ d) range: $(-\infty, 5.5]$
e) Graph opens down $\Rightarrow$ maximum functional value: 5.5 when $x = 2.5$.
f) f is increasing on $(-\infty, 2.5]$ and decreasing on $[2.5, +\infty)$.

By writing the general quadratic function $f(x) = ax^2 + bx + c$ in standard form, we can derive a formula to find the extreme value for a quadratic function.

- Group the x-terms and factor out a: $f(x) = a(x^2 + \frac{b}{a}x \quad) + c$
- Complete the square: Add $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$
- Add and subtract: $f(x) = a(x^2 + bx + \frac{b^2}{4a^2}) + c - a \cdot \frac{b^2}{4a^2}$
- Rewrite: $f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

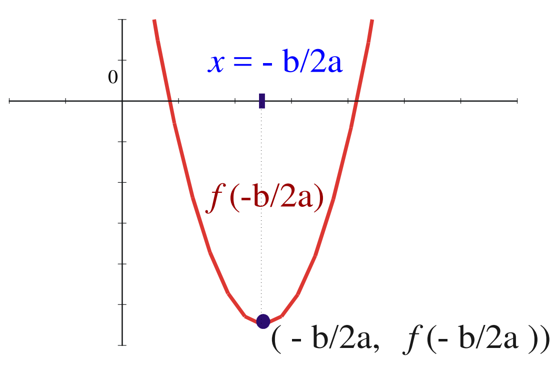
Therefore, the graph of $f(x) = a\left(x - \boxed{\frac{-b}{2a}}\right)^2 + \boxed{c - \frac{b^2}{4a}}$ will be the graph of $y = x^2$ with the following transformations:

Horizontal shift $\frac{-b}{2a}$. Vertical shift $c - \frac{b^2}{4a}$. Vertical stech or shrink by a factor of $|a|$. Graph opens up if a is positive; graph opens down if a is negative. The vertex of the parabola occurs when $x = -\frac{b}{2a}$: $\left(-\frac{b}{2a},\ ?\ \right)$.
To find the y-coordinate of the vertex, evaluate

$$f\left(-\frac{b}{2a}\right) = c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a}$$

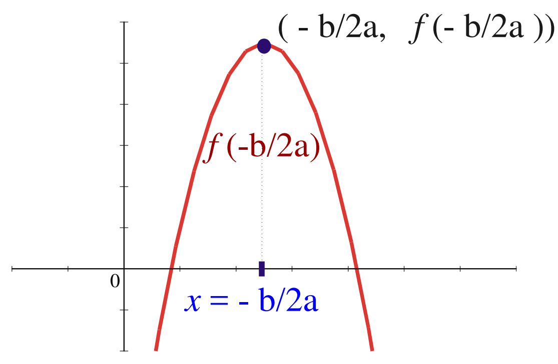
Note this value is the amount of the vertical shift.

If $a > 0$, the parabola will open up.



minimum functional value $f\left(-\frac{b}{2a}\right)$ when $x = -\frac{b}{2a}$

If $a < 0$, the parabola will open down.



maximum functional value $f\left(-\frac{b}{2a}\right)$ when $x = -\frac{b}{2a}$

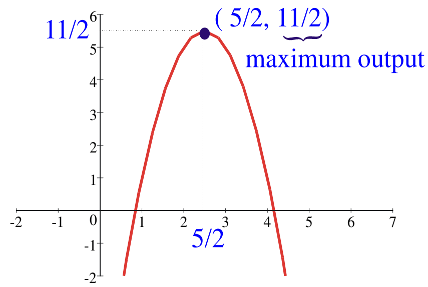
Example 7: Find the maximum/minimum functional value of $f(x) = -2x^2 + 10x - 7$.
Solution: Since $a = -2 < 0$, the parabola will open down. The vertex will be a maximum point on the graph of f.
The x-coordinate of the vertex can be found by $x = -\frac{b}{2a}$.

$$x = -\frac{10}{2(-2)} = -\frac{10}{-4} = \frac{5}{2}.$$

The y-coordinate of the vertex can be found by evaluating $f\left(\frac{5}{2}\right)$.

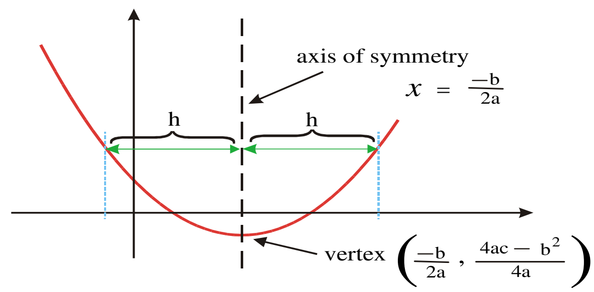
$$f\left(\frac{5}{2}\right) = -2\left(\frac{5}{2}\right)^2 + 10\left(\frac{5}{2}\right) - 7 = \frac{-25}{2} + 25 - 7 = \frac{11}{2}.$$

Therefore, the largest value that $f(x) = -2x^2 + 10x - 7$ will have is $\frac{11}{2}$, which is the output for input $\frac{5}{2}$.



Axis of Symmetry

It may not be clear from the algebraic expression for a quadratic function, $f(x)$, that the vertical line $x = \frac{-b}{2a}$ is an axis of symmetry, but it should be clear from the graph below. By saying that the graph of $f(x)$ is symmetric with respect to this vertical line, we mean that if the graph is rotated 180 degrees about this line, we get the same graph back.



Another way to say that the graph of $f(x)$ is symmetric with respect to the vertical line $x = \frac{-b}{2a}$ is to note that the functional values for $f(x)$ must be the same for values of x which are the same distance away from $\frac{-b}{2a}$. That is, we should have for any number h

$$f\left(\frac{-b}{2a} + h\right) = f\left(\frac{-b}{2a} - h\right)$$

The next lines demonstrate that this is indeed the case for quadratic functions.

$$f\left(\frac{-b}{2a} + h\right) = a\left(\left(\frac{-b}{2a} + h\right) + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = ah^2 - \frac{b^2}{4a} + c$$

$$f\left(\frac{-b}{2a} - h\right) = a\left(\left(\frac{-b}{2a} - h\right) + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = ah^2 - \frac{b^2}{4a} + c$$

Thus, the values of $f(x)$ are the same for points equidistant from $\frac{-b}{2a}$.

Example 8: Let $f(x) = -3x^2 + 5x - 2$. Find the axis of symmetry and the coordinates of the vertex.

Solution: We first use the formulas for determining the axis and vertex. In this function $a = -3$, $b = 5$, and $c = -2$. Thus, we have

$$\text{axis of symmetry : } \frac{-b}{2a} = \frac{-5}{-6} = \frac{5}{6}$$

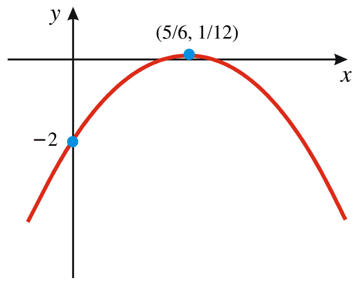
$$\text{vertex : } \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) = \left(\frac{5}{6}, \frac{24 - 25}{-12}\right) = \left(\frac{5}{6}, \frac{1}{12}\right)$$

Since the coefficient of x^2 is negative, the value $\frac{1}{12}$ is the maximum value of $f(x)$. That is, the graph opens downward.

To reinforce these ideas we complete the square once again.

$$\begin{aligned} -3x^2 + 5x - 2 &= -3\left(x^2 - \frac{5}{3}x\right) - 2 \\ &= -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) + \frac{25}{12} - 2 \\ &= -3\left(x - \frac{5}{6}\right)^2 + \frac{1}{12} \end{aligned}$$

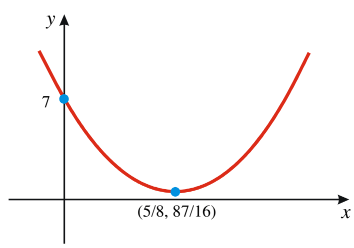
We see once again that $x = \frac{5}{6}$ is the axis of symmtry and the vertex of this parabola is located at $\left(\frac{5}{6}, \frac{1}{12}\right)$. A plot of this quadratic function follows.



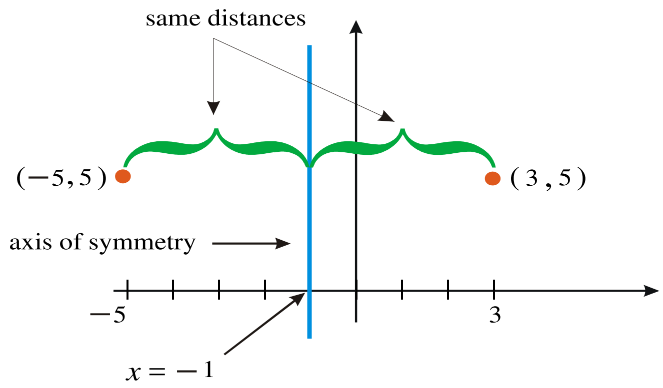
Example 9: Let $f(x) = 4x^2 - 5x + 7$. Determine the axis of symmetry and vertex of this quadratic function.
Solution: We could of course just plug into the formulas, but it is more instructive, and in the long run easier to just complete the square. So one more time

$$\begin{aligned} 4x^2 - 5x + 7 &= 4\left(x^2 - \frac{5}{4}x\right) + 7 \\ &= 4\left(x^2 - \frac{5}{4}x + \frac{25}{64}\right) - \frac{25}{16} + 7 \\ &= 4\left(x - \frac{5}{8}\right)^2 + \frac{87}{16} \end{aligned}$$

Thus, the axis of symmetry is the vertical line $x = \frac{5}{8}$, and the vertex is the point $\left(\frac{5}{8}, \frac{87}{16}\right)$. Since the coefficient of x^2 is positive, the parabola opens upward, and $\frac{87}{16}$ is the smallest value of the function. A plot of $f(x)$ for $0 \leq x \leq 1$ is shown below.



Example 10: If $f(3) = 5, f(-5) = 5$, and $f(x)$ is a quadratic function, what is the axis of symmetry?
Solution: We know that if two points a and b are equidistant from the axis of symmetry, then $f(a) = f(b)$. Here we know that $f(3) = f(-5)$. For a quadratic function, this can only happen if 3 and −5 are the same distance from the axis of symmetry. Thus, the axis of symmetry must lie half way between 3 and −5. That is, $x = \frac{3 + (-5)}{2} = -1$ is the axis of symmetry. We cannot determine what $f(-1)$ equals, but we do know that $f(-1)$ is either the maximum or minimum value of $f(x)$.



Zeros of Quadratic Functions

By a zero of a quadratic function $f(x) = ax^2 + bx + c$, we mean a number x_0 such that $f(x_0) = ax_0^2 + bx_0 + c = 0$.

There are two ways to find the zeros of a quadratic function. The first, and easiest, is to factor the quadratic expression if you can. The second, and this always works, is to use the quadratic formula. Recall, if the expression $ax^2 + bx + c$ equals zero, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 11: Find the zeros of $f(x) = x^2 - x - 2$ by factoring.

Solution: The integer factors of −2 are 1 and 2 with one of them being negative. So we try

$$x^2 - x - 2 = (x-1)(x+2). \text{ (Not correct)}$$

However, when we multiply the two factors together to see if we've got it correct, we compute

$$(x-1)(x+2) = x^2 - x + 2x - 2 = x^2 + x - 2,$$

which is not what we want. The coefficient of x is off by a minus sign. So we try

$$x^2 - x - 2 = (x+1)(x-2),$$

and this is correct. The zeros of $x^2 - x - 2$ are the solutions of the following equation

$$x^2 - x - 2 = (x+1)(x-2) = 0.$$

The only way a product can equal zero is for one of the factors to equal zero. Thus, we have

$$x + 1 = 0 \text{ or}$$
$$x - 2 = 0.$$

From which we conclude that

$$x = -1 \text{ or}$$
$$x = 2$$

Notice that there are exactly two zeros.

Example 12: What are the zeros of $x^2 - 5x + 6$?

Solution: $x^2 - 5x + 6 = (x-2)(x-3)$. Thus the zeros of this quadratic function are

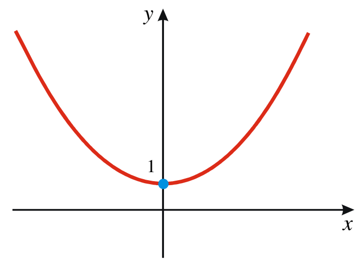
$$x = 2$$
$$x = 3.$$

Example 13: Find the zeros of $f(x) = x^2 - x - 2$ by using the quadratic formula.

