The Free High School Science Texts: A Textbook for High School Students Studying Maths.

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[1] See http://savannah.nongnu.org/projects/fhsst

Contents

Part I

Maths

This book attempts to meet the criteria for the SA "Outcomes Based" syllabus of 2004. A few notes to authors:

All "real world examples" should be in the context of HIV/AIDS, labour disputes, human rights, social, economical, cultural, political and environmental issues. Unless otherwise stated in the syllabus. Where possible, every section should have a practical problem.

The preferred method for disproving something is by counter example. Justification for any mathematical generalisations of applied examples is always desired.

Chapter 1

Numbers

(NOTE: more examples and motivation needed. perhaps drop the proofs for exponents and surds?)

A number is a way to represent quantity. Numbers are not something that we can touch or hold, because they are not physical. But you can touch three apples, three pencils, three books. You can never just touch three, you can only touch three of something. However, you don't need to see three apples in front of you to know that if you take one apple away, that there will be two apples left. You can just think about it. That is your brain representing the apples in numbers and then performing arithmetic on them.

A number represents quantity because we can look at the world around us and quantify it using numbers. How many minutes? How many kilometers? How many apples? How much money? How much medicine? These are all questions which can only be answered using numbers to tell us "how much" of something we want to measure.

A number can be written many different ways and it is always best to choose the most appropriate way of writing the number. For example, the number "a half" may be spoken aloud or written in words, but that makes mathematics very difficult and also means that only people who speak the same language as you can understand what you mean. A better way of writing "a half" is as a fraction $\frac{1}{2}$ or as a decimal number 0,5. It is still the same number, no matter which way you write it.

In high school, all the numbers which you will see are called *real numbers*(NOTE: Advanced: The name "real numbers" is used because there are different and more complicated numbers known as "imaginary numbers", which this book will not go into. Since we won't be looking at numbers which aren't real, if you see a number you can be sure it is a real one.) and mathematicians use the symbol $\mathbb{R}$ to stand for the *set of all real numbers*, which simply means all of the real numbers. Some of these real numbers can be written in a particular way, but others cannot.

This chapter will explain different ways of writing any number, and when each way of writing the number is best.

(NOTE: This intro needs more motivation for different types of numbers, some real world examples and more interesting facts. Lets avoid the whole different numeral systems though... maybe when we do the history edit near release.)

1.1 Letters and Arithmetic

The syllabus requires:

- ~~algebraic manipulation is governed by the algebra of the real numbers~~
- ~~manipulate equations (rearrange for y, expand a squared bracket)~~

(NOTE: "algebra of the Reals". why letters are useful... very simple example, like change from a shop. brackets, squared brackets, fractions, multiply top and bottom. rearranging. doing something to one side and the other.)

When you add, subtract, multiply or divide two numbers, you are performing *arithmetic*[1]. These four basic operations ($+, -, \times, \div$) can be performed on any two real numbers.

Since they work for any two real numbers, it would take forever to write out every possible combination, since there are an infinite(NOTE: Advanced: we **really** need to define what infinite means, nicely!) amount of real numbers! To make things easier, it is convenient to use letters to stand in for any number[2], and then we can fill in a particular number when we need to. For example, the following equation

$$x + y = z \tag{1.1}$$

can find the change you are owed for buying an item. In this equation, x represents the amount of change you should get, z is the amount you payed and y is the price of the item. All you need to do is write the amount you payed instead of z and the price instead of y, your change is then x. But to be able to find your change you will need to *rearrange* the equation for x. We'll find out how to do that just after we learn some more details about the basic operators.

1.1.1 Adding and Subtracting

Adding, subtracting, multiplying and dividing are the most basic operations between numbers but they are very closely related to each other. You can think of subtracting as being the opposite of adding since adding a number and then subtracting the same number will not change what you started with. For example, if we start with a and add b, then subtract b, we will just get back to a again

$$\begin{aligned} a + b - b &= a \qquad (1.2)\\ 5 + 2 - 2 &= 5 \end{aligned}$$

(NOTE: rework these bits into the Negative Numbers section. it needs more attention than we initially thought.) Subtraction is actually the same as adding a *negative number*. A negative number is a number less than zero. Numbers greater than zero are called *positive numbers*. In this example, a and b are positive numbers, but $-b$ is a negative number

$$\begin{aligned} a - b &= a + (-b) \qquad (1.3)\\ 5 - 3 &= 5 + (-3) \end{aligned}$$

[1]Arithmetic is the Greek word for "number"

[2]We will look at this in more detail in chapter 3.

It doesn't matter which order you write additions and subtractions(NOTE: Advanced: This is a property known as *associativity*, which means $a+b=b+a$), but it looks better to write subtractions to the right. You will agree that $a-b$ looks neater than $-b+a$, and it makes some sums easier, for example, most people find $12-3$ a lot easier to work out than $-3+12$, even though they are the same thing.

1.1.2 Negative Numbers

Negative numbers can be very confusing to begin with, but there is nothing to be afraid of. When you are adding a negative number, it is the same as subtracting that number if it were positive. Likewise, if you subtract a negative number, it is the same as adding the number if it were positive. Numbers are either positive or negative, and we call this their *sign*. A positive number has positive sign, and a negative number has a negative sign.

(NOTE: number line here. subtraction is moving to left, adding is moving to the right. maybe something else about negative numbers?)

Table 1.1 shows how to calculate the sign of the answer when you multiply two numbers together. The first column shows the sign of one of the numbers, the second column gives the sign of the other number, and the third column shows what sign the answer will be. So multiplying a negative number by

a	b	$a \times b$
+	+	+
+	-	-
-	+	-
-	-	+

Table 1.1: Table of signs for multiplying two numbers.

a positive number always gives you a negative number, whereas multiplying numbers which have the same sign always gives a positive number. For example, $2 \times 3 = 6$ and $-2 \times -3 = 6$, but $-2 \times 3 = -6$ and $2 \times -3 = -6$.

Adding numbers works slightly differently, have a look at Table 1.2. If you

a	b	$a + b$
+	+	+
+	-	?
-	+	?
-	-	-

Table 1.2: Table of signs for adding two numbers.

add two positive numbers you will always get a positive number, but if you add two negative numbers you will always get a negative number. If the numbers have different sign, then the sign of the answer depends on which one is bigger.

1.1.3 Brackets

In equation (1.3) we used brackets[3] around $-b$. Brackets are used to show the order in which you must do things. This is important as you can get different answers depending on the order in which you do things. For example

$$(5 \times 10) + 20 = 70 \tag{1.4}$$

whereas

$$5 \times (10 + 20) = 150 \tag{1.5}$$

If you don't see any brackets, you should always do multiplications and divisions first and then additions and subtractions[4]. You can always put your own brackets into equations using this rule to make things easier for yourself, for example:

$$\begin{aligned} a \times b + c \div d &= (a \times b) + (c \div d) \\ 5 \times 10 + 20 \div 4 &= (5 \times 10) + (20 \div 4) \end{aligned} \tag{1.6}$$

1.1.4 Multiplying and Dividing

Just like addition and subtraction, multiplication and division are opposites of each other. Multiplying by a number and then dividing by the same number gets us back to the start again:

$$\begin{aligned} a \times b \div b &= a \\ 5 \times 4 \div 4 &= 5 \end{aligned} \tag{1.7}$$

Sometimes you will see a multiplication of letters without the $\times$ symbol, don't worry, its exactly the same thing. Mathematicians are lazy and like to write things in the neatest way possible.

$$abc = a \times b \times c \tag{1.8}$$

It is usually neater to write known numbers to the left, and letters to the right. So although $4x$ and $x4$ are the same thing(NOTE: Advanced: This is a property known as *commutativity*, which means $ab = ba$), it looks better to write $4x$.

If you see a multiplication outside a bracket like this

$$\begin{aligned} &a(b + c) \\ &3(4 - 3) \end{aligned} \tag{1.9}$$

then it means you have to multiply each part inside the bracket by the number outside

$$\begin{aligned} a(b + c) &= ab + ac \\ 3(4 - 3) &= 3 \times 4 - 3 \times 3 = 12 - 9 = 3 \end{aligned} \tag{1.10}$$

[3]Sometimes people say "parenthesis" instead of "brackets".

[4]Multiplying and dividing can be performed in any order as it doesn't matter. Likewise it doesn't matter which order you do addition and subtraction. Just as long as you do any $\times\div$ before any $+-$.

unless you can simplify everything inside the bracket into a single term. In fact, in the above example, it would have been smarter to have done this

$$3(4-3) = 3 \times (1) = 3 \tag{1.11}$$

It can happen with letters too

$$3(4a-3a) = 3 \times (a) = 3a \tag{1.12}$$

If there are two brackets multiplied by each other, then you can do it one step at a time

$$\begin{aligned} (a+b)(c+d) &= a(c+d)+b(c+d) \\ &= ac+ad+bc+bd \\ (a+3)(4+d) &= a(4+d)+3(4+d) \\ &= 4a+ad+12+3d \end{aligned} \tag{1.13}$$

1.1.5 Rearranging Equations

Coming back to the example about change, which we wanted to solve earlier in equation (1.1)

$$x + y = z$$

To recap your memory, z is the amount you (or a customer) payed for something, y is the price and you want to find x, the change. What you need to do is rearrange the equation so only x is on the left.

You can add, subtract, multiply or divide both sides of an equation by any number you want, as long as you always do it to both sides. If you imagine an equation is like a set of weighing scales. (NOTE: diagram here.) If you wish to keep the scales balanced, then when you add something to one side, you must also add something of the same weight to the other side.

So for our example we could subtract y from both sides

$$\begin{aligned} x+y &= z \\ =x+y-y &= z-y \\ x &= z-y \end{aligned} \tag{1.14}$$

so now we can find the change is the amount payed take away the price. In real life we can do this in our head, the human brain is very smart and can do arithmetic without even knowing it.

When you subtract a number from both sides of an equation, it looks just like you moved a positive number from one side and it became a negative on the other, which is exactly what happened. Likewise if you move a multiplied number from one side to the other, it looks like it changed to a divide. This is because you really just divided both sides by that number, and a number divided by itself is just 1

$$\begin{aligned} a(5+c) &= 3a \\ a \div a(5+c) &= 3a \div a \\ 1 \times (5+c) &= 3 \times 1 \\ 5+c &= 3 \\ c &= 3-5=-2 \end{aligned} \tag{1.15}$$

However you must be careful when doing this, as it is easy to make mistakes. **The following is the wrong thing to do**[5].

$$\begin{aligned} 5a + c &= 3a \\ 5 + c &\neq 3a \div a \end{aligned} \tag{1.16}$$

Can you see why it is wrong? The reason why it is wrong is because we didn't divide the c part by a as well. The correct thing to do is

$$\begin{aligned} 5a + c &= 3a \\ 5 + c \div a &= 3 \\ c \div a &= 3 - 5 = -2 \end{aligned} \tag{1.17}$$

1.1.6 Living Without the Number Line

The number line in (NOTE: ref when we do it) is a good way to visualise what negative numbers are, but it can get wery inefficient to use it every time you want to add or subtract negative numbers. To keep things simple, we will write down three rules that you should memorise. These rules will let you work out what the answer is when you add or subtract numbers which may be negative and will also help you keep your work tidy and easier to understand.

Signs Rule 1

If you have an equation which has a negative number on the very far left, it can be confusing. But it doesn't matter where we put the negative number, as long as it is on the left of the equals sign(NOTE: this is confusing). If we move it more to the right, it makes more sense as it just looks like subtracting a positive number.

$$\begin{aligned} -a + b &= b - a \\ -5 + 10 &= 10 - 5 = 5 \end{aligned} \tag{1.18}$$

This makes equations easier to understand. For example, a question like "What is $-7+11$?" looks a lot more complicated than "What is $11-7$?", even though they are exactly the same question.

Signs Rule 2

When you have two negative numbers like $-3 - 7$, you can calculate the answer by simply adding together the numbers as if they were positive and then remembering to put a negative sign in front.

$$\begin{aligned} -c - d &= -(c + d) \\ -7 - 2 &= -(7 + 2) = -9 \end{aligned} \tag{1.19}$$

[5] The $\neq$ symbol says that this is incorrect as it means "not equal to".

Signs Rule 3

In section 1.1.2 we seen that the sign of two numbers added together depends on which one is bigger. This last rule tells us that all we need to do is take the smaller number away from the larger one, and remember to put a negative sign before the answer if the bigger number was subtracted to begin with. In this equation, F is bigger than e.

$$\begin{aligned} e - F &= -(F - e) \qquad (1.20) \\ 2 - 11 &= -(11 - 2) = -9 \end{aligned}$$

You can even combine these rules together, so for example you can use rule 1 on $-10 + 3$ to get $3 - 10$, and then use rule 3 to get $-(10 - 3) = -7$.

Now you know everything there is to know about arithmetic. So try out your skills on the exercises at the end of this chapter and ask your teacher for more questions just like them. You can also try making up your own questions, solve them and try them out on your classmates to see if you get the same answers. Practice is the only way to get good at maths.

1.2 Types of Real Numbers

(NOTE: maybe more intro here, break up the chapter intro and put some here?)

1.2.1 Integers

The *natural numbers* are all the numbers which you can use for counting

$$0, 1, 2, 3, 4 \ldots \qquad (1.21)$$

These are the first numbers learnt by children, and the easiest to understand. Mathematicians use the symbol $\mathbb{N}$ to mean the *set of all natural numbers*. The natural numbers are a *subset* of the real numbers since every natural number is also a real number.

The integers are all of the natural numbers and their negatives

$$\ldots - 4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \qquad (1.22)$$

Mathematicians use the symbol $\mathbb{Z}$ to mean *the set of all integers*. The integers are a subset of the real numbers, since every integer is a real number.

(NOTE: possible analogy... whole fruit on a tree, if you eat some, its not an integer anymore.)

1.2.2 Fractions and Decimal numbers

A fraction is any kind of number divided by another number. There are several ways to write a number divided by another one, such as $a \div b$, a/b and $\frac{a}{b}$. The first way of writing a fraction is very hard to work with, so we will use only the other two. We call the number on the top, the *numerator* and the number on the bottom the *denominator*. e.g.

$$\frac{1}{5} \qquad \frac{\text{numerator} = 1}{\text{denominator} = 5} \qquad (1.23)$$

The *reciprocal* of a fraction is the fraction turned upside down, in other words the numerator becomes the denominator and the denominator becomes the numerator. A fraction times its reciprocal always equals 1 and can be written

$$\frac{a}{b} \times \frac{b}{a} = 1 \tag{1.24}$$

This is because dividing by a number is the same as multiplying by its reciprocal.

A decimal number is a number which has an integer part and a fraction part. The integer and the fraction parts are separated by a *decimal point*, which is written as a comma in South Africa. Every real number can be written as a decimal. For example the number $3 + 14/100$ can be written much more cleanly as 3,14.

All real numbers can be written as a decimal. However, some numbers would take a huge amount of paper (and ink) to write out in full! Some decimal numbers will have a number which will repeat itself, such as 0,33333... where there are an infinite number of 3's. We can write this decimal value by using a dot above the repeating number, so $0{,}\dot{3} = 0{,}33333\ldots$. If there are two repeating numbers such as 0,121212... then you can place dots[6] on each of the repeated numbers $0{,}\dot{1}\dot{2} = 0{,}121212\ldots$. These kinds of repeating decimals are called *recurring decimals*.

(NOTE: should we have a table of, say, the 10 most common fractions?)

1.2.3 Rational Numbers

`The syllabus requires:`

- ~~`identify rational numbers`~~
- ~~`convert between terminating or recurring decimals and their fractional form`~~

A rational number is any number which can be written as a fraction with an integer on the top and an integer on the bottom (as long as the integer on the bottom is not zero) (NOTE: Advanced: This can be expressed in the form $\frac{a}{b}; a, b \in \mathbb{Z}; b \neq 0$ which means "the set of numbers $\frac{a}{b}$ when a and b are integers".).

Mathematicians use the symbol $\mathbb{Q}$ to mean *the set of all rational numbers*. The set of rational numbers contains all numbers which can be written as terminating or repeating decimals. (NOTE: Advanced: All integers are rational numbers with denominator 1.)

An *irrational number* is any real number that is not a rational number. When expressed as decimals these numbers can never be fully written out as they have an infinite number of decimal places which never fall into a repeating pattern, for example $\sqrt{2} = 1{,}41421356\ldots$, $\pi = 3{,}14159265\ldots$. π is a Greek letter and is pronounced just like "pie". We'll mention more about π in chapter 7.1.

You can add and multiply rational numbers and still get a rational number at the end, which is very useful. If we have 4 integers, a, b, c and d(NOTE: Advanced: This can be written formally as $\{a, b, c, d\} \in \mathbb{Z}$ because the $\in$ symbol

[6] or a bar, like $0{,}\overline{12}$

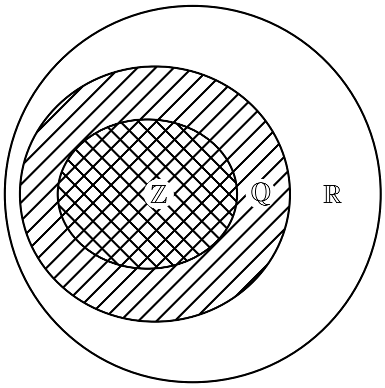


Figure 1.1: Set diagram of all the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$ and the integers $\mathbb{Z}$. The irrational numbers are the numbers not inside the set of rational numbers. All of the integers are also rational numbers, but not all rational numbers are integers. (NOTE: possible question; where is $\mathbb{N}$ in this diagram?)

means *in* and we say that a, b, c and d are *in* the set of integers.), then the rules for adding and multiplying rational numbers are

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \tag{1.25}$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \tag{1.26}$$

Two rational numbers ($\frac{a}{b}$ and $\frac{c}{d}$) represent the same number if $ad = bc$. It is always best to simplify any rational number so that the denominator is as small as possible. This can be achieved by dividing both the numerator and the denominator by the same integer. For example, the rational number 1000/10000 can be divided by 1000 on the top and the bottom, which gives 1/10. $\frac{2}{3}$ of a pizza is the same as $\frac{8}{12}$. (NOTE: maybe a diagram.)

You can also add rational numbers together by finding a *lowest common denominator* and then adding the numerators. Finding a lowest common denominator means finding the lowest number that both denominators are a $f$actor[7] of. A factor of a number is an integer which evenly divides that number without leaving a remainder. The following numbers all have a factor of 3

$$3, 6, 9, 12, 15, 18, 21, 24 \ldots$$

and the following all have factors of 4

$$4, 8, 12, 16, 20, 24, 28 \ldots$$

The common denominators between 3 and 4 are all the numbers that appear in both of these lists, like 12 and 24. The lowest common denominator of 3 and 4 is the number that has both 3 and 4 as factors, which is 12.

[7] Some people say *divisor* instead of factor.

For example, if we wish to add $\frac{3}{4}+\frac{2}{3}$, we first need to write both fractions so that their denominators are the same by finding the lowest common denominator, which we know is 12. We can do this by multiplying $\frac{3}{4}$ by $\frac{3}{3}$[8] and $\frac{2}{3}$ by $\frac{4}{4}$

$$\begin{aligned}
\frac{3}{4}+\frac{2}{3} &= \frac{3}{4}\times\frac{3}{3}+\frac{2}{3}\times\frac{4}{4} \\
&= \frac{3\times 3}{4\times 3}+\frac{2\times 4}{3\times 4} \\
&= \frac{9}{12}+\frac{8}{12} \\
&= \frac{9+8}{12} \\
&= \frac{17}{12}
\end{aligned} \tag{1.27}$$

Dividing by a rational number is the same as multiplying by it's reciprocal, as long as neither the numerator nor the denominator is zero:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}.\frac{d}{c}=\frac{ad}{bc} \tag{1.28}$$

A rational number may also be written as a *proper fraction*, which is the sum of an integer and a rational fraction.

$$A\frac{b}{c}=\frac{Ac+b}{c}=A+\frac{b}{c} \tag{1.29}$$

This notation has the advantage that you can readily tell the approximate size of the fraction, but has the disadvantage that $A\frac{b}{c}$ can be mistaken for $A\times\frac{b}{c}$ instead of $A+\frac{b}{c}$. However, it can be used to help convert rational numbers into decimals and vica-versa. A fraction is called an *improper fraction* if the numerator is bigger than the denominator, meaning that it could be written as a proper fraction.

Converting Decimals into Rational Numbers

If you recall from section 1.2.2 that a decimal number has an integer and a fractional part, then you will notice how similar decimals are to proper fractions. The integer part of a decimal is also the integer part of a proper fraction and each digit after the decimal point is a fraction with denominator in increasing powers of ten, in other words $\frac{1}{10}$ is 0,1, $\frac{1}{100}$is 0,01 and so on. For example, 2,103 is $2+\frac{1}{10}+\frac{0}{100}+\frac{3}{1000}$ which is $2\frac{103}{1000}$ (you could also have just written $\frac{2103}{1000}$).

When the decimal is a recurring decimal, a certain amount of algebraic manipulation is involved in finding the fractional part. Lets have a look at an example to see how this is done. If we wish to write $0,\dot{3}$ in the form $\frac{a}{b}$ (where a

[8] $\frac{3}{3}$ is really just a complicated way of writing 1. Multiplying a number by 1 doesn't change the number.

and b are integers) then we would proceed as follows

$$\begin{aligned}
x &= 0{,}33333\ldots && (1.30)\\
10x &= 3{,}33333\ldots \quad \text{multiply by 10 on both sides} && (1.31)\\
9x &= 3 \quad \text{subtracting (1.30) from (1.31)}\\
x &= \frac{3}{9} = \frac{1}{3}
\end{aligned}$$

And another example would be to write $5{,}\overline{432}$ as a rational fraction

$$\begin{aligned}
x &= 5{,}432432432\ldots && (1.32)\\
1000x &= 5432{,}432432432\ldots && (1.33)\\
999x &= 5427 \quad \text{subtracting (1.32) from (1.33)}\\
x &= \frac{5427}{999} = \frac{201}{37}
\end{aligned}$$

But not all decimal numbers are rational numbers as can be seen in figure 1.1. When possible, you should always use fractions instead of decimals.

Converting Rationals into Decimal Numbers

If you use a calculator, you can simply divide the numerator by the denominator. If there is no calculator at hand, long division will suffice. (NOTE: i did long division in primary school... i don't think it warrants its own intro, so i guess we can assume they know it already. anyone disagree?) For example, we may convert $\frac{1}{4}$ to decimal form by doing the following long division

$$\begin{array}{r@{\,}l}
 & 0{,}25 \\
4\,\big) & 1{,}00 \\
 & \underline{8} \\
 & 20 \\
 & \underline{20}
\end{array} \qquad (1.34)$$

As another example, we can convert $\frac{67}{153}$ to decimal form (with 4 significant digits (NOTE: dependency problem... we mention SFs later than this. should we place this after "accuracy" or reword/remove this?)).

$$\begin{array}{r@{\,}l}
 & 0{,}4379 \\
153\,\big) & 67{,}0000 \\
 & \underline{61\ 2} \\
 & 58\ 0 \\
 & \underline{45\ 9} \\
 & 1210 \\
 & \underline{1071} \\
 & 1390 \\
 & \underline{1377} \\
 & 13 \\
 & \vdots
\end{array} \qquad (1.35)$$

(NOTE: note that this is much easier if the denominator is a multiple of 10)

1.3 Exponents

`The syllabus requires:`

- ~~`simplify expressions using the laws of exponents for rational indices`~~

Exponential notation is a short way of writing the same number multiplied by itself many times. For example, instead of $5 \times 5 \times 5$, we write 5^3 to show that the number 5 is multiplied by itself 3 times, we say it as "5 to the power of 3". Likewise 5^2 is 5×5 and 3^5 is $3 \times 3 \times 3 \times 3 \times 3$.

We will now have a closer look at writing numbers as exponentials (a^n) when n is an integer and a can be any real number[9].

The nth power of a is

$$a^n = 1 \times a \times a \times \ldots \times a \qquad (n \text{ times}) \tag{1.36}$$

with a appearing n times. a is called the *base* and n is called the *exponent*. 1 is here so that it can be seen that any number to the power of zero, is 1. We can also define what it means if $-n$ is a negative integer

$$a^{-n} = 1 \div a \div a \div \ldots \div a \qquad (n \text{ times}) \tag{1.37}$$

If n is an even integer, then a^n will always be positive for any non-zero real number a. For example, although -2 is negative, $(-2)^2 = 1 \times -2 \times -2 = 4$ is positive and so is $(-2)^{-2} = 1 \div -2 \div -2 = \frac{1}{4}$. You should now notice that $a^{-n} = 1/a^n$, which is one of many rules we often use when working with exponentials.

There are several rules we can use to manipulate exponential numbers, making them much easier to work with. We will list all the rules here for easy reference, but we will explain each rule in more detail (NOTE: maybe try to reduce the number here. kids memorise rules and 7 is maybe too much.)

$$a^m a^n = a^{m+n} \tag{1.38}$$

$$a^{-n} = \frac{1}{a^n} \tag{1.39}$$

$$\frac{a^m}{a^n} = a^{m-n} \tag{1.40}$$

$$(ab)^n = a^n b^n \tag{1.41}$$

$$(a^m)^n = a^{mn} \tag{1.42}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \tag{1.43}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{1.44}$$

Exponential Rule 1: $a^m a^n = a^{m+n}$

Our definition of exponential notation shows that

$$\begin{aligned} a^m a^n &= 1 \times a \times \ldots \times a & (m \text{ times}) \\ &\quad \times 1 \times a \times \ldots \times a & (n \text{ times}) \\ &= 1 \times a \times \ldots \times a & (m+n \text{ times}) \\ &= a^{m+n} \end{aligned} \tag{1.45}$$

[9] we can write this as $n \in \mathbb{Z}$, $a \in \mathbb{R}$

This simple rule is the reason why exponentials were originally invented. In the days before calculators, all multiplication had to be done by hand with a pencil and a pad of paper. Multiplication takes a very long time to do and is very tedious. Adding numbers however, is very easy and quick to do. If you look at what this rule is saying you will realise that it means that adding the exponents of two exponential numbers (of the same base) is the same as multiplying the two numbers together. This meant that for certain numbers, there was no need to actually multiply the numbers together in order to find out what their multiple was. This saved mathematicians a lot of time, which they could use to do something more productive.

Exponential Rule 2: $a^{-n} = \frac{1}{a^n}$

Our definition of exponential notation for a negative exponent shows that

$$\begin{aligned} a^{-n} &= 1 \div a \div \ldots \div a \qquad (n \text{ times}) \\ &= \frac{1}{1 \times a \times \cdots \times a} \qquad (n \text{ times}) \\ &= \frac{1}{a^n} \end{aligned} \tag{1.46}$$

This means that a minus sign in the exponent is just another way of writing that the whole exponential number is to be divided instead of multiplied.

Exponential Rule 3: $\frac{a^m}{a^n} = a^{m-n}$

We already realised with rule 2 that a minus sign is another way of saying that the exponential number is to be divided instead of multiplied. Rule 3 is just a more general way of saying the same thing. We can get this rule by just multiplying rule 2 by a^m on both sides and using rule 1.

$$\begin{aligned} \frac{a^m}{a^n} &= a^m a^{-n} \\ &= a^{m-n} \end{aligned} \tag{1.47}$$

Exponential Rule 4: $(ab)^n = a^n b^n$

When real numbers are multiplied together, it doesn't matter in what order the multiplication occurs(NOTE: Advanced: This is a collection of two properties, called *commutativity* (meaning $ab = ba$) and *associativity* (meaning $a(bc) = (ab)c$).). Therefore

$$\begin{aligned} (ab)^n &= a \times b \times a \times b \times \ldots \times a \times b \qquad (n \text{ times}) \\ &= a \times a \times \ldots \times a \qquad (n \text{ times}) \\ &\quad \times b \times b \times \ldots \times b \qquad (n \text{ times}) \\ &= a^n b^n \end{aligned} \tag{1.48}$$

Exponential Rule 5: $(a^m)^n = a^{mn}$

We can find the exponential of an exponential just as well as we can for a number. After all, an exponential number is a real number.

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times \ldots \times a^m \qquad (n \text{ times}) \\ &= a \times a \times \ldots \times a \qquad (m \times n \text{ times}) \\ &= a^{mn}\end{aligned} \tag{1.49}$$

Exponential Rule 6: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Fractions can be multiplied together by separately multiplying their numerators and denominators (remember equation (1.26)). This means

$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \ldots \times \frac{a}{b} \qquad (n \text{ times}) \\ &= \frac{a \times a \times \ldots \times a}{b \times b \times \ldots \times b} \qquad (n \text{ times}) \\ &= \frac{a^n}{b^n}\end{aligned} \tag{1.50}$$

Exponential Rule 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

We say that x is an nth root of b if $x^n = b$. For example, $(-1)^4 = 1$, so -1 is a 4th root of 1. Using rule 5, we notice that

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n}n} = a^m \tag{1.51}$$

therefore $a^{\frac{m}{n}}$ must be an nth root of a^m. We can therefore say

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{1.52}$$

where $\sqrt[n]{a^m}$ is the nth root of a^m (if it exists).

A number may not always have a real nth root. For example, if $n = 2$ and $a = -1$, then there is no real number such that $x^2 = -1$ because x^2 can never be a negative number(NOTE: Advanced: There are numbers which can solve problems like $x^2 = -1$, but they are beyond the scope of this book. They are called *complex numbers*.). It is also possible for more than one nth root of a number to exist. For example, $(-2)^2 = 4$ and $2^2 = 4$, so both -2 and 2 are 2nd (square) roots of 4. Usually if there is more than one root, we choose the positive real solution and move on.

1.4 Surds

`The syllabus requires:`

- ~~`identify between which 2 integers any simple surd lies`~~
- ~~`add, subtract, multiply and divide simple surds`~~

We have already discussed what is meant by the nth root of a number. If such an nth root is irrational, we call it a *surd*. For example, $\sqrt{2}$ and $\sqrt[3]{6}$ are surds, but $\sqrt{4} = 2$ is not a surd, because 2 is a rational number. We will only look at surds of the form $\sqrt[n]{a}$ where a is a positive number. When $n = 2$ we do not write it in and just leave the surd as $\sqrt{a}$, which is much easier to read.

There are several rules for manipulating surds. We will list them all and then explain where each rule comes from in more detail.

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \tag{1.53}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \tag{1.54}$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}} \tag{1.55}$$

Surd Rule 1: $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

It is often enlightening to look at a surd in exponential notation as it allows us to use the rules we learnt in section 1.3. If we write $\sqrt[n]{a}\,\sqrt[n]{b}$ in exponential notation we can see how rule 1 appears

$$\begin{aligned} \sqrt[n]{a}\,\sqrt[n]{b} &= a^{\frac{1}{n}} b^{\frac{1}{n}} \\ &= (ab)^{\frac{1}{n}} \\ &= \sqrt[n]{ab} \end{aligned} \tag{1.56}$$

Surd Rule 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Rule 2 appears by looking at $\sqrt[n]{\frac{a}{b}}$ in exponential notation and applying the exponential rules

$$\begin{aligned} \sqrt[n]{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{n}} \\ &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \\ &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \end{aligned} \tag{1.57}$$

Surd Rule 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Rule 3 appears by looking at $\sqrt[n]{a^m}$ in exponential notation and applying the exponential rules

$$\begin{aligned} \sqrt[n]{a^m} &= (a^m)^{\frac{1}{n}} \\ &= a^{\frac{m}{n}} \end{aligned} \tag{1.58}$$

1.4.1 Like and Unlike Surds

Two surds $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are called *like surds* if $m = n$, otherwise they are called *unlike surds*. For example $\sqrt{2}$ and $\sqrt{3}$ are like surds, however $\sqrt{2}$ and $\sqrt[3]{2}$ are unlike surds. An important thing to realise about the rules we have just learnt is that the surds in the rules are all like surds.

If we wish to use the surd rules on unlike surds, then we must first convert them into like surds. In order to do this we use the formula

$$\sqrt[n]{a^m} = \sqrt[bn]{a^{bm}} \tag{1.59}$$

to rewrite the unlike surds so that bn is the same for all the surds.

1.4.2 Rationalising Denominators

It is useful to work with fractions which have rational denominators instead of surd denominators. It is possible to rewrite any fraction which has a surd in the denominator as a fraction which has a rational denominator. We will now see how this can be achieved.

Any expression of the form $\sqrt{a} + \sqrt{b}$ (where a and b are rational) can be changed into a rational number by multiplying by $\sqrt{a} - \sqrt{b}$ (similarly $\sqrt{a} - \sqrt{b}$ can be rationalised by multiplying by $\sqrt{a} + \sqrt{b}$). This is because

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b \tag{1.60}$$

which is rational (since a and b are rational).

If we have a fraction which has a denominator which looks like $\sqrt{a}+\sqrt{b}$, then we can simply multiply both top and bottom by $\sqrt{a} - \sqrt{b}$ achieving a rational denominator.

$$\begin{aligned} \frac{c}{\sqrt{a}+\sqrt{b}} &= \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} \times \frac{c}{\sqrt{a}+\sqrt{b}} \\ &= \frac{c\sqrt{a}-c\sqrt{b}}{a-b} \end{aligned} \tag{1.61}$$

or similarly

$$\begin{aligned} \frac{c}{\sqrt{a}-\sqrt{b}} &= \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} \times \frac{c}{\sqrt{a}-\sqrt{b}} \\ &= \frac{c\sqrt{a}+c\sqrt{b}}{a-b} \end{aligned} \tag{1.62}$$

1.4.3 Estimating a Surd

(NOTE: has anyone got a better way to do this?) It is sometimes useful to know the approximate value of a surd without having to use a calculator. This involves knowing some roots which have integer solutions.