Solution: We are looking to find those x for which $x^2 - x - 2 = 0$. The solutions of this equation are given by the quadratic formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-2)}}{2}$$
$$= \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}$$
$$= -1 \text{ or } 2.$$

We now discuss why some quadratic functions have no zeros. If we graph the quadratic function $f(x) = x^2 + 1$, we see that it never crosses the x-axis. This means that $f(x)$ never equals zero, or this function has no zeros.



It is instructive to use the quadratic formula to find the zeros of $f(x) = x^2 + 1$.

$$x = \frac{0 \pm \sqrt{0-4}}{2} = \frac{\pm 2\sqrt{-1}}{2} = \pm\sqrt{-1}.$$

Since we cannot take the square root of a negative number and get a real number, we see that there is no real number x for which $x^2 + 1 = 0$. That is $f(x) = x^2 + 1$ has no zeros.

Example 14: Does $f(x) = 3x^2 - 5x + 6$ have any zeros?

Solution: Using the quadratic formula to solve the equation $3x^2 - 5x + 6 = 0$ we have :

$$x = \frac{5 \pm \sqrt{25 - 72}}{2}$$
$$= \frac{2 \pm \sqrt{-52}}{2}$$

Since negative numbers do not have real square roots, this quadratic function has no zeros.

In the table below we summarize the possibilities of zeros for an arbitrary quadratic function.

$f(x) = ax^2 + bx + c$

$b^2 - 4ac > 0$	two zeros
$b^2 - 4ac = 0$	one zero
$b^2 - 4ac < 0$	no zeros

- **Definition**: The expression $b^2 - 4ac$ is called the **discriminant** of the function $f(x) = ax^2 + bx + c$.

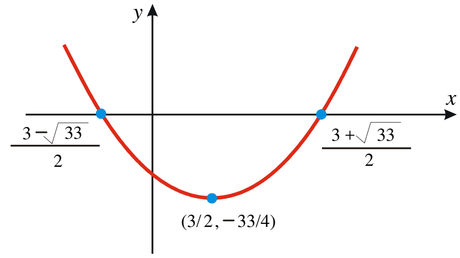
Example 15: Calculate the discriminant of $f(x) = x^2 - 3x - 6$ and plot the function.

Solution:

$$b^2 - 4ac = (-3)^2 - 4(1)(-6)$$
$$= 9 - (-24)$$
$$= 33.$$

Since the discriminant is positive we know that $f(x) = x^2 - 3x - 6$ has two zeros

$$x = \frac{3 \pm \sqrt{33}}{2} = \frac{3 \pm 5.7446}{2} \approx -1.37 \text{ and } 4.37$$

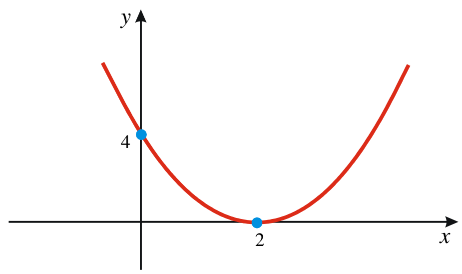


Example 16: Calculate the discriminant of $f(x) = x^2 - 4x + 4$, and plot the function.
Solution:

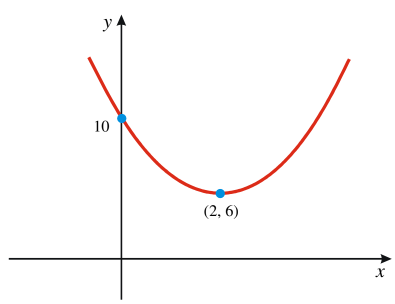
$$b^2 - 4ac = (-4)^2 - 4(1)(4) = 16 - 16 = 0.$$

Since the discriminant is zero, there is only one zero, and it equals

$$x = \frac{-(-4) \pm \sqrt{0}}{2} = 2.$$



Example 17: Calculate the discriminant of $f(x) = x^2 - 4x + 10$, and plot the function.
Solution: $b^2 - 4ac = (-4)^2 - 4(1)(10) = 16 - 40 = -24$.
Since the discriminant is negative, this quadratic function has no zeros.



Applications

Situations abound in many fields where the need to know what the maximum/minimum value of a function is, and for which value of the independent variable it occurs. If the function is a quadratic function, the extreme values can be found by using $x = \frac{-b}{2a}$ to find the value of x where the extreme point occurs. The maximum or minimum functional value can be found by evaluating $f\left(\frac{-b}{2a}\right)$.

Example 1: University T-shirts has determined that they sell fewer " Blinn College" t-shirts as they increase the price p. In fact, the equation $p = -0.02x + 30$, where x represents the number of shirts sold, models the relation between the number of shirts sold and the price of the shirts. The total revenue R for the t-shirts can be written as

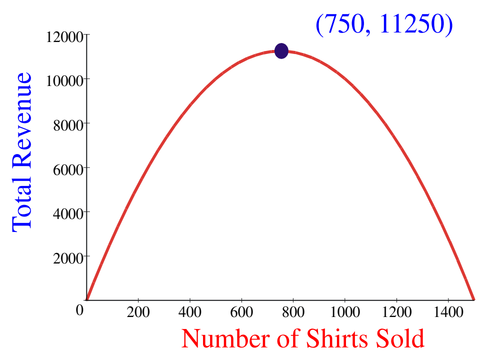
$$R(x) = xp$$
$$= x(-0.02x + 30)$$

For how many shirts sold will the maximum revenue occur? What price should be placed on the shirts to receive the maximum revenue?

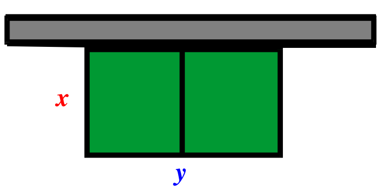
Solution: Since $R(x) = x(-0.02x + 30) = -0.02x^2 + 30x$ is a quadratic function which opens down $(a = -0.02 < 0)$, R will indeed have a maximum functional value, and it will occur when $x = \frac{-b}{2a}$.

$x = \frac{-30}{2(-.02)} = 750$ ⇒the maximum revenue will occur when 750 shirts are sold. The maximum revenue will be $R(750) = -0.02(750)^2 + 30(750)$ =\$11250. The price p that should be charged to maximize the revenue is $p(750) = -0.02(750) + 30 = 15 \Rightarrow \$15$.

Confirm Graphically:



Example 2: A farmer wishes to fence a rectangular area adjacent to a straight brick wall. He doesn't need to fence along the wall, but he wants to divide the fenced area into two smaller areas as shown below. If he has a total of 4000 feet of fencing material, find the maximum area he can enclose.



4000 feet of fencing⇒ $4000 = y + 3x \Rightarrow y = 4000 - 3x$

Solution: Write the area (the variable that we want to maximize) as a function of one of the sides.

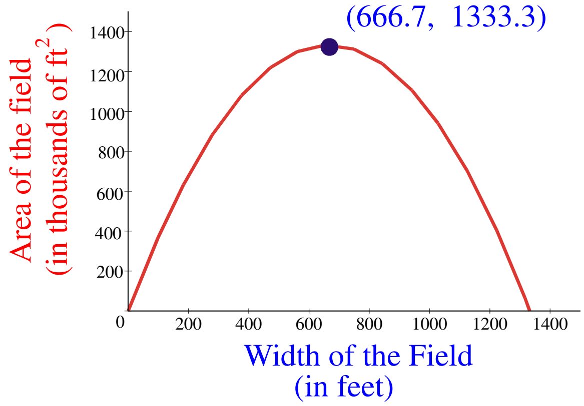
- Let x = the number of feet in one side of the rectangle, y = the number of feet in the other side, and A = the number of square feet in the area. Then $A = xy$.
- Substitute $y = 4000 - 3x$ for y: $A = x(4000 - 3x) \Rightarrow A(x) = 4000x - 3x^2$.
- The function A is a quadratic function–a parabola that opens down $(a = -3 < 0)$
  $x = \frac{-b}{2a} \Rightarrow x = \frac{-4000}{2(-3)} = \frac{2000}{3} \approx 666.67$ ⇒The maximum area occurs when the width is

$\frac{2000}{3}\ ft \approx 666.67\ ft.$

$y = 4000 - 3\left(\frac{2000}{3}\right) = 2000 \Rightarrow$ The length of the fence is 2000 ft.

- The maximum area that can be fenced is $A\left(\frac{2000}{3}\right) = \frac{2000}{3}(2000) = \frac{4000000}{3} \approx 1{,}333{,}333.33\ ft^2$

Confirm Graphically:



Note that the area is graphed in thousands of feet.

Example 3: During a Fourth of July fireworks show, a rocket was launched from the ground at a velocity of 60 ft/sec. If its distance in feet from the ground at time t in seconds is represented by $s(t)$. Then

$$s(t) = -16t^2 + 60t\ .$$

Find the maximum height the rocket reaches, and the number of seconds it takes to reach that height.

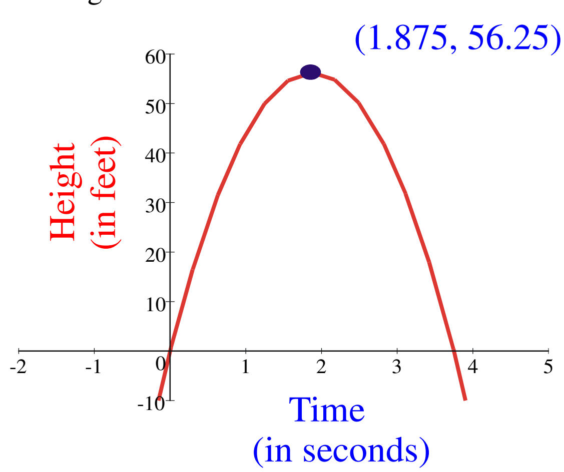
Solution: Since s is a parabola that opens down, it has a maximum height which occurs when $t = \frac{-b}{2a}$.

$$t = \frac{-60}{2(-16)} = \frac{15}{8} = 1.875$$

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) = \frac{225}{4} = 56.25$$

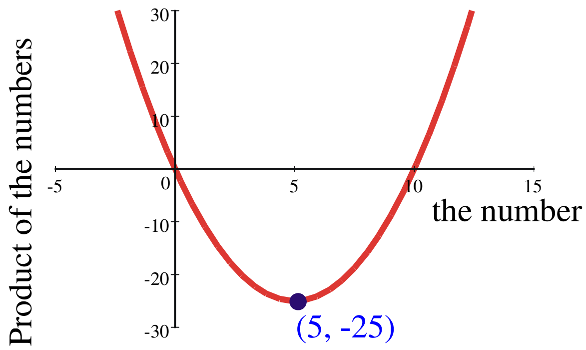
The rocket reaches a height of 56.25 feet in 1.875seconds.

Support graphically:



Example 4: What is the smallest product of two numbers if their difference is 10?

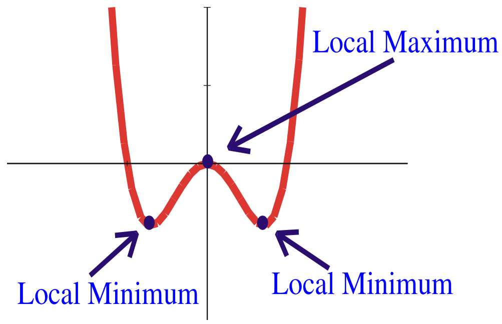
Solution: Let x = one number, y = the other number, P = the product of the two numbers. Since we want to minimize the product, we will write the function for P in terms of x: $P(x)$. We know that $P = xy$ so we use the fact that $x - y = 10$ to write y in terms of x: $y = x - 10$. Substituting for y: $P(x) = x(x - 10)$ or $P(x) = x^2 - 10x$. The graph of P is a parabola that opens up $\Rightarrow$ P has a minimum at $x = \dfrac{-b}{2a} = \dfrac{-(-10)}{2(1)} = 5$ and $P(5) = 5(5 - 10) = -25$ $\Rightarrow$The minimum product is -25 when $x = 5$. Note that $y = 5 - 10 = -5$.



Local Maximum and Minimum Values

Observe the graph below. If we were walking along the graph, moving from left to right, we would be traveling downhill at the beginning. We would reach a low-point and begin travelling uphill until we reached a high point from which we would travel downhill again. After the next low point we continue uphill for the rest of the way.

Critical points occur where the graph turns around. These are not necessarily the lowest or highest functional values for the entire graph, but they are high points or low points for a particular area of the graph. These points are called generically **local extreme points**, and may either be **local maximum** or **local minimum** points.



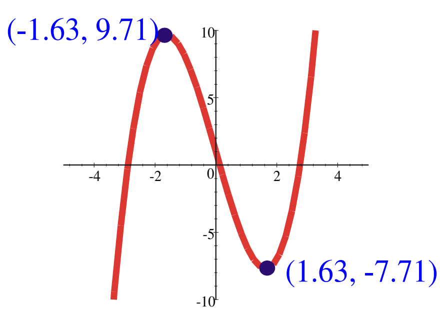
The graph changes from decreasing to increasing at local minimum points.