For a surd $\sqrt[n]{a}$, find an integer smaller than a with an integer as its nth root and then find the next highest integer (which should also be larger than a) with an integer nth root. The surd which you are trying to estimate will be between those two integers. (NOTE: this paragraph sucks, rewrite it so that it can be understood.)

For example, when given the surd $\sqrt[3]{52}$ you should be able to tell that it lies somewhere between 3 and 4, because $\sqrt[3]{27} = 3$ and $\sqrt[3]{64} = 4$ and 52 is between 27 and 64. In fact $\sqrt[3]{52} = 3.73\ldots$ which is indeed between 3 and 4.

The easiest arithmetic procedure[10] to find the square root of any number N is to choose a number x that is close to the square root, find $\frac{N}{x}$ and then use $x' = \frac{x+\frac{N}{x}}{2}$ for the next choice of x. x converges rapidly towards the actual value of the square root - the number of significant digits doubles each time. (NOTE: arithmetic? procedure? converges? rapidly? this language is not basic enough!)

We will now use this method to find $\sqrt{55}$. We know that $7^2 = 49$, therefore the square root of 55 must be close to 7.

$$
\begin{aligned}
\text{Let } x &= 7 && (1.63)\\
\text{then, } \frac{55}{7} &= 7{,}8571\ldots && (1.64)\\
\therefore x' = \frac{7+7{,}8571\ldots}{2} &= 7{,}4285\ldots && (1.65)\\
\frac{55}{7.4285\ldots} &= 7{,}4038\ldots && (1.66)\\
\therefore x' &= \frac{7{,}4285\ldots+7{,}4038\ldots}{2} = 7{,}4162\ldots && (1.67)
\end{aligned}
$$

Using a calculator we find that $\sqrt{55} = 7{,}416198\ldots$, which is very close to our approximation.

1.5 Accuracy

`The syllabus requires:`

- ~~`write irrational (and rational) solutions rounded to a specified degree of accuracy`~~
- ~~`know when to approximate an irrational by a terminating rational, and when not to`~~
- ~~`express large and small numbers in scientific or engineering notation`~~

We already mentioned in section 1.2.2 that certain numbers may take an infinite amount of paper and ink to write out. Not only is that impossible, but writing numbers out to a high accuracy (too many decimal places) is very inconvenient and rarely gives better answers. For this reason we often estimate the number to a certain number of decimal places or to a given number of *significant figures*, which is even better.

(NOTE: the notes on rounding need to be better. this is not very good.) Approximating a decimal number to a given number of decimal places is the quickest way to approximate a number. Just count along the number of places you have been asked to approximate the number to and then forget all the numbers after that point. You *round up* the final digit if the number you cut off was greater or equal to 5 and *round down* (leave the digit alone) otherwise. For example, approximating 2,6525272 to 3 decimal places is 2,653 because the final digit is rounded up.

[10]This procedure is known as *Heron's Method*, which was used by the Babylonians over 4000 years ago.

(NOTE: more on the difference between DP and SF needed) In a number, each non-zero digit is a significant figure. Zeroes are only counted if they are between two non-zero digits or are at the end of the decimal part. For example, the number 2000 has 1 significant figure, but 2000,0 has 5 significant figures. Estimating a number works by removing significant figures from your number (starting from the right) until you have the desired number of significant figures, rounding as you go. For example 6,827 has 4 significant figures, but if you wish to write it to 3 significant figures it would mean removing the 7 and rounding up, so it would be 6,83.

It is important to know when to estimate a number and when not to. It is usually good practise to only estimate numbers when it is absolutely necessary, and to instead use symbols to represent certain irrational numbers (such as π); approximating them only at the very end of a calculation. If it is necessary to approximate a number in the middle of a calculation, then it is often good enough to approximate to a few decimal places.

1.5.1 Scientific Notation

In science one often needs to work with very large or very small numbers. These can be written more easily in scientific notation, which has the general form

$$a \times 10^m \tag{1.68}$$

where a is a decimal number between 1 and 10. The m is an integer and if it is positive it represents how many zeros should appear to the right of a. If m is negative then it represents how many times the decimal place in a should be moved to the left. For example $3{,}2 \times 10^3$ represents 32000 and $3{,}2 \times 10^{-3}$ represents 0,0032.

If a number must be converted into scientific notation, we need to work out how many times the number must be multiplied or divided by 10 to make it into a number between 1 and 10 (i.e. we need to work out the value of the exponent m) and what this number is (the value of a). We do this by counting the number of decimal places the decimal point must move. It is usually enough to estimate a to only a few decimal places.

1.5.2 Worked Examples

Worked Example 1 **: Manipulating Rational Numbers**

Question:
Simplify the following expressions

a) $\frac{7}{8} + \frac{5}{2}$ b) $\frac{11}{27} \times \frac{20}{3}$ c) $\frac{73}{69} \div \frac{73}{69}$

Answer:
a)
Step 1 : Rule of addition
Write out the rule of addition for rational numbers (1.25)

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Step 2 : Fill in the values
Fill in the values for a,b,c and d. Here you can read off that $a = 7$, $b = 8$, $c = 5$ and $d = 2$

$$\frac{7}{8} + \frac{5}{2} = \frac{7 \times 2 + 8 \times 5}{8 \times 2} = \frac{54}{16}$$

Step 3 : Minimise the denominator
$\frac{54}{16}$ is the correct answer, but it is not the simplest way to write it. We can see that both 54 and 16 can be divided by 2, so we divide both by 2 and get $\frac{27}{8}$, which cannot be simplified any further.

b)
Step 1 : Rule of multiplication
Write out the rule of multiplication for rational numbers (1.26)

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Step 2 : Fill in the values
Fill in the values for a,b,c and d. Here you can read off that $a = 11$, $b = 27$, $c = 20$ and $d = 3$

$$\frac{11}{27} \times \frac{20}{3} = \frac{11 \times 20}{27 \times 3} = \frac{220}{81}$$

There is no number which will divide into both 220 and 81, so $\frac{220}{81}$ is the simplest form of the answer.

c)
Step 1 : Use the division rule
Calculate the reciprocal of $\frac{73}{69} = \frac{69}{73}$, and write out the division rule (1.28)

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Step 2 : Fill in the values
We can read off that $a = c = 73$ and $b = d = 69$ so

$$\frac{73 \times 69}{69 \times 73} = \frac{5037}{5037} = 1$$

This question could also have been answered in one single line by noticing that the two fractions are the same, and any number divided by itself is one.

1.5.3 Exercises

TODO

Chapter 2

Patterns in Numbers

(NOTE: SH notes:at the moment, this whole chapter needs a lot more inline examples. i think it is too complicated for 16 year olds without examples. even some of the equations might be over their heads. the use of indices is also inconsistent... the use of letters like i, n, m should not be interchanged so much as it is only leading to confusion.)

(NOTE: also, i think we are aiming too high. it is possible that when the syllabus says "prove", it really means "show explicitly the first few terms and assume the rest of the sequence is the same". so perhaps we should drop a few of the proofs.)

2.1 Sequences

The syllabus requires:

- investigate number patterns, be able to conjecture a pattern and prove those conjectures
- ~~recognise a linear pattern when there is a constant difference between consecutive terms~~
- ~~recognise a quadratic pattern when there is a constant 2nd difference~~
- identify ''not real'' numbers and how they occur (NOTE: i think this would be best taught in quadratic equations as that is the only place they occur in this syllabus)
- ~~(grade 12) arithmetic and geometric sequences~~

Can you spot any patterns in the following lists of numbers?

$$2,4,6,8,\ldots \tag{2.1}$$
$$1,2,4,7,\ldots \tag{2.2}$$
$$1,4,9,16,\ldots \tag{2.3}$$
$$5,10,20,40,\ldots \tag{2.4}$$
$$3,1,4,1,5,9,2,\ldots \tag{2.5}$$

The first is a list of the even numbers, the numbers in the second list first differ by one, then by two, then by three. The third list contains the squares of all the integers. In the fourth list, every term is equal to the previous term times two and the last list contains the digits of the number π. These lists are all examples of *sequences*. In this section we will be studying sequences and how they can be described mathematically. (NOTE: a few real world examples here wouldn't go amiss.)

A sequence is a list of objects (in our case numbers) which have been ordered. We could take as an example a sequence of books. If you put all your books in alphabetical order by the author, that would be a sequence because it is a list of things in order. Someone could look at the sequence and work out how you ordered them if they knew the alphabet. You could rearrange the collection so that it was ordered alphabetically by title. That would be a different sequence because the order is different. Similarly the sequence of numbers 1,2,3 is different to 3,2,1. You could even shuffle up all the books so that the order they were in didn't follow a pattern, but they would still make a sequence.

Notice that not all sequences have to continue forever - what characterises a sequence is that it is a list which is ordered. In the alphabetised books example, someone who didn't know the alphabet would not be able to work out how you had ordered the books. How would you be able to find your seats at the theatre or at a stadium if the seats were not ordered ? Likewise if you are shown a sequence of numbers, you may not be able to work out what pattern relates them. That might be because there is no pattern, or it might just be that you can't see it straight away.

We will be thinking about sequences in this chapter and it is useful to be able to talk in general about them. We will want to talk about the numbers in the sequence, so rather than having to say something longwinded like "the second term in the sequence is related to the first term by this rule....", we give each term in the sequence a name. The first term of a sequence is named a_1, the second term is named a_2 and the nth term is named a_n. Now we can say "a_2 is related to a_1 by this rule...".

The small n or number like 1 or 2 beside the letter is called a *subscript* or *index* but we will refer to it as the *subscript* It helps us keep everything tidy by using the same letter (in this example, a) for all the terms in a sequence.

A sequence does not have to follow a pattern, but when it does we can often write down a formula for the nth term, a_n. In the example above, 2.3 where the sequence was of all square numbers, the formula for the nth term is $a_n = n^2$. You can check this by looking at $a_1 = 1^2 = 1, a_2 = 2^2 = 4, a_3 = 3^2 = 9, \ldots$

2.1.1 Arithmetic Sequences

> *Definition*: A linear arithmetic sequence is a sequence in which each successive term differs by the same amount.

Each term is equal to the previous term plus a constant number. $T_n = T_{n-1} + k_1$ where k_1 is some constant. We will see in the next example what this constant is and how to determine it.

Say you and 3 friends decide to study for maths and you are seated at a square table. A few minutes later, 2 other friends join you and would like to sit at your table and help you study. Naturally you move another table and add it

Figure 2.1: Tables moved together

to the existing one. Now six of you sit at the table. Another two of your friends join your table and you take a third table and add it to the existing tables. Now 8 of you can sit comfortably. Let assume this pattern continues and we tabulate what is happening.

(NOTE: Insert pictures here.)

No. Tables (n)	No of people seated	Formula
1	4 = 4	= 4 + 2(0)
2	4+2 = 6	= 4 + 2(1)
3	4+2+2 = 8	= 4 + 2(2)
4	4+2+2+2 = 10	= 4 + 2(3)
⋮	⋮	⋮
n	4+2+2+2+...+2	= 4 + 2(n - 1)

We can see for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two the whole time. Thus for each table added, the number of persons increase with two. Thus, 4,6,8,... is a sequence and each term (table added), differs by the same amount (two).

More formally, the number we start out with is called a_1 and the difference between each successive term is d. Now our equation for the nth term will be:

$$a_n = a_1 + d(n - 1) \tag{2.6}$$

The general linear sequence looks like $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots$, using the general formula **??** How many people can sit in this case around 12 tables ? By simply using the derived equation we are looking for where $n = 12$ and thus a_{12}

$$\begin{aligned}
a_n &= a_1 + d(n - 1) \\
a_{12} &= 4 + 2(12 - 1) \\
&= 4 + 2(11) \\
&= 4 + 22 \\
&= 26
\end{aligned}$$

OR

How many tables would you need for 20 people ?

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ 20 &= 4 + 2(n-1) \\ 20 - 4 &= 2(n-1) \\ 16 \div 2 &= n - 1 \\ 8 + 1 &= n \\ n &= 9 \end{aligned}$$

A simple test for an arithmetic sequence is to check that $a_2 - a_1 = a_3 - a_2 = d$

This is quite an important equation and is a definitive test for an arithmetic sequence. If this condition does not hold, the sequence is not an arithmetic sequence.

It is also important to note the difference between n and a_n. n can be compared to a place holder while a_n is the value at the place 'held' by n. Like our study table above.Table 1 holds 4 people thus at place n=1 the value of $a_1 = 4$.

n	1	2	3	4	...
a_n	4	6	8	10	...

2.1.2 Quadratic Sequences

(NOTE: maybe put in a note about the quadratic equations section, and 1st/2nd differences in terms of differentiating wrt n.) A quadratic sequence is a sequence in which the differences between each consecutive term differ by the same amount, called a *constant second difference*. In the example sequences in the introduction, equation (2.2) is a quadratic sequence because the difference between each term differs by one each time. We can look at the difference between each term and see that the differences form a linear sequence:

$$\begin{aligned} a_2 - a_1 &= 2 - 1 &&= 1 \\ a - 3 - a_2 &= 4 - 2 &&= 2 \\ a_4 - a_3 &= 7 - 4 &&= 3 \end{aligned}$$

Here you can see clearly that the difference between each difference is 1. We call this the *constant second difference* and in this case is $= 1$. The general form of this example of a quadratic sequence (a sequence with constant second difference) is

$$a_n = \frac{1}{2}(n^2 - 1) - \frac{1}{2}(n - 1) + 1 \tag{2.7}$$

For a general quadratic sequence with constant second difference D the formula for a_n is

$$a_n = \frac{D}{2}(n^2 - 1) + d(n - 1) + a_1 \tag{2.8}$$

The difference between a_n and a_{n-1} is $Dn + d$.

Check for yourself that $a_n - a_{n-1} = Dn + d$. *(Use the formula for a_n and then again for a_{n-1} (setting $n = n - 1$ in the formula) and then work out what $a_n - a_{n-1}$ is.)* Make up your own quadratic sequences with a constant second difference not equal to 1.

Figure 2.2: Tree diagram of series

2.1.3 Geometric Sequences

> *Definition*: A geometric sequence is a sequence in which every number in the sequence is equal to the previous number in the sequence, multiplied by another constant number.

This means that the *ratio* between consecutive numbers in the sequence is a constant. We will explain what we mean by ratio after looking at this example.

What is influenza (flu)? Influenza, commonly called 'the flu', is caused by the influenza virus, which infects the respiratory tract (nose, throat, lungs). It can cause mild to severe illness, that most of us get during winter time The main way that influenza viruses are spread is from person to person in respiratory droplets of coughs and sneezes. (This is called 'droplet spread.') This can happen when droplets from a cough or sneeze of an infected person are propelled (generally up to 3 feet) through the air and deposited on the mouth or nose of people nearby. It is good practise to cover your mouth when you cough or sneeze to not infect others around you when you have the flu.

Lets assume you have the flu virus and you forgot to cover your mouth when two friends came to visit while you were sick in bed. They leave and the next day, they also have the flu. Lets assume that they in turn spread the virus to two of their friends by the same droplet spread the following day. Lets assume this pattern continues and each person infected, infects 2 other friends. We can represent these events in the following manner:

(NOTE: Insert pictures here.)

Again we can tabulate the events and formulate an equation for the general case:

# day (n)	# Carrier	# Recipients/Carrier	Formula
1	You spread virus	2	$2 = 2$
2	2	4	$4 = 2$ x $2 = 2$ x 2^1
3	4	8	$8 = 2$ x $4 = 2$ x 2x2 $= 2$ x 2^2
4	8	16	$16 = 2$ x $8 = 2$ x 2x2x2 $= 2$ x 2^3
5	16	32	$32 = 2$ x $16 = 2$ x 2x2x2x2 $= 2$ x 2^4
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	...	...	$= 2$ x 2x2x2x...x2 $= 2$ x 2^{n-1}

You sneeze and the virus is carried over to 2 people who start the chain ($a_1 = 2$). The next day, each one then infects 2 of their friends. Now 4 people are infected. Each of them infects 2 people the third day and 8 people are infected etc. These events can be written as a geometric sequence: 2,4,8,16,32,... Note the common factor between the events. Recall from the linear arithmetic sequence how the common difference between terms were established. In the geometric sequence we can determine the common factor, r by

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r \tag{2.9}$$

Or more general

$$\frac{a_{n+1}}{a_n} = \frac{a_{n+2}}{a_{n+1}} = r \tag{2.10}$$

$\frac{a_2}{a_1}$ is called the ratio and is used to describe the 'factor difference' between the elements of the series. i.e. The ratio between a_1 and a_2 is 2

From the question in the above example we know $a_1 = 2$ and $r = 2$ and we have seen from the table that the nth term is given by $a_n = 2 \times 2^{n-1}$. Thus in general,

$$a_n = a_1 r^{n-1} \tag{2.11}$$

So if we want to know how many people has been infected after 10 days, we need to work out a_{10}

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 2^9 \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

Or, how many days would pass before 16384 people are infected with the flu virus ? (NOTE: I'm not sure if SURDs and exponents have been done at this stage. check first. This chapter should be taught AFTER exponents and SURDs because the techniques are used !!)

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ 16384 &= 2 \times 2^{n-1} \\ 16384 \div 2 &= 2^{n-1} \\ 8192 &= 2^{n-1} \\ 2^{13} &= 2^{n-1} \\ 13 &= n - 1 \\ n &= 14 \end{aligned}$$

2.1.4 Recursive Equations for sequences

When discussing linear and quadratic sequences we noticed that the difference between two consecutive terms in the sequence could be written in a general way. For linear sequences, where the constant difference between two consecutive terms was d, we can write this information as $a_n - a_{n-1} = d$ for any term in the sequence. We can rearrange this to $a_n = a_{n-1} + d$. This is an expression for a_n in terms of a_{n-1}, which is called a *recursive equation*. So the recursive equation for a linear sequence of constant difference d is

$$a_n - a_{n-1} = d \tag{2.12}$$

We can do the same thing for quadratic sequences. There we noticed that $a_n - a_{n-1} = Dn + d$. Then the recursive equation for a quadratic sequence with constant second derivative D is

$$a_n - a_{n-1} = Dn + d \tag{2.13}$$

(NOTE: Here we haven't said explicitly what d is or how to work it out. This bothers me. I think to be honest that you need more information. like maybe $a_n - a_{n-2}$ wouldn't include d.)

It is not always possible to find a recursive equation for a sequence, even when you know the general way to write down any term a_n. Can you find a recursive equation for a geometric sequence? *This is not supposed to be easy! It's just to get you to have a go at working things out.*

Recursive equations are extremely powerful: you can work out every term in the series just by knowing the previous one, and as you can see for the example above, working out a_n from a_{n-1} can be a much simpler computation than working out a_n from scratch using a general formula. This means that using a recursive formula when programming a computer to work out a sequence would mean the computer would finish its calculations significantly quicker. (NOTE: Real world example of this?)

2.1.5 Extra

(NOTE: Jacques had sections on arithmetic/geometric means. i want to add that back in, but make it clear it is non-syllabus. No questions are usually asked in the final exam as far as I can see, but it is part of the syllabus.)

2.2 Series *(Grade 12)*

`The syllabus requires:`

- `(grade 12) prove and calculate the following sums`

$$\begin{aligned}
\sum_{i=1}^{n} 1 &= n \\
\sum_{i=1}^{n} i^2 &= \frac{n(2n+1)(n+1)}{6} \\
\sum_{i=1}^{n} a + (i-1)d &= \frac{n}{2}(2a + (n-1)) \\
\sum_{i=1}^{n} a.r^{i-1} &= \frac{a(r^n - 1)}{r - 1} \\
\sum_{i=1}^{\infty} a.r^{i-1} &= \frac{a}{1-r} \qquad -1 < r < 1
\end{aligned}$$

`(NOTE: equation 3 is not correct in the official syllabus. there is a` d `missing.)`

When we *sum* terms in a sequence, we get what is called a *series*. If we only sum a finite amount of terms, we get a *finite series*. We use the symbol S_n to mean the sum of the first n terms of a sequence. For example, the sequence of numbers 3,1,4,1,5,9,2,... has a finite series S_4 which is simply the first 4 terms added together, $3 + 1 + 4 + 1 = 9$.

If we sum infinitely many terms of a sequence, we get an *infinite series*.

A sum may be written out using the summation symbol $\sum$ (Sigma). This symbol is the capital "S" (for Sum) in the Greek alphabet. It indicates that you must sum the expression to the right of it

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n \tag{2.14}$$

a_i are the terms in a sequence and here we sum from $i = m$ (as indicated below the summation symbol) up until $i = n$ (as indicated above). We usually just sum from $n = 1$, which is the first term in the sequence. In which case we can use either S_n or $\sum$ notation since they mean the same thing

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n \tag{2.15}$$

For example, in the following sum

$$\sum_{i=1}^{5} i \tag{2.16}$$

we have to add together all the terms in the sequence $a_i = i$ from $i = 1$ up until $i = 5$

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15 \tag{2.17}$$

which gives us 15.

2.2.1 Finite Arithmetic Series

When we sum a finite number of terms in an arithmetic sequence, we get a finite arithmetic series. The simplest arithmetic sequence is when $a_1 = 1$ and $d = 0$ in the general form (**??**), in other words all the terms in the sequence are one.

$$\begin{aligned} a_i &= d(i-1) + a_1 \\ &= 0(i-1) + 1 \\ &= 1 \\ a &= 1,1,1,1,1,\ldots \end{aligned} \tag{2.18}$$

If we wish to sum this sequence from $i = 1$ to any integer n, we would write

$$\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \ldots \qquad n \text{ times} \tag{2.19}$$

Since all the terms are equal to one, it means that if we sum to an integer n we will be adding n number of ones together, which is equal to n.

$$\sum_{i=1}^{n} 1 = n \tag{2.20}$$

Another simple arithmetic sequence is when $a_1 = 1$ and $d = 1$, which is the sequence of positive integers

$$\begin{aligned} a_i &= d(i-1) + a_1 \\ &= (i-1) + 1 \\ &= i \\ a &= 1,2,3,4,5,\ldots \end{aligned} \tag{2.21}$$

If we wish to sum this sequence from $i = 1$ to any integer n, we would write

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n \tag{2.22}$$

This is an equation with a very important solution as it gives the answer to the sum of positive integers[1]. We notice that the largest number may be added to the smallest, then the second largest added to second smallest, giving the same number. If we keep doing this we find that all the numbers may be paired up together like this until we reach the middle and there are no more numbers left to pair off

$$a_1 + a_2 + \ldots + a_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \ldots \qquad \frac{n}{2} \text{ times} \tag{2.23}$$

If there are an odd number of numbers, then we must not forget to add the unpaired number to the answer at the end. For example

$$\begin{aligned} 1 + 2 + 3 + 4 + 5 &= (1+5) + (2+4) + 3 \\ &= 6 + 6 + 3 \\ &= (3+3) + (3+3) + 3 \\ &= 15 \end{aligned} \tag{2.24}$$

We can write this down in general as

$$\sum_{i=1}^{n} i = \frac{n}{2}(n+1) \tag{2.25}$$

If we wish to sum any arithmetic sequence, there is no need to work it out term for term as we just have for these examples. We will now show what the general form of a finite arithmetic series is by starting with the general form of an arithmetic sequence and summing it from $i = 1$ to any integer n.

[1] A famous mathematician named Carl Friedrich Gauss discovered this proof when he was only 8 years old. His teacher had decided to give his class a problem which would distract them for the entire day by asking them to add all the numbers from 1 to 100. Young Carl realised how to do this almost instantaneously and shocked the teacher with the correct answer, 5050.

Writing out the sum of a sequence and then substituting in the general form for an arithmetic sequence gives us

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} d(i-1) + a_1 \tag{2.26}$$

If there is a sum inside a sum, we can break it into two separate sums and calculate each part separately.

$$\sum_{i=1}^{n} d(i-1) + a_1 = \sum_{i=1}^{n} \underbrace{di}_{\text{left}} + \underbrace{(a_1 - d)}_{\text{right}} \tag{2.27}$$

If a sum is multiplied by a constant, we can take the constant outside of the $\sum$. The term on the right is a sum of $a_1 - d$, which is a constant, so we may rewrite that term as

$$\begin{aligned} \sum_{i=1}^{n} a_1 - d &= (a_1 - d)\sum_{i=1}^{n} 1 \\ &= (a_1 - d)n \end{aligned} \tag{2.28}$$

Here we used equation (2.20) to arrive at the solution. The term on the left of equation (2.27) is also quite simple. Firstly we can take the constant d out of the sum

$$\sum_{i=1}^{n} di = d\sum_{i=1}^{n} i \tag{2.29}$$

and then we can use equation (2.25) to find

$$d\sum_{i=1}^{n} i = \frac{dn}{2}(n+1) \tag{2.30}$$

Adding together the solutions to the left and right terms (equations (2.28) and (2.30)) we get the general form of a finite arithmetic series

$$\sum_{i=1}^{n} d(i-1) + a_1 = \frac{n}{2}(2a_1 + d(n-1)) \tag{2.31}$$

For example, if we wish to know the series S_{20} for the arithmetic sequence $a_i = 7(i-1)+3$, we could either calculate each term individually and sum them

$$\begin{aligned} \sum_{i=1}^{20} 7(i-1) + 3 &= 3 + 10 + 17 + 24 + 31 + 38 + 45 + 52 \\ &\quad +59 + 66 + 73 + 80 + 87 + 94 + 101 \\ &\quad +108 + 115 + 122 + 129 + 136 \\ &= 1390 \end{aligned} \tag{2.32}$$

or more sensibly, we could use equation (2.31) noting that $d = 7$, $a_1 = 3$ and $n = 20$ so that

$$\begin{aligned} \sum_{i=1}^{20} 7(i-1) + 3 &= \frac{20}{2}(2 \times 3 + 7 \times 19) \\ &= 1390 \end{aligned} \tag{2.33}$$

In this example, it is clear that using (2.31) is beneficial.

2.2.2 Finite Squared Series

When we sum a finite number of terms in a quadratic sequence, we get a finite quadratic series. The general form of a quadratic series is quite complicated, so we will only look at the simple case when $D = 1$ and $d = a_0 = 0$ in the general form (**??**). This is the sequence of squares of the integers

$$\begin{aligned} a_i &= i^2 \\ &= 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, \ldots \\ &= 1, 4, 9, 16, 25, 36 \ldots \end{aligned} \tag{2.34}$$

If we wish to sum this sequence and create a series, then we write

$$S_n = \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \ldots + n^2 \tag{2.35}$$

which can be written in general as (NOTE: the syllabus requires that we prove this result! any ideas, without confusing the hell out of a 16 year old? i thought even the other ones were a bit too hard for this level, to be honest.)

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6} \tag{2.36}$$

2.2.3 Finite Geometric Series

When we sum a finite number of terms in a geometric sequence, we get a finite geometric series. We know from (**??**) that we can write out each term of a geometric sequence in a general form. By simply adding together the first n terms in the general form we are actually writing out the series

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} \tag{2.37}$$

We may multiply this by r on both sides, giving us

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n \tag{2.38}$$

You may notice that all the terms are the same in (2.37) and (2.38), except the first and last. If we subtract (2.37) from (2.38) we are left with just

$$\begin{aligned} rS_n - S_n &= a_1 + a_1 r^n \\ S_n(r-1) &= a_1(1 + r^n) \end{aligned} \tag{2.39}$$

dividing by $(r-1)$ on both sides, we have the general form of a geometric sequence since $S_n = \sum_{i=1}^{n} a.r^{i-1}$

$$\sum_{i=1}^{n} a.r^{i-1} = \frac{a(r^n - 1)}{r - 1} \tag{2.40}$$

2.2.4 Infinite Series

Thus far we have been working only with finite sums, meaning that whenever we determined the sum of a series, we only considered the sum of the first n terms. It is the subject of this section to consider what happens when we add infinitely many terms together. You might think that this is a silly question - surely one will get to infinity when one sums infinitely many numbers, no matter how small they are? The surprising answer is that in some cases one will reach infinity (like when you try to add all the integers together), but in some cases one will get a finite answer. If you don't believe this, try doing the following sum on your calculator or computer: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$. You might think that if you keep adding more and more terms you will eventually get larger and larger numbers, but in fact you won't even get past 1 - try it and see for yourself!

There is a special sigma notation for infinite series: we write $\sum_{i=1}^{\infty} i$ to indicate the infinite sum $1+2+3+4+.....$ When we sum the terms of a series, and the answer we get after each summation gets closer and closer to some number, we say that the series *converges*. If a series does not converge, we say that it *diverges*.

There is a rule for knowing instantly which geometric series converge and which diverge. When r, the common ratio, is strictly between -1 and 1, i.e. $-1<r<1$, the infinite series will converge, otherwise it will diverge. There is also a formula for working out what the series converges to. The sum of an infinite series, symbolised by S_∞, is given by the formula

$$S_\infty=\sum_{i=1}^{\infty} a_1.r^{i-1}=\frac{a_1}{1-r} \qquad -1<r<1 \tag{2.41}$$

where a_1 is the first term of the series, and r is the common ratio. (NOTE: the syllabus requires us to PROVE this series! how can we do that without a notion of a limit? again the syllabus talks nonsense.) We can see how this comes about by looking at (2.40), as $-1<r<1$ and $n=\infty$. We can ignore the r^n term since a small number raised to the power of infinity is infinitely small. Try this yourself by typing in a number between -1 and 1 into your calculator and square it, continuing to square the answers thereafter; your calculator will eventually decide that the answer is zero.

2.3 Worked Examples

(NOTE: I think maybe the worked examples should follow the relevant section. Check if the general layout is set.)

1. Classify the following as arithmetic sequence or geometric sequence: 15,19,23,...

For arithmetic sequence, We have to check for a common difference or a common ratio.

$a_2-a_1=a_3-a_2=d$
$a_2-a_1=19-15=4$
$a_3-a_2=23-19=4$

Thus, $a_2 - a_1 = a_3 - a_2 = 4$ and we can say that 15,19,23, ... is an arithmetic sequence and $d = 4$

2. Classify the following as arithmetic sequence or geometric sequence: 5,10,20, ...

For arithmetic sequence, We have to check for a common difference or a common ratio.
$a_2 - a_1 = a_3 - a_2 = d$
$a_2 - a_1 = 10 - 5 = 5$
$a_3 - a_2 = 20 - 10 = 10$

Thus, $a_2 - a_1 \neq a_3 - a_2$ and we can say that 5,10,20, ... is not an arithmetic sequence.

Test for geometric sequence:
$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r$
$\frac{a_2}{a_1} = \frac{10}{5} = 2$
$\frac{a_3}{a_2} = \frac{20}{10} = 2$

Thus, $\frac{a_2}{a_1} = \frac{a_3}{a_2}$ and $r = 2$ and we can say that 5,10,20, ... is a geometric sequence.

3. Determine d and a_9 for the following arithmetic sequence:
17,14,11, ...

It is given that 17,14,11, ... is an arithmetic sequence, thus
$a_2 - a_1 = a_3 - a_2 = d$
$14 - 17 = 11 - 14 = -3$
$d = -3$

To determine a_9 we use $a_n = a_1 + d(n-1)$ with $n = 9$
Thus:

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ a_9 &= 17 + (-3)(9-1) \\ &= 17 - 3(8) \\ &= 17 - 24 \\ &= -7 \end{aligned}$$

4. Determine r and a_7 for the following geometric sequence: $81, -27, 9, \ldots$

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r$$
$$\frac{a_2}{a_1} = \frac{-27}{81} = -\frac{1}{3}$$
$$\frac{a_3}{a_2} = \frac{9}{-27} = -\frac{1}{3}$$

To determine a_7 we use $a_n = a_1 r^{n-1}$ with $n = 7$
Thus:

$$\begin{aligned}
a_n &= a_1 r^{n-1} \\
a_7 &= (81)(-\frac{1}{3})^{7-1} \\
&= -(3^4)(3^{-1(6)} \\
&= -(3^4.3^{-6}) \\
&= -(3^{4-6}) \\
&= -(3^{-2}) \\
&= -\frac{1}{9}
\end{aligned}$$

5. The third term of a geometric sequence is equal to 1 and the 5^{th} term is 16. Find r and the seventh term

Given: $a_3 = 1$ and $a_5 = 16$
We also know
$a_n = a_1 r^{n-1}$
Thus:

$$\begin{aligned}
a_3 &= a_1 r^{3-1} \\
1 &= a_1 r^2
\end{aligned}$$

Also:

$$\begin{aligned}
a_5 &= a_1 r^{5-1} \\
16 &= a_1 r^4
\end{aligned}$$

Dividing (2) by (1):

$$\begin{aligned}
\frac{16}{1} &= \frac{a_1 r^4}{a_1 r^2} \\
16 &= r^2 \\
r &= 4
\end{aligned}$$

To find a_7 we use $a_7 = a_1 r^{7-1}$ but first we need a_1. From (1) we know:

$$\begin{aligned}
1 &= a_1 r^2 \\
1 &= a_1.(4)^2 \\
a_1 &= \frac{1}{16} \\
\\
a_7 &= a_1 r^{7-1} \\
&= \frac{1}{16}.4^6 \\
&= \frac{4096}{16} \\
&= 256
\end{aligned}$$

6. The fourth term of an arithmetic sequence is $1\frac{1}{2}$ and the the 8^{th} term is $\frac{1}{2}$. Find the second term.