The graph changes from increasing to decreasing at local maximum points.

It takes the methods of calculus to determine these points without a calculator; however, we can approximate these values using a graphing calculator.

Example 1: Use a graphing calculator to find the local extrema rounded to the nearest hundredth for the function $f(x) = x^3 - 8x + 1$. Use the extreme points to determine the intervals where f is increasing and decreasing.

Solution:



local maximum functional value of 9.71 when $x \approx -1.63$

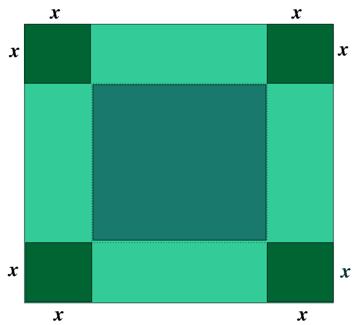
local minimum functional value of -7.71 when $x \approx 1.63$.

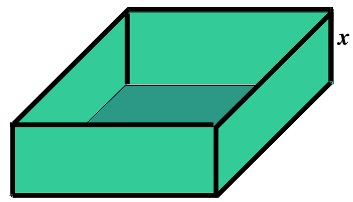
Increasing: $(-\infty, -1.63) \cup (1.63, +\infty)$

Decreasing: $(-1.63, 1.63)$

Local Extreme Values: Applications

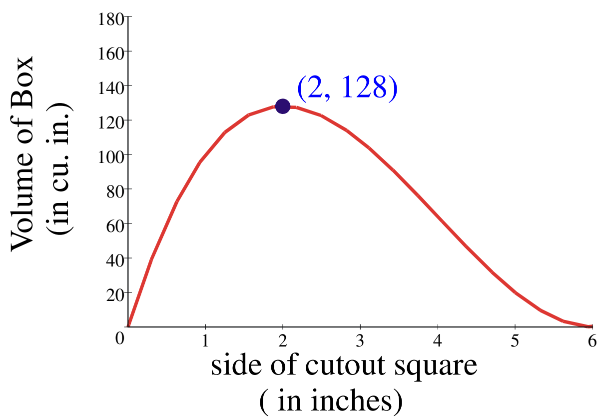
Example 1: An open box is constructed from a 12-inch square piece of cardboard by cutting squares of equal length from each corner and turning up the sides. What should the dimensions of the cutout square be for the box to have a maximum volume?





Solution: Express volume as a function of the length of the cutout square:

- Let x = number of inches on one side of the cutout square
  V = number of cubic inches in volume of the box
  Then $V(x) = x(12-2x)^2$
- The domain must be restricted to $[0,6]$ because the value of x represents a length and must be non-negative. Moreover, since the side of the piece of cardboard is 12 inches and we can cut no more than half of 12, we must also have $x \le 6$.
- Enter the function $x(12-2x)^2$ into a graphing calculator over the restricted domain $[0,6]$.



- The maximum volume of the box is 128 in^3 if a 2-inch square is cut out of each corner of the cardboard.

Question: Why couldn't we analyze this problem as we did the previous problems?

Answer: The function whose extreme value we wanted to find is a cubic polynomial. At this time we only know how to analyze quadratic polynomials. You will learn how to analyze cubics and even more complicated functions if you study calculus.

Exercises for Chapter 4D - Maximum/Minimum Function Values

1. Graph $f(x) = -(x+2)^2 - 3$ by transforming $y = x^2$. Use your graph to find the vertex, axis of symmetry, domain, range, maximum or minimum functional value, intervals where f is increasing and decreasing.

2. Determine by inspecting the function whether it will have a maximum or minimum functional value. If necessary, write the function in standard form $f(x) = a(x-h)^2 + k$ to find the extreme value and the value of x where it occurs.

- **a**. $f(x) = x^2 - 3$
- **b**. $f(x) = -(x+2)^2$
- **c**. $f(x) = 2(x-1)^2 + 3$
- **d**. $f(x) = x^2 + 6x - 3$
- **e**. $f(x) = 2x^2 + 8x + 5$
- **f**. $f(x) = -x^2 - 2x - 3$
- **g**. $f(x) = -3x^2 - 9x + 2$

3. Determine by inspection whether the function will have a minimum or maximum functional value. Use $x = \frac{-b}{2a}$ to find the extreme value.

- **a**. $f(x) = -3x^2 + 12x + 5$
- **b**. $f(x) = 5x^2 - 35x + 1$
- **c**. $f(x) = x^2 - 6x + 3$
- **d**. $f(x) = -x^2 - 6x + 5$

4. Which of the following functions are quadratic?
$\frac{1}{x^2 - 2x}$, $2x - 3$, $x^3 - x^2 - x + 4$, $-3x^2$, $5x^2 - 4$

5. Let $f(x) = 5x^2$. Compute $f(2), f(-1), f(h), f(a+h)$

6. Let $f(x) = -2x^2 + 3x - 1$. Compute $f(2), f(-1), f(h), f(a+h)$

7. Plot the function $f(x) = 2 - x^2$. How many times does the graph cross the x-axis?

8. How many times does the graph of the function $f(x) = x^2$ cross the x-axis ?

9. How many times does the graph of the function $f(x) = 1 + x^2$ cross the x-axis ?

10. If $f(x) = ax^2 + bx + c$ is a quadratic function for which $f(0) = -1$, $f(3) = 2$, and $f(4) = 6$, find the coefficients a, b, and c

11. If $f(x) = -2x^2 + 5$, what is the largest possible value of $f(x)$?

12. What quadratic function has a graph which contains the three points $(1,-2)$, $(3,4)$, and $(5,9)$?

13. If $f(x) = x^2 - 3x + 7$ compute $f(1+h) - f(1)$

14. If $g(x) = -3x^2 - 5x$ compute the largest value of $g(x)$ and its zeros.

15. What is the difference between the graphs of $f(x) = x^2$ and $f(x) = -x^2$?

16. What determines if the graph of a quadratic function opens up or down?

17. If $f(x) = x^2 + x$ where is the axis of symmetry ?

18. If the axis of symmetry of the quadratic function is the line $x = 5$, and $f(6) = 7$, what must $f(4)$ equal?

19. What is the axis of symmetry of $f(x) = -3x^2 - 7x + 1$?

20. What is the axis of symmetry of $f(x) = x^2$?

21. Find the quadratic function which has a maximum value of 5 at $x = 2$, and for which $f(3) = -8$

22. If the vertex of the quadratic function $f(x)$ lies at the point $(3,6)$ and $f(1) = 8$, what is $f(5)$? Does the graph of $f(x)$ open up or down?

23. Find the zeros of the quadratic function $f(x) = 2 - 3x + 5x^2$

24. Solve the equation $f(x) = 2x^2 - 2x + 2 = 0$

25. How many zeros does the quadratic function $f(x) = 2x^2 - 5x$ have?

26. Calculate the discriminate of each of the following quadratic functions:

- **a**. $2x^2 - 5x$
- **b**. $5 - 3x + 7x^2$
- **c**. $6x^2 - 9$

27. Can you tell from the discriminate whether the graph of a quadratic function opens up or down?

28. The monthly profit at Brookside Apartment is given by $P(x) = -10x^2 + 1600x - 45,000$, where x is the number of apartments rented. How many should be rented to maximize profits? Support your findings graphically.

29. A baseball is thrown upward at a velocity of 88 ft/sec. If it is released at a point 7 feet from the ground, its distance s from the ground at t seconds is given by $s(t) = -16t^2 + 88t + 7$. When will the baseball reach its maximum height and how high will it go? Support your findings graphically.

30. Amanda Richards is planning to build a dog run for her chihuahua by enclosing a small rectangular area under a tree in her backyard. If she has 100 feet of fencing, what is the maximum area she can enclose? Support your findings graphically.

31. What is the maximum rectangular area Farmer Jones can enclose with 800 feet of fencing if one side of the enclosed area is bordered by a straight river and requires no fence?

32. Find two positive numbers whose sum is 20 if the sum of their squares is a minimum.

33. Use a graphing utility to find the local maximum and minimum functional values of each function and the values of x for which they occur. Use your findings to write the intervals where the function is increasing and decreasing. State each answer correct to the nearest two decimal places.

- **a**. $f(x) = x^3 - 4x$
- **b**. $f(x) = -3x^4 + 9x^3 + x^2 - 4x + 3$
- **c**. $f(x) = 9x^2 - x^3$
- **d**. $f(x) = 3.5x^4 - 5.3x^3 + 1.9x^2 + 3x - 2$
- **e**. $f(x) = 20x - 5\sqrt{x} + 2$
- **f**. $f(x) = \dfrac{2x+1}{x^2+2}$
- **g**. $f(x) = \dfrac{2}{1+x^2}$

34. Alcon Electrics manufactures small dorm-sized refrigerators. If the daily production costs for their mid-sized model 2015 is $C(x) = 0.0002x^3 - 0.29x^2 + 100x + 500$, how many refrigerators should be manufactured and sold to minimize the daily costs? Assume the company desires to produce more than 300 refrigerators. Support your findings graphically.

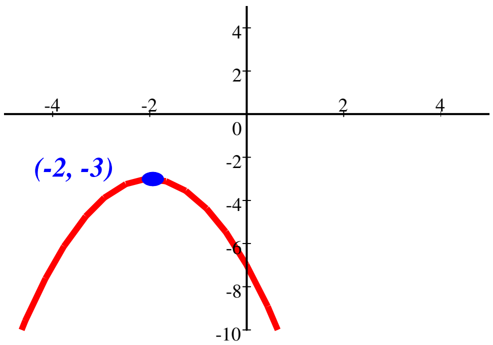
35. The number of murders in the city of Harris from 1989 – 1998 is approximated by $M(t) = -0.1t^3 + 1.1t^2 + 60$, where t is the number of years since 1989 and $0 \le t \le 9$. In what year did the most murders occur and how many were there?

36. Weekly production costs for peanut brittle at Betty's Brittle Factory are given by $C(x) = 0.0001x^3 - 0.03x^2 + 0.5x + 400$, where x is the number of pounds of peanut brittle produced and sold. How many pounds should be produced to minimize the production costs? If this occurs, what will her production costs be?

37. Find the maximum volume of an open box that can be made from an 8" by 10" sheet of cardboard by cutting equal squares out of each corner and turning up the sides.

Answers to Exercises for Chapter 4D - Maximum/Minimum Function Values

1. Vertex: $(-2,-3)$. Axis of Symmetry: $x = -2$. Domain: $(-\infty,+\infty)$. Range: $(-\infty,-3]$.
Maximum Functional Value -3 when $x = -2$. Increasing: $(-\infty,-2)$. Decreasing: $(-2,+\infty)$



2.

- **a.** Minimum value: -3 when $x = 0$
- **b.** Maximum value: 0 when $x = -2$
- **c.** Minimum value: 3 when $x = 1$
- **d.** Minimum value: -12 when $x = -3$
- **e.** Minimum value: -3 when $x = -2$
- **f.** Maximum value: -2 when $x = -1$
- **g.** Maximum value: $\frac{35}{4}$ when $x = -\frac{3}{2}$

3.

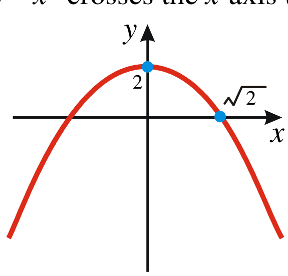
- **a.** Maximum value: 17 when $x = 2$
- **b.** Minimum value: $\frac{13}{20}$ when $x = \frac{3}{10}$
- **c.** Minimum value: -6 when $x = 3$
- **d.** Maximum value: 14 when $x = -3$

4. Of the five functions listed only $-3x^2$ and $5x^2 - 4$ are quadratic.

5. $f(2) = 20,\quad f(-1) = 5,\quad f(h) = 5h^2,\quad f(a+h) = 5(a+h)^2$

6. $f(2) = -2(2)^2 + 3(2) - 1 = -3,\quad f(-1) = -2(-1)^2 + 3(-1) - 1 = -6,$
$f(h) = -2h^2 + 3x - 1,\quad f(a+h) = -2(a+h)^2 + 3(a+h) - 1$

7. The graph of the function $f(x) = 2 - x^2$ crosses the x-axis twice.



8. The graph crosses the axis once.

9. The graph never crosses the x-axis.

10. If $f(x) = ax^2 + bx + c$, then: $-1 = f(0) = c$

$$2 = f(3) = 9a + 3b + c$$
$$6 = f(4) = 16a + 4b + c$$

From the first equation we see that $c = -1$. Substituting this value into the second and third equations we have: $2 = 9a + 3b - 1$

$$6 = 16a + 4b - 1$$

This system of equations leads to the system:

$$3 = 9a + 3b$$
$$7 = 16a + 4b$$

Multiply the first equation by 4 and the second equation by 3.

$$12 = 36a + 12b$$
$$21 = 48a + 12b$$

Now subtract the first equation from the second.

$$9 = 12a \text{ or } a = \frac{3}{4}$$

Place this value of a into either equation and solve for b. This gives $b = -\frac{5}{4}$. Thus, the quadratic function $f(x)$ equals

$$f(x) = \frac{3}{4}x^2 - \frac{5}{4}x - 1.$$

11. The term $-2x^2$ can never be positive. It is either zero or negative and its largest value is 0. Thus, we have

$$f(x) = -2x^2 + 5 \leq 0 + 5 = 5.$$

Moreover since $f(0) = 5$, the number 5 is the largest value which $f(x)$ can attain.

12. If $f(x) = ax^2 + bx + c$ is the quadratic function, then we have

$$-2 = f(1) = a + b + c$$
$$4 = f(3) = 9a + 3b + c$$
$$9 = f(5) = 25a + 5b + c$$

Solving this system of equations (Gausian elimination) for a, b, and c we have

$$c = -\frac{43}{8}, b = \frac{7}{2}, a = -\frac{1}{8}.$$

Thus, $f(x) = -\frac{1}{8}x^2 + \frac{7}{2}x - \frac{43}{8}$. Let's evaluate $f(x)$ at the given points to verify that we do indeed have the correct quadratic function

$$f(1) = -2$$
$$f(3) = 4$$
$$f(5) = 9$$

13. $f(1+h) - f(1) = \left[(1+h)^2 + 4 - 3h\right] - [5] = [5 - h + h^2] - [5] = -h + h^2$

14. The zeros of $g(x)$ are easily found by factoring $g(x)$. $g(x) = -3x^2 - 5x = -x(3x+5)$. Hence the zeros are located at $x = 0$ and $x = -\frac{5}{3}$. To compute the largest value of $g(x)$ we complete the square.

$$\begin{aligned} g(x) &= -3x^2 - 5x \\ &= -3\left(x^2 + \frac{5}{3}x\right) \\ &= -3\left(x^2 + \frac{5}{3}x + \frac{25}{36}\right) + \frac{25}{12} \\ &= -3\left(x + \frac{5}{6}\right)^2 + \frac{25}{12} \end{aligned}$$

The largest value of $g(x)$ occurs when $x = -\frac{5}{6}$ and equals $\frac{25}{12}$.

15. The graph of x^2 opens upward, while the graph of $-x^2$ opens downward.

16. The coefficient of x^2 is the determining factor in deciding if the graph opens up or down. In $ax^2 + bx + c$, if $a > 0$, the graph opens up, and if $a < 0$, the graph opens down.

17.

$$\begin{aligned} x^2 + x &= \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \end{aligned}$$

The vertex of the graph of $x^2 + x$ is at the point $\left(-\frac{1}{2}, -\frac{1}{4}\right)$. Hence the line $x = -\frac{1}{2}$ is the axis of symmetry. A second way to see that $x = -\frac{1}{2}$ is the axis is symmetry is to factor $f(x)$.

$$\begin{aligned} f(x) &= x^2 + x \\ &= x(x+1) \end{aligned}$$

Since $f(0) = f(-1)$, the axis of symmetry must be halfway between 0 and -1 or at $-\frac{1}{2}$.

18. Since the distance from 6 to 5 (axis of symmetry) is the same as the distance from 4 to 5, we must have $f(4) = f(6) = 7$.

19.

$$\begin{aligned} -3x^2 - 7x + 1 &= -3(x^2 + \frac{7}{3}x + \frac{49}{36}) + \frac{49}{12} + 1 \\ &= -3\left(x + \frac{7}{6}\right)^2 + \frac{61}{12} \end{aligned}$$

The vertex of the graph of $-3x^2 - 7x + 1$ is at the point $\left(-\frac{7}{6}, \frac{61}{12}\right)$. Hence the line $x = -\frac{7}{6}$ is the axis of symmetry.

20. The axis of symmetry is the line $x = 0$. Notice that $f(-2) = f(2)$, and in general $f(-x) = f(x)$.

21. Since $f(x)$ is a quadratic function which has its maximum at $x = 2$, we know that

$$f(x) = -a(x-2)^2 + b.$$

Moreover, since the maximum value is 5, when $x = 2$, this tells us that $b = 5$. Thus,

$$f(x) = -a(x-2)^2 + 5$$

We also know that $-8 = f(3) = -a + 5$. Hence $a = -13$, and

$$f(x) = -13(x-2)^2 + 5.$$

22. Since the vertex of $f(x)$ lies on the point $(3, 6)$, the axis of symmetry is the line $x = 3$. Both 1 and 5 are the same distance from $x = 3$. Thus,

$$f(5) = f(1) = 8.$$

Since 8 is larger than 6, which is the minimum value for $f(x)$, the graph has to upen up.

23. Setting $2 - 3x + 5x^2$ equal to zero, and using the quadratic formula we have

$$5x^2 - 3x - 2 = 0.$$

Thus,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 5 \cdot (-2)}}{2 \cdot 5}$$

$$x = \frac{3 \pm \sqrt{49}}{10} = \frac{3 \pm 7}{10}$$

$$x = \frac{-1}{2} \text{ or } x = 1$$

24. Since the discriminate of $2x^2 - 2x + 2$ equals

$$(-2)^2 - 4(2)(2) = -12,$$

there are no solutions.

25. $2x^2 - 5x = x(2x - 5)$. If we set this expression equal to zero, we get 2 solutions: $x = 0$ and $x = \frac{5}{2}$.

26. Calculate the discriminate of each of the following quadratic functions:

a. The discriminate of $2x^2 - 5x$ is $(-5)^2 - 4(2)(0) = 25$.

b. The discriminate of $5 - 3x + 7x^2$ is $(-3)^2 - 4(7)(5) = -131$.

c. The discriminate of $6x^2 - 9$ is $(0)^2 - 4(6)(-9) = 216$.

27. No, you cannot determine from the discriminate whether the graph opens up or down. The discriminates of x^2 and $-x^2$ are both zero, yet one graph opens up and the other opens down.

28. Maximum profits: $\$19,000$ when $x = 80$ apartments are rented.

29. Maximum height is 128 feet 2.75 seconds after launch.

30. She can fence a maximum of 625 ft^2 with 100 ft of fencing with dimensions $25'x25'$.

31. He can fence a maximum of $80,000$ ft^2. Dimensions will be $200'x400'$

32. The numbers are 10 and 10.

33.

a. Local maximum: 3.08 when $x = -1.15$. Local minimum: -3.08 when $x = 1.15$. Increasing: $(-\infty, -1.15) \cup (1.15, +\infty)$. Decreasing: $(-1.15, 1.15)$

b. Local maximum values: 4.11 when $x = -0.39$ and 24.69 when $x = 2.26$. Local minimum value: 2.06 when $x = 0.38$. Increasing: $(-\infty, -0.39) \cup (0.38, 2.26)$. Decreasing: $(-0.39, 0.38) \cup (2.26, +\infty)$

c. Local maximum value: 108 when $x = 6$. Local minimum value: 0 when $x = 0$. Increasing: $(0, 6)$. Decreasing: $(-\infty, 0) \cup (6, +\infty)$

d. Minimum value: -8.23 when $x = -0.51$. Increasing: $(-0.51, +\infty)$. Decreasing: $(-\infty. -0.51)$

e. Minimum value: 2 when $x = 0$. Increasing: $(0, +\infty)$

f. Local minimum: -0.5 when $x = -2$. Local maximum: 1 when $x = 1$. Increasing: $(-0.5, 1)$. Decreasing $(-\infty, -0.5) \cup (1, +\infty)$

g. Maximum value: 2 when $x = 0$. Increasing: $(-\infty, 0)$. Decreasing: $(0, +\infty)$

34. To minimize production costs at $\approx \$10,870$ about 667 refrigerators should be manufactured.

35. The most murders in the years covered were 80 occurring in 1996.

36. The minimum cost of $97 occurs when 191.3 pounds are produced and sold.

37. Maximum volume of the box is 56.5 in^3 when a $1.47"$ square is cut out of each corner.

Chapter 4E - Combinations of Functions

Sum, Difference, Product and Quotient

We can perform algebraic operations on functions much as we do on numbers.
For example, if $f(x) = x^2 + 1$ and $g(x) = x - 4$ then $f(x) + g(x) = (x^2 + 1) + (x - 4) = x^2 + x - 3$.
This new function is called the sum of f and g, denoted by $f + g$. The difference, product, and quotient of two functions are defined in a similar manner.

If f and g are functions with domains A and B respectively, we define the following operations on functions:

1.	SUM:	$(f+g)(x) = f(x) + g(x)$	Domain: $A \cap B$
2.	DIFFERENCE:	$(f-g)(x) = f(x) - g(x)$	Domain: $A \cap B$
3.	PRODUCT:	$(fg)(x) = f(x) \cdot g(x)$	Domain: $A \cap B$
4.	QUOTIENT:	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0.$	Domain: $A \cap B - \{x \mid g(x) = 0\}$

Example 1: Let $f(x) = 2x^2 + x + 5$ and $g(x) = 3x^2 + 2x - 1$, find the following:

a) domain of f b) domain of g
c) $(f+g)(x)$ d) domain of $f+g$
e) $(f-g)(x)$ f) domain of $f-g$
g) $(fg)(x)$ h) domain of fg
i) $\left(\frac{f}{g}\right)(x)$ j) domain of $\frac{f}{g}$

Solution: Since there are no fractions or radicals in either f or g,

a) Domain of $f = (-\infty, +\infty)$
b) Domain of $g = (-\infty, +\infty)$

c) $(f+g)(x) = f(x) + g(x) = (2x^2 + x + 5) + (3x^2 + 2x - 1) = 5x^2 + 3x + 4$
d) Domain of $f + g = (-\infty, +\infty)$

e) $(f-g)(x) = f(x) - g(x) = (2x^2 + x + 5) - (3x^2 + 2x - 1) = -x^2 - x + 6$
f) Domain of $f - g = (-\infty, +\infty)$

g) $(fg)(x) = f(x) \cdot g(x) = (2x^2 + x + 5) \cdot (3x^2 + 2x - 1) = 6x^4 + 7x^3 + 15x^2 + 9x - 5$
h) Domain of $fg = (-\infty, +\infty)$

i) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{(2x^2 + x + 5)}{(3x^2 + 2x - 1)} = \frac{(2x^2 + x + 5)}{(3x - 1)(x + 1)}, \quad x \neq \frac{1}{3}, -1$
j) Domain of $\frac{f}{g} = (-\infty, +\infty) - \{\frac{1}{3}, -1\} = (-\infty, -1) \cup \left(-1, \frac{1}{3}\right) \cup \left(\frac{1}{3}, +\infty\right)$
(That is, all real numbers except those that cause the denominator to be 0, namely $\frac{1}{3}$ and -1)

Example 2: Let $f(x) = \sqrt{x-3}$ and $g(x) = x-5$, find $f+g$, $f-g$, fg, $\frac{f}{g}$, and their domains.

Solution:

- Domain $[f] = [3,+\infty)$
  $x - 3 \geq 0$
  $x \geq 3$

  Domain$[g] = (-\infty,+\infty)$
  No fractions or radicals.

- $(f+g)(x) = \sqrt{x-3} + (x-5)$
  Domain$[f+g] = [3,+\infty) \cap (-\infty,+\infty) = [3,+\infty)$

- $(f-g)(x) = \sqrt{x-3} - (x-5) = \sqrt{x-3} - x + 5$
  Domain$[f-g] = [3,+\infty)$

- $(fg)(x) = \left(\sqrt{x-3}\right)(x-5) = x\sqrt{x-3} - 5\sqrt{x-3}$
  Domain$[fg] = [3,+\infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-3}}{x-5}, \quad x \neq 5$
  Domain$\left[\frac{f}{g}\right] = [3,5) \cup (5,+\infty)$. Since $g(5) = 0$, we must omit 5 from the domain of $\frac{f}{g}$.

Example 3: Let $f(x) = \sqrt{1+x}$ and $g(x) = \sqrt{1-x}$, find $f+g$, $f-g$, fg, $\frac{f}{g}$, and their domains.