Given: $a_4 = \frac{3}{2}$, $a_8 = \frac{1}{2}$ and this is an arithmetic sequence.
Thus we can use $a_n = a_1 + d(n-1)$

$a_4 = \frac{3}{2} = a_1 + d(4-1) = a_1 + 3d$ and
$a_8 = \frac{1}{2} = a_1 + d(8-1) = a_1 + 7d$

Subtract the one equation from the other to get rid of a_1 and solve for d:
$\frac{3}{2} - \frac{1}{2} = (a_1 + 3d) - (a_1 + 7d)$
$1 = -4d$
$d = -\frac{1}{4}$

$\frac{1}{2} = a_1 + (-\frac{1}{4})(7)$
$\frac{1}{2} = a_1 - \frac{7}{4})$
$a_1 = \frac{9}{4}$

$a_2 = \frac{9}{4} - (\frac{1}{4})(2-1)$
$a_2 = \frac{8}{4} = 2$

2.4 Exercises

1. Classify the following as arithmetic sequence or geometric sequence:

$$\begin{aligned}
&i) && \frac{1}{3},\frac{1}{6},0,-\frac{1}{6},-\frac{1}{3},\ldots \\
&ii) && 1,-1,1,\ldots \\
&iii) && \frac{3}{2},\frac{1}{2},\frac{1}{6},\ldots \\
&iv) && 2^2,2,1,\ldots \\
&v) && \frac{1}{2},\frac{5}{8},\frac{3}{4},\ldots
\end{aligned}$$

2. Find a_7 for each of the series above.
3. Determine which term in the series $14,8,2,\ldots$ is equal to -34 ? 4. Which

term in 2,6,18, . . . is equal to 486 ?
5. In a geometric series, $a_4 = \frac{2}{3}$ and $a_6 = \frac{3}{2}$. Find a_2.
6. In an arithmetic series, $a_3 = -2$ and $a_8 = 23$. Determine a_1 and d.
7. The third term of a Geometric series is equal to minus three eighths and the seventh term is equal to $\frac{3}{128}$. Find a_5.
8. Insert 4 numbers between 4 and 972 to form a geometric series.

Chapter 3

Functions

3.1 Functions and Graphs

The syllabus requires:

- (grade 12) formal definition of the function concept
- able to switch between words, tables, graphs and formula to represent the relation between variables
- generate graphs using point to point plotting to test conjectures on relations between x and y for the situations (NOTE: this list has changed in the latest syllabus... be warned. we do the trig functions in the trig section. it makes more sense this way.)

$$\begin{aligned} y &= ax+b \\ y &= a^x+b \qquad a>0, a\neq 1 \\ y &= \frac{a}{x}+b \\ y &= \frac{a}{x+b}+c \\ y &= a^{x+b}+c \end{aligned}$$

  (NOTE: page 25 of the syllabus lists some more situations, but they are corrupted. we need to get an uncorrupted version of the syllabus and add them to this list)

- identify the domain and range, axes intercepts, turning points (max/min), asymptotes, shape and symmetry, periodicity and amplitude, rates of change, increasing/decreasing ranges and continuity. and can sketch graphs using these characteristics
- (grade 12) can generate graphs of function inverses. in particular

$$\begin{aligned} y &= ax+b \\ y &= a^x \qquad a>0, a\neq 1 \\ y &= ax^2 \\ y &= \sin(x) \end{aligned}$$

- `(grade 12) decide which inverses are functions and if necessary the restriction to make it a function`

(NOTE: functions are neither conceptually simple nor very interesting - so this intro needs to be very sexy (and i know mine probably isn't, so rewriting is good). try to reword so as to not use 1st and 3rd person.)

Most people don't know it but they've come across functions all their lives. In fact, our very existence is tied to certain, very special functions called the laws of nature. Even ignoring those, though, it would be difficult to go through a day without coming into contact with all sorts of functions. We can say that the idea of a function is one of the most basic and powerful ideas in the mathematics.

Functions everywhere? But who's ever heard of such a thing? Where does the word even come from? Well, they are everywhere, and, once you begin to see them, functions will be the easiest concept in mathematics. Where are these functions? Well, the menu in a restaurant is a function. So are the prices in a supermarket. Today's temperature. Your height, your age, your weight. These are all functions.

A function is just a way of attaching or relating one thing to another. A menu attaches prices to the food in a restaurant, and a supermarket attaches prices to the things it sells. We need to notice one very important fact, which is that these functions can give only one price to each item. We would certainly get angry if a restaurant charged two different prices for the same dish. However, it's perfectly natural for a restaurant to charge the same price for different dishes. Similarly, one person cannot have two different heights, but two people can have the same height.

3.1.1 Variables, Constants and Relations

A *variable* is a label which we allow to change and become any element of some set of numbers. For example, on a menu in a restaurant "price" is a variable on the set of real numbers, since for any menu item the manager can choose any price he or she feels like (with the aim of staying in business). Most often, a variable will be a letter which can take on any value in some set of numbers. In this textbook we will only use *real variables*, which may take on the value of any real number. Though a variable is free to vary, if we wish we can specify that the variable takes on a specific value, in which case we say that we *assign* a value to the variable. In fact, we do this all the time when working with variables. When we say "what if we set the price to R50", we are just assigning the value "R50" to the variable "price". You have probably already done this quite frequently in algebra, when you say "let x be 1".

A *constant* is a variable which is fixed. We may not know the value of this constant, but this is a number which does not change throughout any problem. The "speed of light" is a variable which is *always* 300 000km per second, i.e. it is a constant. Such constant variables occur most frequently in the laws of physics.

Variables on their own are very abstract, so don't worry if it is slightly confusing. They become much more understandable when we start to relate them to each other. "Price" on a menu may not be a constant, but it must be tied to the items on that menu. For each item, we have a specified price. We can think of “item” as a variable in its own right, and then the menu does nothing but tell us the relationship between the two variables “item” and “price”.

In general, a *relation* is an equation which relates two variables. For example, $y = 5x$ and $y^2 + x^2 = 5$ are relations. In both examples x and y are variables and 5 is a constant, but for a given value of x the value of y will be very different in each relation.

Our example of a restaurant menu shows that relations between variables take on varied representations. Besides writing them as formulae, we most often come across relations in words, tables and graphs. Instead of writing $y = 5x$, we could also say “y is always five times as big as x”. We could also give the following table:

(NOTE: Working on a Latex-less machine, so table will come later)

(I put in a table but not sure if it’s ok - Jothi)

x	$y = 5x$
2	10
6	30
8	40
13	65
15	75

Some of you may object that this table isn’t very satisfactory, as the same table could represent almost any relation between x and y. However, when using tables we normally cheat and just assume that the obvious relationship in the table is *the* relationship.

Finally, we look at graphs (NOTE: surely thisneeds to wait until later? sorry - structuring major headache here)

3.1.2 Definition of a Function (grade 12)

A *function* is a relation for which there is only one value of y corresponding to any value of x. We sometimes write $y = f(x)$, which is notation meaning ’y is a function of x’. This definition makes complete sense when compared to our real world examples — each person has only one height, so height is a function of people; on each day, in a specific town, there is only one average temperature.

However, some very common mathematical constructions are not functions. For example, consider the relation $x^2 + y^2 = 4$. This relation describes a circle of radius 2 centred at the origin, as in figure 3.1. If we let $x = 0$, we see that $y^2 = 4$ and thus either $y = 2$ or $y = -2$. Since there are two y values which are possible for the same x value, the relation $x^2 + y^2 = 4$ is **not** a function.

There is a simple test to check if a relation is a function, by looking at its graph. This test is called the *vertical line test*. If it is possible to draw any vertical line (a line of constant x) which crosses the relation more than once, then the relation is **not** a function. If more than one intersection point exists, then the intersections correspond to multiple values of y for a single value of x.

We can see this with our previous example of the circle by looking at its graph again in figure 3.1. We see that we can draw a vertical line, for example the dotted line in the drawing, which cuts the circle more than once. Therefore this is **not** a function.

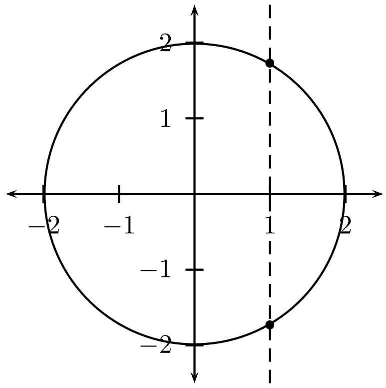


Figure 3.1: Graph of $y^2 + x^2 = 4$

In a function $y = f(x)$, y is called the *dependent variable*, because the value of y depends on what you choose as x. We say x is the *independent variable*, since we can choose x to be any number.

3.1.3 Domain and Range of a Relation

The *domain* of a relation is the set of all the x values for which there exists at least one y value according to that relation. The *range* is the set of all the y values, which can be obtained using at least one x value. If the relation is of height to people, then the domain is all living people, while the range would be about 0.1 to 3 metres — no living person can have a height of 0m, and while strictly it's not impossible to be taller than 3 metres, no one alive is. An important aspect of this range is that it does not contain *all* the numbers between 0.1 and 3, but only six billion of them (as many as there are people).

As another example, suppose x and y are real valued variables, and we have the relation $y = 2^x$. Then for *any* value of x, there is a value of y, so the domain of this relation is the whole set of real numbers. However, we know that no matter what value of x we choose, 2^x can never be less than or equal to 0. Hence the range of this function is all the real numbers strictly greater than zero.

These are two ways of writing the domain and range of a function, *set notation* and *interval notation*. (NOTE: the syllabus does not say which notation method to use. we should find out, and only use the one if possible. there is no need to add further confusion. then move the unused notation to Extra.)

Set Notation

First we introduce the symbols $>$, $<$, $\leq$, $\geq$. $>$ means 'is greater than' and $\geq$ means 'is greater than or equal to'. So if we write $x > 5$, we say that x is greater than 5 and if we write $x \geq y$, we mean that x can be greater than or equal to y. Similarly, $<$ means 'is less than' and $\leq$ means 'is less than or equal to'. Instead of saying that x is between 6 and 10, we often write $6 < x < 10$. This directly means 'six is less than x which in turn is less than ten'.

A set of certain x values has the following form:

$$\{x : \text{conditions, more conditions}\} \tag{3.1}$$

We read this notation as "the set of all x values where all the conditions are satisfied". For example, the set of all positive real numbers can be written as $\{x : x \in \mathbb{R}, x > 0\}$ which reads as "the set of all x values where x is a real number and is greater than zero". (NOTE: have we even explained what $>$, $<$, $\leq$, $\geq$ mean yet? remember... this book must not assume that anyone has seen this stuff before.)

We use the same notation (with the letter y instead of x) for the range of the function.

Interval Notation

(NOTE: rewrite this subsubsection. first describe what the brackets mean, and then introduce the concept of a union. all these concepts are new... so we must describe everything in detail.)

Here we write an interval in the form '*lower bracket, lower number, comma, upper number, upper bracket*'. We can use two types of brackets, square ones [,] or round ones (,). A square bracket means including the number at the end of the interval whereas a round bracket means excluding the number at the end of the interval. It is important to note that this notation can only be used for all real numbers in an interval. It cannot be used to describe integers in an interval or rational numbers in an interval.

So if x is a real number greater than 2 and less than or equal to 8, then x is any number in the interval

$$(2,8] \tag{3.2}$$

It is obvious that 2 is the lower number and 8 the upper number. The round bracket means 'excluding 2', since x is greater than 2, and the square bracket means 'including 8' as x is less than or equal to 8.

Now we come to the idea of a **union**, which is used to combine things. The symbol for **union** is $\cup$. Here we use it to combine two or more intervals. For example, if x is a real number such that $1 < x \leq 3$ or $6 \leq x < 10$, then the set of all the possible x values is

$$(1,3] \cup [6,10) \tag{3.3}$$

where the $\cup$ sign means the union(or combination) of the two intervals. We use the set and interval notation and the symbols described because it is easier than having to write everything out in words.

3.1.4 Example Functions

In this section we will look at several examples of functions. Here we will let go of our real-world examples, and look exclusively at real valued functions, because only in such cases do we see the full use and power of functional mathematics. While it is instructive to see a menu or people's height as a function, it is not very interesting. On the other hand, all of advanced physics and statistics depend on real valued functions. Very little is more important than gaining an intuitive grasp of real functions, and we will spend the remainder of this chapter doing just that.

When considering real valued functions, our major tool is drawing graphs. In the first place, if we have two real variables, x and y, then we can assign values to them simultaneously. That is, we can say "let x be 5 and y be 3". Just as we write "let $x = 5$" for "let x be 5", we have the shorthand notation "let $(x, y) = (5, 3)$" for "let x be 5 and y be 3". We usually think of the real numbers as an infinitely long line, and picking a number as putting a dot on that line. If we want to pick *two* numbers at the same time, we can do something similar, but now we must use two dimensions. What we do is use two lines, one for x and one for y, and rotate the one for y, as in diagram (NOTE: insert diagram). We call this the *Cartesian plane*.

(NOTE: This whole y and $f(x)$ thing needs to be cleared up — I would do it here, but then it's also discussed above in the definition of a function. I think the problem comes with the varying uses of y, and I think a physicist would be better than a mathematician to clear this up. Personally, rigorously, I don't really know *what's* going on with this notation.)

The great beauty of doing this is that it allows us to "draw" functions, in a very abstract way. Let's say that we were investigating the function $f(x) = 2x$. We could then consider all the points (x, y) such that $y = f(x)$, i.e. $y = 2x$. For example, $(1, 2), (2.5, 5)$, and $(3, 6)$ would all be such points, whereas $(3, 5)$ would not since $5 \neq 2 \times 3$. If we put a dot at each of those points, and then at every similar one for all possible values of x, we would obtain the graph shown in (NOTE: put in).

The form of this graph is very pleasing — it is a simple straight line through the middle of the plane. Now some of you may have guessed this graph long before we plotted it, but the point is that the technique of "plotting", which we have followed here, is *the* key element in understanding functions. To show you why, we will now consider whole *classes* of functions, and we will relate them by the simple fact that their graphs are nearly identical.

Straight Line Functions

These functions have the general form

$$f(x) = ax + b \tag{3.4}$$

where a and b are constants. The value of a is called the *gradient* or *slope* and tells us how steep the line is (the larger the number, the steeper the line). If a is greater than zero it means the line increases from left to right (slopes upwards), if it is smaller than zero the line increases from right to left (slopes downwards). b is called the y-*intercept* and tells us where the line goes through the y-axis.

For example the function $f(x) = 2x + 3$ has a gradient of 2 and a y-intercept of 3. This means that the line cuts through the y-axis at a value of 3 and slopes upwards. We can calculate the values of y for certain values of x and then plot them in a graph (see figure 3.2).

x :	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = 2x + 3$:	-7	-5	-3	-1	1	3	5	7	9	11	13

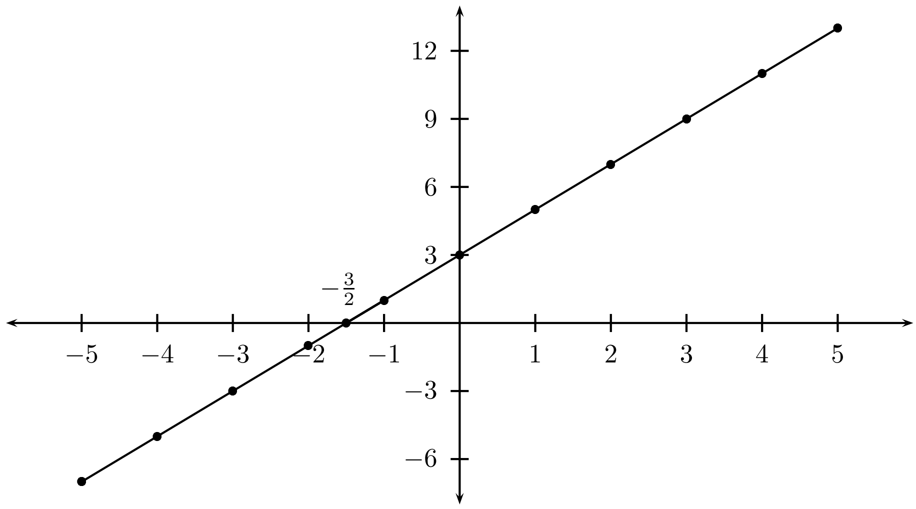


Figure 3.2: Graph of $f(x) = 2x + 3$

However we only need two points to plot a straight line graph. The easiest points to use are the x-*intercept* (where the line cuts the x-axis) and the y-intercept. The x-intercept occurs when $y = 0$, so it is always equal to $-\frac{a}{b}$. So if asked to plot a straight line, there is no need to calculate lots of y values, you just need to find the x and y intercepts and draw a line through them.

Parabolic Functions

A parabola looks like a hill, either upside down (for a "positive" parabola) or right way up (for a "negative" one), which is the same on both sides, as in the diagrams (NOTE: put in):

You may have noted that when we say the parabola is "the same on both sides", we are just stating that these functions are horizontally symmetric. This means that if you flip them from left to right along a specific line, which is called the line of symmetry, they look the same. This line of symmetry is sometimes called the axis of symmetry.

Parabolic functions are functions of the form

$$f(x) = ax^2 + bx + c \tag{3.5}$$

where a, b and c are constants. The a involves the shape of the parabola and says how steep the curves are. If a is positive, then the hill is upside-down. If a is negative, then the hill is the right way up. c is the y-intercept, which is where the parabola cuts the y axis. b has to do with the shift in the parabola to the left or the right. Two important features of the parabola are its turning point and line of symmetry(described above). The turning point says how high the hill is. If the hill is the right way up, then the turning point is the maximum value of the parabola, and if it is upside-down, then the turning point is the minimum value.

Now the above form of the parabola is the standard form. It can also be written in the form

$$f(x) = a(x - p)^2 + q \tag{3.6}$$

where the two new constants p and q give the turning point (p,q) of the parabola. This form of the parabola can be obtained from the standard form by completing the square (See algebra). The value p of the turning point is actually the line of symmetry. So if $p = 3$, then $x = 3$ is the line of symmetry(which is always vertical for the parabola). q is the maximum or minimum value of the function(that is the maximum or minimum value of y).

At first it might seem difficult to sketch the graph of a parabola but once a simple procedure is followed, then it becomes easier. When sketching the graph, we need to use some information about it. The only information we have are its shape, x and y-intercepts and its turning point. We start off by seeing whether the parabola is an hill that is the right way up or upside-down. Recall that we can find this out from the sign of a. Next we calculate the x-intercepts by setting $y = 0$ and solving the equation

$$0 = ax^2 + bx + c \tag{3.7}$$

which doesn't always have a solution, meaning that not all parabolas cut the x-axis. These would be hills which never quite make it to the x-axis, or upside-down hills which are never low enough to touch the x-axis. However, if there is one solution, and it is not zero, then because of the symmetry there must be two solutions which can be both positive, or negative or a plus and a minus one. (NOTE: max/min turning points.)

The y-intercept is just c. Last we find the turning point of the parabola. One way is to write the equation of the parabola in the form 3.6 and we have found p and q. Another way is to calculate $x = \frac{-b}{2a}$, the line of symmetry and also the value of p. q is found by putting p in the equation of the parabola. So now we are able to plot some points and join them up to form a parabola.

We can create a table of x and y values for the parabola $f(x) = x^2 - 9$ and then plot them (see figure 3.3). Note that you could spot the symmetry of the graph by examining the table alone, where we see that $x = 1$ and $x = -1$ give the same value for y. Note also that for this parabola $b = 0$, so the line of the symmetry is the y-axis since the parabola looks the same on both sides of the y-axis.

x :	-4	-3	-2	-1	0	1	2	3	4
$y = x^2 - 9$:	7	0	-5	-8	-9	-8	-5	0	7

Hyperbolic Functions

Hyperbolas look like 2 parabolas on their side which are mirror reflections of each other around the diagonal (NOTE: sketch). Hyperbolic functions look like

$$f(x) = \frac{a}{x} + b \tag{3.8}$$

where a and b are constants. Just like for parabolas, a tells us how steep the curves are and b tells us how high the curves are.

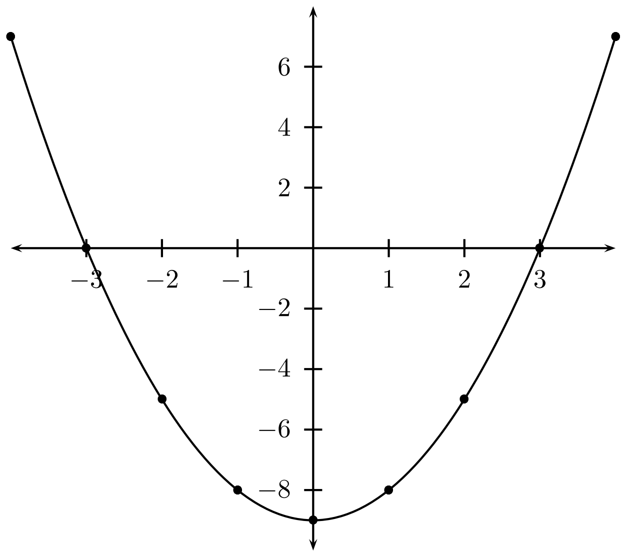


Figure 3.3: Graph of the parabola $f(x) = x^2 - 9$

Since we cannot divide by zero[1], it is not possible to have $x = 0$, so there is no y-intercept. When you go far enough away from the y-axes, the curves start to look like straight lines, and we call them *asymptotes.*

For example we can create a table of x and y values for the hyperbolic function $f(x) = \frac{4}{x}$ and plot them (see figure 3.4)

x :	-8	-4	-2	-1	-$\frac{1}{2}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \frac{4}{x}$:	-$\frac{1}{2}$	-1	-2	-4	-8	8	4	2	1	$\frac{1}{2}$

Exponential Functions

$$y = ab^x + c \qquad b > 0 \tag{3.9}$$

3.2 Exponentials and Logarithms

`The syllabus requires:`

- `(grade 12) switch between log and exp form of an equation`
- `(grade 12) derive and use the laws of logs`

(NOTE: need an intro. this should have lots of stuff about how people used exp/logs to multiply numbers by adding them. with a few examples... to show that you don't need a calculator.)

[1](NOTE: i'm sure there are interesting facts about dividing by zero (not that i know of — luke))

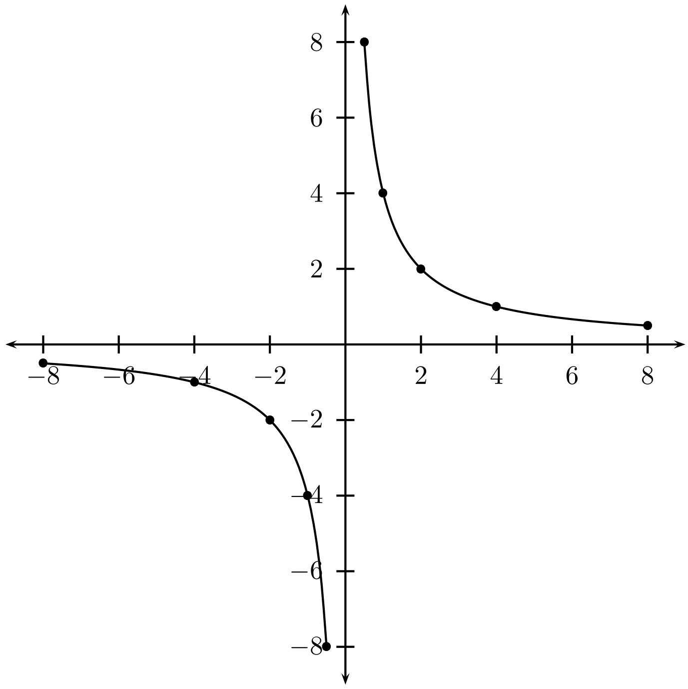


Figure 3.4: Graph of the hyperbola $f(x) = \frac{4}{x}$

3.2.1 Exponential Functions

(NOTE: need an intro. we already covered exponentials in "numbers", but maybe we should move it here instead.)

3.2.2 Logarithmic Functions

(NOTE: these Laws need introduced properly with more detailed derivations and examples of their use, highlight each ones importance. rewrite the intro to not include so many new terms... and to read better for a 16 year old.)

Logarithms, commonly referred to as Logs, are the algebraic inverse of exponents. When we say "inverse function" we mean that the answer becomes the question and the question becomes the answer. For example, in the expression $a^b = x$ the "question" is "what is a raised to the b power." The answer is "x." The inverse function would be $log_a x = b$ or "by what power must we raise a to obtain x." The answer is "b." Many students find logarithms difficult. For now you can be successful if you learn the terminology and come to understand the relationships of the terms.

(NOTE: this next graph needs more explanation)

Law 1

Since $a^0 = 1$, $log_a 1 = 0$

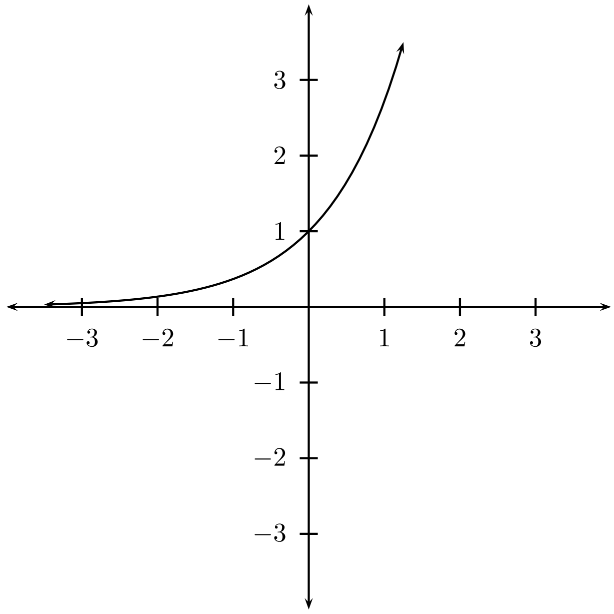


Figure 3.5: The Exponential Function $f(x) = e^x$

Law 2

Since $a^1 = a$, $log_a a = 1$

Law 3

This one is a bit trickier to see. The law is that $log_a a^x = x$. If we re-write it as $log_a(a^x) = x$ we can see that it is $a^x = (a^x)$, which is, of course, true.

We can also then say that $log_a a^x = x \cdot log_a a = x(1) = x$. The upshot being that any exponent of the (operand?) can simply be moved to simple multiplication by the log.

Law 4

The laws of exponents $a^m \cdot a^n = a^{m+n}$ and $\frac{a^m}{a^n} = a^{m-n}$ translate to the laws of logarithms $log_a(m \cdot n) = log_a m + log_a n$ and $log_a(\frac{m}{n}) = log_a m - log_a n$ respectively.

Base

In the previous examples a is the *base*. We generally use the "common" base, 10, or the *natural* base, e.

The number e is an irrational number between 2.71 and 2.72. It comes up surprisingly often in Mathematics, but for now suffice it to say that it is one of the two common bases.

While the notation $log_{10}(x)$ and $log_e(x)$ may be used, $log_{10}(x)$ is often styled $log(x)$ in Science and $log_e(x)$ is normally written as $ln(x)$ in both Science and Mathematics.

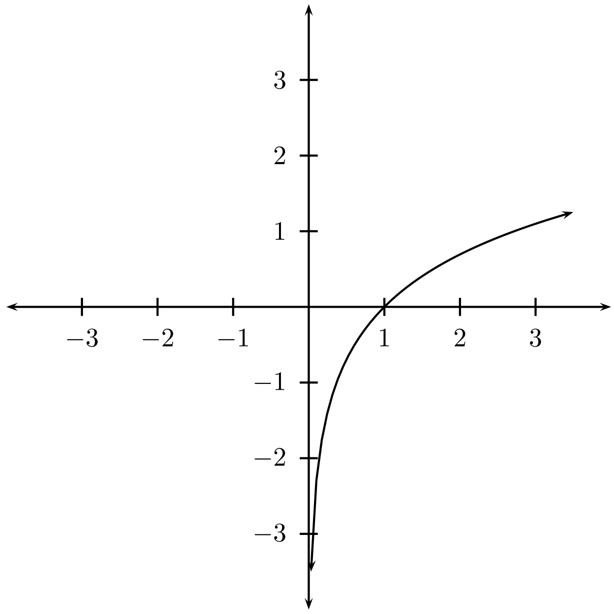


Figure 3.6: The Logarithmic Function $f(x) = ln(x)$

It is often necessary or convenient to convert a log from one base to another. An Engineer might need an approximate solution to a log in a base for which he does not have a table or calculator function, or it may be algebraically convenient to have two logs in the same base.

To affect a change of base, apply the change of base formula:

$$log_a x = \frac{log_b x}{log_b a} \tag{3.10}$$

where b is any base you find convenient. Normally a and b are known, therefore $log_b a$ is normally a known, if irrational, number.

3.3 Extra

(NOTE: this is non-syllabus content on absolute value functions, but perhaps the absolute operator should be worked into the main text and this section deleted, as it is quite important.)

3.3.1 Absolute Value Functions

(NOTE: i'm pretty sure this is not on the syllabus)

The absolute value of x has the following definition

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{3.11}$$

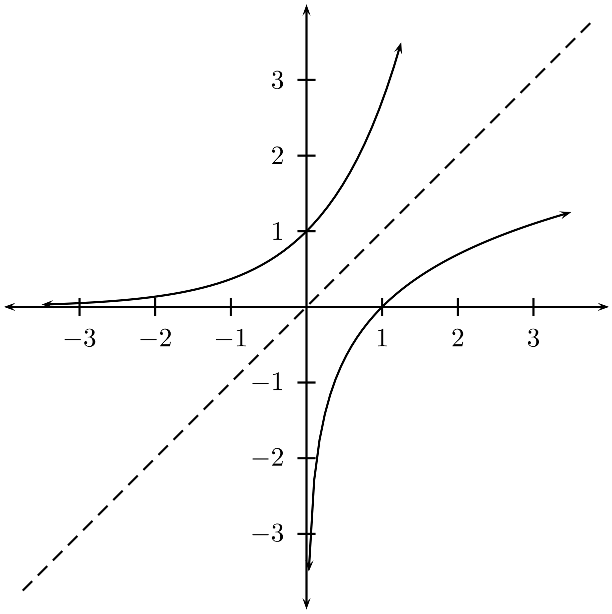


Figure 3.7: The Functions $f(x) = ln(x)$ and $f(x) = e^x$ are symmetrical about the origin.

In other words, the absolute value sign makes the term inside this sign positive. If it is already positive, then there is no change, and otherwise the sign of this term changes.

Now an absolute value function has the following general form

$$f(x) = a|x - b| + c \tag{3.12}$$

where a, b and c are constants.

Let us again consider an absolute value function with the general form $y = a|x - b| + c$. We must consider two cases separately:

$\underline{x \geq b}$:

Now, since $x \geq b$ and thus $x - b \geq 0$, the term inside the absolute value sign is positive and therefore $|x - b| = x - b$. Thus

$$y = a(x - b) + c = ax + (c - ab) \tag{3.13}$$

In other words, this is a straight line with slope a and y-intercept $c - ab$.

$\underline{x < b}$:

In this case, the term in the absolute value sign is negative and thus $|x-b| = -(x - b) = -x + b$. Therefore

$$y = a(-x + b) + c = -ax + (c + ab) \tag{3.14}$$

which is a straight line with slope $-a$ and y-intercept $c + ab$.

Now at $x = b$ the function value is $y = a|0| + c = c$. Therefore function consists of half of two straight lines with slopes $-a$ and a, which meet at the turning point (b,c).

The function has the axis of symmetry $x = b$. In other words, the part of the function on one side of the vertical line $x = b$ is the same as the reflection about this line of the part of the function on the other side. We can see this as follows:

Consider the function values at the points $x = b + z$ and $x = b - z$, where $z > 0$ (these are two point the same distance from the line $x = b$). Now the function values at these two points are

$$\begin{aligned} f(b+z) &= a|(b+z)-b|+c && (3.15)\\ &= a|z|+c && (3.16)\\ &= az+c && (3.17)\end{aligned}$$

and

$$\begin{aligned} f(b-z) &= a|(b-z)-b|+c && (3.18)\\ &= a|-z|+c && (3.19)\\ &= az+c && (3.20)\end{aligned}$$

These function values are the same. Therefore, whether we move to the left or the right of the line $x = b$, the function values remain the same. Therefore $x = b$ is an axis of symmetry.

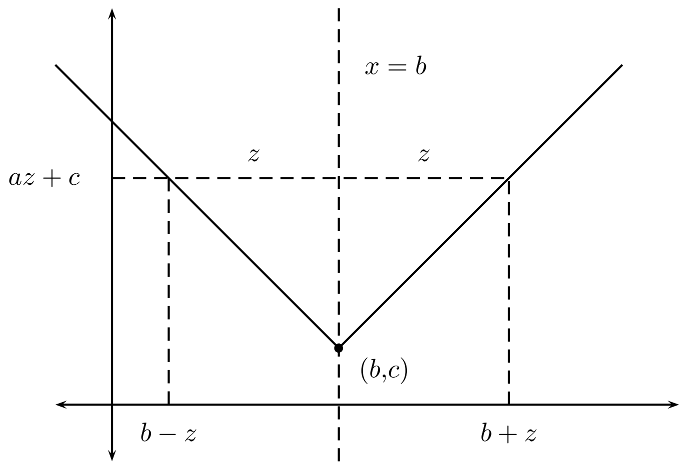


Figure 3.8: Graph of $f(b+z)$ and $f(b-z)$ where $f(x) = a|x| + c$; with the line of symmetry $x = b$

Now let us consider two cases: $a < 0$ and $a > 0$.

If a is positive, then the line on the left of the turning point (with slope $-a$) will have a negative slope and the line on the right (with slope a) will have a positive slope. Thus the graph will be shaped like a V.

Otherwise, if a is negative, then the line on the left has the positive slope $-a$ and the line on the right has the negative slope a. Therefore the graph is an upsidedown V.

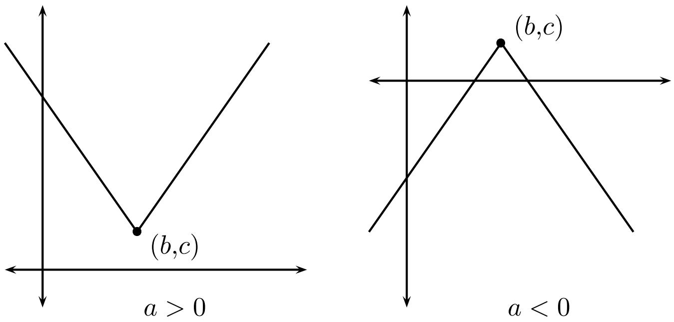


Figure 3.9: Graph of $f(b+z)$ and $f(b-z)$ where $f(x) = a|x| + c$. One case is for $a > 0$ the other for $a < 0$.

Notice also that an absolute value function does not necessarily have x-intercepts. It these do exist, then they will be the x-intercepts of the two straight lines making up the absolute value function.

Chapter 4

Numerics

4.1 Optimisation

`The syllabus requires:`

- `Linear Programming (Grade 11)`

  1. `Solve linear programming problems by optimising a function in two variables, subject to one or more linear constraints, by numerical search along the boundary of the feasible region.`
  2. `Solve a system of linear equations to find the co-ordinates of the vertices of the feasible region.`

In everyday life people are interested in knowing the most efficient way of carrying out a task or achieving a goal. For example, a farmer might want to know how many crops to plant during a season in order to maximise yield (produce) or a stock broker might want to know how much to invest in stocks in order to maximise profit. These are examples of **optimisation** problems, where by optimising we mean finding the maxima or minima of a function. This function we wish to optimise (i.e. maximise or minimise) is called the **objective function** (we will only be looking at objective functions which are functions of two variables). In the case of the farmer, the objective function is the yield and it is dependent on the amount of crops planted. If the farmer has two crops then we can express the yield as $f(x,y)$ where the variable x represents the amount of the first crop planted and y the amount of the second crop planted. For the stock broker, assuming that there are two stocks to invest in, $f(x,y)$ is the amount of profit earned by investing x rand in the first stock and y rand in the second.

In practice it is often that **constraints**, or restrictions, are placed on x and y. The most common of these constraints is the non-negativity constraint. That is, we might require that $x \geq 0$ and $y \geq 0$. For the farmer, it would make little sense if we were to speak of planting a negative amount of crops and so when optimising $f(x,y)$ the constraints $x \geq 0$ and $y \geq 0$ must be considered. Other constraints might be that the farmer cannot plant more of the second crop than the first crop and that no more than 20 units of the first crop can be planted; these constraints translate into the inequalities $x \geq y$ and $x \leq 20$. Constraints mean that we can't just take any x and y when looking for the x and y that optimise our objective function. If we think of the variables x and y as a point (x,y) in the xy plane then we call the set of all points in the xy plane that satisfy our constraints the **feasible region**. Any point in the feasible region is called a **feasible point**.

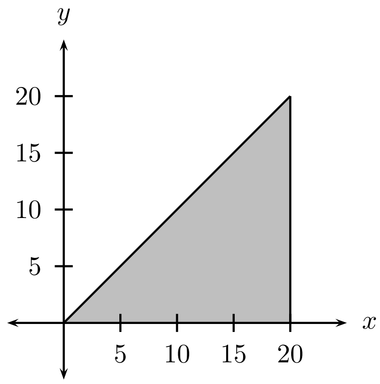


Figure 4.1: The feasible region corresponding to the constraints $x \geq 0$, $y \geq 0$, $x \geq y$ and $x \leq 20$.

For example, the non-negativity constraints $x \geq 0$ and $y \geq 0$ mean that every (x,y) we can consider must lie in the first quadrant of the xy plane. The constraint $x \geq y$ means that every (x,y) must lie on or below the line $y = x$ and $x \leq 20$ means that x must lie on or to the left of the line $x = 20$. For these constraints the feasibility region is illustrated as the shaded region in Figure 4.1.

Constraints that have the form $ax + by \leq c$ or $ax + by = c$ are called **linear** constraints. Examples of linear constraints are $x + y \leq 0$, $-2x = 7$ and $y \leq \sqrt{2}$; a constraint being linear just means that it requires that any feasible point (x,y) lies on one side of or on a line. Interpreting constraints as graphs in the xy plane is very important since it allows us to construct the feasible region such as in Figure 4.1. We have the following rule for any linear constraint:

$ax + by = c$	If $b \neq 0$, feasible points must lie *on* the line $y = -\dfrac{a}{b}x + \dfrac{c}{b}$.
	If $b = 0$, feasible points must lie *on* the line $x = c/a$
$ax + by \leq c$	If $b \neq 0$, feasible points must lie *on* or *below* the line $y = -\dfrac{a}{b}x + \dfrac{c}{b}$.
	If $b = 0$, feasible points must lie *on* or *to the left of* the line $x = c/a$

Once we have determined the feasible region the **solution** of our problem will be the feasible point where the objective function is a maximum/ minimum. Sometimes there will be more than one feasible point where the objective function is a maximum/minimum — in this case we have more than one solution.

4.1.1 Linear Programming

The objective function is called **linear** if it looks like $f(x,y) = ax + by$ where the coefficients a and b are real numbers. For example, $f(x,y) = 10x - y$ is a linear objective function. If the objective function and all of the constraints are linear then we call the problem of optimising the objective function subject to these constraints a **linear program**. All optimisation problems we will look at will be linear programs.

The major consequence of the constraints being linear is that *the feasible region is always a polygon.* This is evident since the constraints that define the feasible region all contribute a line segment to its boundary (see Figure 4.1). It is also always true that the feasible region is a convex polygon.

The objective function being linear means that *the feasible point(s) that gives the solution of a linear program always lies on one of the vertices of the feasible region.* This is very important since, as we will soon see, it gives us a way of solving linear programs. (NOTE: Should I mention that a linear objective function defines a plane? This is crucial to the fact that optimal solutions are obtained at the vertices, though. Do Grade 11s know the equation of a plane? I would like to use the idea that the level sets of planes are lines and in so doing justify the "ruler" method.)

We will now see why the solutions of a linear program always lie on the vertices of the feasible region. Firstly, note that if we think of $f(x,y)$ as lying on the z axis, then the function $f(x,y) = ax+by$ (where a and b are real numbers) is the definition of a plane. If we solve for y in the equation defining the objective function then

$$\begin{aligned} f(x,y) &= ax + by \\ \therefore \quad y &= \frac{-a}{b}x + \frac{f(x,y)}{b} \end{aligned} \tag{4.1}$$

What this means is that if we find all the points where $f(x,y) = c$ for any real number c (i.e. $f(x,y)$ is constant with a value of c), then we have the equation of a line. This line we call a **level line** of the objective function (NOTE: Should I use this terminology?). Consider again the feasible region described in Figure 4.1. Lets say that we have the objective function $f(x,y) = x - 2y$ with this feasible region. If we consider Equation 4.1 corresponding to $f(x,y) = -20$ then we we get the level line $y = \frac{1}{2}x + 10$ which has been drawn in Figure 4.2. Level lines corresponding to $f(x,y) = -10$ ($y = \frac{x}{2} + 5$), $f(x,y) = 0$ ($y = \frac{x}{2}$), $f(x,y) = 10$ ($y = \frac{x}{2} - 5$) and $f(x,y) = 20$ ($y = \frac{x}{2} - 10$) have also been drawn in. It is very important to realise that these aren't the only level lines; in fact, there are infinitely many of them and they are *all parallel to each other*. Remember that if we look at any one level line $f(x,y)$ has the *same* value for every point (x,y) that lies on that line. Also, $f(x,y)$ will always have different values on different level lines.

If a ruler is placed on the level line corresponding to $f(x,y) = -20$ in Figure 4.2 and moved down the page parallel to this line then it is clear that the ruler will be moving over level lines which correspond to *larger* values of $f(x,y)$. So if we wanted to maximise $f(x,y)$ then we simply move the ruler down the page until we reach the "lowest" point in the feasible region—this point will then be the feasible point that maximises $f(x,y)$. Similarly, if we wanted to minimise $f(x,y)$ then the "highest" feasible point will give the minimum value of $f(x,y)$.

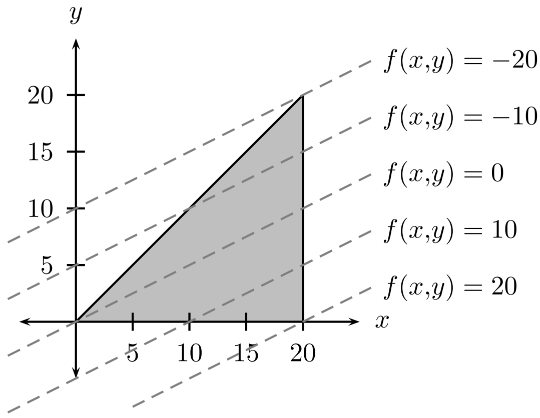


Figure 4.2: The feasible region corresponding to the constraints $x \geq 0$, $y \geq 0$, $x \geq y$ and $x \leq 20$ with objective function $f(x,y) = x - 2y$. The dashed lines represent various level lines of $f(x,y)$.

Since our feasible region is a polygon, these points *will always lie on vertices in the feasible region*. (NOTE: We could have infinitely many solutions if the gradient of a constraint = the gradient of the level lines... should I mention this?). The fact that the value of our objective function along the line of the ruler increases as we move it down and decreases as we move it up depends on this particular example. Some other examples might have that the function increases as we move the ruler up and decreases as we move it down. It is a general property, though, of linear objective functions that they will consistently increase or decrease as we move the ruler up or down. Knowing which direction to move the ruler in order to maximise/minimise $f(x,y) = ax + by$ is as simple as looking at the sign of b (i.e. "is b negative, positive or zero?"). If b is *positive*, then $f(x,y)$ *increases* as we move the ruler up and $f(x,y)$ *decreases* as we move the ruler down. The opposite happens for the case when b is negative: $f(x,y)$ *decreases* as we move the ruler up and $f(x,y)$ *increases* as we move the ruler down. If $b = 0$ then we need to look at the sign of a. If a is positive then $f(x,y)$ increases as we move the ruler to the right and decreases if we move the ruler to the left. Once again, the opposite happens for a negative. If we look again at the objective function mentioned earlier, $f(x,y) = x - 2y$ ($a = 1$ and $b = -2$), then we should find that $f(x,y)$ increases as we move the ruler down the page since $b = -2 < 0$. This is exactly what we found happening in Figure 4.2.

The main points about linear programming we have encountered so far are

- The feasible region is always a polygon.
- Solutions occur at vertices of the feasible region.
- Moving a ruler parallel to the level lines of the objective function up/down to the top/bottom of the feasible region shows us which of the vertices is the solution.
- The direction in which to move the ruler is determined by the sign of b and also possibly by the sign of a.

(NOTE: I would like to mention ∇f to determine 'the direction in which to move the ruler'. Even if I neglect the fact that students certainly know nothing about partial differentiation, I'm still not sure whether I can mention and work with vectors in the plane...)

These points are sufficient to determine a method for solving any linear program. If we wish to maximise the objective function $f(x,y)$ then:

1. Find the gradient of the level lines of $f(x,y)$ (this is always going to be $-\frac{a}{b}$ as we saw in Equation 4.1)
2. Place your ruler on the xy plane, making a line with gradient $-\frac{a}{b}$ (i.e. b units on the x-axis and $-a$ units on the y-axis)
3. The solution of the linear program is given by appropriately moving the ruler. Firstly we need to check whether b is negative, positive or zero.
   (a) If $b > 0$, move the ruler up the page, keeping the ruler parallel to the level lines all the time, until it touches the "highest" point in the feasible region. This point is then the solution.
   (b) If $b < 0$, move the ruler in the opposite direction to get the solution at the "lowest" point in the feasible region.
   (c) If $b = 0$, check the sign of a
      i. If $a < 0$ move the ruler to the "leftmost" feasible point. This point is then the solution.
      ii. If $a > 0$ move the ruler to the "rightmost" feasible point. This point is then the solution.

(NOTE: Point 3 is essentially trying to work with ∇f without actually knowing what it is or that it exists!)

4.2 Gradient

`The syllabus requires:`

- `Investigate numerically the average gradient between two points on a curve and develop an intuitive understanding of the concept of the gradient of a curve at a point (NOTE: this is undefined if this should be numerical, or an intro to differentiation. perhaps it is best to have it spread over both)`

4.3 Old Content (please delete when finished)

4.3.1 Problems

We often have to solve problems in which there are several variables, which we can change to suit us. We can now develop a method of dealing with such problems.

Worked Example 1:

Q: A farmer grows wheat and maize. He has 20 fields of available land on which he can plant crops. He must grow at least 5 fields of maize. Also he cannot grow more than twice as much maize as wheat. Draw a graph to show the feasible region showing the possible number of fields of wheat and maize the farmer can plant. What is the maximum number of fields of wheat the farmer can plant?

A: Step 1: Analyse the problem and assign the variables x and y.

Let x be the number of fields of wheat the farmer plants.
Let y be the number of fields of maize the farmer plants.

Step 2: Write down the inequalities which are the restrictions on x and y.

$$x + y \leq 20 \quad \text{(the farmer only has 20 fields)} \tag{4.2}$$
$$y \geq 5 \quad \text{(at least 5 fields of maize must be planted)} \tag{4.3}$$
$$x \geq 0 \quad \text{(it is not possible to have a negative number of fields of wheat)} \tag{4.4}$$
$$y \leq 2x \quad \text{(the farmer cannot plant more than twice as much maize as wheat)} \tag{4.5}$$

Remember that every piece of information you are given is important, so check that you have not left out an inequality. Also note that often variables cannot be negative, which give further inequalities as in the case of $x \geq 0$.

Step 3: Solve for y in terms of x where possible.

$$y \leq -x + 20 \tag{4.6}$$
$$y \geq 5 \tag{4.7}$$
$$x \geq 0 \tag{4.8}$$
$$y \leq 2x \tag{4.9}$$

Step 4: Plot a graph and find the feasible region.

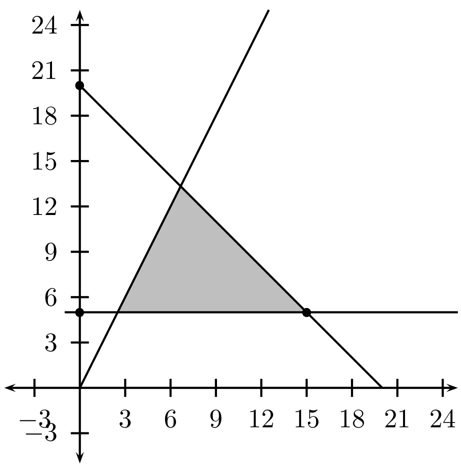


Figure 4.3: Graph of TODO

Step 5: Answer the original question.

We need to find the maximum number of fields of wheat which can be planted. This is the maximum x value which is in the feasible region. This occurs at the point (15,5). Thus the maximum x value is 15. Remembering to give the answer in terms of the original question:

The farmer can plant a maximum of 15 fields of wheat.

4.3.2 Maximising or Minimising the Objective Function

The objective function is a function of x and y. We are usually told to maximise or minimise this function.

Worked Example 2:

Q: Consider the same situation as in worked example 1. The farmer can make a profit of R100 on every field of wheat and R200 on every field of maize that he grows. How many fields of wheat and maize must the farmer plant to maximise his profit and what is this maximum profit?

Steps 1 - 4 are as in worked example 1.

A: Step 5: Define the objective function.

The objective function, in this case, is the profit in terms of the number of fields of wheat and maize (the variables x and y). This is given by

$$P = 100x + 200y \tag{4.10}$$

Step 6: Solve for y.

$$y = -\frac{1}{2}x + \frac{P}{200} \tag{4.11}$$

Step 7: Maximise/minimise the objective function.

In this case we need to maximise the objective function which is the profit P. The greater P the larger the y intercept of the straight line of y as a function of x. However, the slope of the line will always be $-\frac{1}{2}$ (line A is an example of such a line).

Now to maximise P we need the y-intercept to be as large as possible, but the line must still pass through the feasible region. Thus take a ruler and move it parallel to line A (keeping the slope the same). Move the ruler outwards until it is at the edge of the feasible region. This is line B, which is the line of maximum P. The point on the feasible region through which this line passes (in this case $(\frac{20}{3},\frac{40}{3})$) is the point giving this profit (so $x = \frac{20}{3} = 6\frac{2}{3}$ and $y = \frac{40}{3} = 13\frac{1}{3}$). The profit can be calculated from the objective function as $P = R3333$.

Step 8: Give the answer in terms of the question.

For a maximum profit of R3333, the farmer must plant $6\frac{2}{3}$ fields of wheat and $13\frac{1}{3}$ fields of maize.

(NOTE: Further examples need to be included here, particularly add an example which uses discreet variables.)

Worked Example 3:

Q: A delivery company delivers wood to client A and bricks to client B. The company has a total of 5 trucks. A truck cannot travel more than 8 hours per day and it takes 4 hours make the trip to and back from client A and 2 hours for

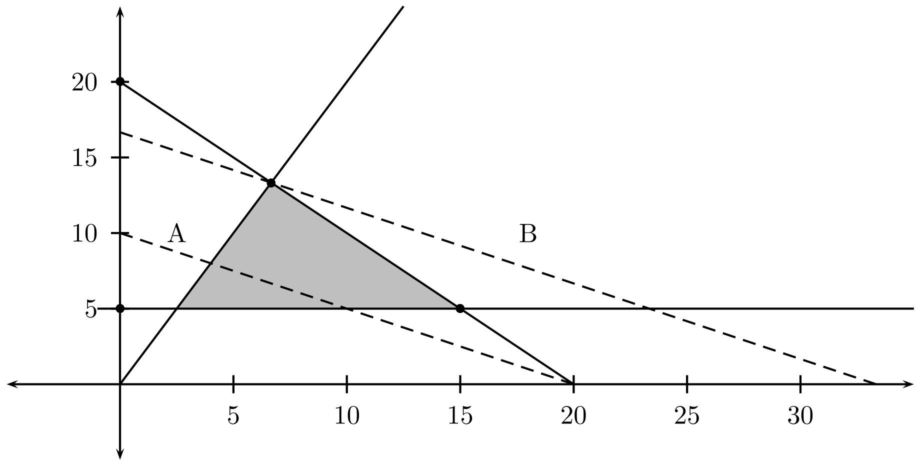


Figure 4.4: Graph of TODO

client B. To honour an agreement with client B, at least 2 truck loads of bricks must be delivered per day. Also client A needs no more than 9 truck loads of wood per day.

The delivery company makes a profit of R100 per truck load of wood and R150 per truck load of bricks delivered. How many truck loads of wood and bricks should be delivered per day so as to maxmise the profit? What is this maximum profit?

Note: Client A is in the opposite direction to client B, so each truck can only deliver a full truck load to A or B (a truck cannot take half a load to A and the other half to B).

A: Step 1:

Let x be the number of truck loads of wood the company delivers to client A per day.
Let y be the number of truck loads of bricks the company delivers to client B per day.

Step 2:

Firstly, the total number of hours of delivery time available is 5×8 hours $= 40$ hours, since there are 5 trucks, which cannot be driven more than 8 hour per day. Delivery to client A takes 2 hours and delivery to client B takes 4 hours. Therefore

$$2x + 4y \leq 40 \tag{4.12}$$

The other inequalities are

$$\begin{aligned}
y &\geq 2 && \text{(the company must delivered at least 2 truck loads of bricks per day)} && (4.13)\\
x &\leq 9 && \text{(client A needs no more than 9 truck loads of wood per day)} && (4.14)\\
x,y &\geq 0 && \text{(we cannot have a negative number of truck loads)} && (4.15)\\
& && && (4.16)
\end{aligned}$$

Furthermore, we know that x and y must be integers (in other words we cannot have a fractional truck load). These are therefore called discreet variables.

Step 3:

$$y = -\frac{x}{2} + 10 \tag{4.17}$$
$$y \geq 2 \tag{4.18}$$
$$x \leq 9 \tag{4.19}$$
$$x,y \geq 0 \tag{4.20}$$

Step 4:

We now plot the constraints and the feasible region (see the graph at the end). The region enclosed by the constraints is the shaded region, but since the variables x and y can only take on positive integer values, the feasible region actually consists of the collection of dots showing the integer values in the shaded region.

Step 5:

The objective function is the profit (in Rands), which is

$$P = 100x + 150y \tag{4.21}$$

Step 6:

Solving for y gives

$$y = -\frac{2}{3}x + \frac{P}{150} \tag{4.22}$$

Step 7:

We need to maximise the profit P and therefore we need to maximise the y-intercept of the previously defined straight line. Line A shows an arbitrary straight line with slope $-\frac{2}{3}$, which is drawn, for convenience, with intercepts $x = 9$ and $y = 6$. If a ruler is moved outwards parallel to this line (i.e. keeping the slope fixed) to the edge of the feasible region, we obtain line B, which passes through the point (8,6).(NOTE: RULERS??? is this really the best way to do this? can we please have some equations of lines!)

Therefore the maximum profit occurs when $x = 8$ and $y = 6$. This profit (in Rands) is

$$\begin{aligned} P &= 100x + 150y && (4.23)\\ &= 100(8) + 150(6) && (4.24)\\ &= 1700 && (4.25) \end{aligned}$$

Note: We cannot use the point $(9, 5\frac{1}{2})$, which is actually the point at the edge of the shaded region, because this is $5\frac{1}{2}$ is not an integer (we cannot have a half a truck load).

Step 8:

The maximum profit is R1700 per day, which is obtained when the company delivers 8 truck loads of wood to client A and 6 truck loads of bricks to client B per day.

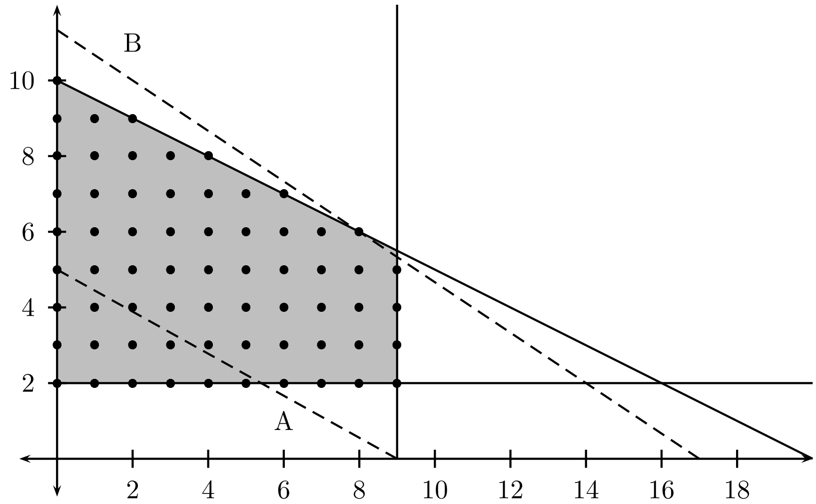


Figure 4.5: Graph of TODO

Essay 1: Differentiation in the Financial World

Author: Fernando Durrell

I lived in Cape Town (South Africa) all my life. I attended Thomas Wildschudtt Junior and Senior Primary Schools. I then attended St. Owen's Senior School in Retreat up to half way through Grade 11 at which point I left for St. Joseph's Senior School in Rondebosch (since St. Owen's closed permanently at the end of my Grade 11 year). It was always one of my ambitions to attend the University of Cape Town (UCT) because it is a prestigious university. I applied to study medicine at UCT, but was not accepted, and so I enrolled for a science degree at UCT and have never regretted it. (I can stand only so much visible blood.) I wasn't sure about what I wanted to do with my life so I enrolled for Mathematics, Applied Mathematics, Chemistry and Physics in my first year at university. By the end of my first year at UCT, I wanted to continue with the Mathematics stream. I completed by Bachelor of Science (BSc) degree with majors (main subjects) Mathematics and Applied Mathematics and then completed my BSc (Honours) degree in Applied Mathematics. I completed by Master of Science degree in Financial Mathematics and am currently registered for the degree of Doctor of Philosophy in Mathematics.

Differentiation in the Financial World

Most of us don't really think about saving our money to buy something in the future - we have to spend it now! Our parents unfortunately (and one day when we're older most probably we too) have to save for the future: for that possible time when they don't have a job; they may save to purchase furniture for the house; or a present for your birthday. The most important thing adults save for is retirement - this is when they decide that they want to stop working. Their kids may not be able to care for them because they too may have families or may not have jobs themselves. So, adults have to save money while they are working to support them when they retire. The amount of money they receive after they retire is called their pension - so adults save so that they can receive a pension when they retire.

Suppose James is paying R100 every month toward his pension. When James retires, he wants every month to receive a bit more than the R100 he contributed toward his pension (while he was working). If he doesn't get a bit more than R100 pension every month (when he retires), then he may as well save his money under his bed until he retires. Now, there are many adults like James who are saving for their pension. To whom do all these adults pay their monthly pension savings? They pay their monthly pensions savings to a pension fund. Suppose there are ten million adults paying R100 every month to a pension fund. That means that each month the pension fund receives R100x10 000 000 = R1b (i.e. one billion rand) in total each month.

Now, each adult, like James, will want to receive a monthly pension which is greater than R100 when they retire. So, the pension fund must ensure that, when a pension fund contributor retires, he/she receives more than R100 pension every month. There are many pension funds in the world, so, if the pension fund James is saving with is going to give him a R110 monthly pension and another pension fund is going to give him a R120 monthly pension, then he is going to save with the latter pension fund. So, pension funds can't give pensioners too little pension. In fact, they have to give pensioners as big a pension as possible. Now, the pension fund can't just put the monthly R1bn in the bank and let it earn interest and divide this amongst all pensioners. The government takes a lot of the interest earned by pension funds as tax. So, the pension funds have to make more money, and so they turn to the stock market. But the thing about the stock market is that one can lose a lot of money very quickly if one is not careful. The advantage about putting one's money in the bank is that, when you come back the next day, your money will still be there. If you invest in the stock market today and you come back tomorrow, then you could have lost a substantial amount of money, but you could also have made a lot of money. So, for the pension fund, depositing the monthly R1b with the bank is appealing, but so is the stock market. The pension has to find the best combination of the two (the bank and the stock market). That involves, finding out how much money to deposit with the bank and how much to invest in the stock market so that the pension fund makes as much money as possible. To solve this problem, involves differentiation, which is the topic of the next chapter. This is just one way in which differentiation is used in the financial world.

Chapter 5

Differentiation

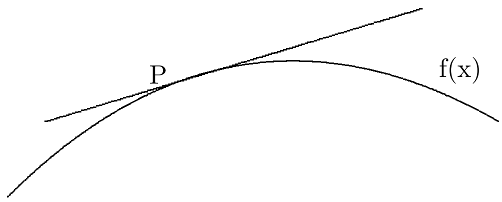
5.1 Limit and Derivative

Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is more dynamic and less static. It is concerned with change and motion. It deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study. In this section we give a glimpse of some of the main ideas of calculus by showing how limits arise when we attempt to solve a variety of problems.

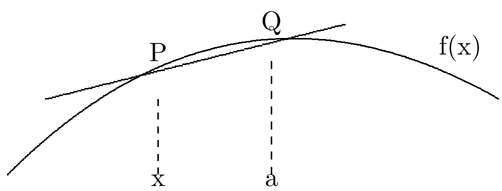
5.1.1 Gradients and limits

A traditional slingshot is essentially a rock on the end of a string, which you rotate around in a circular motion and then release. When you release the string, in which direction will the rock travel? Many people mistakenly believe that the rock will follow a curved path. Newton's First Law of Motion tells us that the path is straight. In fact, the rock follows a path along the tangent line to the circle, at the point of release. If we wanted to determine the path followed by the rock, we could do so, as tangent lines to circles are relatively easy to find. Recall, from elementary geometry that a tangent line to a circle is a line that intersects the circle in exactly one point. In this chapter we will be concerned with tangent lines to a variety of functions, as the tangent line gives us the slope of a function at a point.

Now let us consider the problem of trying to find the equation of the tangent line t to a curve with equation $y = f(x)$ at a given point P.



Since we know that the point P lies on the tangent line, we can find out the equation of t if we know its slope m. The problem is that we need two points to compute the slope and we only have one, namely P on t. To get around the problem we first find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ.



From the figure we see that

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} \tag{5.1}$$

Now imagine that Q moves along the curve toward P.The secant line approaches the tangent line as its limiting position. This means that the slope m_{PQ} of the secant line becomes closer and closer to the slope m of the tangent line as Q approaches P. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and we say that m is the limit of m_{PQ} as Q approaches P along the curve. Since x approaches a as Q approaches P, we could also use Equation (5.1) to write

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{5.2}$$

The tangent problem has given rise to the branch of calculus called differential calculus.

5.1.2 Differentiating $f(x) = x^n$

The central concept of differential calculus is the derivative. After learning how to calculate derivatives, we use them to solve problems involving rates of change.

Definition: The **derivative of a function f at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a\ +\ h) - f(a)}{h} \tag{5.3}$$

if this limit exists.

Let us use this definition to calculate the derivative of $f(x) = x^2$, where n is a positive integer.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x\ +\ h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} 2x + h \\ &= 2x \end{aligned}$$

You should repeat this calculation for $f(x) = x^3$ and (if you haven't spotted a pattern yet!) for $f(x) = x^4$. Then see if you can generalise what you are seeing to write down a formula for $f'(x)$ where $f(x) = x^n$. (This isn't a valid mathematical way of arriving at a formula, but if you want to prove the general case you need to use the binomial theorem, which is outside the scope of your syllabus.)

Hopefully you calculated the derivative of $f(x) = x^3$ to be $3x^2$, and shortly after that spotted the pattern for powers of x:

$\frac{d}{dx}\ (x^n) = nx^{n-1}$

5.1.3 Other notations

If we use the traditional notation $y = f(x)$ to indicate that the dependent variable is y and the independent variable is x, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. It is very important that you learn to identify these different ways of denoting the derivative, and that you are consistent in your usage of them when answering questions.

Note

Though we choose to use a fractional form of representation, $\frac{dy}{dx}$ is a limit and **IS NOT** a fraction, i.e. $\frac{dy}{dx}$ does not mean $dy \div dx$. $\frac{dy}{dx}$ means y differentiated with respect to x. Thus, $\frac{dp}{dx}$ means p differentiated with respect to x. The '$\frac{d}{dx}$' is the "operator", operating on some function of x.

`The syllabus requires:`

- ~~(grade 12) understand the limit concept in the context of approximating the rate of change or gradient of a function at a point~~
- ~~(grade 12) establish derivatives of $f(x) = x^n$ from 1st principles and then generalise to obtain the derivative of~~

$$\begin{aligned} f(x) &= b \\ f(x) &= x^3 \\ f(x) &= x^2 \\ f(x) &= \frac{1}{x} \end{aligned}$$

5.2 Rules of Differentiation

In order to avoid differentiating functions from first principles, we can establish certain rules.

Rule 1

If f is a constant function, $f(x) = c$, then $f'(x) = 0$

Rule 1 may also be written as

$$\frac{d}{dx}\ c = 0 \tag{5.4}$$

This result is geometrically evident if one considers the graph of a constant function. This is an horizontal line, which has slope 0.

Rule 2: The Power Rule

If $f(x) = x^n$, where n is an integer, then

$$f'(x) = nx^{n-1} \tag{5.5}$$

The **Power Rule** may also be written as

$$\frac{d}{dx}\ (x^n) = nx^{n-1} \tag{5.6}$$

This rule applies when n is a negative number. For example, the derivative of $f(x) = \frac{1}{x}$ is $f'(x) = -x^{-2}$, remembering that $\frac{1}{x} = x^{-1}$.

Rule 3: Linearity of Differentiation

If c is a constant and both f and g are differentiable, then

$$\frac{d}{dx}\ (cf) = c\frac{df}{dx} \tag{5.7}$$

$$\frac{d}{dx}\ (f+g) = \frac{df}{dx} + \frac{dg}{dx} \tag{5.8}$$

5.2.1 Summary

$\frac{d}{dx}\ c = 0$ $\quad$ $\frac{d}{dx}\ (x^n) = nx^{n-1}$ $\quad$ $\frac{d}{dx}\ (cf) = c\frac{df}{dx}$ $\quad$ $\frac{d}{dx}\ (f+g) = \frac{df}{dx} + \frac{dg}{dx}$

5.3 Using Differentiation with Graphs

`The syllabus requires:`

- ~~`(grade 12) find equation of a tangent to a graph`~~
- ~~`(grade 12) sketch graph of a cubic function using diff to determine stationary points and their nature. use factor theorem to determine x-axis intercept`~~

5.3.1 Finding Tangent Lines

In section 5.1.1 we saw that finding the tangent to a function is the same as finding its slope at a particular point. The slope of a function at a point is just its derivative.

If we want to find a general formula for a tangent to a function, we differentiate the function. To find the slope of the tangent at a particular point, we substitute that point's x value into the function's derivative. This will give us a single value, which is the slope of a straight line. We'll look at one of these problems in the Worked Examples (section 5.4).

5.3.2 Curve Sketching

Suppose we are given that $f(x) = ax^3 + bx^2 + cx + d$ and we are asked to sketch the graph of this function. We will use our newfound knowledge of differentiation to solve this problem. There are **FIVE** steps to be followed:

1. If $a > 0$, then the graph is increasing from left to right, and has a maximum and then a minimum. As x increases, so does $f(x)$.
   If $a < 0$, then the graph decreasing is from left to right, and has first a minimum and then a maximum. $f(x)$ decreases as x increases.
2. Determine the value of the y-intercept by substituting $x = 0$ into $f(x)$
3. Determine the x-intercepts by factorising $ax^3 + bx^2 + cx + d = 0$ and solving for x. First try to eliminate constant common factors, and to group like terms together so that the expression is expressed as economically as possible. Use the factor theorem if necessary.
4. Find the turning points of the function by working out the derivative $\frac{df}{dx}$ and setting it to zero, and solving for x.
5. Determine the y-coordinates of the turning points by substituting the x values obtained in the previous step, into the expression for $f(x)$.
6. Step 6 of 5, Draw a neat sketch.