Solution:

- Domain$[f] = [-1,+\infty)$
  $1 + x \geq 0$
  $x \geq -1$

  Domain$[g] = (-\infty,1]$
  $1 - x \geq 0$
  $-x \geq -1$
  $x \leq 1$

- $(f+g)(x) = \sqrt{1+x} + \sqrt{1-x}$
  Domain$[f+g] = [-1,+\infty) \cap (-\infty,1] = [-1,1]$

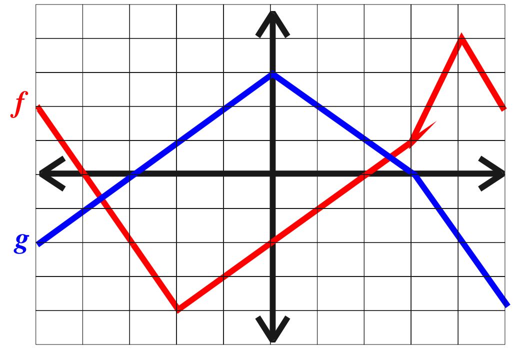
- $(f-g)(x) = \sqrt{1+x} - \sqrt{1-x}$
  Domain$[f-g] = [-1,1]$

- $(fg)(x) = \left(\sqrt{1+x}\right)\left(\sqrt{1-x}\right) = \sqrt{(1+x)(1-x)} = \sqrt{1-x^2}$
  Domain$[fg] = [-1,1]$

- $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{1+x}}{\sqrt{1-x}}, \quad x \neq 1$
  Domain$\left[\frac{f}{g}\right] = [-1,1] - \{1\} = [-1,1)$. Since $g(1) = 0$, we must omit 1 from the domain of $\frac{f}{g}$.

Sums, differences, products, and quotients of functions are defined pointwise. For example, the sum $(f+g)(x)$ is found by evaluating $f(x)$ and $g(x)$ and adding the resulting functional values.

Example 4: Find the pointwise sum of the graphs shown below.



a) Use the graphs of f and g to complete the table of values.

x	$f(x)$	$g(x)$	$(f+g)(x)$
–5			
–4			
–3			
–2			
–1			
0			
1			
2			
3			
4			
5			

b) Use the points to draw the graph of $f+g$.

Solution:

a)

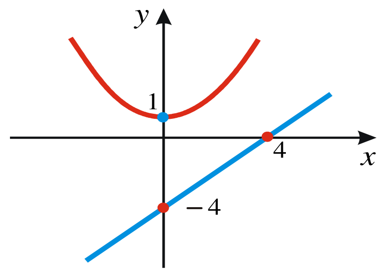
x	$f(x)$	$g(x)$	$(f+g)(x)$
–5	2	–2	0
–4	0	–1	–1
–3	–2	0	–2
–2	–4	1	–3
–1	–3	2	–1
0	–2	3	1
1	–1	2	1
2	0	1	1
3	1	0	1
4	4	–2	2
5	2	–4	–2

b) Plot the points $(x,(f+g)(x))$ from your table and draw the lines to get the graph of $f+g$.

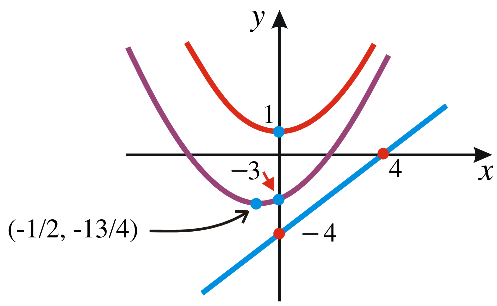
Example 5: Use graph paper to graph the functions $f(x) = x^2 + 1$ and $g(x) = x - 4$ on the same axes .

a) Use point-wise addition to find the graph of their sum.
b) Determine the rule for $f + g$ algebraically.
c) What shape should the graph of the $f + g$ have?

Solution: **red**: $f(x) = x^2 + 1$ **blue**: $g(x) = x - 4$



a) **red**: $x^2 + 1$ **blue**: $x - 4$ maroon: $f + g$



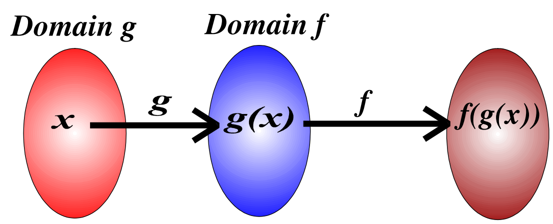
b) $(f + g)(x) = f(x) + g(x) = (x^2 + 1) + (x - 4) = x^2 + x - 3.$

c) A parabola.

Function Composition

Another way that functions are combined is called function composition. This is a "function of a function". For example, if $y = f(u) = u^2$ and $u = g(x) = 2x - 5$, by substituting $g(x)$ for u, we get $y = f(g(x)) = (2x-5)^2$.

- **Definition**: For functions f and g, the **composition of functions** $f \circ g$ is defined as $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is $\{x \in \text{Dom}[g] \mid g(x) \in \text{Dom}[f]\}$.



Example 1: Let $f(x) = x^2 - 2$ and $g(x) = 2x + 3$.
a) Find $(f \circ g)(2)$ and $(f \circ g)(x)$
b) Find $(g \circ f)(2)$ and $(g \circ f)(x)$
c) Find the domains of $f \circ g$ and $g \circ f$.

Solution:

a) $(f \circ g)(2) = f(g(2)) = f(2(2)+3) = f(7) = (7)^2 - 2 = 47$
$(f \circ g)(x) = f(g(x)) = f(2x+3) = (2x+3)^2 - 2 = 4x^2 + 12x + 7$

b) $(g \circ f)(2) = g(f(2)) = g(2^2 - 2) = g(2) = 2(2) + 3 = 7$
$(g \circ f)(x) = g(f(x)) = g(x^2 - 2) = 2(x^2 - 2) + 3 = 2x^2 - 1$

c) $\text{Domain}[f] = (-\infty, +\infty)$ and $\text{Domain}[g] = (-\infty, +\infty)$

$$\begin{aligned}\text{Domain}[f \circ g] &= \{x \in \text{Domain}[g] \mid g(x) \in \text{Domain}[f]\} \\ &= \{x \in (-\infty,+\infty) \mid (2x+3) \in (-\infty,+\infty)\} \\ &= (-\infty,+\infty)\end{aligned}$$

$$\begin{aligned}\text{Domain}[g \circ f] &= \{x \in \text{Domain}[f] \mid f(x) \in \text{Domain}[g]\} \\ &= \{x \in (-\infty,+\infty) \mid (x^2-2) \in (-\infty,+\infty)\} \\ &= (-\infty,+\infty)\end{aligned}$$

Example 2: Let $f(x) = x + 2$ and $g(x) = \frac{1}{x}$, find $f \circ g$ and $g \circ f$ and their domains.

Solution: $\text{Domain}[f] = (-\infty,+\infty)$ $\quad$ $\text{Domain}[g] = (-\infty,0) \cup (0,+\infty) \Leftrightarrow x \neq 0$

- $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right) + 2$

  $\text{Domain}[f \circ g] = \{x \neq 0 \mid \frac{1}{x} \text{ is a real number}\} \Rightarrow \text{Domain}[f \circ g] = (-\infty,0) \cup (0,+\infty)$

  [start with the domain of g; then, $g(x) = \frac{1}{x}$ must be in domain of f]

- $(g \circ f)(x) = g(f(x)) = g(x+2) = \dfrac{1}{x+2}$

  $\text{Domain}[g \circ f] = \{x \in (-\infty,+\infty) \mid x + 2 \neq 0\} = (-\infty,-2) \cup (-2,+\infty)$

  [start with the domain of f; then $f(x) = x + 2$ must be in the domain of g]

Example 3: Let $f(x) = x^2 + 2$ and $g(x) = \sqrt{x-2}$, find $f \circ g$ and $g \circ f$ and their domains.

Solution: $\text{Domain}_f = (-\infty, +\infty)$ $\text{Domain}_g = [2, +\infty) \Leftrightarrow x \geq 2$

- $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x-2}\right) = \left(\sqrt{x-2}\right)^2 + 2 = x - 2 + 2 = x$
  $\text{Domain}[f \circ g] = \left\{x \geq 2 \middle| \sqrt{x-2} \text{ is a real number}\right\} \Rightarrow \text{Domain}[f \circ g] = [2, +\infty)$
  Notice that the domain of $(f \circ g)(x)$ is not all real numbers, even though it equals the identity function $h(x) = x$ on the set $[2, \infty)$.
- $(g \circ f)(x) = g(f(x)) = g(x^2 + 2) = \sqrt{(x^2+2) - 2} = \sqrt{x^2} = |x|$
  $\text{Domain}[g \circ f] = \{x \in (-\infty, +\infty) \mid x^2 + 2 \geq 2\} = (-\infty, +\infty)$
  $[x^2 + 2 \geq 2 \quad \Rightarrow \quad x^2 \geq 0$ which is true for all real numbers.]

Example 4: Let $(f \circ g)(x) = (3x+2)^2 - 5(3x+2)$. Find the component functions f and g.

Solution: $(f \circ g)(x) = f\left(\boxed{g(x)}\right) = \left(\boxed{3x+2}\right)^2 - 5\left(\boxed{3x+2}\right)$

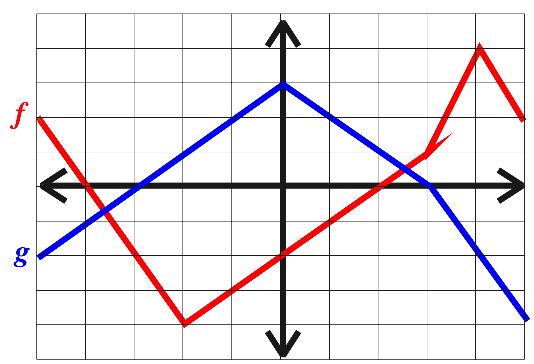
$$f(x) = \boxed{x}^2 - 5\boxed{x}$$

Therefore, $g(x) = 3x + 2$ and $f(x) = x^2 - 5x$.

Check your answer: For $g(x) = 3x + 2$ and $f(x) = x^2 - 5x$,
$(f \circ g)(x) = f(g(x)) = f(3x+2) = \underbrace{(3x+2)^2 - 5(3x+2)}_{\text{same as original problem}}$

Therefore, our work is correct.

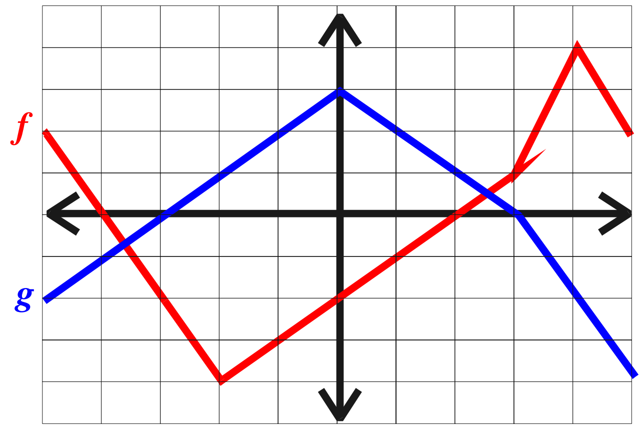
Consider the graphs of f and g below.



Example 5: Use the graphs to evaluate $(f \circ g)(-4)$.
Solution: We know that $(f \circ g)(-4) = f(g(-4))$ so we must fiirst locate $x = -4$ and then find $g(-4)$ which is -1. Then we find $f(-1)$ which is -3. Therefore, $(f \circ g)(-4) = f(g(-4)) = f(-1) = -3$.

Question: Referring to the plot above, does the domain of $f \circ g$ equal the domain of g ?
Answer: Yes, the range of the function g is the closed interval $[-4, 3]$, and the domain of the function f is the closed interval $[-5, 5]$. Thus, $g(x) \in \text{domain}[f]$ for all $x \in \text{domain}[g]$.

Example 6: Use the graph below to evaluate the following:



a) $(f \circ g)(2)$ b) $(f \circ g)(-3)$ c) $(g \circ f)(0)$ d) $(f \circ g)(0)$
e) $(g \circ f)(4)$ f) $(f \circ g)(-2)$ g) $(f \circ g)(1)$ h) $(g \circ f)(-2)$

Solution:

a) $(f \circ g)(2) = f(g(2)) = f(1) = -1$
b) $(f \circ g)(-3) = f(g(-3)) = f(0) = -2$
c) $(g \circ f)(0) = g(f(0)) = g(-2) = 1$
d) $(f \circ g)(0) = f(g(0)) = f(3) = 1$
e) $(g \circ f)(4) = g(f(4)) = g(4) = -2$
f) $(f \circ g)(-2) = f(g(-2)) = f(1) = -1$
g) $(f \circ g)(1) = f(g(0)) = f(3) = 1$
h) $(g \circ f)(-2) = g(f(-2)) = g(-4) = -1$

Exercises for Chapter 4E - Combinations of Functions

1. Let $f(x) = x + 5$ and $g(x) = x^2 - 2x + 1$. Find each of the following functions and their domains.
- **a.** $f + g$
- **b.** $f - g$
- **c.** fg
- **d.** $\frac{f}{g}$

2. Let $f(x) = \sqrt{x+3}$ and $g(x) = 3x$. Find each of the following functions and their domains.
- **a.** $f + g$
- **b.** $f - g$
- **c.** fg
- **d.** $\frac{f}{g}$

3. Let $f(x) = \frac{1}{x}$ and $g(x) = x + 2$. Find each of the following functions and their domains.
- **a.** $f + g$
- **b.** $f - g$
- **c.** fg
- **d.** $\frac{f}{g}$
- **e.** $\frac{g}{f}$

4. Let $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{4+x}$.
- **a.** Use your knowledge of transforming functions to graph f and g on the same axes and determine their domains.
- **b.** Use pointwise addition to graph the sum $f + g$.