`The syllabus requires:`

- ~~`(grade 12) use the differentiation rules`~~

$$\begin{aligned} D_x[f(x) \pm g(x)] &= D_x[f(x)] \pm D_x[g(x)] \\ D_x[k.f(x)] &= k.D_x[f(x)] \end{aligned}$$

5.4 Worked Examples

Worked Example 2 **: Finding derivatives from first principles**

Question:
Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Answer:

Step 1 : Write out the definition
From definition (5.3) we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Step 2 : Fill in the function $f(x)$ and multiply out

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{[(a+h)^2-8(a+h)+9]-[a^2-8a+9]}{h} \\ &= \lim_{h\to 0}\frac{a^2+2ah+h^2-8a-8h+9-a^2+8a-9}{h} \end{aligned}$$

Step 3 : Simplify

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{2ah+h^2-8h}{h} \\ &= \lim_{h\to 0}\frac{h(2a+h-8)}{h} \\ &= 2a-8 \end{aligned}$$

And you're done!

***Worked Example 3* : Finding and using derivatives from first principles**

Question:
If $f(x) = 4x + 2x^2$, find $f'(x)$ from first principles and hence calculate $f'(2)$.

Answer:

Step 1 : Write out definition (5.3) and fill in $f(x)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{4(x+h)+2(x+h)^2-(4x+2x^2)}{h}\right] \end{aligned}$$

Step 2 : Multiply out and simplify

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left[\frac{4h+4xh+2h^2}{h}\right] \\ &= \lim_{h\to 0}[4+4x+2h] \\ &= 4+4x \end{aligned}$$

Step 3 : Substitute the value of x into $f'(x)$
Since

$$f'(x) = 4+4x$$

then

$$f'(2) = 4+4(2) = 12$$

***Worked Example 4* : Using Notation and Rules of Differentiation**

Question:
Differentiate the following using the Rules of Differentiation listed above:
a) $y = t^4$ b)$y = x^{1000}$ c) $h(x) = x^6 + x^4$ d)$\frac{d}{dr}(5r^3)$
e) $D_u(u^m)$

Answer:
a)
Step 1 : Write out the Power Rule, equation (5.5)
$\frac{d}{dt}\left(t^n\right) = nt^{n-1}$

Step 2 : In this case $n = 4$ *so...*
$y' = 4t^{3-1}$

Step 3 : Simplify
$\frac{dy}{dt} = 4t^3$

b)
Step 1 : Write out the Power Rule, equation (5.5)
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Step 2 : In this case $n = 1000$ *so...*
$y' = 1000x^{999-1}$

Step 3 : Simplify
$y' = 1000x^{999}$

c)
Step 1 : Write out the Second Linearity Rule, equation (5.8)
$\frac{d}{dx}\left(f + g\right) = \frac{df}{dx} + \frac{dg}{dx}$

Step 2 : Write out the Power Rule, equation (5.5)
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Step 3 : Identify f and g, and differentiate them separately using the Power Rule
$f(x) = x^6$ so $f'(x) = 6x^{6-1} = 6x^5$

$g(x) = x^4$ so $g'(x) = 4x^{4-1} = 4x^3$

Step 4 : Add the derivatives of f and g
$h'(x) = 6x^5 + 4x^3$

d)

Step 1 : Write out the First Linearity Rule, equation (5.7)
$\frac{d}{dx}(cf) = c\frac{df}{dx}$

Step 2 : Write out the Power Rule, equation (5.5)
$\frac{d}{dr}(r^n) = nr^{n-1}$

Step 3 : In this case $n = 3$ and $c = 5$ so...
$\frac{d}{dr}(5r^3) = 5 \times 3r^{3-1}$

Step 4 : Simplify
$\frac{d}{dr}(5r^3) = 15r^2$

e)
Step 1 : Write out the Power Rule, equation (5.5)
$D_u(u^n) = nu^{n-1}$

Step 2 : In this case $n = m$ so...
$D_u(u^m) = mu^{m-1}$ which cannot be simplified further

***Worked Example 5* : Finding tangent lines**

Question:
Find the slope of the tangent to the graph of $y(x) = 3x^2 + 4x + 1$ at $x = 5$.

Answer:

Step 1 : Differentiate y to get a general equation for the tangent to the graph
$y'(x) = 6x + 4$

Step 2 : Substitute the value $x = 5$ into the tangent equation just calculated
$y'(5) = 6 \times 5 + 4 = 34$

So the slope of the tangent line to $y(x)$ at $x = 5$ is 34.

***Worked Example 6* : Drawing graphs**

Question: Draw the graph of $f(x) = x^3 + 3x^2$.

Answer:
Step 1 : Basic shape of graph
a is positive so from left to right, the graph has first a maximum and then a minimum

Step 2 : y intercept
$y = x^3 + 3x^2$ therefore $y(0) = 0$.

Step 3 : x intercepts

$$\begin{aligned} x^3 + 3x^2 &= 0 \\ x^2(x+3) &= 0 \\ x = 0 \quad \text{or} \quad x &= -3 \end{aligned}$$

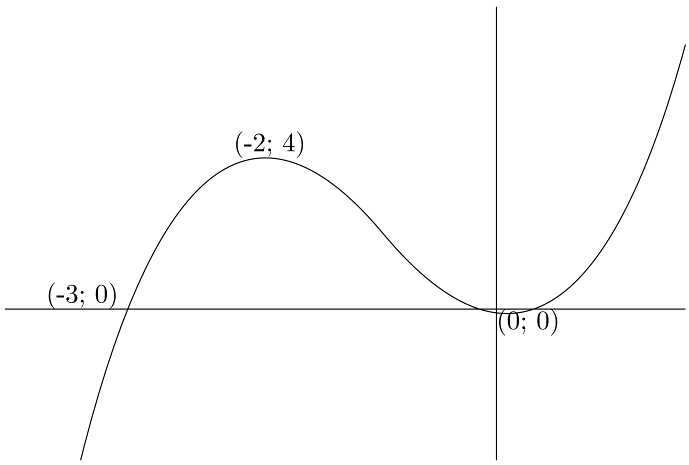
Step 4 : Turning points

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 + 6x \quad \text{set this to zero} \\ 0 &= 3x^2 + 6x \\ 0 &= 3x(x+2) \\ x &= 0 \quad \text{or} \quad x = -2 \end{aligned}$$

Step 5 : y-coordinate of the turning points

$$y(0) = 0 \text{ and } y(-2) = (-2)^3 + 3(-2)^2 = 4$$
$$\text{Local max at } (-2; 4) \text{ and local min at } (0; 0)$$

Step 6 : Draw a neat sketch



5.5 Exercises

1. Draw the graph of $y = x^2 + x - 6$ for $-5 \leq x \leq 6$. Draw the tangents to this curve at $x = 3$, $x = 1$ and $x = -2$, and hence find a value for the gradient of the curve at each of these points.

2. Draw the graph of

$$y = \frac{x^2 - 4x}{4}$$

for $0 \leq x \leq 6$. Draw tangents to the curve at $x = 4$, $x = 3$ and $x = 2$ and hence find a value for the gradient of the curve at each of these points.

3. Differentiate each of the following from first principles to find $\frac{dy}{dx}$

   (a) $y = 5x$
   (b) $y = 9x + 5$
   (c) $y = 3x^2$
   (d) $y = x^3$
   (e) $y = x^2 + 3x$
   (f) $y = 5x - x^2 + 7$
   (g) $y = \frac{1}{x}$
   (h) $y = \frac{1}{x^2}$

4. If $f(x) = 3x - 2x^2$ find $f'(x)$ from first principles and hence evaluate $f'(4)$ and $f'(-1)$

5. If $f(x) = 2x^2 + 5x - 3$ find $f'(x)$ from first principles and hence evaluate $f'(-1)$ and $f'(-2)$

6. If $f(x) = x^3 - 2x$ find $f'(x)$ from first principles and hence evaluate $f'(1)$,$f'(0)$ and $f'(-1)$

1. Differentiate the following functions with respect to x:

   (a) x^5
   (b) x^3
   (c) $12x^2$
   (d) $5x^4$
   (e) $3x^2$
   (f) 7
   (g) $x^{5/3}$
   (h) $x^{3/4}$
   (i) $x^{2/5}$
   (j) $8x^{1/4}$
   (k) $\sqrt{x}$
   (l) $\sqrt{x^3}$

(m) $2/x$

(n) $3/x^2$

2. Find the gradient function $\frac{dy}{dx}$ for each of the following:

   (a) $y = x^2 + 7x - 4$
   (b) $y = x - 7x^2$
   (c) $y = x^3 + 7x^2 - 2$
   (d) $y = 3x^2 + 7x - 4 + \frac{1}{x}$
   (e) $y = (x + 3)(x - 1)$
   (f) $y = (2x + 3)(x + 2)$

3. Find the gradient of the following lines at the points indicated:

   (a) $y = x^2 + 4x$ at (0,0)
   (b) $y = 5x - x^2$ at (1,4)
   (c) $y = 3x^3 - 2x$ at (2,20)
   (d) $y = 5x + x^3$ at $(-1, -6)$
   (e) $y = 3x + \frac{1}{x}$ at (1,4)
   (f) $y = 2x^2 - x + \frac{4}{x}$ at (2,8)

4. Find the coordinates of the point(s) on the following lines where the gradient is given:

   (a) $y = x^2$, gradient 8
   (b) $y = x^2$, gradient -8
   (c) $y = x^2 - 4x + 5$, gradient 2
   (d) $y = 5x - x^2$, gradient 3
   (e) $y = x^4 + 2$, gradient -4
   (f) $y = x^3 + x^2 - x + 1$, gradient 0

5. If $f(x) = x^3 + 4x$ find

   (a) $f(1)$
   (b) $f'(x)$
   (c) $f'(1)$
   (d) $f''(x)$
   (e) $f''(1)$

Chapter 6

Geometry

(NOTE: we need motivation and history of geometry here. real world examples (and obscure figures) and some interesting facts (this is rich... e.g. architecture, computer graphics, manufacturing... like carpentry). What is a degree... why is it out of 360... etc.)

6.1 Polygons

`The syllabus requires:`

- `develop conjectures related to triangles, quadrilaterals and other polygons. attempt to justify, explain or prove these conjectures using any logical method`
- ~~`define various polygons (isosceles, equilateral, right angled triangles, trapezium, isosceles trapezium, kite, parallelogram, rectangle, rhombus, square and the regular polygons)`~~
- ~~`can tell when polygons are similar.`~~ `equilateral triangles are similar`
- ~~`the line drawn parallel to one side of a triangle divides the other 2 sides proportionally`~~

A *polygon* is a shape or figure with many straight sides. A polygon has interior angles. These are the angles that are inside the polygon. The number of sides of a polygon equals the number of interior angles. If a polygon has equal length sides and equal interior angles then the polygon is called a *regular polygon*.

(NOTE: the language used in this notation section sometimes aims too high; words like "denote" and "line segment" may be daunting when simpler words may be used. the audience is only 15 and all of this is new to them. everything needs explained in detail, using simple language... and diagrams help too.)

We denote a line segment that extends between a point A and some point B by line AB. The length of this line is just AB. So if we say, $AB = CD$ we mean that the length of the line segment from A to B is equal to the length of the line segment from C to D.

$\vec{AB}$ is the line segment with length AB and direction from point A to point B. Similarly, $\vec{BA}$ is the line segment with length $AB = BA$ and direction from point B to point A.

Suppose we have two line segments AB and BC that join at a point B. We denote the angle B between the line segments by $\hat{B}$.

A line of symmetry divides a shape in such a way that it appears the same on both sides of the line. For example, if you divide a square along its one diagonal then you divide it into two triangles that are exactly the same i.e. they fit perfectly on each other when the square is folded along the diagonal. If a line AB bisects a line CD then AB divides CD into half.

A stop sign is in the shape of an octagon, an eight-sided polygon. Some coins are heptagonal and hexagonal. In the UK there are two heptagonal coins. The honeycomb of a beehive consist of hexagonal cells. (NOTE: these are true examples, but maybe best left till the list of names of polygons. examples here should try to motivate the study of polygons... how can we actually use the study of polygons to enrich our lives.)

6.1.1 Triangles

A triangle is a three-sided polygon. The sum of the angles of a triangle is 180°. The exterior angle of any corner of a triangle is equal to the sum of the two opposite interior angles (NOTE: a diagram for these 2 rules). We have the following triangles:

Equilateral

All 3 sides are equal and each angle is 60°.

(NOTE: need an example diagram)

Isosceles

Two equal angles occur opposite two equal sides and vice versa.

(NOTE: need an example diagram)

Right-angled

This triangle has a right angle. The side opposite this angle is called the *hypotenuse*. Pythagoras's Theorem is often applied to this type of triangle (NOTE: the students have not been subjected to Pythagoras yet... so place in a reference to the relevant chapter/section.)

(NOTE: need an example diagram)

Scalene

This is any other triangle where the sides have different lengths and angles are different sizes. (NOTE: this is not on the syllabus but its small, and need an example diagram)

6.1.2 Quadrilaterals

Quadrilaterals are four-sided polygons. The basic quadrilaterals are the trapezium, parallelogram, rectangle, rhombus, square and kite,.

Trapezium

This quadrilateral has one pair of parallel opposite sides. It may also be called a *trapezoid*. If the other pair of opposite sides is also parallel then the trapezium is a *parallelogram*. Another type of trapezium is the *isosceles trapezium*, where one pair of opposite sides is parallel, the other pair of sides is equal and the angles at the ends of each parallel side are equal. An isosceles trapezium has one line of symmetry and its diagonals are equal in length. (NOTE: need an example diagram)

Parallelogram

A parallelogram is a special type of trapezium. It is a quadrilateral with two pairs of opposite sides equal. Squares, rectangles and rhombuses are parallelograms. We have the following properties of parallelograms. Both pairs of opposite sides are parallel. Both pairs of opposite sides are equal in length. (NOTE: what does equal mean? is it in length, or direction, or both? we must be more precise in our wording) Both pairs of opposite angles are equal. Both diagonals bisect each other (i.e. they cut each other in half). There are not always lines of symmetry. (NOTE: what is a line of symmetry? we haven't mentioned them before here. is this really a property?)

(NOTE: need an example diagram)

Rectangle

- This is a parallelogram with 90° angles.

Both pairs of opposite sides are parallel. Both pairs of opposite sides are equal. All angles are equal to 90°. Both diagonals bisect each other. Diagonals are equal in length. There are two lines of symmetry.

(NOTE: need an example diagram)

Rhombus

- This is a parallelogram with adjacent sides equal.

Both pairs of opposite sides are parallel. All sides are equal in length. Both pairs of opposite angles equal. Both diagonals bisect each other at 90°. Diagonals of a rhombus bisect both pairs of opposite angles. There are two lines of symmetry.

(NOTE: need an example diagram)

Square

- This is a rhombus with all four angles equal to 90° or a rectangle with adjacent sides equal.

In a square both pairs of opposite sides are parallel. All sides are equal in length. All angles are equal to 90°. Both diagonals bisect each other at right-angles. Diagonals are equal in length and bisect both pairs of opposite angles. There are four lines of symmetry.

(NOTE: need an example diagram)

Kite

- A kite is a parallelogram with two pairs of adjacent sides equal.

Other properties of a kite are that the two pairs of adjacent sides are equal. One pair of opposite angles are equal where the angles must be between unequal sides. One diagonal bisects the other diagonal and one diagonal bisects one pair of opposite angles. Diagonals intersect at right-angles. There is one line of symmetry.

(NOTE: need an example diagram)

6.1.3 Other polygons

There are many other polygons, some of which are given in the table below. (NOTE: need an example diagram)

6.1.4 Similarity of Polygons

If two polygons are *similar*, one is an enlargement of the other. This means that the two polygons will have the same angles and their sides will be in the same proportion. (NOTE: expand this with quick examples.)

We can use the symbol $\sim$ to mean *is similar to*.

Two polygons are similar if and only if either or both of the following are true:

- Corresponding (NOTE: define corresponding) angles are equal.
- Corresponding sides are all in proportion.

For example, $\triangle ABC \sim \triangle DEF$ if $\hat{A} = \hat{D}, \hat{B} = \hat{E}, \hat{C} = \hat{F}$ and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (NOTE: need diagram here)

(NOTE: we need a lot more examples here, specifically that all equilateral triangles are similar.)

6.1.5 Midpoint Theorem

(NOTE: this section could really do with some interesting facts and examples.) Line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side. Given $\triangle ABC$ with midpoints M of AB and N of AC.

(NOTE: although proofs are nice... it is often more important to push the result itself, and why it is useful. if a proof is confusing, the student will simply skip the section because it is too hard... even though application may be easy.) Prove that MN is parallel to BC and $MN = \frac{1}{2}BC$. (NOTE: diagram 'join the dots style' is in comments in the source file.) Extend MN by its own length to a point P. Join AP, CP and MC. $MN = NP$ by construction. $AN = NC$ given N midpoint of AC. So APCM is a parm diags bisect each other. So $CP = MA$ opp sides of parm equal. But $BM = AM$ given M midpoint of AB. So $CP = MB$. Now $CP \parallel AB$ APCM is a parm So $CP \parallel MB$ AB is a line segment. So BMPC is a parm 1 pair of opp sides equal and parallel. So MN parallel to BC opp sides of parm parallel. Now $MN = NP$ by construction. So $MN = \frac{1}{2} * MP$. But $MP = BC$ opp sides of parm equal. So $MN = \frac{1}{2} * BC$

6.1.6 Extra

Angles of regular polygons

We have a formula to calculate the size of the interior angle of a regular polygon.

$$\hat{A} = \frac{n-2}{n} \times 180^\circ \tag{6.1}$$

where n is the number of sides and $\hat{A}$ is any angle.

Areas of Polygons

Area of triangle: $\frac{1}{2}\times$ base $\times$ perpendicular height

Area of trapezium: $\frac{1}{2}\times$ (sum of $\parallel$ sides) $\times$ perpendicular height

Area of parallelogram and rhombus: base $\times$ perpendicular height

Area of rectangle: length $\times$ breadth

Area of square: length of side $\times$ length of side

(NOTE: everything from here on in Extra is probably acceptable syllabus material, but it is here for now so i can see what needs to be brought back in to the main text. the theorems are not on the syllabus, but we should maybe include them since they use basic geometry techniques... but do not call them theorems, rather use them as in-line examples or worked examples.)

Parallelograms

To show that a quadrilateral is a parallelogram, show any one of the first four properties or that one pair of opposite sides are equal and parallel. **Theorem 1:** Given a parallelogram ABCD (with both pairs of opposite sides parallel), prove that the opposite sides and angles are equal. (figure 4 here) **Proof:** Join AC. In $\triangle ABC$ and $\triangle ADC$: 1. $\angle DAC = \angle ACB$ alternate angles $=, AD \parallel BC$ 2. $\angle BAC = \angle ACD$ alternate angles $=, AB \parallel CD$ 3. $AC = AC$ common sides So $\triangle ABC \equiv \triangle CDA$ (AAS) So $AB = CD$ and $DA = BC$ corresponding sides in congruent triangles $\hat{B} = \hat{D}, \hat{A} = \hat{C}$ corresponding angles in congruent triangles Hence opposite sides equal and opposite angles equal. **Theorem 2:** Given parallelogram ABCD with AC and BD joined and denote their intersection by O. Prove that AC and BD bisect each other. (figure 5 here) **Proof:** In $\triangle AOB$ and $\triangle COD$: 1. $\angle BAC = \angle ACD$ alternate angles $=, AB \parallel CD$ 2. $\angle ABD = \angle BDC$ alternate angles $=, AB \parallel CD$ 3. $AB = CD$ opposite sides of parm equal So $\triangle ABO \equiv \triangle CDO$ (AAS) So $AO = OC, BO = OD$ corresp sides in congruent triangles Hence AC and BD bisect each other.

Rectangles

To prove that a quadrilateral is a rectangle you can first prove that it is a parallelogram and then prove that it has a right-angle. Or you can directly prove that it has four right-angles. **Theorem :** Given rectangle ABCD prove that diagonals are equal in length. (figure 6 here) **Proof:** In $\triangle ACD$ and $\triangle BCD$: 1. $AD = BC$ opposite sides equal 2. $DC = DC$ common sides are equal 3. $\angle D = \angle C$ all angles equal So $\triangle ADC \equiv \triangle BCD$ (RHS) So $AC = BD$ corresp sides in congruent triangles Hence diagonals are equal in length.

Rhombuses

To prove that a quadrilateral is a rhombus you can first prove that it is a parallelogram. Then you can prove: all four sides equal, diagonals intersect at right-angles or diagonals bisect corner angles. **Theorem :** Given rhombus ABCD with diagonals intersecting at point O. Prove that the diagonals intersect at right-angles and that they bisect the corner angles. (figure 7 here) **Proof:** In $\triangle AOB$ and $\triangle AOD$: 1. $AB = AD$ all sides of rhombus equal 2. $AO = AO$ common sides are equal 3. $OB = OD$ diags bisect each other So $\triangle AOB \equiv \triangle AOD$ (SSS) So $\angle AOB = \angle AOD$ corresp angles in congruent triangles But BD is a straight line So $\angle AOB = \angle AOD = 90°$ sum of angles is 180° So AC and BD intersect therefore diagonals intersect Now $\angle BAO = \angle DAO$ corresp angles in congruent triangles Similarly, $\triangle AOB \equiv \triangle COB \Rightarrow \angle ABO = \angle CBO$ $\triangle COD \equiv \triangle AOD \Rightarrow \angle CDO = \angle ADO$ $\triangle BOC \equiv \triangle DOC \Rightarrow \angle BCO = \angle DCO$ Hence diagonals also bisect the corner angles.

Squares

To prove that a quadrilateral is a square you can prove that it is a rhombus and then prove that it has four right-angles or equal diagonals. You can also prove that it is a rectangle and then prove all four sides equal, diagonals intersect at right-angles or diagonals bisect corner angles.

6.2 Solids

`The syllabus requires:`

- `analyse, describe and represent the properties and relationships of geometric solids by calculating surface area, volume and the effect on these by scaling one or more dimension by` k.
- `estimate volume of everyday objects`
- `solids to consider: sphere, hemisphere, combinations with cylinders`
- `(grade 12) solids to consider: right circular cone, tetrahedron, pyramid`
- `classify geometric solids in various ways (including regular polyhedra)`
- `investigate the effect of a plane cutting the regular polyhedra in various ways`

- (grade 12) plane cutting right circular cone

6.3 Coordinates

The syllabus requires:

- use coordinate systems to represent geometric figures and derive for any 2 points a formula for distance between points (NOTE: call it the metric), gradient of line between points and the coordinates of the midpoint of the line joining points (NOTE: SH: i assume they expect this to all be done in $\mathbb{E}^2$ or $\mathbb{E}^3$. the syllabus author seems oblivious to the non-triviality of doing this on any old surface)

6.4 Transformations

The syllabus requires:

- generalise the effect of the following rigid transformations to a point: translations, reflections in x,y, and $x = y$
- recognise when an object is similar to another object under some transformation. conjecture and prove such similarities
- rotation of a point through 180^0
- vertices of a polygon after enlargement by factor k
- vertices of a polygon after shearing (base on x axis, opposite side parallel)
- emphasise that rigid transformations (trans, ref, rot, glide ref) preserve shape and size. enlargement preserves shape but not size and shearing preserves area
- (grade 12) generalise the effect on the point of stretch by k (NOTE: first i have ever heard of stretching a point) and rotation about the origin by an angle α^0
- (grade 12) identify and classify geometric border patterns and tessellations in terms of line symmetry, glide reflection symmetry, rotational symmetry and point symmetry

6.4.1 Shifting, Reflecting, Stretching and Shrinking Graphs:

Shifting Graphs

Let us assume that we know some function $f(x)$. What would happen if we defined the function $y = f(x - a)$, where a is some positive constant? Well, the value of y at x is really just the value of the function f at $x - a$ moved to the point x. In other words, this defines the graph we would get if we shifted the function $f(x)$ by a to the right. (NOTE: Shifting is not really a good mathematical term... can we call it translation instead?)

Similarly, the function $y = f(x+a)$, is the result of shifting the function by a to the left (the function value at $x+a$ is moved to x).

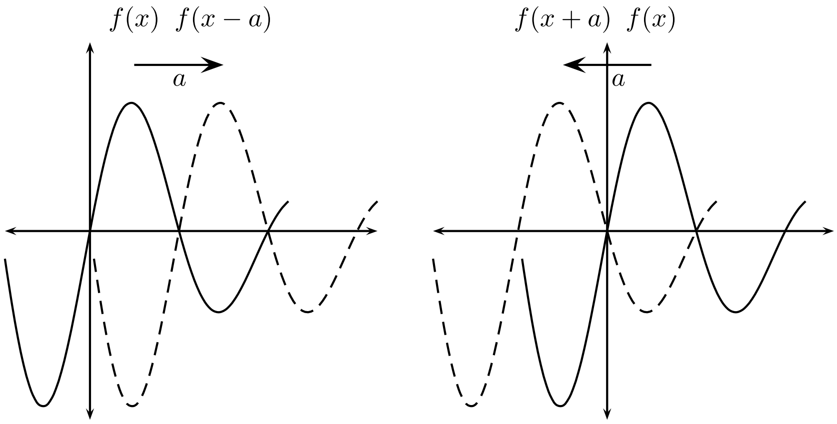


Figure 6.1: Graph of a function with the translations $x \to x + a$ and $x \to x - a$. Note that this is just a simple shift either left or right of the entire graph.

Now let us look at the function $y - b = f(x)$, where b is some positive constant, which is the result of replacing y by $y - b$ in the function $y = f(x)$. This gives us that $y = f(x) + b$. The function is thus shifted upwards by the constant b.

We can also replace y by $y + b$, which again just results in movement in the opposite direction. We can see this because $y + b = f(x)$ implies that $y = f(x) - b$, which shows that $f(x)$ has been shifted downwards by b.

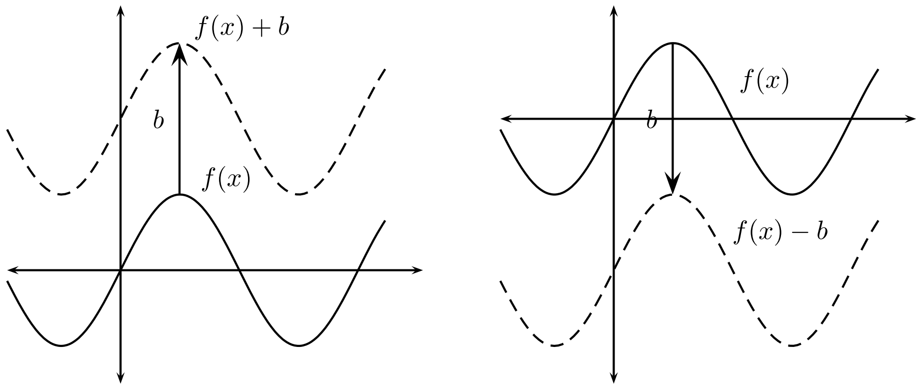


Figure 6.2: Graph of $\sin(x)$ with the translations $y \to y + b$ and $b \to y - b$. Note that this is just a simple shift either up or down of the entire graph.

Now, what about relations in general? If we take a relation, which depends on x and y, and replaced x by $x - a$ and y by $y - b$ (where a and b are positive constants), then what would happen? Well, the relation value at the point $(x-a, y-b)$ would be moved to the point (x, y). Therefore the graph of the relation would be shifted right by a and upwards by b. Similarly, if we replaced x by $x+a$ and y by $y+b$, then the function would be moved left by a and downwards by b.

Reflections

Now consider defining $y = f(-x)$, where $f(x)$ is a known function. This takes the function value at the point $-x$ to the point x. In other words, the function values on one side of the y-axis are moved to the other side of the y-axis. Thus the function is reflected about the y-axis.

Alternatively we can look at the function $-y = f(x)$. This is the same as saying $y = -f(x)$, which reflects the function about the x-axis (every positive function value is changed to the corresponding negative function value and vice versa).

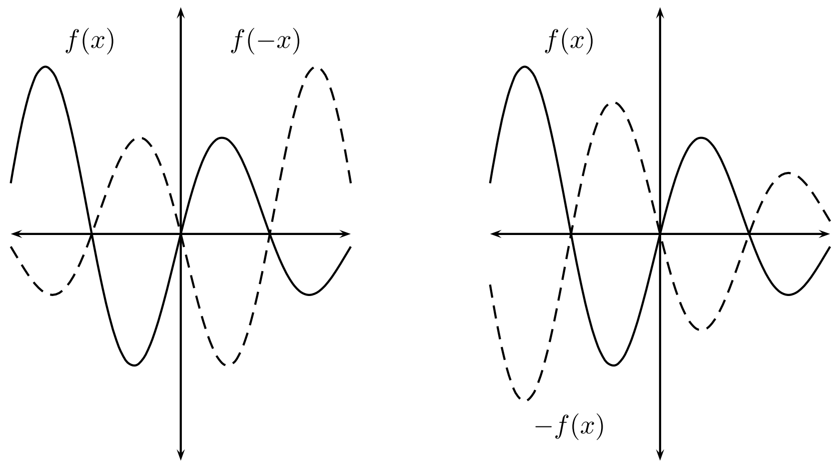


Figure 6.3: Graph of a function with the reflections $x \to -x$ and $f(x) \to -f(x)$. Note that these are just reflections in the vertical and horizontal axes, respectively.

As before, we can generalise these idea to deal with relations. In all cases, if we change x to $-x$, then the relation will be reflected about the y-axis and if we replace y by $-y$ then there will be a reflection about the x-axis.

Note: We say that $f(x)$ is symmetric about the y axis if $f(x) = f(-x)$ (in other words, the function and its reflection about the y-axis are the same). Similarly, if $f(x) = -f(x)$, then we say that $f(x)$ is symmetric about the x-axis.

6.5 Stretching and Shrinking Graphs

(NOTE: The figures are correct, but i think the figures are negating the truth.)
We shall now look at what happens to $f(x)$ if we consider the function $y = f(ax)$, where a is a positive constant and $a > 1$. The point ax on the x-axis is further

from the y-axis than x (since $|ax| > |x|$). Now the function value at ax is moved to x, so the function is moved towards the x-axis by a factor of a. Thus the effect is to shrink $f(x)$ horizontally by a factor of a.

The function $y = f(\frac{x}{a})$ has the opposite effect. The point $\frac{x}{a}$ on the x-axis is closer to the y-axis than x (as $|\frac{x}{a}| < |x|$). The function value at $\frac{x}{a}$ is moved to x so the function is stretched horizontally by a factor of a.

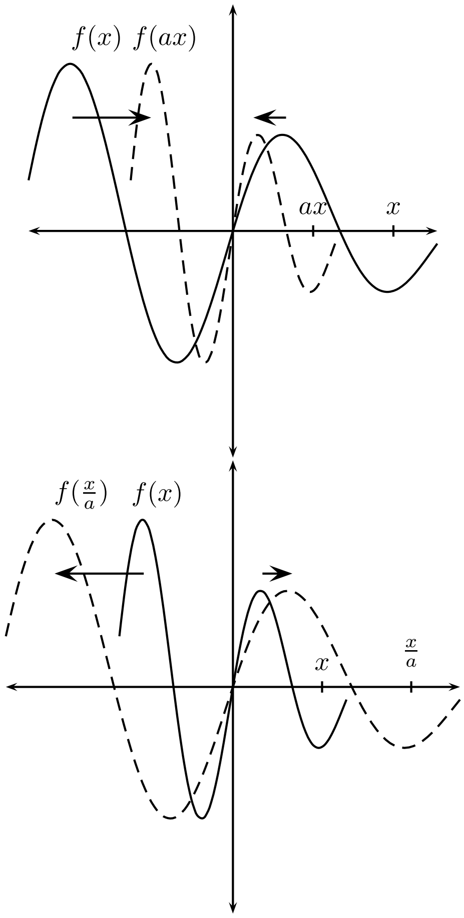


Figure 6.4: Graph of a function with the rescaling $x \to ax$ and $x \to \frac{x}{a}$. Note that these are just a shrinking and stretching in the horizontal axis.

(NOTE: this is correct. vertical and horizontal rescalings are different... i think the author got confused and thought the same thing happened in each. this needs fixed.) Replacing y by by, where b is a positive constant and $b > 1$, gives $by = f(x)$ and thus $y = \frac{1}{b}f(x)$. The function value at any point x is reduced by a factor of b. Therefore the graph shrinks vertically by a factor of b.

Similarly, if we replace y with $\frac{y}{b}$ to give $\frac{y}{b} = f(x)$, we obtain the function $y = bf(x)$. At each x value the function value is increased by a factor of b so the function is stretched vertically.

Again these result can be used to deal with relations as well. If x and y are

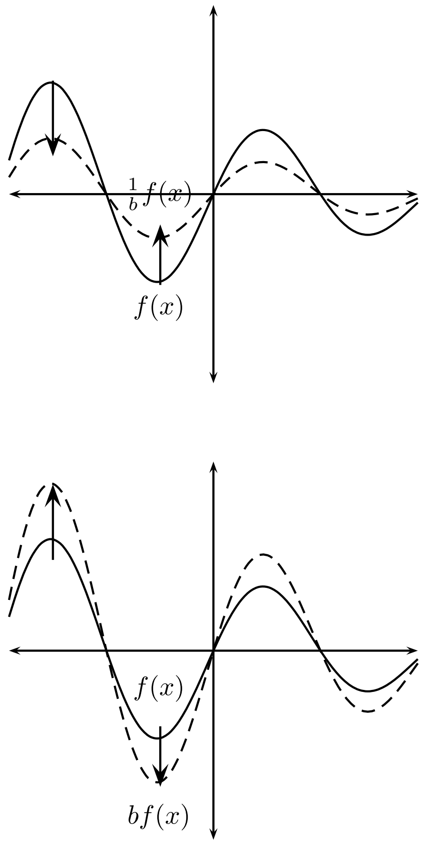


Figure 6.5: Graph of a function with the rescaling $f(x) \to \frac{1}{b}f(x)$ and $f(x) \to bf(x)$. Note that these are just a shrinking and stretching in the vertical axis.

replaced by ax and by in any relation, the effect is to shrink the graph of this relation by a factor of a horizontally and by a factor of b vertically. Similarly, changing x to $\frac{x}{a}$ and y to $\frac{y}{b}$ causes the relation to be stretched horizontally and vertically by factors of a and b respectively.

6.6 Mixed Problems

If we perform many of these transformations on a given function, then we must combine the different effects. However, it is very important that we effect these changes in the **right order**. Here are some examples of mixed problems.

6.7 Equation of a Line

`The syllabus requires:`

- derive formula for the equation of a line when given 2 points
- derive formula for the line parallel to a given line and passing through a point
- derive formula for the inclination of a line

6.8 Circles

The syllabus requires:

- (grade 12) tangent is perpendicular to the radius
- (grade 12) the line from the centre of a circle perpendicular to a chord bisects the chord
- (grade 12) angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle
- (grade 12) the opposite angle of a cyclic quadrilateral are supplementary the tangent chord theorem (NOTE: really... thats what it says, word for word)

6.8.1 Circles & Semi-circles

Circles:

A circle centered at the origin with radius r is described by the relation

$$x^2 + y^2 = r^2 \tag{6.2}$$

We can see that a circle is not a function, since both $(0,r)$ and $(0,-r)$ satisfy the relation (in other words, the line $x = 0$ will always cut the circle at two points).

Now, since $x^2 = r^2 - y^2$ and y^2 is never negative, it follows that $x^2 \leq r^2$ and thus $-r \leq x \leq r$. Therefore the domain of the relation is $[-r,r]$. Similarly, $y^2 = r^2 - x^2$ and therefore $-r \leq y \leq r$. Thus the range of the relation is $[-r,r]$.

Semi-Circles:

The equation for a circle $x^2 + y^2 = r^2$ can also be written as

$$y = \pm\sqrt{r^2 - x^2} \tag{6.3}$$

Now let us we consider the positive and negative square roots separately. These describe semi-circles on either side of the x-axis. Thus the equations for two types of semi-circles are as follows:

$$y = \sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2} \tag{6.4}$$

The domain of each of these semi-circles is $[-r,r]$ and the range is $[0,r]$ (for the first semi-circle) and $[-r,0]$ (for the second semi-circle).

Note: These semi-circles are functions, since there is only one y value corresponding to each x value.

6.9 Locus

`The syllabus requires:`

- `(grade 12) derive the equation of the locus of all points; equidistant from a given point, equidistant from 2 given points, equidistant from a given point and a line parallel to the x or y axis`

6.10 Other Geometries

`The syllabus requires:`

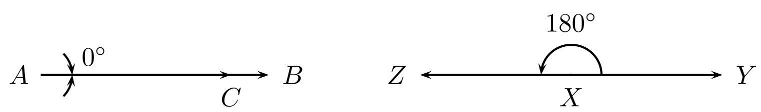
- `basic knowledge of spherical geometry, taxicab geometry and fractals`

6.11 Unsorted

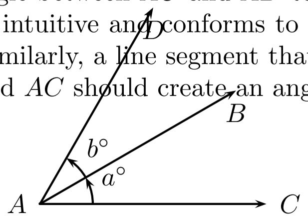
6.11.1 Fundamental vocabulary terms

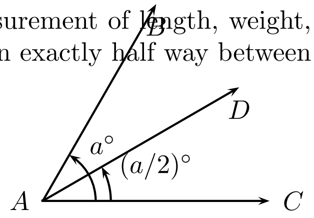
Measuring angles

The magnitude of an angle does not depend on the length of its sides; it only depends on the relative direction of the two sides. E.g. the adjacent edges of a postcard are at an angle of 90° with respect to each other. But so is the Empire State Building in New York City with respect to Fifth Ave. and 34^{th} Street (NOTE: its SA... maybe this should be the baobab tree with respect to the Limpopo river (i kid... i kid)). So we can't use notions of length to measure angles. How do we measure angles then? We begin by finding a way to enumerate angles. What is the smallest angle you can draw? Two lines subtending almost no angle. Two coincident line segments pointing in the same direction subtend an angle of 0°, e.g. , lines AB and AC in the figure below. Now if we keep one line fixed and move the other while still pivoted at the common vertex, we can obtain any other angle. Two line segments with a common vertex and facing in opposite direction are said to form an angle of 180°, e.g. , XY and XZ in the figure below. ZXY is a straight line.

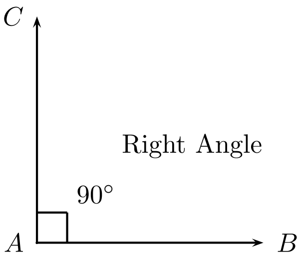


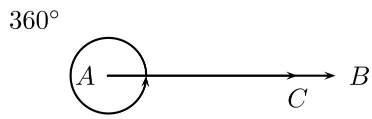
The choice of a measure of 180° for the angle subtended by the segments of a straight line is a matter of historical convention. Once we have decided what the measure of an angle formed by a straight line is, we have also fixed the measure of all the other angles. This is because we would like the angles to obey some desirable properties. E.g. if we have an angle $a°$ between two lines AB and AC, and another angle $b°$ between AB and AD, we would like the angle between AC and AD to be $(a + b)°$. This makes the measurement of angles intuitive and conforms to our notion of measurement of length, weight, etc. Similarly, a line segment that defines a direction exactly half way between AB and AC should create an angle of $(a/2)°$.





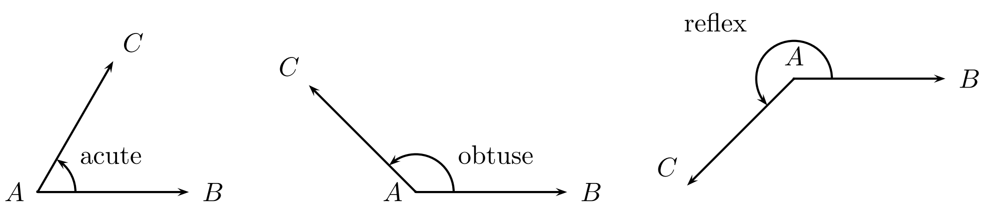
An angle of 90° is termed a right angle. A right angle is half the measure of the angle subtended by a straight line (180°). An angle twice the measure of a straight line is 360°. An angle measuring 360° looks identical to an angle of 0°, except for the labelling. All angles after 360° also look like we have seen them before. Angles that measure more than 360° are largely for mathematical convenience to maintain continuity in our enumeration of angles.





We define some other terms at this point. These are simply labels for angles in particular ranges.

- Acute angle: An angle $\geq 0°$ and $< 90°$.
- Obtuse angle: An angle $> 90°$ and $< 180°$.
- Straight angle: An angle measuring 180°.
- Reflex angle: An angle $> 180°$ and $< 360°$.



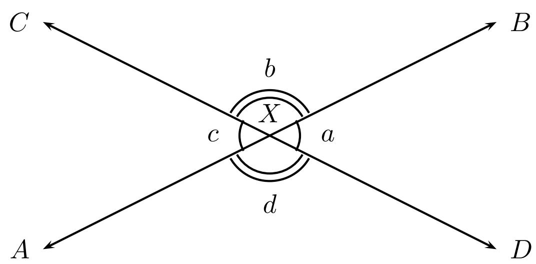
Once we can number or measure angles, we can also start comparing them. E.g. all right angles are 90°, hence equal. An obtuse angle is larger than an acute angle, etc.

An alternative measure of angles is used on a compass. E.g. if North(N) is 0°, North-East(NE) is 45°, NNE is 22.5°, etc.

Special Angle Pairs

In the previous section, we classified angles based on measurement. In this section we'll examine some interesting properties of angles formed by a pair of intersecting lines.

First we consider a single straight line, AB. There's a point on the line AB called X. The measure of angle AXB is 180° as defined in the previous section. Now let us draw another straight line intersecting the first. Without loss of generality, let the point of intersection be X. We call the four angles formed with X as the vertex a, b, c and d. At this point, we introduce some definitions for convenience.



Vertical angles

Definition: The angles formed by two intersecting straight lines that share a vertex but do not share any sides are called vertical angles. E.g. , a and c in the figure above are vertical angles. b and d are also vertical angles.

Adjacent Angles

Definition: Two angles that share a common vertex and a common side are called adjacent angles. E.g. , (a,b) and (c,d) are adjacent angles.

Linear pairs

Definition: The adjacent angles formed by two intersecting straight lines are said to form a linear pair. E.g. (a,b), (b,c), (c,d) and (d,a) all form linear pairs. Since the non common sides of a linear pair are part of the same straight line, the total angle formed by the linear pair is 180° by definition. E.g. $a+b = 180°$, etc.

What can we say about the vertical angles? Looking at figure above, it seems like the vertical angles are equal to one another. We can prove the following result.

Theorem: The vertical angles formed by intersection of two straight lines are equal.
Proof: Since a and b form a linear pair, $a + b = 180°$. Similiarly, b and c form a linear pair, so, $b+c = 180°$. Thus, $a+b = b+c$. Since the angle b contributes equally to both sides of the equation, it can be cancelled out leaving, $a = c$. The proof for the pair (b,d) is identical.

This proof confirms our intuition that vertical angles are equal in magnitude. This result will later be used when proving properties for parallel lines.

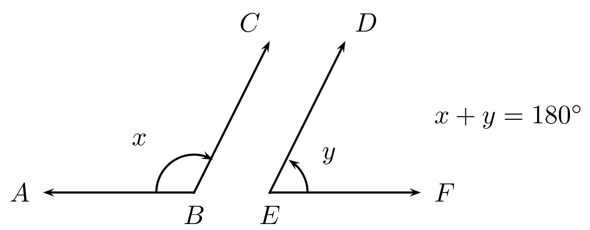
We end this section with some more definitions.

Supplementary and Complementary pairs

Definition: Two angles are called supplementary if their sum equals 180°. E.g. angles that constitute a linear pair are supplementary.

Definition: Two angles are called complementary if their sum equals 90°.

Note that in order to be labelled supplementary or complementary, the two angles being considered need not be adjacent. E.g. x and y in the figure below are supplementary, but they are not adjacent and thus do not form a linear pair.



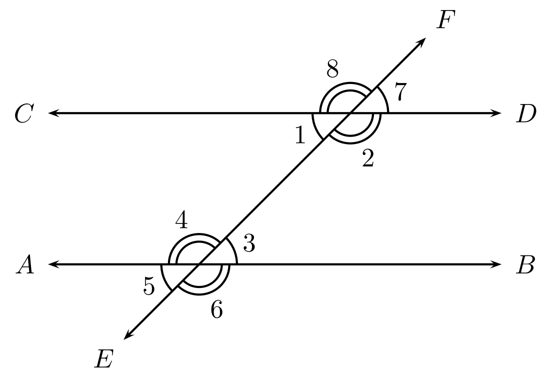
6.11.2 Parallel lines intersected by transversal lines

Two lines are said to intersect if there is a point that lies on both lines. Informally, two lines intersect if they meet at some point when extended indefinitely in either direction. E.g. at a traffic intersection, two or more streets intersect; the middle of the intersection is the common point between the streets.

It is possible that two lines that lie on the same plane never intersect even when extended to infinity in either direction. Such lines are termed parallel lines. E.g. the tracks of a straight railway line are parallel lines. We wouldn't want the tracks to intersect as that would be catastrophic for the train! A section of the Australian National Railways Trans-Australian line is perhaps one of the longest pairs of man-made parallel lines.

> **Longest Railroad Straight** (Source: www.guinnessworldrecords.com) The Australian National Railways Trans-Australian line over the Nullarbor Plain, is 478 km. (297 miles) dead straight, from Mile 496, between Nurina and Loongana, Western Australia, to Mile 793, between Ooldea and Watson, South Australia.

A transversal of two or more lines is a line that intersects these lines. E.g. in the figure below, AB and CD are two lines and EF is a transversal. We are interested in the properties of the angles formed by these intersecting lines, so we'll introduce some definitions for various angle pairs.



Definitions:

- Interior angles: When two lines are intersected by a transversal, the angles that lie between the two lines are called interior angles. E.g. in the figure above, 1, 2, 3 and 4 are interior angles.
- Exterior angles: When two lines are intersected by a transversal, the angles formed that lie outside the two lines are called exterior angles. E.g. 5, 6, 7 and 8 are exterior angles.
- Alternate interior angles: When two lines are intersected by a transversal, the interior angles that lie on opposite sides of the transversal are termed alternate interior angles. E.g. in the above example, 1 and 3 are are a pair of alternate interior angles. 2 and 4 are also alternate interior angles.
- Interior angles on the same side: As the name suggests, these are interior angles that lie on the same side of the transversal. E.g. (1,4) and (2,3).
- Corresponding angles: The angles on the same side of the transversal and the same side of the two lines are called corresponding angles. E.g. (1,5), (4,8) and (3,7), etc. , are pairs of corresponding angles.

In order to prove relationships between the angles defined above, we will assume the following postulate regarding parallel lines.

Euclid's Parallel Line Postulate:
Postulate: If a straight line falling on two straight lines makes the two interior angles on the same side less than two right angles (180°), the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

The above is one of the fundamental postulates of Euclidean geometry and has no proof based on the other postulates. Now we'll use the above postulate to prove some other properties.

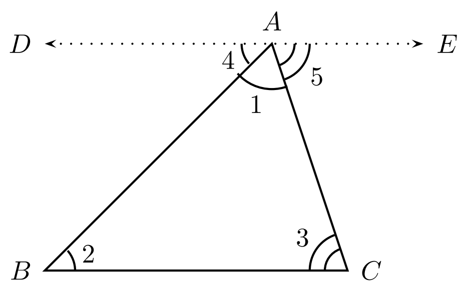
Theorem 1: If two parallel lines are intersected by a transversal, the sum of interior angles on the same side of the transversal is two right angles (180°).
Proof: Consider parallel lines AB and CD intersected by the transversal EF in the figure above. Suppose that the sum of the interior angles is less than 180° on one side of the transveral, e.g. $1 + 4 < 180°$. Then Euclid's

Parallel Line Postulate implies that the AB and CD meet on that side of the transversal and are not parallel. This contradicts the assumption that the lines are parallel.
Now suppose that the sum of the interior angles 1 and 4 is greater than $180°$. Now, $(2) = 180° - (1)$ and $(3) = 180° - (4)$. So $(2+3) = 360° - (1+4)$. Since $(1 + 4) > 180°$, $(2 + 3) < 180°$. Thus the parallel line postulate implies that the lines will meet on that side of the transveral and are not parallel. Thus both pairs of interior angles on the same side need to sum up to $180°$ for the lines to be parallel.

Theorem 2: If two parallel lines are intersected by a transversal, the alternate interior angles are equal.
Proof: In the figure above, using Theorem 1,
$(1 + 4) = 180°$
Also, since AB is a straight line, 1 and 4 are supplementary.
$(4 + 3) = 180°$
Thus, $1 = 3$. Similarly, $2 = 4$.

Theorem 3: If two parallel lines are intersected by a transversal, the corresponding angles are equal.
Proof: Again using Theorem 1, in the figure above,
$(1 + 4) = 180°$
Also, since EF is a straight line,
$(4 + 5) = 180°$
So $1 = 5$, etc.

Theorem 4: The sum of the three angles in a triangle is 180.
Proof: Consider triangle ABC shown in the figure below. The three angles are denoted 1, 2 and 3. We have to show that $1 + 2 + 3 = 180°$. Consider a straight line DE through point A that is parallel to BC. We denote the angles between DE and the sides of the triangle as 4 and 5.

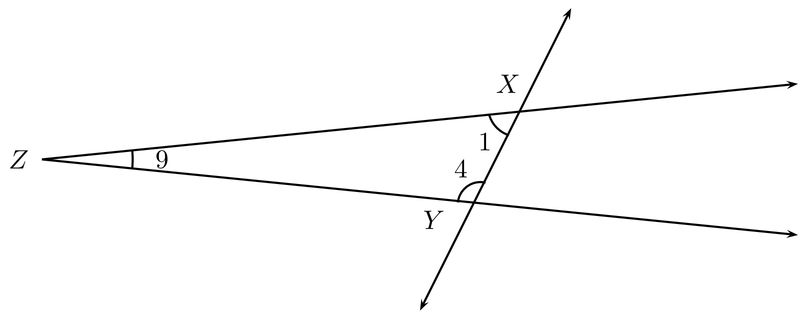


Since DE is a straight line, $1 + 4 + 5 = 180°$.
Now DE is parallel to BC, and 2 and 4 are alternate interior angles (AB is the transversal). So $2 = 4$. Similarly, $3 = 5$.
So substituting these in the first equation, $1 + 2 + 3 = 180°$.

The following theorems help in determining when two lines are parallel to each other.

Theorem 5: If two lines are intersected by a transversal such that any pair of interior angles on the same side is supplementary, then the two

lines are parallel.
Proof: We'll prove that if two lines are not parallel, the interior angles on the same side are not supplementary. We'll prove this by contradiction. Assume that two non-parallel distinct lines are intersected by a transversal such that interior angles 1 and 4 are supplementary.
$1 + 4 = 180°$ (Eq. (i))
Since the lines are not parallel, they have to intersect at some point Z. Since the two lines are distinct, they have to form a non-zero angle at their point of intersection.



XYZ is a triangle. So $1 + 4 + 9 = 180°$, using Theorem 4.
But using Eq. (i), $1 + 4 = 180°$, so $9 = 0°$. This contradicts the fact that distinct intersecting lines create a non-zero angle at their point of intersection. So our original assumption is not supportable and the interior angles 1 and 4 cannot be supplementary.

Theorem 6: If two lines are intersected by a transversal such that a pair of alternate interior angles are equal, the lines are parallel.
Proof: Left as an exercise.

Theorem 7: If two lines are intersected by a transversal such that a pair of alternate corresponding angles are equal, the lines are parallel.
Proof: Left as an exercise.

Theorem 8: Prove that if a line AB is parallel to CD, and AB is parallel to EF, then CD is parallel to EF.
Proof: Left as an exercise. (Hint: We can prove this in two steps:

1. Prove that if two lines are parallel, then a line that intersects one also intersects the other.
2. Use the equivalence of corresponding angles to get the result.)

Sides	Name
5	pentagon
6	hexagon
7	heptagon
8	octagon
10	decagon
15	pentadecagon

Table 6.1: Table of some polygons and their number of sides.

Chapter 7

Trigonometry

7.1 Syllabus

7.1.1 Triangles

The syllabus requires:

- similarity of triangles is the basis of trig functions (NOTE: perhaps this is best left in geometry)
- ~~solve problems in 2D by constructing and interpreting geometric and trig models including scale drawings, maps and building plans~~

7.1.2 Trigonometric Formulæ

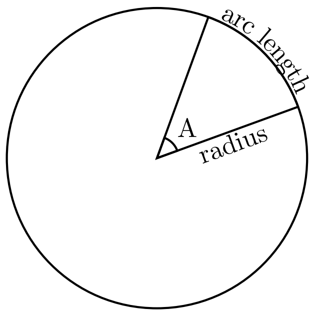
The syllabus requires:

- some history from various cultures
- ~~derive reduction formulæ for trig ratios~~
- recognise equivalence of trig expressions by reduction
- ~~solve 2D problems by establishing sin/cos/area rules~~ (NOTE: perhaps just do the rules here, and do the problems in section 8.5)
- ~~use trig for height and distances~~ (NOTE: SH isn't this already done in 7.1.1?)

7.2 Radian and Degree Measure

You should be familiar with the idea of measuring angles from geometry but have you ever stopped to think why there are 360 degrees in a circle?

The reason is purely historical [1]. There are, in fact, many different ways of measuring angles. The two most commonly used are degrees (the one you have been using up to now) and radians.



The radian measure of an angle is defined as the ratio of the arc length subtending the angle to the radius of the circle.

$$A = \frac{arclength}{radius} \tag{7.1}$$

We know from geometry that the circumference of a circle is found using the equation $c = 2\pi r$. If we divide through by the radius of the circle, r, we find that the radian angle subtended by the complete circumference, (or in other words the number of radians in a full circle) is $\frac{2\pi r}{r} = 2\pi$. This means that 2π radians is the same as $360°$.
With this in mind we can easily work out how to convert between degrees and radians.

Definition: $\theta(rad) = \theta(°) \times \frac{2\pi}{360}$	or	$\theta(°) = \theta(rad) \times \frac{360}{2\pi}$

Using these formulae we can express common angles in radians. It is worth learning these as questions may be asked using either degrees, radians or a mixture of both.

Degrees	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

7.2.1 The unit of radians

You may be wondering what the unit of radians is. The answer is that it doesn't have one. This is because a radian is the ratio of two lengths: the arc length divided by the radius. Now, both of these will have the

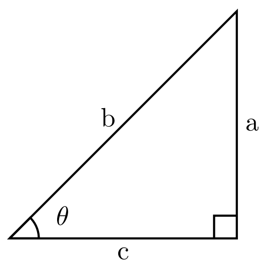
[1] There are 360 degrees in a circle because the ancient Babylonians had a number system with base 60. A base is the number you count up to before you get an extra digit. The number system that we use everyday is called the decimal system (the base is 10), but computers use the binary system (the base is 2). $360 = 6 \times 60$ so for them it make sense to have 360 degrees in a circle.

same unit, so when you divide them, the units simply disappear! This is what is known as a dimensionless quantity. Sometimes we write radians (or simply *rad*) after the number to emphasise that we are using radians, but this is not necessary.
In general, if an angle is expressed in terms of π it is meant to be in radians. Be careful though. If the question does not explicitly say whether the angle is measured in degrees or radians you need to use common sense to decide which to use.

7.3 Definition of the Trigonometric Functions

7.3.1 Trigonometry of a Right Angled Triangle

Consider a right-angled triangle.



We define

$$\sin\theta = \frac{a}{b} \quad (7.2)$$

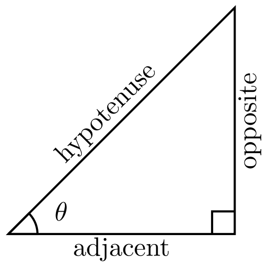
$$\cos\theta = \frac{c}{b} \quad (7.3)$$

$$\tan\theta = \frac{a}{c} \quad (7.4)$$

These are abbreviations for sine, cosine and tangent. These functions, known as trigonometric functions, relate the lengths of the sides of a triangle to its interior angles.

How to remember the definitions

Different people have different ways of remembering these ratios. One way involves defining opposite to be side of the triangle opposite to the angle, hypotenuse to be the side opposite to the right-angle (just like we use the term in geometry) and adjacent to be the side next to the angle, which is not the hypotenuse. This is illustrated in the following picture, where we show the adjacent, opposite and hypotenuse for the angle θ.



So, using these definitions we have:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \tag{7.5}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{7.6}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \tag{7.7}$$

There is a mnemonic to remember these:

S	**S**ine
O	**O**pposite
H	**H**ypotenuse
C	**C**os
A	**A**djacent
H	**H**ypotenuse
T	**T**an
O	**O**pposite
A	**A**djacent

Another mnemonic that is perhaps easier to remember goes as follows:

Silly **O**ld **H**ens	**S**in = $\frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}$
Cackle **A**nd **H**owl	**C**os = $\frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}$
Till **O**ld **A**ge	**T**an = $\frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$

CAUTION! The definitions of opposite, adjacent and hypotenuse only make sense when you are working with right-angled triangles! Always check to make sure your triangle has a right angle before you use them, otherwise you will get the wrong answer. We will find ways of working with the trigonometry of non right-angled triangles later in the chapter.
By using the appropriate triangles it is possible to work out the following values of the sine, cosine and tangent functions for a number of common angles.

	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-$	0

These values are useful to remember as they often occur in questions. They are also a good way of helping us to visualise the graphs of the sine, cosine and tangent functions.

7.3.2 Trigonometric Graphs

Sine and Cosine Graphs

Let us look back at our values for $\sin\theta$ –

	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

As you can see, the function $\sin\theta$ has a value of 0 at $\theta = 0°$. Its value then smoothly increases until $\theta = 90°$ when its value is 1. We then know that it later decreases to 0 when $\theta = 180°$. Putting all this together we can start to picture the full extent of the sine graph. The sine graph is shown in figure 7.1.

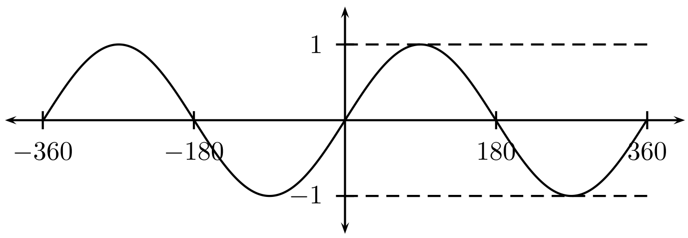


Figure 7.1: The sine graph.

Let us now look back at the values of cosine–

	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1

If you look carefully you will notice that the cosine of an angle θ is the same as the sine of the angle $90° - \theta$. Take for example,

$$\cos 60° = \frac{1}{2} = \sin 30° = \sin(90° - 60°)$$

This tells us that in order to create the cosine graph all we need to do is to shift the sine graph 90° to the left[2]. The cosine graph is shown in figure 7.2.

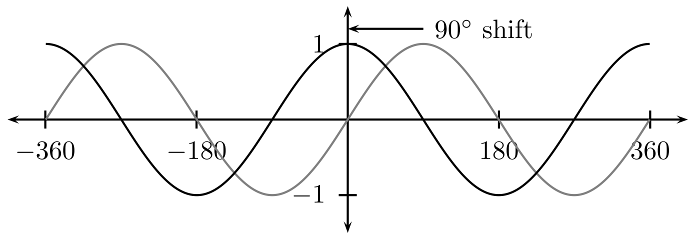


Figure 7.2: The cosine graph (in black) with the sine graph (in gray).

Tangent graph

Now that we have the sine and cosine graphs there is an easy way to visualise the tangent graph. Let us look back at our definitions of $\sin\theta$ and $\cos\theta$ in a right angled triangle.

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$$

This is the first of an important set of equations called trigonometric identities. An identity is an equation which holds true for any value which is put into it. In this case we have shown that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

for any value of θ.
So we know that for values of θ for which $\sin\theta = 0$, we must also have $\tan\theta = 0$. Also, if $\cos\theta = 0$ our value of $\tan\theta = 0$ is undefined as we cannot divide by 0. The complete graph[3] is shown in figure 7.3.

[2]You may have noticed that the transformation we are using is in fact a translation of 90° followed by a reflection in the y axis due to a negative sign in front of the θ. However, because cosine is an even function (*i*.e. symmetric about the y axis) this reflection doesn't really matter!

[3]The dotted lines in the tangent graph are known as asymptotes and the graph is said to display asymptotic behaviour. This means that as θ approaches 90°, $\tan\theta$ approaches infinity. In other words, there is no defined value of the function at the asymptote values. Another graph which displays asymptotic behaviour is $y = \frac{1}{x}$ whose asymptotes are the x and y axes themselves.

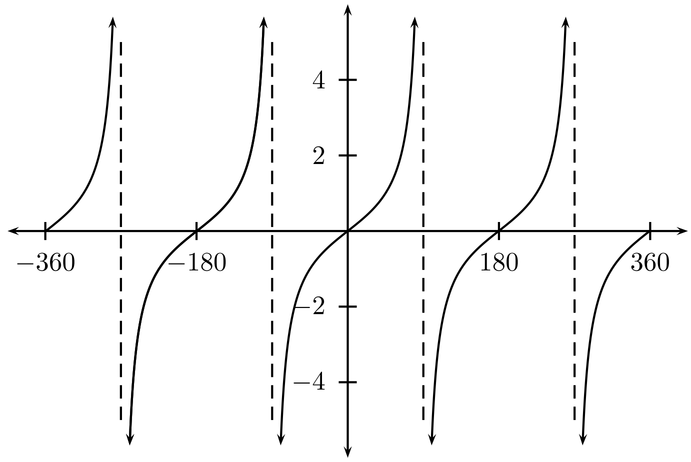


Figure 7.3: The tangent graph.

7.3.3 Secant, Cosecant, Cotangent and their graphs

In the sections that follow it will often be useful to define the reciprocal functions of sine, cosine and tangent. We shall define them as follows–

$$\csc\theta = \frac{1}{\sin\theta} \quad = \quad \frac{\text{hypotenuse}}{\text{opposite}} \tag{7.8}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad = \quad \frac{\text{hypotenuse}}{\text{adjacent}} \tag{7.9}$$

$$\cot\theta = \frac{1}{\tan\theta} \quad = \quad \frac{\text{adjacent}}{\text{opposite}} \tag{7.10}$$

The graphs of these functions are shown in figures 7.4–7.6. There are a number of points worth noting about these graphs. Firstly, since $|\sin\theta|$ and $|\cos\theta|$ are always less than or equal to 1 their reciprocal functions $|\csc\theta|$ and $|\sec\theta|$ must always be greater than or equal to 1. Secondly notice that the sectant graph can be obtained from the cosecant graph by performing a 90° shift, just like we did with sine and cosine. Notice also that these graphs have asymptotes whenever their reciprocal function is 0.

One important feature of all these trigonometric functions is that they are periodic with a period of 360°. This is most easily understood by looking back at a circle.

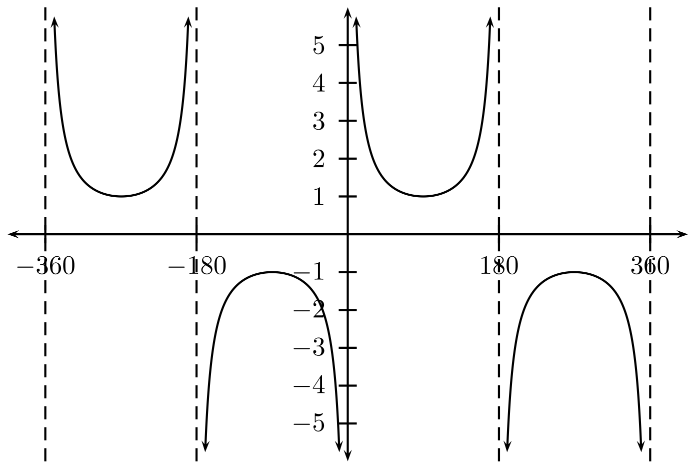


Figure 7.4: The cosecant graph.

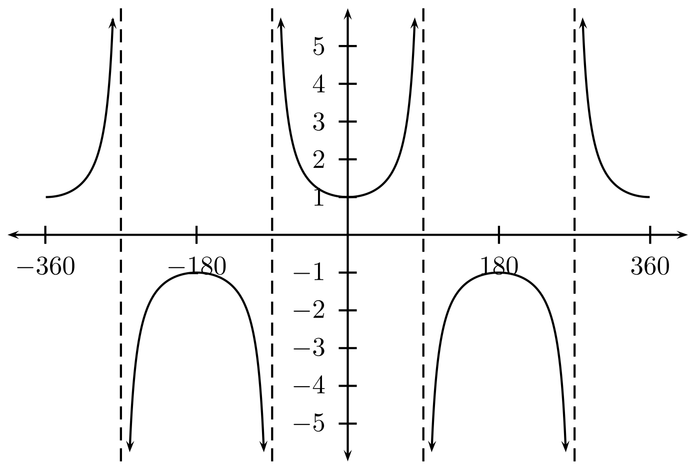


Figure 7.5: The sectant graph.

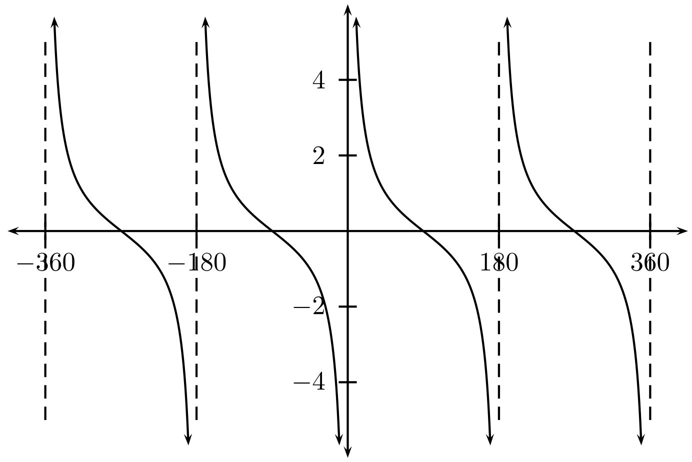
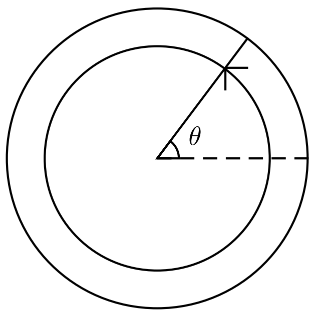


Figure 7.6: The cotangent graph.



Imagine that we are measuring the angle θ on this circle. Now let us add $360°$ to our angle so that our line sweeps all the way around the circle and ends up back where it started as indicated in the diagram. There is no way of knowing whether we have swept around the circle in this way as everything ends up exactly where it started. In other words, if we add $360°$ to an angle we effectively have the same angle we started with. Since our diagram is the same after the rotation the values of our trigonometric functions also remain unchanged. This is the reason that all trigonometric functions have a period of $360°$ – adding $360°$ degrees to an angle does nothing more than sweep it all the way round a circle back to where it began, so all of our functions must have the same value for θ and $\theta \pm 360°$.