5. Let $f(x) = x^2 - x + 1$ and $g(x) = 2x + 3$. Find $f \circ g$ and $g \circ f$ and their domains.

6. Let $f(x) = \sqrt{x+2}$ and $g(x) = x - 3$. Find $f \circ g$ and $g \circ f$ and their domains.

7. Let $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x}$. Find $f \circ g$ and $g \circ f$ and their domains.

8. Let $f(x) = x^2 + 2$ and $g(x) = \sqrt{x-2}$. Find $f \circ g$ and $g \circ f$ and their domains.

9. If $(f \circ g)(x) = (3x+2)^2 - 5(3x+2)$, find f and g.

10. If $(f \circ g)(x) = 2\sqrt{x+1} - 3$, find f and g.

11. If $(f \circ g)(x) = \dfrac{2(x^2-1)^3 + 5}{(x^2-1) + 4}$, find f and g.

Answers to Exercises for Chapter 4E - Combinations of Functions

1. **a.** $(f+g)(x) = x^2 - x + 6$

Domain: $(-\infty, +\infty)$

b. $(f-g)(x) = -x^2 + 3x + 4$

Domain: $(-\infty, +\infty)$

c. $(fg)(x) = x^3 + 3x^2 - 9x + 5$

Domain: $(-\infty, +\infty)$

d. $\left(\frac{f}{g}\right)(x) = \frac{x+5}{x^2 - 2x + 1} = \frac{x+5}{(x-1)^2}$

2. Domain: $(-\infty, 1) \cup (1 + \infty)$

a. $(f+g)(x) = \sqrt{x+3} + 3x$

Domain: $[-3, +\infty)$

b. $(f-g)(x) = \sqrt{x+3} - 3x$

Domain: $[-3, +\infty)$

c. $(fg)(x) = 3x\sqrt{x+3}$

Domain: $[-3, +\infty)$

d. $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+3}}{3x}$

3. Domain: $[-3, 0) \cup (0, +\infty)$

a. $(f+g)(x) = \frac{1}{x} + x + 2$

Domain: $(-\infty, 0) \cup (0, +\infty)$

b. $(f-g)(x) = \frac{1}{x} - x - 2$

Domain: $(-\infty, 0) \cup (0, +\infty)$

c. $(fg)(x) = 1 + \frac{2}{x}$

Domain: $(-\infty, 0) \cup (0, +\infty)$

d. $\left(\frac{f}{g}\right)(x) = \frac{1}{x(x+2)}$

Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$

e. $\frac{g}{f} = \frac{x+2}{1/x} = x^2 + 2x.$

Looking at expression $x^2 + 2x$ one would think that the domain of $\frac{g}{f}$ is all real numbers.

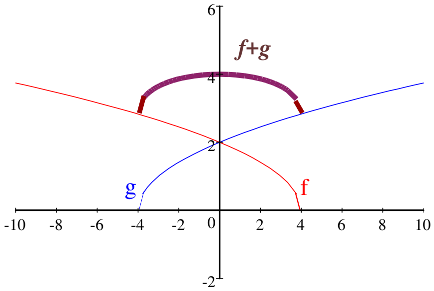
However, that is not correct. The domain of $\frac{g}{f}$ equals

$$[\text{Domain}(f) \cap \text{Domain}(g)] - \{x{:}f \neq 0\}$$

Since the function f is never 0, the domain of $\frac{g}{f}$ equals

$$\text{Domain}(f) \cap \text{Domain}(g) = \{x{:}x \neq 0\}$$

4. Domain f: $(-\infty, 4]$ Domain g: $[-4, +\infty)$ Domain $f + g$: $[-4, 4]$



5. $(f \circ g)(x) = (2x+3)^2 - (2x+3) + 1$

Domain $f \circ g$: $(-\infty, +\infty)$

$(g \circ f)(x) = 2(x^2 - x + 1) + 3$

Domain $g \circ f$: $(-\infty, +\infty)$

6. $(f \circ g)(x) = \sqrt{x-1}$

Domain $f \circ g$: $[1, +\infty)$

$(g \circ f)(x) = \sqrt{x+2} - 3$

Domain $g \circ f$: $[-2, +\infty)$

7. $(f \circ g)(x) = \dfrac{1}{x^2} + 1$

Domain $f \circ g$: $(-\infty, 0) \cup (0, +\infty)$

$(g \circ f)(x) = \dfrac{1}{x^2 + 1}$

Domain $g \circ f$: $(-\infty, +\infty)$

8. $(f \circ g)(x) = x$

Domain $f \circ g$: $[2, +\infty)$

$(g \circ f)(x) = |x|$

Domain $g \circ f$: $(-\infty, +\infty)$

9. $f(x) = x^2 - 5x$ $g(x) = 3x + 2$

10. $f(x) = 2x - 3$ $g(x) = \sqrt{x+1}$

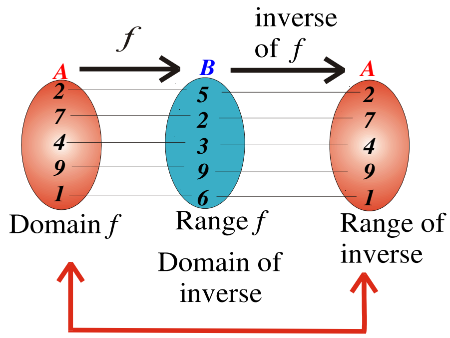
11. $f(x) = \dfrac{2x^3 + 5}{x + 4}$ $g(x) = x^2 - 1$

Chapter 4F - Inverse Functions

Inverse Relations

Recall that a function *f* pairs an output *y* with each input *x*. The inverse of *f* reverses the process. That is, the inverse pairs the original *y* with the original *x*. That is if $(a,b) \in f$, then $(b,a) \in$ inverse of *f*.

Domain f	**Range** f	$\Rightarrow$	**Domain** f^{-1}	**Range** f^{-1}
2	5	$\Rightarrow$	5	2
7	2	$\Rightarrow$	2	7
4	3	$\Rightarrow$	3	4
9	9	$\Rightarrow$	9	9
1	6	$\Rightarrow$	6	1



Example 1: Find the inverse for the function below. Is the inverse a function? Group of friends with the make of car they drive:

Domain	**Range**
Heather	Mazda
Emily	Chevrolet
Cole	Ford
Andre	Honda
Jose	BMW

Solution: For the original function, the friends (drivers of the car) are the input values and the types of cars they drive is the output. The inverse relation of *f* is found by interchanging the inputs and outputs. The input values are the types of cars, and the outputs are the persons that drive that car.

Domain	**Range**
Mazda	Heather
Chevrolet	Emily
Ford	Cole
Honda	Andre
BMW	Jose

The inverse is a function because every input (type of car) is paired with one output (driver of the car).

Example 2: Find the inverse for the function below. Is the inverse a function? Group of friends with the color of their car:

Domain	**Range**
Heather	white
Emily	red
Cole	blue
Andre	white
Jose	blue

Solution: Inverse of f:

Domain	**Range**
white	Heather
red	Emily
blue	Cole
white	Andre
blue	Jose

The inverse is NOT a function. White is paired with Heather and Andre. Also, blue is paired with both Cole and Jose.

Example 3: Find the inverse for each function below. Is the inverse of each function itself a function?

1. The set of ordered pairs (x, y): $\{(2,5), (3,7), (4,6), (-2,0), (-4,-2)\}$

2. The function:

Domain	**Range**
-2	3
-3	7
1	4
5	3
9	8

Solution:

1. Interchanging the x and y coordinates of the pairs in function f, we get the inverse relation: $\{(5,2), (7,3), (6,4), (0,-2), (-2,-4)\}$

The inverse is a function because no two ordered pairs have the same first element.

2. Interchanging the input and output values we get the inverse relation:

Domain	**Range**
3	-2
7	-3
4	1
3	5
8	9

The inverse is NOT a function because 3 is paired with more than one output: -2 and 5.

In your own words: How can you determine if the inverse of a function will be a function by merely inspecting the original function?

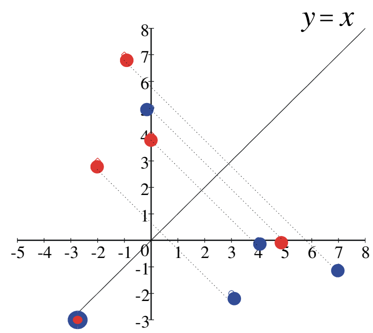
Consider the graphs of the function below (red) and its inverse (blue).

f :

Domain	Range
–2	3
–1	7
0	4
–3	–3
5	0

inverse of f :

Domain	Range
3	–2
7	–1
4	0
–3	–3
0	5

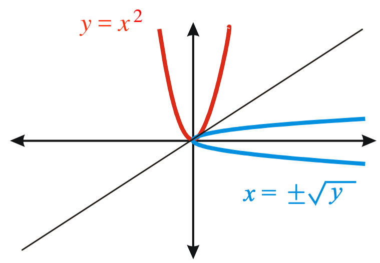


Note the following:

- The domain of f is the range of its inverse
- The range of f is the domain of its inverse.
- Corresponding points have symmetry about the line $y = x$.
- x-intercept of f is the y-intercept of its inverse
- y-intercept of f is the x-intercept of its inverse.
- Graphs of a function and its inverse are symmetric about the line $y = x$.

Example 4: Use a calculator to graph the function $y = x^2$ and its inverse on the same viewing window. Also include the graph of $y = x$. [Be sure to use a square window.] Find the domain, range, and intercepts of each. Is the inverse a function?

Solution:



Domain of f: $(-\infty, +\infty)$
Range of f: $[0, +\infty)$
x- intercept: $(0, 0)$
y-intercept: $(0, 0)$

Domain of the inverse of f: $[0, +\infty)$
Range of the inverse of f: $(-\infty, +\infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$

Example 5: On a piece of graph paper, graph each function and its inverse on the same coordinate system using different colors for each set. Draw in the line $y = x$. Use the plot of the inverse to determine if it is a function.

1. F: $\{(2,5), (3,7), (4,6), (-2,0), (-4,-2)\}$

Inverse of F : $\{(5,2), (7,3), (6,4), (0,-2), (-2,-4)\}$

2. G:

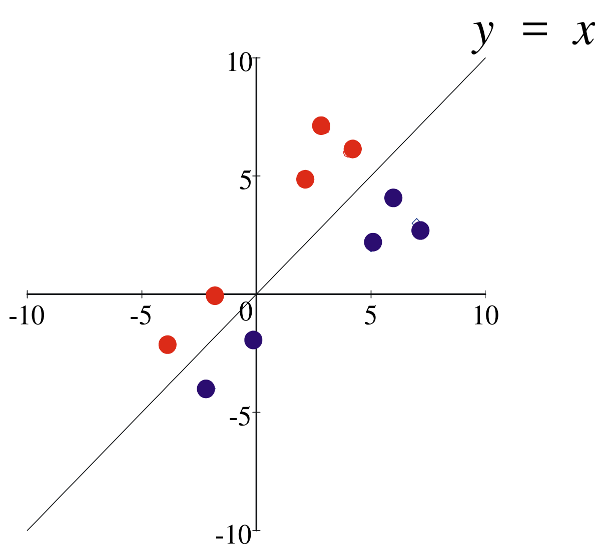
Domain	Range
−2	3
−3	7
1	4
5	3
9	8

Inverse of G :

Domain	Range
3	−2
7	−3
4	1
3	5
8	9

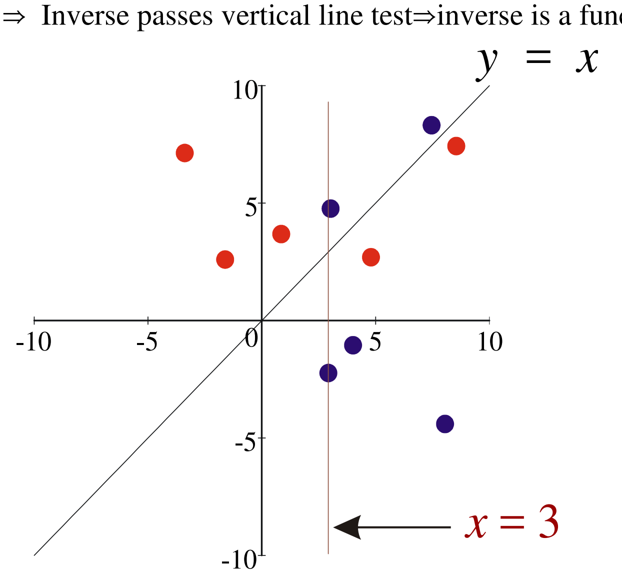
Solution:

1.