7.3.4 Inverse trigonometric functions

Like all functions the trigonometric functions have inverses. These functions take in ratios (such as $\frac{\text{opposite}}{\text{hypotenuse}}$ in the case of inverse sine) and give out the angle it corresponds to. However, due to the periodicity of the

trigonometric functions there are many possible angles for one ratio. For example, both $\sin 30°$ and $\sin 150°$ give values of $\frac{1}{2}$ so $\sin^{-1}\frac{1}{2}$ can be either 30°, 150° or any one of an infinite amount of other possibilities.
Two notations are commonly used for the inverse functions – $\sin^{-1}\theta$ and $\arcsin\theta$. Both the $^{-1}$ and arc versions can be used for any of the six trigonometric functions. However, you must be careful not to confuse the reciprocal functions (csc, sec and cot) with the inverse functions (arcsin, arccos and arctan). They are different functions with different meanings and will give different answers[4].

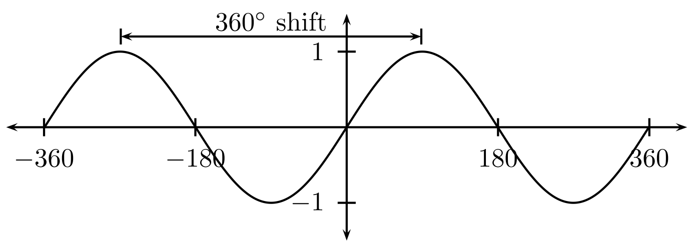
7.4 Trigonometric Rules and Identities

7.4.1 Translation and Reflection

We found earlier that all trigonometric functions are periodic, with a period of 360°. We can express this more formally by writing–

$$\sin(\theta \pm 360°) = \sin\theta$$

This identity states that the sine of an angle is unchanged if we add or subtract 360°. Another way to think of this is as a translation of the sine graph by 360° to the right or left.



As you can see, if we shift the whole graph by 360° left or right it will end up back on top of itself. The sine of an angle is therefore completely unchanged. Identities of this form can be very useful. We shall consider a few such identities here using the ideas of chapter (NOTE: Add in correct \ref for transformations in geometry chapter).
First let us consider reflecting the sin graph in the x and y axes. We know that if we reflect in the x axis we will get the graph of $-\sin\theta$. Figure 7.7 shows the original sine graph (gray) and its reflection in the x axis. Just by looking at the graph we can see that reflecting sine in the y axis would give the same result as reflecting in the x axis did[5]. Mathematically, a

[4]Remember – you can have inverse reciprocal functions such as arccsc and $\sec^{-1}$.
[5]If you have a mirror you can check this by putting it along the y axis.

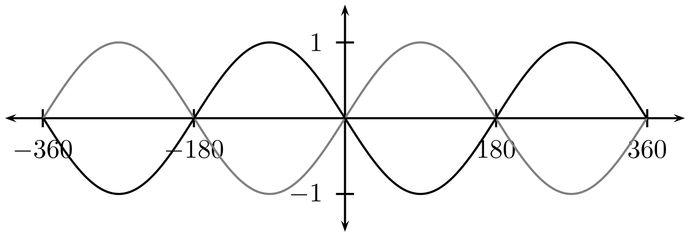


Figure 7.7: The sine graph reflected in the x axis.

reflection in the y axis gives us the function $\sin(-\theta)$. This gives us a second identity–

$$\sin(-\theta) = -\sin\theta$$

We could also obtain the black function in figure 7.7 by translating the gray sine graph 180° to the right or left. This tells us that–

$$\sin(-\theta) = -\sin\theta = sin(\theta \pm 180^\circ)$$

Let us now look at the cosine function. Again we can use the fact that cosine is periodic with period 360° to give–

$$\cos(\theta \pm 360^\circ) = \cos\theta$$

This time our reflections are a little more complicated. Firstly let's reflect the cosine function in the y axis to generate $\cos(-\theta)$. Since the cosine graph is symmetric about the y axis the reflection will not change our graph. This tells us that –

$$\cos(-\theta) = \cos\theta$$

Reflecting in the x axis we obtain figure 7.8. The first point of interest

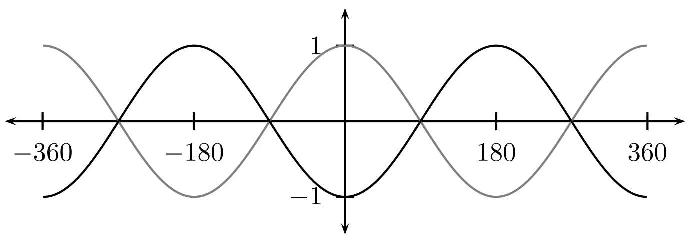


Figure 7.8: The cosine graph reflected in the x axis.

is that unlike with sine we get different graphs by reflecting in the x and

y axes. We can however still consider the black graph as a translation of 180° to the left or right. As before this gives us an identity–

$$-\cos\theta = cos(\theta \pm 180°)$$

One final and very important class of translation identity are those that convert sine into cosine and vice versa. We have already seen one of these when we looked at the graphs of sine and cosine–

$$\cos\theta = \sin(90° - \theta)$$

This can be written in many ways by using the identities we have already proven. One more sensible version is–

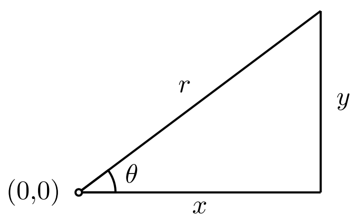
$$\cos(\theta - 90°) = \sin\theta$$

It is now easier to see that the sine graph is just the cosine graph moved 90° to the right. We shall prove that these two identities are the same in the worked example at the end of this section.

There are many more identities such as these for sine and cosine as well as tangent and the reciprocal functions. One way to find such identities is to uses the addition and subtraction formulae which we will derive in section 7.4.6. Always remember to check if your expressions can be simplified using these identities.

7.4.2 Pythagorean Identities

Consider a right angled triangle–



Let us use our definitions of sine and cosine in a right–angled triangle on the angle θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \tag{7.11}$$

$$\tag{7.12}$$

$$\Rightarrow y = r\sin\theta \quad \text{and} \quad x = r\cos\theta \tag{7.13}$$

Using Pythagorases theorem we can say that–

$$x^2 + y^2 = r^2$$

Substituting in our expressions for x and y–

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

If we divide both sides through by r^2 we arrive at–

$$\cos^2\theta + \sin^2\theta = 1$$

This is known as a Pythagorean identity. We can express the identity in terms of the other trigonometric functions by dividing through by either $\sin^2\theta$ or $\cos^2\theta$ on both sides–

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad \Rightarrow \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \Rightarrow \qquad \tan^2\theta + 1 = \sec^2\theta$$

These 3 versions of the Pythagorean Identity can be used in any question. They do not need to be used in a triangle they will work for any angle in any situation.

Definition: Pythagorean Identities–

$$\cos^2\theta + \sin^2\theta = 1$$

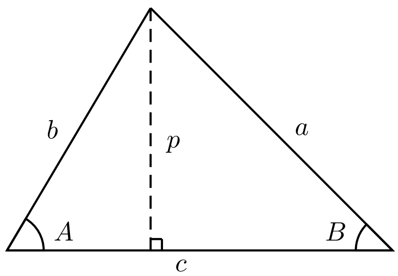
$$1 + \cot^2\theta = \csc^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

7.4.3 Sine Rule

So far we have only dealt with the trigonometry of right angled triangles where we are able to use our definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$. There are some rules which we can derive that hold true for any triangle. One of these is the sine rule.

Consider a scalene triangle (*i*.e. one with all sides different lengths and all angles different)–



Note that the order in which the angles of the triangle are labelled is important. The three sides are labelled a, b, and c in any order. Then we define angle A as the angle opposite side a and so on.
We can split it into two right-angled triangles by choosing a perpendicular, p, to one side (in this case side c) which passes through the opposite vertex, as shown. You may want to draw a few triangles and convince yourself that we can always do this regardless of the shape of the triangle.
Now, since we have right angled triangles we can use our old definition of $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ to find–

$$\sin A = \frac{p}{b} \quad \text{and} \quad \sin B = \frac{p}{a}$$

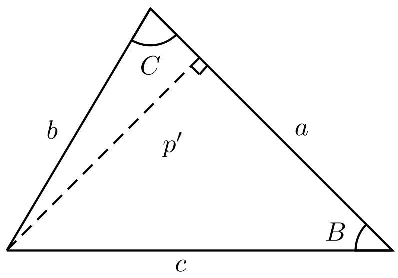
or, rearranging slightly–

$$p = b\sin A \quad \text{and} \quad p = a\sin B$$

If we now set these two equations for p equal to each other (to eliminate the p) and rearrange again (by dividing through both sides by ab) we get the following–

$$\begin{aligned} p = b\sin A &= a\sin B \\ \Rightarrow \frac{\sin A}{a} &= \frac{\sin B}{b} \end{aligned} \tag{7.14}$$

We could have chosen our perpendicular to go through any side of our triangle so let us repeat the proof with the perpendicular through a.



Now we have–

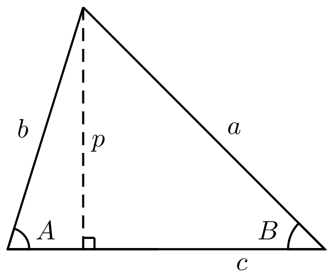
$$\begin{aligned} p' = b\sin C &= c\sin B \\ \Rightarrow \frac{\sin B}{b} &= \frac{\sin C}{c} \end{aligned} \tag{7.15}$$

Check this result for yourself to make sure you understand where it came from!
Putting equations 7.14 and 7.15 together we obtain the sine rule–

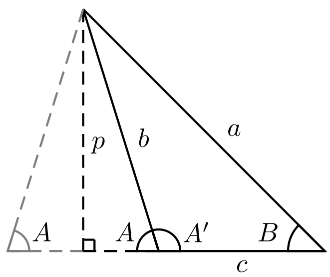
Definition: Sine Rule –

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Many books now tell us that we have not proved the sine rule fully. Consider the following triangle–



This type of triangle which has all of its angles smaller than 90° is called an acute triangle. We have already proved the sine rule for acute triangles. If we 'fold' this acute triangle along the perpendicular we obtain the following–



This kind of triangle, which contains an angle greater than 90°, is called an obtuse triangle. Notice that we now have no way of finding a perpendicular through side c which goes through a vertex of the new triangle. Instead, the perpendicular must be drawn outside of the triangle. This is the reason that some books say that our proof is incomplete – the perpendicular which we used for acute triangles does not always exist in obtuse ones! We can either repeat our proof from the start for obtuse triangles(which is messy!) or we can use our knowledge of trigonometry to prove these other books wrong.
Notice that the lengths of a, b and p have not changed, neither has the value of angle B. The only change is that we have replaced the old angle, A, with a new angle, A'. If we look at A and A' we can see that they lie on a straight line so from our knowledge of geometry we can say that $A + A' = 180°$ or alternatively that $A' = 180° - A$. We have already shown that–

$$\sin(180 - \theta) = \sin\theta$$

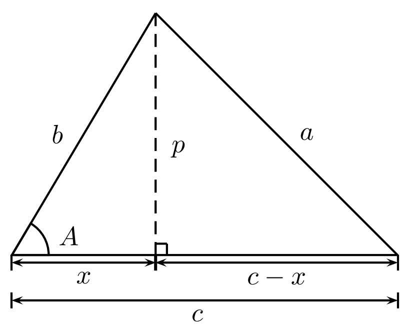
so we can say that–

$$\sin A' = \sin(180 - A) = \sin A$$

We now know that although angle A has changed, its sine has stayed the same. Since we are only interested in the sine of the angle our equation must still be valid! Any obtuse triangle can be formed by 'folding' an acute triangle in this way so the sine rule must be true for any triangle we choose.

7.4.4 Cosine Rule

Let us return to our scalene triangle–



We have defined a perpendicular, p, just as we did before. This perpendicular divides side c into two parts. One part lies between vertex A and the perpendicular. We shall call its length x. The other part between the perpendicular and vertex B must, therefore, have a length of $c - x$. Using the usual definitions of sine and cosine on the left had section of the triangle we find–

$$\sin A = \frac{p}{b} \quad \text{and} \quad \cos A = \frac{x}{b}$$
$$\Rightarrow p = b \sin A \quad \text{and} \quad x = b \cos A$$

Now we will use Pythagorases theorem on the right hand side of the triangle.

$$a^2 = p^2 + (c - x)^2 \tag{7.16}$$
$$= p^2 + c^2 - 2cx + x^2 \tag{7.17}$$

Substituting in $p = b \sin A$ and $x = b \cos A$

$$a^2 = b^2 \sin^2 A + c^2 - 2cb \cos A + b^2 \cos^2 A \tag{7.18}$$
$$= b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A \tag{7.19}$$

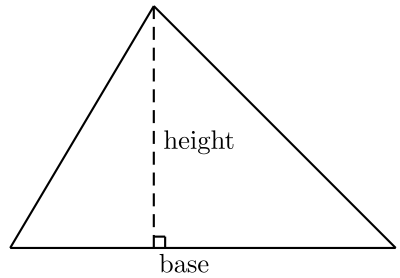
Using the Pythagorean identity $\sin^2 A + \cos^2 A = 1$ we get the cosine rule.

Definition: Cosine rule –

$$a^2 = b^2 + c^2 - 2bc\cos A$$

7.4.5 Area Rule

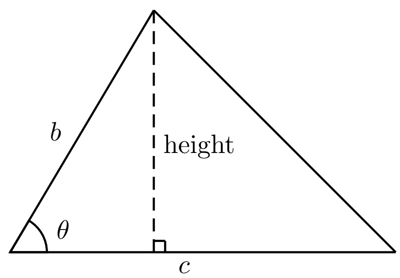
One last triangle rule is an extension of the area formula for a triangle.



From geometry–

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}$$

where height is measured perpendicular to the base.



From trigonometry–

$$\sin\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{height}}{b}$$

$$\Rightarrow \text{height} = b\sin\theta$$

Calling the base c we can write–

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}bc\sin\theta$$

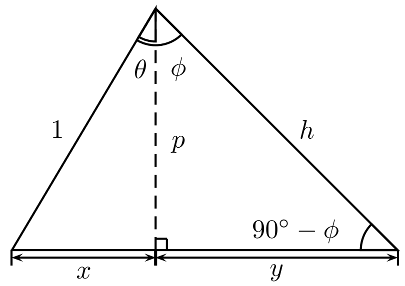
where θ must be the angle between sides b and c.

Definition: Area rule –

$$\text{area} = \frac{1}{2}bc\sin\theta$$

7.4.6 Addition and Subtraction Formulae

Let us return to our scalene triangle complete with a perpendicular as we had before.



The perpendicular, p, divides the top angle into two pieces, θ and ϕ. We can use the fact that the internal angles of a triangle sum to 180° to find that the bottom right angle must be $90° - \phi$. If we define one side of the triangle to be of legnth 1 for simplicity we can easily see from the left hand triangle that–

$$\sin\theta = \frac{x}{1} = x \quad \Rightarrow \quad x = \sin\theta \tag{7.20}$$

$$\cos\theta = \frac{p}{1} = p \quad \Rightarrow \quad p = \cos\theta \tag{7.21}$$

From the right hand triangle we can see that–

$$\cos\phi = \frac{p}{h} \quad \Rightarrow \quad h = \frac{p}{\cos\phi} \tag{7.22}$$

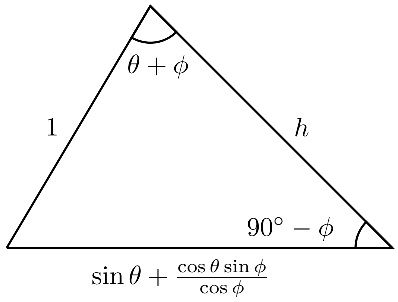
$$\sin\theta = \frac{y}{h} \quad \Rightarrow \quad y = h\sin\phi = \frac{p\sin\phi}{\cos\phi} \tag{7.23}$$

Substituting in our expression for p we get–

$$y = \frac{\cos\theta\sin\phi}{\cos\phi}$$

Putting these expressions togeher we find that the base of the triangle, $x + y$, has a length–

$$x + y = \sin\theta + \frac{\cos\theta\sin\phi}{\cos\phi}$$



We can now use the sine rule on the two angles that we know–

$$\frac{\sin(\theta+\phi)}{\sin\theta + \frac{\cos\theta\sin\phi}{\cos\phi}} = \frac{\sin(90^\circ - \phi)}{1}$$

As we learnt in section 7.4.1, $\sin(90^\circ - \theta) = \cos\theta$. Using this identity and cross multiplying we get–

$$\frac{\sin(\theta+\phi)}{\sin\theta + \frac{\cos\theta\sin\phi}{\cos\phi}} = \frac{\cos\phi}{1} \tag{7.24}$$

$$\Rightarrow \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{7.25}$$

This is the sine addition formula. If we take this formula and replace θ with $(90^\circ - \theta)$ (remember, we can only do this with identities as they must be true for any value of their variables) we get–

$$\sin(90^\circ - \theta + \phi) = \sin(90^\circ - \theta)\cos\phi + \cos(90^\circ - \theta)\sin\phi$$

As before we can use the identities

$$\sin(90^\circ - \phi) = \cos\phi \tag{7.26}$$

$$\cos(90^\circ - \phi) = \sin\phi \tag{7.27}$$

Using these identities we can obtain the cosine subtraction formula–

$$\begin{aligned} \sin(90^\circ - (\theta - \phi)) = \cos(\theta-\phi) &= \sin(90^\circ - \theta)\cos\phi + \cos(90^\circ - \theta)\sin\phi && (7.28) \\ &= \cos\theta\cos\phi + \sin\theta\sin\phi && (7.29) \end{aligned}$$

We are missing two identities, the sine subtraction and cosine addition formulae. To obtain the sine subtraction we replace ϕ with $-\phi$ in the sine addition formula. Since we know that $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = \cos\phi$ from section 7.4.1 we find that

$$\begin{aligned} \sin(\theta - \phi) &= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi) && (7.30) \\ &= \sin\theta\cos\phi - \cos\theta\sin\phi && (7.31) \end{aligned}$$

Proof of the cosine addition formula can be done in the same way, starting with the subtraction formula.

$$\begin{aligned} \cos(\theta + \phi) &= \cos\theta\cos(-\phi) + \sin\theta\sin(-\phi) && (7.32) \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi && (7.33) \end{aligned}$$

Similar equations can be found for the tangemt function. Proof of these is left to the reader[6].

Definition: Addition and Subtraction Formulae–

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{7.34}$$
$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \tag{7.35}$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{7.36}$$
$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \tag{7.37}$$
$$\tan(\theta+\phi) = \frac{\tan\phi+\tan\theta}{1-\tan\theta\tan\phi} \tag{7.38}$$
$$\tan(\theta-\phi) = \frac{\tan\phi-\tan\theta}{1+\tan\theta\tan\phi} \tag{7.39}$$

7.4.7 Double and Triple Angle Formulae

Let us remind ourselves of the addition formulae for sine and cosine–

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{7.40}$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{7.41}$$

If we set ϕ to be equal to θ we get the following equations–

$$\cos(\theta+\theta) = \cos(2\theta) = \cos\theta\cos\theta - sin\theta\sin\theta = \cos^2\theta - \sin^2\theta \tag{7.42}$$
$$\sin(\theta+\theta) = \sin(2\theta) = \sin\theta\cos\theta + cos\theta\sin\theta = 2\cos\theta\sin\theta \tag{7.43}$$

These are known as the double angle formulae. By using the phythagorean identity $\cos^2\theta + \sin^2\theta = 1$ we can substitute into the cosine double angle formula for $\cos^2\theta$ or $\sin^2\theta$ to get different forms. We can do the same for the tangent function.

Definition: Double angle formulae–

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{7.44}$$
$$= 2\cos^2\theta - 1 \tag{7.45}$$
$$= 1 - 2\sin^2\theta \tag{7.46}$$
$$\sin(2\theta) = 2\cos\theta\sin\theta \tag{7.47}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{7.48}$$

[6] HINT–Remember, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ so you can use the sine and cosine addition formulae to find the tangent one. To get it into the same form as in the definition you need to divide everything through by a factor. The fact there is a 1 in the denominator should give you a clue as to what the factor is! Once you have the addition forumula you need to remember that $\tan(-\theta) = -\tan\theta$ to find the subtraction formula.

Now that we have the double angle formulae it is easy to find higher order multiple angle formulae. We shall derive the cosine triple angle formulae here. We start by taking the cosine addition faormula and setting $\phi = 2\theta$–

$$\cos(\theta + 2\theta) = \cos(3\theta) = \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta)$$

We now substitute in the double angle formulae for $\cos(2\theta)$ and $\sin(2\theta)$. We have a choice of forms for the $\cos(2\theta)$ formula. We shall choose $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

$$\begin{aligned} \cos(3\theta) &= \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta) && (7.49)\\ &= \cos\theta(\cos^2\theta - \sin^2\theta) - \sin\theta(2\cos\theta\sin\theta) && (7.50)\\ &= \cos^3\theta - 3\sin^2\theta\cos\theta && (7.51) \end{aligned}$$

The corresponding sine triple angle formula is–

$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$$

7.4.8 Half Angle Formulae

We can rearrange the double angle formulae to find the half angle formulae. We shall start by rearranging the cosine double angle formula of the form $\cos(2\theta) = 2\cos^2\theta - 1$.

$$\begin{aligned} 2\cos^2\theta - 1 &= \cos(2\theta) && (7.52)\\ \Rightarrow 2\cos^2\theta &= 1 + \cos(2\theta) && (7.53)\\ \Rightarrow \cos^2\theta &= \frac{1 + \cos(2\theta)}{2} && (7.54)\\ \Rightarrow \cos\theta &= \pm\sqrt{\frac{1 + \cos(2\theta)}{2}} && (7.55) \end{aligned}$$

Another way to write this is to halve both of the angles (we can do this because it is an identity, so must be valid for <u>any</u> angle) –

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

Using the same method to rearrange the identity $\cos(2\theta) = 1 - 2\sin^2\theta$ we obtain–

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

We can now use the ratio identity to find the tangent half angle formula–

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \pm\frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$$

As with all identities the half angle formulae can be expressed in a number of ways. Some of these will be proven in the worked example and more given in the summary of identities at the end of the chapter.

Definition: Half Angle Formulae

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{7.56}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{7.57}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \tag{7.58}$$

7.4.9 'Product to Sum' and 'Sum to Product' Identities

For completeness we include a brief comment on the 'product to sum' and 'sum to product' identities. They can be derived from the addition and subtraction formulae. We shall only derive one of each type here as their derivations are broadly similar.
We start by proving a product to sum identity. This identity us an experssion linking the product of two cosine functions ($\cos\theta\cos\phi$) to a sum of cosine functions. The derivation is as follows–

$$\begin{aligned}
\cos(\theta+\phi)+\cos(\theta-\phi) &= (\cos\theta\cos\phi-\sin\theta\sin\phi)+(\cos\theta\cos\phi+\sin\theta\sin\phi) && (7.59)\\
&= 2\cos\theta\cos\phi && (7.60)\\
& && (7.61)\\
\Rightarrow \cos\theta\cos\phi &= \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)] && (7.62)
\end{aligned}$$

Sum to product identities are messier to prove. Here we prove the identity linking the sum of two cosines by exchange of variables. We substitute $\theta = \theta'+\phi'$ and $\phi = \theta'-\phi'$ into the product to sum identity above (the primes just prevent us getting confused, we shall drop them later).

$$\begin{aligned}
2\cos(\theta'+\phi')\cos(\theta'-\phi') &= \cos((\theta'+\phi')+(\theta'-\phi'))+\cos((\theta'+\phi')-(\theta'-\phi')) && (7.63)\\
&= \cos(2\theta')+\cos(2\phi') && (7.64)
\end{aligned}$$

As always with identities we can divide all our variables by 2 for convenience (since it must be true for <u>any</u> angle) and drop out the primes to give–

$$\cos\theta+\cos\phi = 2\cos\left(\frac{\theta'+\phi'}{2}\right)\cos\left(\frac{\theta'-\phi'}{2}\right)$$

7.4.10 Solving Trigonometric Identities

A standard type of question in an exam is of the form "show that $\frac{\sin(2\theta)}{\tan\theta} = 2\cos^2\theta$". As well as being important in examinations being able to prove

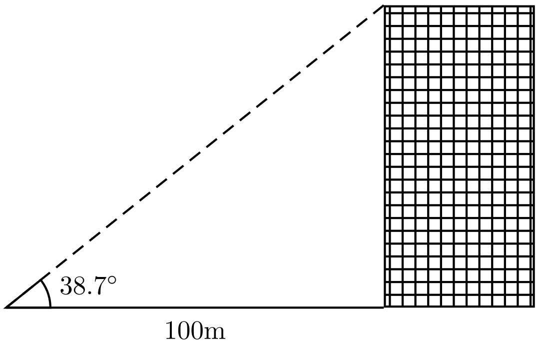


Figure 7.9: Determining the height of a building using trigonometry.

identities is a key mathematical skill. Most identities can be proven by using the standard identities we have already learnt earlier in this section. There are three ways that the two sides of an identity can differ–

1. The <u>functions</u> are different
   e.g. $\sin\theta\cot\theta = \cos\theta$
2. The <u>operations</u> are different
   e.g. $\sin\theta\cos^3\theta = \sin\theta\cos\theta(1-\sin^2\theta)$
3. The <u>angles</u> are different
   e.g. $\sin\theta = \frac{\sin(2\theta)}{2\cos\theta}$

Of course, real identities (and even the examples above) contain a mixture of these three differences, but they can be solved by dealing with each of the differences, one at a time.

7.5 Application of Trigonometry

Trigonometry is very important in many areas of every day life. In this section we shall learn to use trigonometry to solve problems which would otherwise require very complicated solutions.

7.5.1 Height and Depth

One simple task is to find the height of a building using trigonometry. We could just use tape measure lowered from the roof but this is impractical (and dangerous) for tall buildings. It is much more sensible to measure a distance along the ground and use trigonometry to find the height of the building.
Figure 7.9 shows a building whose height we do not know. We have walked 100m away from the building and measured the angle up to top. This angle is found to be 38.7°. We call this angle the angle of elevation. As you can see from figure 7.9 we now have a right angled triangle, one side of which

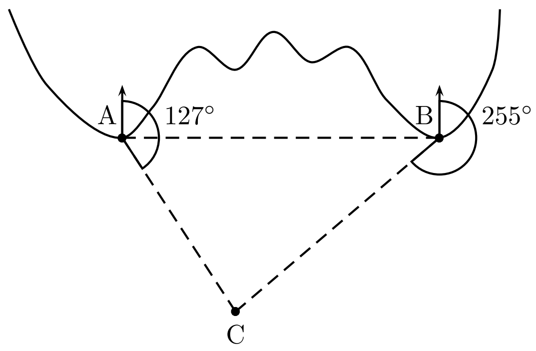


Figure 7.10: Two lighthouses, A and B, and a boat, C.

is the height of the building, which also includes our 100m distance and the angle of elevation. Using the standard definition of tangent–

$$\begin{aligned}
\tan 38.7^\circ &= \frac{\text{opposite}}{\text{adjacent}} && (7.65)\\
&= \frac{\text{height}}{100} && (7.66)\\
\Rightarrow \text{height} &= 100 \times \tan 38.7^\circ && (7.67)\\
&= 80\text{m} && (7.68)
\end{aligned}$$

7.5.2 Maps and Plans

Maps and plans are usually scale drawings. This means that they are an enlagement (usually with a negative scale factor so that they are smaller than the original) so all angles are unchanged. We can use this to make use of maps and plans by adding information from the real world.
Let us imagine that there is a coastline with two lighthouses, one either side of a beach. This is shown in figure 7.10. The two lighthouses are 0.67km apart and one is exactly due east of the other. Let us suppose that no boat may get closer that 200m from the lighthouses in case it runs aground. How can the lighthouses tell how close the boat is?
Both lighhouses take bearings to the boat (remember – a bearing is an angle measured clockwise from north). These bearings are shown on the map in figure 7.10. We can see that the two lighthouses and the boat form a triangle. Since we know the distance between the lighthouses and we have two angles we can use trigonometry to find the remaining two sides of the triangle, the distance of the boat from the two lighthouses.
Figure 7.11 shows this triangle more clearly. We need to know the legnths of the two sides $\overline{AC}$ and $\overline{BC}$. We can choose to use either sine or cosine rule to find our missing legnths. We shall use both here. Using the sine rule–

$$sin \qquad (7.69)$$

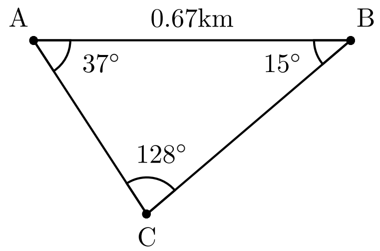


Figure 7.11: Two lighthouses, A and B, and a boat, C.

7.6 Trigonometric Equations

Trigonomeric equations often look very simple. Consider solving the equation $\sin\theta = 0.7$. We can take the inverse sine of both sides to find that $\theta = \sin^{-1}(0.7)$. If we put this into a calculator we find that $\sin^{-1}(0.7) = 44.42°$. This is true, however, it does not tell the whole story. As you

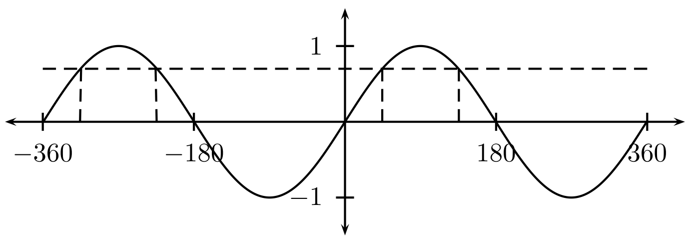


Figure 7.12: The sine graph. The dotted line represents $\sin\theta = 0.7$.

can see from figure 7.12, there are four possible angles with a sine of 0.7 between $-360°$ and $360°$. If we were to extend the range of the sine graph to infinity we would in fact see that there are an infinite number of solutions to this equation! This difficulty (which is caused by the periodicity of the sine function) makes solving trigonometric equations much harder than they may seem to be.

Any problem on trigonometric equations will require two pieces of information to solve. The first is the equation itself and the second is the range in which your answers must lie. The hard part is making sure you find all of the possible answers within the range. Your calculator will always give you the smallest answer (*i.*e. the one that lies between $-90°$ and $90°$ for tangent and sine and one between $0°$ and $180°$ for cosine). Bearing this in mind we can already solve trigonometric equations within these ranges.

Worked Example 7 :

Question: Find the values of x for which $\sin 3x = 0.5$ if it is given that $0 < x < 90°$.
Answer: Because we are told that x is an acute angle, we can simply apply an inverse trigonometric function to both sides.

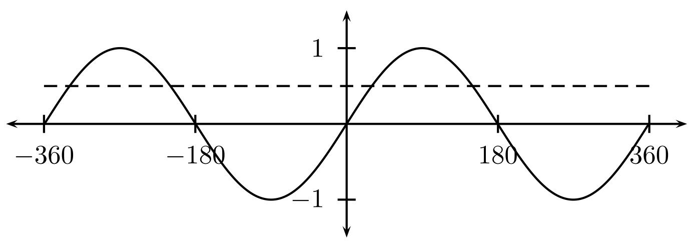
$$\begin{aligned} \sin x &= 0.5 && (7.70)\\ \Rightarrow x &= \arcsin 0.5 && (7.71)\\ \Rightarrow x &= 30° && (7.72)\\ & && (7.73) \end{aligned}$$

We can, of course, solve trigonometric equations in any range by drawing the graph.

Worked Example 8 :

Question: For what values of x does $\sin x = 0.5$, when $-360° < x < 360°$?
Answer:

Step 1 : Draw the graph
We take look at the graph of $\sin x = 0.5$ on the interval [-360, 360]. We want to know when the y value of the graph is 0.5, so we draw in a line at y=0.5.



Step 2 :
Notice that this line touches the graph four times. This means that there are four solutions to the equation.
Step 3 :
Read off those x values from the graph as $x = -330°$,$-210°$,$30°$ and $150°$.

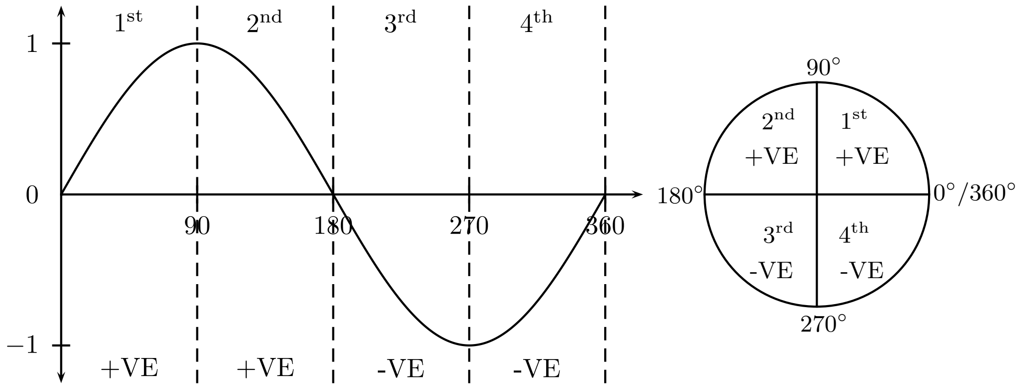
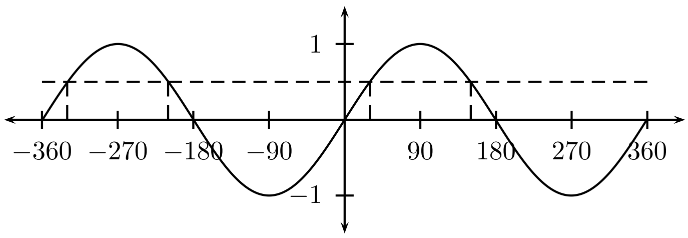


Figure 7.13: The graph and unit circle showing the sign of the sine function.