⇒ Inverse passes vertical line test⇒inverse is a function.

2.



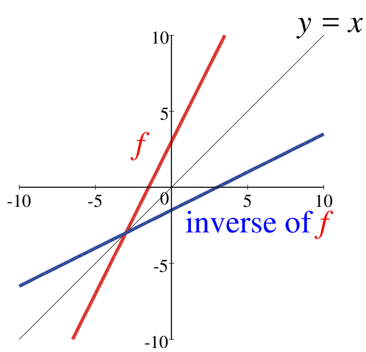
⇒ Inverse is not a function, as the graph fails the vertical line test.

Example 6: Graph each function. Use symmetry about the line $y = x$ to graph the inverse relation. Determine if the inverse is a function.

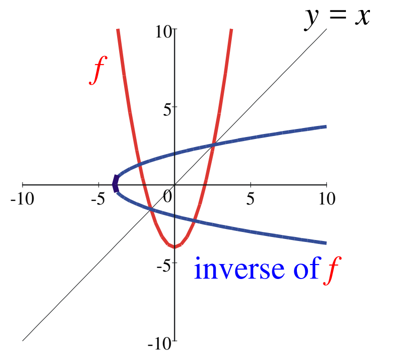
a) $f(x) = 2x + 3$
b) $f(x) = x^2 - 4$
c) $f(x) = \sqrt{x - 1}$
d) $f(x) = x^3 + 1$

Solution:

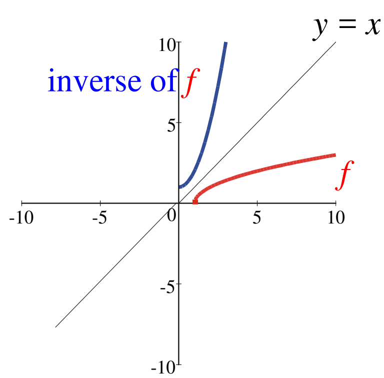
a) Inverse is a function.



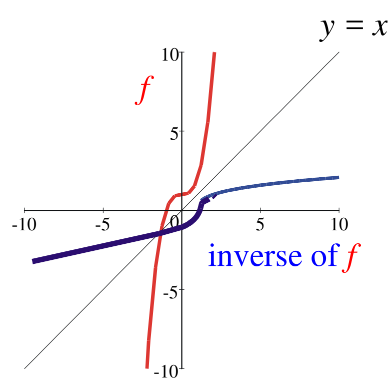
b) Inverse is not a function.



c) Inverse is a function.



d) Inverse is a function.

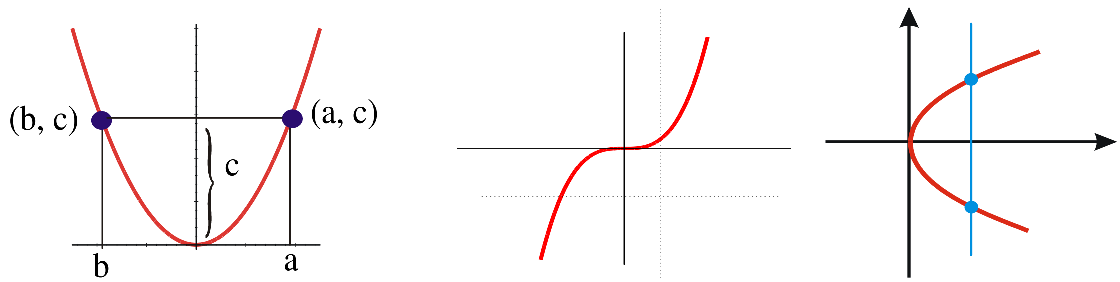


One-to-One Functions

In the previous section we have seen that all functions have inverse relations, but not all inverse relations were functions themselves. For a function f to have an inverse function, each x in the domain must be paired with a different y in the range. We call functions of this type one-to-one functions.
To determine graphically if the function is one-to-one we use the horizontal line test.

Horizontal Line Test:

A function is one-to-one if and only if no horizontal line intersects the graph in more then one point.



Function–not one-to-one
Fails horizontal line test

One-to-one function
Passes vertical and horizontal line tests

Not a function
Fails vertical line test

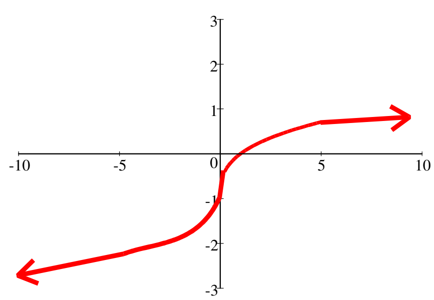
To determine algebraically whether a function is one-to-one, we use the definition:

- **Definition**: A function f is said to be **one-to-one** if and only if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. That is, two inputs cannot have the same output unless they were equal to begin with.

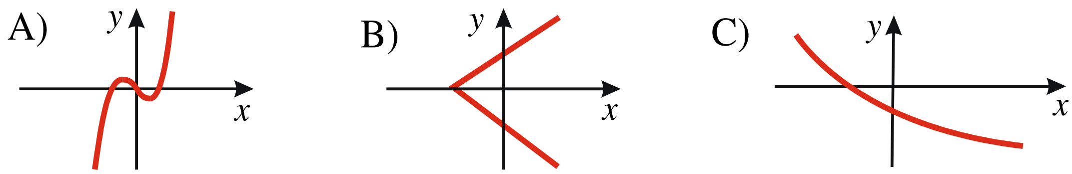
Example 1: Prove algebraically that $y = \sqrt[3]{x} + 2$ is a one-to-one function. Support this conclusion graphically.

Solution: Let $f(x_1) = f(x_2)$. (We must show that x_1 must equal x_2).
Subtract 2 : $\sqrt[3]{x_1} + 2 = \sqrt[3]{x_2} + 2$
Cube both sides: $\sqrt[3]{x_1} = \sqrt[3]{x_2}$
$x_1 = x_2$ Therefore, f is a one-to-one function.

This conclusion is supported graphically because the graph of the function passes the horizontal line test.



Example 2: Which of the following graphs represent a function, a one-to-one function, or not a function. Explain your answer.



Solution:

A represents a function because it passes the vertical line test. But it is NOT one-to-one because a horizontal line can intersect it in more than one point.

B is NOT a function because there exists a vertical line that intersects the graph more than once.

C is a one-to-one function. It passes both the vertical line test (function) and the horizontal line test (one-to-one).

Example 3: Determine algebraically whether each function is a one-to-one function.

a) $f(x) = \sqrt{x+5}$ b) $f(x) = x^2 - 2$

Solution:

a) We must show that if the functional values of x_1 and x_2 are equal, then $x_1 = x_2$. The use of different subscripts is the same as using different variables.

Assume: $f(x_1) = f(x_2)$

Substitute into f : $\sqrt{x_1 + 5} = \sqrt{x_2 + 5}$

Square both sides: $x_1 + 5 = x_2 + 5$

Subtract 5: $x_1 = x_2$ $\Rightarrow$ f is a one-to-one function.

b) Let $f(x_1) = f(x_2)$

Substitute $(x_1)^2 - 2 = (x_2)^2 - 2$

Add 2: $(x_1)^2 = (x_2)^2$

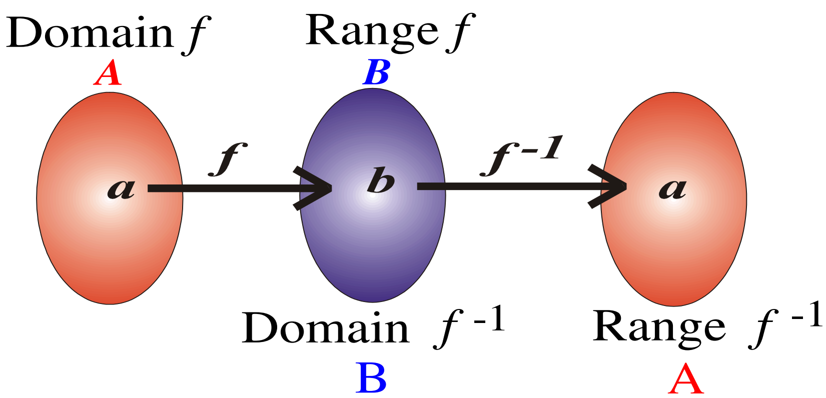
Take square root: $x_1 = \pm x_2$. $\Rightarrow$ f is NOT one-to-one. Two different inputs can have the same output: $x_1 = x_2$ OR $x_1 = -x_2$.

For example, if $f(x) = 7$, then $x^2 - 2 = 7$ $\Rightarrow$ $x^2 = 9$ $\Rightarrow$ $x = \pm 3$. That is, both 3 and –3 have the output of 7, so that f is not a one-to-one function.

Definition of Inverse Function

For a function f to have an inverse function, f must be a one-to-one function. We call the inverse function of f "f-inverse", and denote it with f^{-1}.

- **Definition**: If f is a one-to-one function with domain A and range B, then , for every b in B, its **inverse function**, f^{-1}, is defined by $f^{-1}(b) = a$ if and only if $f(a) = b$. That is, if $(a,b) \in f$, then $(b,a) \in f^{-1}$.



Note: The domain of f^{-1} is the range of f.

The range of f^{-1} is the domain of f.

Example 1: If f is a one-to-one function, answer the following:

a) If $f(2) = 5$, then $f^{-1}(5) =$_____?
b) If $f^{-1}(3) = 7$, then $f(7) =$_______?
c) If the domain of f is $[0,+\infty)$, then the range of f^{-1} =________?
d) If the range of f is $(-3,2]$, then the domain of f^{-1} =__________?

Solution:
a) 2 b) 3 c) $[0,+\infty)$ d) $(-3,2]$

To verify that functions f and g are inverses, we must show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$

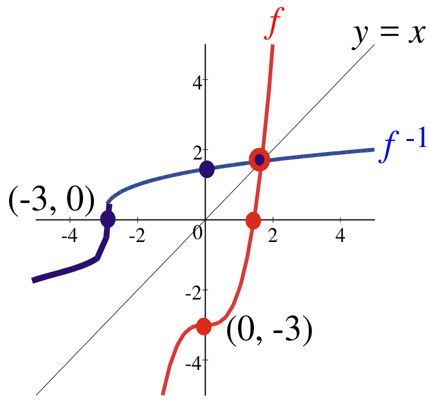
Example 2: Show that $f(x) = \sqrt[3]{x+3}$ and $g(x) = x^3 - 3$ are inverse functions.

Solution: $(f \circ g)(x) = f(g(x)) = f(x^3 - 3) = \sqrt[3]{(x^3-3)+3} = \sqrt[3]{x^3} = x$

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt[3]{x+3}\right) = \left(\sqrt[3]{x+3}\right)^3 - 3 = x + 3 - 3 = x$

Therefore, f and g are inverse functions.

The two functions from the previous example, along with the line $y = x$, have been graphed below using a square viewing window. The graphs of f and g appear to have symmetry about the line $y = x$.
Note that we cannot **prove** that the functions are inverses using the graphs. We need algebra for that.



Example 3: Use the property of inverses to verify that $f(x) = \frac{2}{x} - 1$ and $f^{-1}(x) = \frac{2}{x+1}$ are inverse functions.

Solution: We must show that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f\ )(x) = x$.

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{2}{x+1}\right) = \frac{2}{\left(\frac{2}{x+1}\right)} - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x.$

$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{2}{x} - 1\right) = \frac{2}{\left(\frac{2}{x} - 1\right) + 1} = \frac{2}{\frac{2}{x}} = 2\left(\frac{x}{2}\right) = x.$

Therefore f and f^{-1} are inverse functions.

Example 4: Use the property of inverses to verify that $f(x) = 4x + 3$ and $f^{-1}(x) = \frac{x-3}{4}$ are inverse functions.

Solution: We must show that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f\ )(x) = x$.

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-3}{4}\right) = 4\left(\frac{x-3}{4}\right) + 3 = x - 3 + 3 = x.$

$(f^{-1} \circ f)(x) = f^{-1}(4x + 3) = \frac{(4x+3) - 3}{4} = \frac{4x + 3 - 3}{4} = \frac{4x}{4} = x.$

Therefore f and f^{-1} are inverse functions.

Finding The Inverse of a Function

In the beginning of this section when we defined functions by giving a table of data we found the inverse by interchanging the two columns of data.

If a function is described by an expression, to find the rule that defines a function's inverse we interchange the x's and the y's in the expression, and then solve for y. For the inverse to be a function we must guarantee that the original function is one-to-one.

Example 1: Find f^{-1} for $f(x) = 3x - 2$. Verify your results using the property of inverse functions. Support your conclusion graphically.

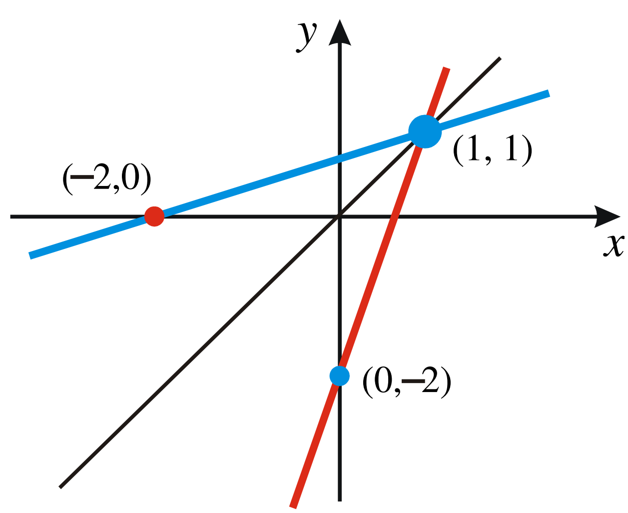
Solution:

- We know that the graph of the line $y = 3x - 2$ passes the horizontal line test , and is therefore a one-to-one function. To prove that f is one-to-one algebraically, we use the definition of one-to-one to show that $f(x_1) = f(x_2) \Rightarrow 3x_1 - 2 = 3x_2 - 2 \Rightarrow x_1 = x_2$. Therefore, f is one-to-one.
- Write the function as $y = 3x - 2$.
- Interchange the x and the y: $x = 3y - 2$
- Solve for y: $x + 2 = 3y$
  $y = \frac{x+2}{3} \Rightarrow f^{-1}(x) = \frac{x+2}{3}$ [The resulting y is f^{-1}.]
- Verify your results by showing that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$:

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x+2}{3}\right) = 3\left(\frac{x+2}{3}\right) - 2 = x + 2 - 2 = x$$

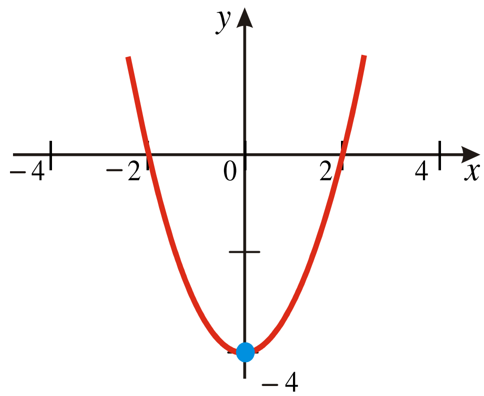
$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x-2) = \frac{3x - 2 + 2}{3} = \frac{3x}{3} = x$$

- Support graphically by showing that the graphs of f and f^{-1} have symmetry about the line $y = x$.



Example 2: The function $f(x) = x^2 - 4$ is not a one-to-one function. Restrict the domain of f so that its inverse will be a function.

Solution: Note from the graph of f, the part of the graph on either side of the line $x = 0$ is one-to-one. Therefore, we will restrict the domain to be $[0, +\infty)$.



Domain of f is $[0, +\infty)$: $x \geq 0$
Range of f is $[-4, +\infty)$: $y \geq -4$

Example 3: In the above example we restricted the domain of $f(x) = x^2 - 4$ so that f has an inverse function. Find f^{-1} and its domain and range.

Solution: The domain of f was restricted to $x \geq 0$, and the range is $y \geq -4$ so that f is one-to-one.

- We write: $y = x^2 - 4,\quad x \geq 0 \quad y \geq -4$
- Interchange x and y: $x = y^2 - 4,\quad y \geq 0 \quad x \geq -4$ [Interchange domain and range.]
- Solve for y: $y^2 = x + 4 \quad y \geq 0 \quad x \geq -4$

  $y = \pm\sqrt{x+4} \quad y \geq 0 \quad x \geq -4$

  Since $y \geq 0$, we use only the positive root. $f^{-1}(x) = \sqrt{x+4}$
- Domain of f^{-1} : $x \geq -4 \Rightarrow [-4, +\infty)$

  Range of f^{-1} : $y \geq 0 \Rightarrow [0, +\infty)$

Exercises for Chapter 4F - Inverse Functions

1. Find the inverse relation of each of the following and determine whether the inverse is a function:

a. $\{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$

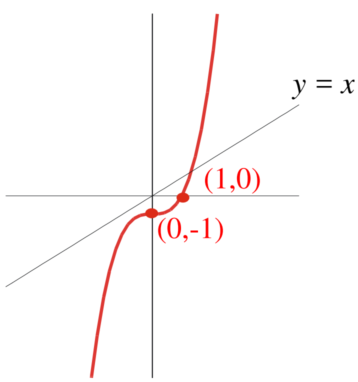
b.

Dog's Name	Breed
Casey	Schnauser
Tilly	Shepherd
Madison	Lab
Shelby	Schnauser
Scooter	Mutt
Freckles	Beagle

c.

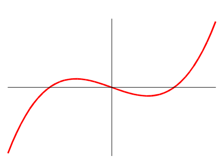
x	y
4	1
7	2
9	3
1	4
3	5
6	6

2. Use the fact that inverse functions have symmetry about the line $y = x$ to sketch the graph of the inverse of the function graphed below.

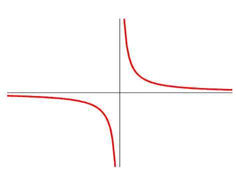


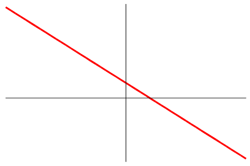
3. Determine which of the following functions is one-to-one using the horizontal line test.

a.



b.



c. 

4. Determine by graphing if $f(x) = x^2 - 3$ is a one-to-one function. Confirm your answer algebraically.

5. Determine graphicallly if $f(x) = \sqrt{x+2}$ is a one-to-one function. Confirm your answer algebraically.

6. Restrict the domain of each of the following functions to obtain a one-to-one function.

a. $f(x) = x^2 - 3$

b. $f(x) = |x+2|$

c. $f(x) = (x-5)^2$

7. Use the property of inverses to determine whether $f(x) = \sqrt[3]{x} + 2$ and $g(x) = (x-2)^3$ are inverse functions.

8. Use the property of inverses to determine whether $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1}{x} + 2$ are inverse functions.

9. Use the property of inverses to determine whether $f(x) = 2x - 3$ and $g(x) = -3x + 2$ are inverse functions.

10. Find f^{-1} for each of the following functions. If the function is not one-to-one, restrict the domain. State the domain and range for each function and its inverse.

a. $f(x) = 2x - 1$

b. $f(x) = x^3 + 5$

c. $f(x) = \dfrac{1}{x-5}$

d. $f(x) = x^2 + 3$

e. $f(x) = \sqrt{x-1}$

11. Verify that f and f^{-1} obtained above are inverses using the property of inverses.

12. Graph f and f^{-1} obtained above on the same axes to show they have symmetry about $y = x$.

Answers to Exercises for Chapter 4F - Inverse Functions

1. Find the inverse relation of each of the following and determine whether the inverse is a function::

a. $\{(4,-2),(1,-1),(0,0),(1,1),(4,2)\}$ Not a function.

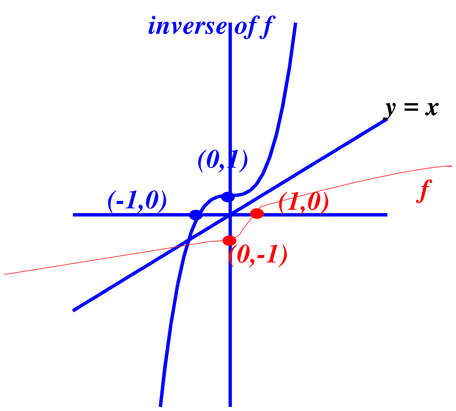
b. Not a function.

Breed	Dog's Name
Schnauser	Casey
Shepherd	Tilly
Lab	Madison
Schnauser	Shelby
Mutt	Scooter
Beagle	Freckles

c. Is a function.

x	y
1	4
2	7
3	9
4	1
5	3
6	6

2.



3.

a. Function, but not one-to-one

b. Is a one-to-one function

c. Is a one-to-one function

4. f is not a one-to-one function. $f(x_1) = f(x_2) \Rightarrow x_1 = \pm x_2$. The graph is $y = x^2$ shifted down 3 which does not pass the horizontal line test for one-to-one functions.

5. Square root function shifted to the left 2 $\Rightarrow$ f is a one-to-one function.
$\sqrt{x_1+2} = \sqrt{x_2+2} \Rightarrow x_1 = x_2$

6.

a. Restrict to $x \geq 0$: $[0,+\infty)$. Other answers possible.

b. Restrict to $[-2,+\infty)$. Other answers possible.

c. Restrict to $[5, +\infty)$. Other answers possible.

7. $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. Therefore, f and g are inverses.

8. $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. Therefore, f and g are inverses.

9. No. $(f \circ g)(x) = -6x + 1$

10.

a. Linear function is one-to-one. $f^{-1}(x) = \dfrac{x+1}{2}$

Domain f: $(-\infty, +\infty)$ Range f: $(-\infty, +\infty)$

Domain f^{-1}: $(-\infty, +\infty)$ Range f^{-1}: $(-\infty, +\infty)$

b. $y = x^3$ shifted up 5 is one-to-one. $f^{-1}(x) = \sqrt[3]{x-5}$

Domain f: $(-\infty, +\infty)$ Range f: $(-\infty, +\infty)$

Domain f^{-1}: $(-\infty, +\infty)$ Range f^{-1}: $(-\infty, +\infty)$

c. $y = \dfrac{1}{x}$ shifted to the right 5 is one-to-one. $f^{-1}(x) = \dfrac{1}{x} + 5$

Domain f: $(-\infty, 5) \cup (5, +\infty)$ Range f: $(-\infty, 0) \cup (0, +\infty)$

Domain f^{-1}: $(-\infty, 0) \cup (0, +\infty)$ Range f^{-1}: $(-\infty, 5) \cup (5, +\infty)$

d. $f(x) = x^2 + 3$ is a parabola $\Rightarrow f$ is not one-to-one. Restrict the domain to $x \ge 0$. Since $y = x^2$ is shifted up 3 units, the range of f is $y \ge 3$.

$y = x^2 + 3 \quad x \ge 0, \quad y \ge 3.$

$x = y^2 + 3 \quad x \ge 3, \quad y \ge 0$

$f^{-1}(x) = \sqrt{x-3}$ domain: $[3, +\infty)$ range: $[0, +\infty)$

e. f is the square root function shifted to the right 1. Therefore f is one-to-one with domain $[1, +\infty)$ and range $[0, +\infty)$.

$y = \sqrt{x-1}$ domain $[1, +\infty)$ range $[0, +\infty)$

$x = \sqrt{y-1}$ domain $[0, +\infty)$ range $[1, +\infty)$

$f^{-1}(x) = x^2 + 1$ domain $[0, +\infty)$ range $[1, +\infty)$

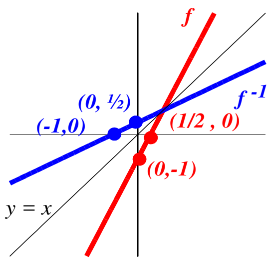
11. a) $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ for all x. Therefore, f and f^{-1} are inverses.

b) $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ for all x. Therefore, f and f^{-1} are inverses.

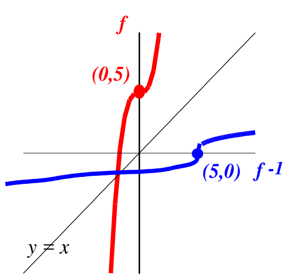
c) $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ for all x in the restricted domain. Therefore, f and f^{-1} are inverses over the domain of each.

d) $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ for all x in the restricted domain. Therefore, f and f^{-1} are inverses over the domain of each.

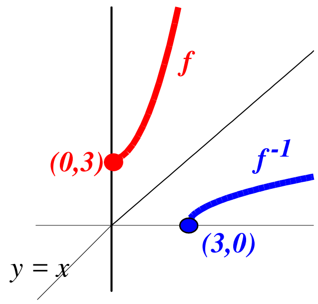
12. a)



b)



c)



d)