This method can be time consuming and inexact. We shall now look at how to solve these problems algebraically.

7.6.1 Solution using CAST diagrams

The Sign of the Trigonometric Function

The first step to finding the trigonometry of any angle is to determine the sign of the ratio for a given angle. We shall do this for the sine function first.

In figure 7.13 we have split the sine graph unto four quadrants, each 90° wide. We call them quadrants because they correspond to the four quadrants of the unit circle. We notice from figure 7.13 that the sine graph is positive in the 1^{st} and 2^{nd} quadrants and negative in the 3^{rd} and 4^{th}.
Figure 7.14 shows similar graphs for cosine and tangent. All of this can be summed up in two ways. Table 7.1 shows which trrigonometric fuctions are positive and which are negative in each quadrant. A more convenient way of writing this is to note that all fuctions are positive in the 1^{st} quadrant, only sine is positive in the 2^{nd}, only tangent in the 3^{rd} and only

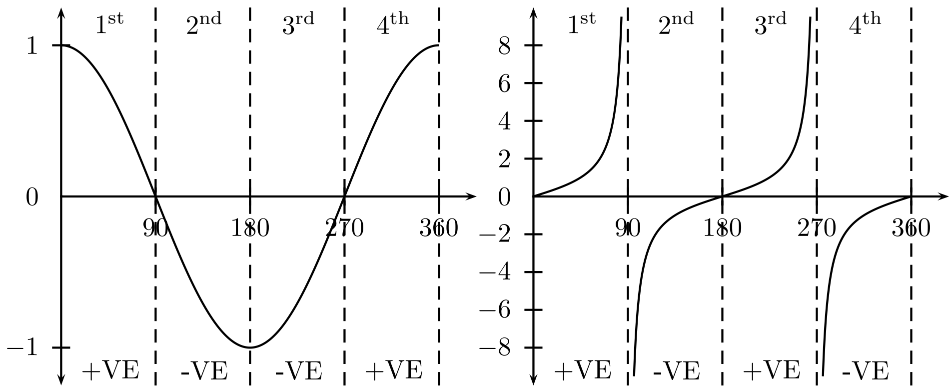
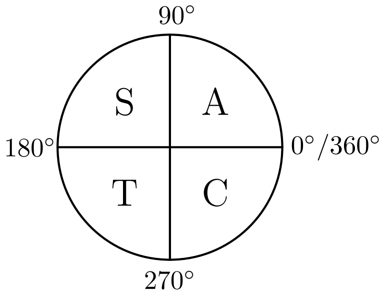


Figure 7.14: Graphs showing the sign of the cosine and tangent functions.

	1st	2nd	3rd	4th
sin	+VE	+VE	-VE	-VE
cos	+VE	-VE	-VE	+VE
tan	+VE	-VE	+VE	-VE

Table 7.1: The signs of the three basic trigonometric functions in each quadrant.

cosine in the 4^{th}. We express this using the CAST digram (figure 7.15). This diagram is known as a CAST diagram as the letter, taken anticlock-



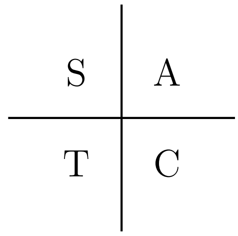


Figure 7.15: The two forms of the CAST diagram.

wise from the bottom right, read C-A-S-T. The letter in each quadrant tells us which trigonometric functions are positive in that quadrant. The 'A' in the 1^{st} quadrant stands for all (meaning sine, cosine and tangent are all positive in this quadrant). 'S', 'C' and 'T' ,of course, stand for sine, cosine and tangent.
The diagram is shown in two forms. The version on the left shows the CAST diagram including the unit circle. This version is useful for equations which lie in large or negative ranges. The simpler version on the right is useful for ranges between 0° and 360°.

Magnitude of the trigonometric functions

Now that we know which quadrants our solutions lie in we need to know which angles in these quadrants satisfy our equation.
Calculators give us the smallest possible answer (sometimes negative) which satisfies the equation. For example, if we wish to solve $\sin\theta = 0.3$ we can apply the inverse sine function to both sides of the equation to find–

$$\begin{aligned}\theta &= \arcsin 0.3\\ &= 17.46^\circ\end{aligned}$$

However, we know that this is just one of infinitely many possible answers. We get the rest of the answers by finding relationships between this small angle, θ, and answers in other quadrants.
To do this we need to condider the modulus[7] of the sine graph.
As you can see in figure 7.16 there is a solution to the equation $|\sin\theta| = 0.3$ in every quadrant. The 1^{st} quadrant solution is of course 17.46° as our calculator told us. The 2^{nd} quadrant solution can be seen to be $180^\circ - \theta$. Another way to see this is to look at the identity

$$\sin\theta = \sin(180^\circ - \theta)$$

[7]This means we plot only the magnitude of the function. This is the same as reflecting negative sections in the x axis.

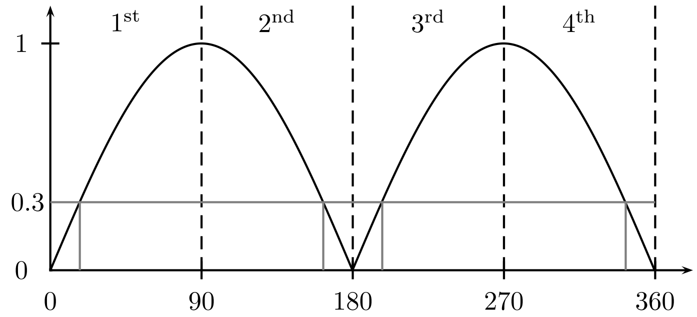


Figure 7.16: The modulus of the sine graph.

proved in section 7.4. Using the same logic the 3$^{\text{rd}}$ quadrant solution can be seen to be $(180° + \theta)$ and the 4$^{\text{th}}$ quadrant solution $(360° - \theta)$. It is now left to the reader to show, using similar graphs for cosine and tangent, that these relationships are true for all three of the trigonometric functions.
These rules can be expressed in a simpler way. If we define the solution lying between 0° and 90° as ϕ–

Definition:

- If we are in the 1$^{\text{st}}$ or 3$^{\text{rd}}$ quadrants our solution is the lower boundary of the quadrant plus ϕ.
- If we are in the 2$^{\text{nd}}$ or 4$^{\text{th}}$ quadrants our solution is the upper boundary of the quadrant minus ϕ.

7.6.2 Solution Using Periodicity

Up until now we have only solved trigonometric equations where the argument (the bit after the function, e.g. the θ in $\cos\theta$ or the $(2x - 7)$ in $\tan(2x - 7)$) has been θ. If there is anything more complicated than this we need to be a little more careful.
Let us try to solve $\tan(2x) = 2.5$ in the range $0° \leq x \leq 360°$. We want solutions for positive tangent so using our CAST diagram we know to look in the 1$^{\text{st}}$ and 3$^{\text{rd}}$ quadrants. Our calulator tells us that $\arctan(2.5) = 68.2°$. This is our first quadrant solution for $2x$. Our 3$^{\text{rd}}$ quadrant lies between 180° and 270° so our solution is $180° + 68.2°$. Putting this together–

$$\begin{aligned} 2x &= 68.2° \quad \text{or} \quad 248.2° \\ \Rightarrow x &= 34.1° \quad \text{or} \quad 124.1° \end{aligned}$$

Notice that we did not divide by the 2 until we had found our answers. Now try to put $x = 214.1$ into the equation. This gives $2x = 428.2$ and we find that $\tan(428.2) = 2.5$! This solution, $x = 214.1$, lies within the range $0° \leq x \leq 360°$ so we should have included it in our answer. Why did we not find this solution before?
The answer is that when we halved our solutions for $2x$ to find x we also halved our range. We looked for solutions for $0° \leq \mathbf{2x} \leq 360°$ so, after halving, our final answer gave us solutions in the range $0° \leq x \leq 180°$. There are two ways of dealing with this. We could redo the problem looking the the range $0° \leq 2x \leq 720°$. This will work but there is a simpler method.
We know that all the trigonometric functions are periodic with a period of 360°. This means we can add (or subtract) a factor of $360n°$ (where n is an integer) our solution to find another equally valid solution. Let us try this with $\tan(2x) = 2.5$. If $n = 0$ we regain our original answers–

$$2x = 68.2° \quad \text{or} \quad 248.2°$$

Adding 360° ($n = 1$) to our solutions for $2x$ we find the next two solutions–

$$\begin{aligned} 2x &= 68.2° + 360° \quad \text{or} \quad 248.2° + 360° \\ \Rightarrow &= 428.2° \quad \text{or} \quad 608.2° \\ \Rightarrow x &= 214.1° \quad \text{or} \quad 304.1° \end{aligned}$$

7.6.3 Linear Triginometric Equations

Just like with regular equation solving without trigonometric functions the equations can become a lot more complicated. You should solve these just like normal equations and once you have a signal trigonometric ratio isolated, then you follow the strategy outlined in the previous section.(ADD AN EXAMPLE HERE)

7.6.4 Quadratic and Higher Order Trigonometric Equations

The simplest quadratic trigonometric equation is of the form–

$$\sin^2 x - 2 = -1.5$$

This type of equation can be easily solved by rearranging to get a more familiar linear equation–

$$\sin^2 x = 0.5 \tag{7.74}$$

$$\Rightarrow \sin x = \pm\sqrt{0.5} \tag{7.75}$$

This gives two linear trigonometric equations. The solutions to either of these equations will satisfy the original quadratic. (ADD AN EXAMPLE HERE)
The next level of complexity comes when you need to solve a trinomial

which contains trig functions. Here you can make you life a lot easier if you use temporary variables. Consider solving–

$$\tan^2(2x+1) + 3\tan(2x+1) + 2 = 0$$

Here you should notice that $\tan(2x+1)$ occurs twice in the equation, hence we let $y = \tan(2x+1)$ and rewrite:

$$y^2 + 3y + 2 = 0$$

That should look rather more familiar so that you can immediately write down the factorised form and the solutions:

$$(y+1)(y+2) = 0$$
$$\Rightarrow y = -1 \quad \text{OR} \quad y = -2$$

Next one just substitutes back for the temporary variable:

$$\tan(2x+1) = -1 \quad \text{or} \quad \tan(2x+1) = -2$$

And then we are left with two linear trigonometric equations. Be careful: sometimes one of the two solutions will be outside the range of the trigonometric function. In that case you need to discard that solution. For example sonsicer the same equation with cosines instead of tangents–

$$\cos^2(2x+1) + 3\cos(2x+1) + 2 = 0$$

Using the same method we find that–

$$\cos(2x+1) = -1 \quad \text{or} \quad \cos(2x+1) = -2$$

The second solution cannot be valid as cosine must lie between -1 and 1. We must, therefore, reject the second equation. Only solutions to the first equation will be valid.

7.6.5 More Complex Trigonometric Equations

Here are two examples on the level of the hardest trig equations you are likely to encounter. They require using everything that you have learnt in this chapter. If you can solve these, you should be able to solve anything! (ADD AN EXAMPLE HERE)

7.7 Summary of the Trigonomertic Rules and Identities

Pythagorean Identities	Reciprocal Identities	Ratio Identities
$\cos^2\theta + \sin^2\theta = 1$	$\csc\theta = \frac{1}{\sin\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$1 + \cot^2\theta = \csc^2\theta$	$\sec\theta = \frac{1}{\cos\theta}$	
$\tan^2\theta + 1 = \sec^2\theta$	$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

Odd/Even Identities	Periodicity Identities	Cofunction Identities
$\sin(-\theta) = -\sin\theta$	$\sin(\theta \pm 360°) = \sin\theta$	$\sin(90° - \theta) = \cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos(\theta \pm 360°) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(\theta \pm 180°) = \tan\theta$	$\tan(90° - \theta) = \cot\theta$
$\cot(-\theta) = -\cot\theta$	$\cot(\theta \pm 180°) = \cot\theta$	$\cot(90° - \theta) = \tan\theta$
$\csc(-\theta) = -\csc\theta$	$\csc(\theta \pm 360°) = \csc\theta$	$\csc(90° - \theta) = \sec\theta$
$\sec(-\theta) = \sec\theta$	$\sec(\theta \pm 360°) = \sec\theta$	$\sec(90° - \theta) = \csc\theta$

Double Angle Identities	Addition/Subtraction Identities	Half Angle Identities
$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$	
	$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$	$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$
$\cos(2\theta) = 2\cos^2\theta - 1$	$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$	
$\cos(2\theta) = 1 - 2\sin^2\theta$		
$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$	$\tan(\theta + \phi) = \frac{\tan\phi+\tan\theta}{1-\tan\theta\tan\phi}$	$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$
	$\tan(\theta - \phi) = \frac{\tan\phi-\tan\theta}{1+\tan\theta\tan\phi}$	

Sine Rule	Area Rule	Cosine Rule
	$\text{Area} = \frac{1}{2}bc\cos A$	$a^2 = b^2 + c^2 - 2bc\cos A$
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	$\text{Area} = \frac{1}{2}ac\cos B$	$b^2 = a^2 + c^2 - 2ac\cos B$
	$\text{Area} = \frac{1}{2}ab\cos C$	$c^2 = a^2 + b^2 - 2ab\cos C$

Product to Sum Identities	Sum to Product Identities
$\cos\theta\cos\phi = \frac{1}{2}\left[\cos(\theta+\phi)+\cos(\theta-\phi)\right]$	$\sin\theta+\sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$
$\sin\theta\sin\phi = \frac{1}{2}\left[\cos(\theta-\phi)-\cos(\theta+\phi)\right]$	$\sin\theta-\sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$
$\sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta+\phi)+\sin(\theta-\phi)\right]$	$\cos\theta+\cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$
	$\cos\theta-\cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$

Chapter 8

Solving Equations

8.1 Linear Equations

`The syllabus requires:`

- `solve linear equations`

8.1.1 Introduction

Let's imagine you have a friend called Joseph. He picked up your test results from the Biology class and now he refuses to tell what you scored, or what he scored! Obviously you are trying everything to get him to tell you, and he decides to tease you and makes you work it out for yourself. He says the following:

"I have 2 marks more than you and the sum of both our marks is equal to 14. How much did we get?"

Now if the numbers are simple like in the example, you might be able to work it out in your head. Can you? But to make it easier, you can use a linear equation!

This is how it works:

We use a placeholder for your amount and that placeholder is x. So:

$You = x$

Then we need a placeholder for Joseph

$Joseph = y$

BUT the trick is that we have some information about Joseph's mark, which is that Joseph has 2 more than you. We need to use that, so how about:

$Joseph = you + 2$
Or
$Joseph = (x + 2)$
Or
$y = (x + 2)$

Now we need to use the last bit of information we have and that is:

$You + Joseph = 14$

Or using placeholders
$x + y = 14$

Or substituting y
$x + (x + 2) = 14$

What we have here is the actual linear equation.

You already know what an equation is but what does linear mean? Linear means the highest power of the unknown variable, usually called x, is one.

8.1.2 Solving Linear equations - the basics

To find out what your test result is we need to now simplify this equation until we only have the x on the one side of the equal sign and a value on the other side. There are a few rules on how to simplify these equations to get a value for x. They can be organized into 3 groups. Once we have worked through them and we are sure about them, then we can attempt to find out what the answer to our problem is. So here they are:

Rule one - Addition or subtraction

You are allowed to subtract or add any amount as long as you do it on both sides of the equal sign:

Example 1:

$$x + 5 = -6 \tag{8.1}$$
$$\Rightarrow \quad x + 5 - 5 = -6 - 5 \tag{8.2}$$
$$\Rightarrow \quad x = -11 \tag{8.3}$$

Example 2:

$$
\begin{aligned}
& x - \frac{1}{2} = 7 && (8.4)\\
\Rightarrow\quad & x - \frac{1}{2} + \frac{1}{2} = 7 + \frac{1}{2} && (8.5)\\
\Rightarrow\quad & x = \frac{15}{2} && (8.6)
\end{aligned}
$$

Rule two - Multiply or divide

The same principle applies for multiplication and division:

Example 1:

$$
\begin{aligned}
& 2x = 9 && (8.7)\\
\Rightarrow\quad & \frac{2x}{2} = \frac{9}{2} && (8.8)\\
\Rightarrow\quad & x = \frac{9}{2} && (8.9)
\end{aligned}
$$

Example 2:

$$
\begin{aligned}
& \frac{x}{4} = 5 && (8.10)\\
\Rightarrow\quad & \frac{x}{4} \times 4 = 5 \times 4 && (8.11)\\
\Rightarrow\quad & x = 20 && (8.12)
\end{aligned}
$$

Rule three - Fractions

If x is multiplied by a fraction we need to divide both sides of the equal sign with that fraction to get x alone. We do that by flipping the fraction around and then multiplying both sides with it

Example 1:

$$
\begin{aligned}
& (\frac{3}{2})x = 7 && (8.13)\\
\Rightarrow\quad & (\frac{3}{2})x(\frac{2}{3}) = 7(\frac{2}{3}) && (8.14)\\
\Rightarrow\quad & x = \frac{14}{3} && (8.15)
\end{aligned}
$$

These are the basic rules to apply when simplifying a linear equation. But most linear equations will require a few combinations of these before x is sitting alone on the one side of the equal sign. That means we might have to make use of the rules above a number of times one after the other. Let's do a few examples where we will use multiple steps to solve the equation:

8.1.3 Solving linear equations - Combining the basics in a few steps

TIP: Start with eliminating the terms without x, that way you avoid having to calculate too many fracti

Example 1:

$$7 + 5x = 62 \tag{8.16}$$
$$\Rightarrow \quad 7 + 5x - 7 = 62 - 7 \tag{8.17}$$
$$\Rightarrow \quad 5x = 55 \tag{8.18}$$
$$\Rightarrow \quad \frac{5x}{5} = \frac{55}{5} \tag{8.19}$$
$$\Rightarrow \quad x = 11 \tag{8.20}$$

Example 2:

$$55 = 5x + \frac{3}{4} \tag{8.21}$$
$$\Rightarrow \quad 55 - \frac{3}{4} = 5x + \frac{3}{4} - \frac{3}{4} \tag{8.22}$$
$$\Rightarrow \quad 54(\frac{1}{4}) = 5x \tag{8.23}$$
$$\Rightarrow \quad \frac{217}{4} = 5x \tag{8.24}$$
$$\Rightarrow \quad \frac{217}{4} \times \frac{1}{5} = 5x \times \frac{1}{5} \tag{8.25}$$

Doing that is the the same as dividing by 5

$$\Rightarrow \frac{217}{20} = x \tag{8.26}$$

TIP: Start by moving all the terms with x to the one side and all the terms without x to the opposite side of the equal sign. Remember we can do that by changing the sign of the term

Example 3:

$$
\begin{aligned}
& 5x = 3x + 45 && (8.27)\\
\Rightarrow\ & 5x - 3x = 45 && (8.28)\\
\Rightarrow\ & 2x = 45 && (8.29)\\
\Rightarrow\ & \frac{2x}{2} = \frac{45}{2} && (8.30)\\
\Rightarrow\ & x = 22\frac{1}{2} && (8.31)
\end{aligned}
$$

Example 4:

$$
\begin{aligned}
& 23x - 12 = 6 + 2x && (8.32)\\
\Rightarrow\ & 23x - 2x - 12 = 6 && (8.33)\\
\Rightarrow\ & 23x - 2x = 6 + 12 && (8.34)\\
\Rightarrow\ & 21x = 18 && (8.35)\\
\Rightarrow\ & x = \frac{21}{18} && (8.36)
\end{aligned}
$$

Example 5:

$$
\begin{aligned}
& 12 - 6x + 34x = 2x - 24 - 64 && (8.37)\\
\Rightarrow\ & -6x + 34x = 2x - 24 - 64 - 12 && (8.38)\\
\Rightarrow\ & -6x + 34x - 2x = -24 - 64 - 12 && (8.39)\\
\Rightarrow\ & 26x = -100 && (8.40)\\
\Rightarrow\ & x = -\frac{100}{26} && (8.41)\\
\Rightarrow\ & x = -\frac{50}{13} && (8.42)
\end{aligned}
$$

We simplified the answer - but this is not necessarily a required step

TIP: If there are parentheses (brackets) in the equation, start by removing them - multiply with the

Example 6:

$$
\begin{aligned}
& -3(3x - 4) = 8 && (8.43)\\
\Rightarrow\ & -9x + 12 = 8 && (8.44)\\
\Rightarrow\ & -9x = 8 - 12 && (8.45)\\
\Rightarrow\ & -9x = -4 && (8.46)\\
\Rightarrow\ & x = \frac{4}{9} && (8.47)
\end{aligned}
$$

see the term is now positive - do you remember why?

Example 7:

$$6x + 3x = 4 - 5(2x - 3) \tag{8.48}$$

lets start with the parentheses - don't forget the minus!

$$\Rightarrow 6x + 3x = 4 - 10x + 15 \tag{8.49}$$

next we move the like terms to their own sides

$$\Rightarrow \quad 6x + 3x + 10x = 4 + 15 \tag{8.50}$$
$$\Rightarrow \quad 19x = 19 \tag{8.51}$$
$$\Rightarrow \quad x = 1 \tag{8.52}$$

Example 8:

$$8(3x - 14) - 34 = 2(4x - 22) - 5(3 + 2x) \tag{8.53}$$

Looks like a big one? Lets take it step by step

$$\Rightarrow 24x - 112 - 34 = 8x - 44 - 15 - 10x \tag{8.54}$$

that's all the brackets gone

$$\Rightarrow \quad 24x - 112 - 34 - 8x + 10x = -44 - 15 \tag{8.55}$$
$$\Rightarrow \quad 24x - 8x + 10x = -44 - 15 + 112 + 34 \tag{8.56}$$

and now solve!

$$\Rightarrow \quad 26x = 87 \tag{8.57}$$
$$\Rightarrow \quad x = \frac{87}{26} \tag{8.58}$$

And that is it for our examples. This covers all the types of linear equations you can be expected to solve. It's the best to always keep your priority of steps in mind and then just simply do them one by one. If you are unsure about your answer you can just substitute it into your original equation and see if you get the same value for both sides:

Example :

$$5(x - 3) = 5 \tag{8.59}$$
$$\Rightarrow \quad 5x - 15 = 5 \tag{8.60}$$
$$\Rightarrow \quad 5x = 5 + 15 \tag{8.61}$$
$$\Rightarrow \quad 5x = 20 \tag{8.62}$$
$$\Rightarrow \quad x = 4 \tag{8.63}$$

Test:

$$5(4-3) = 5 \tag{8.64}$$
$$\Rightarrow \quad 5(1) = 5 \tag{8.65}$$
$$\Rightarrow \quad 5 = 5 \tag{8.66}$$

and there we see it works!

Now lets get back to our original problem of your test results! The linear equation was:

$$x + (x+2) = 14 \tag{8.67}$$
$$\Rightarrow \quad x + x + 2 = 14 \tag{8.68}$$
$$\Rightarrow \quad x + x = 14 - 2 \tag{8.69}$$
$$\Rightarrow \quad 2x = 12 \tag{8.70}$$
$$\Rightarrow \quad x = 6 \tag{8.71}$$

You scored 6 for your Biology test and Joseph scored 6 +2 = 8!

8.2 Quadratic Equations

`The syllabus requires:`

- `solve quadratic equations by factorisation, completing the square and quadratic formula`
- `identify ''not real'' numbers and how they occur.  (see 2.1)`

8.2.1 The Quadratic Function

(NOTE: these notes have just been copied and pasted from the older structure notes and have not been written to the syllabus. it needs a serious edit and the worked example methodology needs redone (we now do inline examples and analogies with worked examples and exercises at the end of the chapter.)) A quadratic or parabolic function is a function of the form $f(x) = ax^2 + bx + c$.

(NOTE: very quick notes by sam for simple quadratics... this would be best as a decision tree. make the student appreciate that its basically trial and error to get the answer, but you can do some detective work first to eliminate most possibilities

- write the problem in the form $ax^2 + bx + c = 0$ (with a positive)
- write down two brackets, with an x in each, leaving room for a number on each side

$$(\quad x \quad)(\quad x \quad) \tag{8.72}$$

- write out your options (in a table at the side) for multiplying two numbers together to give a. these numbers should go in front of the xs in your brackets.
- if c is positive, then the other two numbers you need are either both positive or both negative. they are both negative if b is negative, and both positive id b is positive. if c is negative, it means only one of your numbers is negative, the other one beng positive.
- your two numbers should multiply to give c, so write out your options (in a table off to the side). if each number is multiplied by the number in front of the x in the other bracket, then added together it gives you b. so try different combinations of the numbers you have written). if it doesn't work, go back to the 3rd step and try a different combination of numbers to give you a.
- once you get an answer, multiply out your brackets again just to make sure it works (sanity check).

damn thats long winded!)

Worked Example:

Q: Draw a graph of the quadratic function $y = x^2 - x - 6$.

A: First let us set up a table of x and y values:

$x:$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y:$	24	14	6	0	-4	-6	-6	-4	0	6	14

The graph of this function is shown in figure 8.1. Notice that the function can also be written as $y = (x+2)(x-3)$. This shows that the x-intercepts (where $y = 0$) are $x = -2$ and $x = 3$, which agrees with the graph. The y-intercept (where $x = 0$) is at $y = -6$.

8.2.2 Writing a quadratic function in the form $f(x) = a(x-p)^2 + q$.

Consider the general form of the quadratic function $y = ax^2+bx+c$. Now adding and subtracting the same factor $\frac{b^2}{4a}$ from this expression does not change anything. Therefore

$$y = ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c \quad (8.73)$$

Taking out a factor of a then gives

$$y = a(x^2 + \frac{b}{a} + (\frac{b}{2a})^2) + c - \frac{b^2}{4a} \quad (8.74)$$

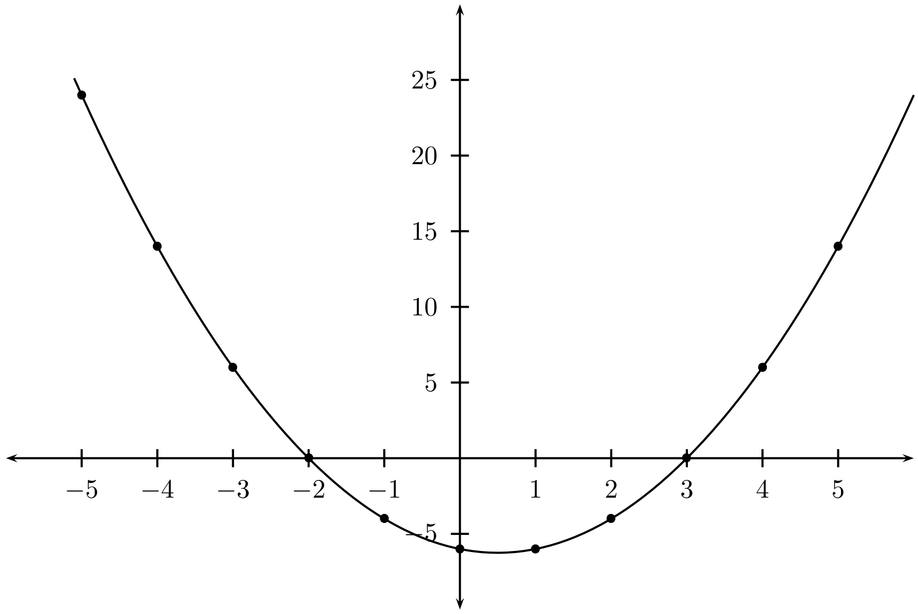


Figure 8.1: Graph of $y = x^2 - x - 6$

The expression in brackets is then a perfect square so that

$$y = a(x + (\frac{b}{2a}))^2 + c - \frac{b^2}{4a} \quad (8.75)$$

$$= a(x - (-\frac{b}{2a}))^2 + (c - \frac{b^2}{4a}) \quad (8.76)$$

which can be written in the form

$$y = a(x - p)^2 + q \quad (8.77)$$

where $p = -\frac{b}{2a}$ and $q = c - \frac{b^2}{4a}$.

Since $(x-p)^2$ is a perfect square and therefore always positive, $(x-p)^2$ is at a minimum of 0 when $x = p$. This means that y is minimum (if $a0$) or maximum (if $a < 0$) when $x = p$ and $y = q$. This point (p,q) is therefore called the turning point.

Now notice that the quadratic function is symmetric about $x = p$. In other words $f(p+v) = f(p-v) = av^2 + q$ for any real number v. This means that the part of the quadratic to the right of the vertical line $x = p$ looks like the part to the left of $x = p$ flipped about this line. Therefore we call the line $x = p$ the axis of symmetry of the parabola.

Worked Example:

Q: Consider the quadratic function $f(x) = -x^2+6x-5$. Put this function into the form $f(x) = a(x-p)^2 + q$ and thus find the turning point and axis of symmetry. Plot a graph of $f(x)$ showing all the intercepts.

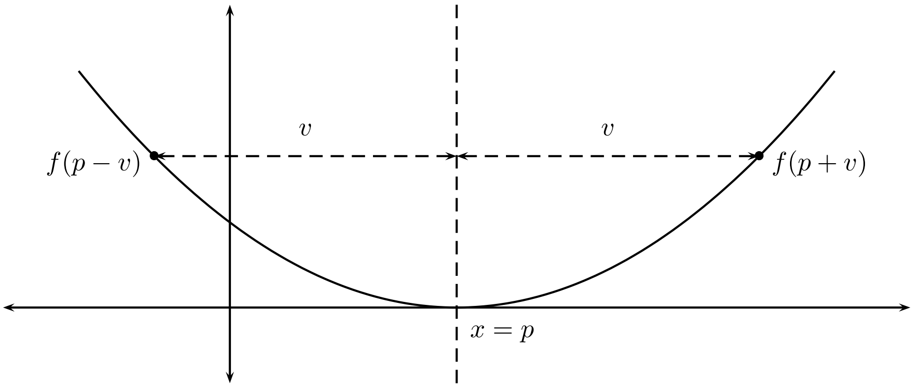


Figure 8.2: Graph of TODO (NOTE: original BMP (qf-gen1.bmp) missing, i made this up on the fly)

A: First we shall write the quadratic in the form $f(x) = a(x-p)^2 + q$.

$$
\begin{aligned}
f(x) &= -x^2 + 6x - 5 && (8.78)\\
&= -(x^2 - 6x + 5) && (8.79)\\
&= -(x^2 - 6x + 9 - 9 + 5) && (8.80)\\
&= -(x^2 - 6x + 9) + 4 && (8.81)\\
&= -(x^2 - 3)^2 + 4 && (8.82)
\end{aligned}
$$

Therefore the turning point is (3,4) and the axis of symmetry is $x = 3$ (in other words $p = 3$ and $q = 4$).

Now to plot the graph we need to know the intercepts. The y-intercept is $y = f(0) = -5$. The x-intercepts can be found by solving the equation $f(x) = 0$ as follows:

$$
\begin{aligned}
& -x^2 + 6x - 5 = 0 && (8.83)\\
\Rightarrow\quad & x^2 - 6x + 5 = 0 && (8.84)\\
\Rightarrow\quad & (x-1)(x-5) = 0 && (8.85)\\
\Rightarrow\quad & x = 1 \quad \text{or} \quad x = 5 && (8.86)
\end{aligned}
$$

Note that the technique of writing $x^2 - 6x + 5 = (x-1)(x-5)$ is called factorisation. We shall learn more about this in the following chapter. For now just check that this is true by multiplying out the brackets.

Thus the graph of the quadratic function $f(x) = -x^2 + 6x - 5$ is

8.2.3 What is a Quadratic Equation?

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation. Solving this equation for x is the same as finding the roots (x-intercepts) of the quadratic function $f(x) = ax^2 + bx + c$.

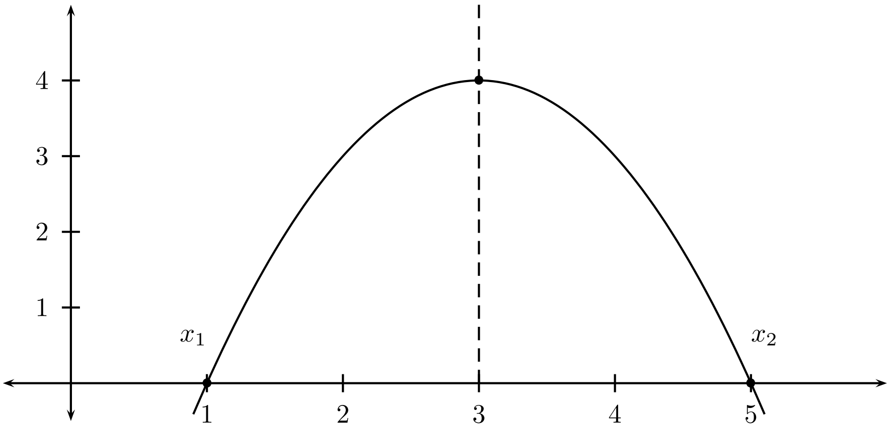


Figure 8.3: Graph of TODO (NOTE: original BMP (qf-eg2.bmp) missing, i made this up on the fly)

8.2.4 Factorisation

We have already seen examples of solving for the roots of a quadratic function by writing this function as the multiple of two brackets. For example, $x^2 - x - 6 = (x+2)(x-3) = 0$ means that either $x = -2$ or $x = 3$. This is called factorising the quadratic function.

Knowing how to factorise a quadratic takes some practice, but here some general ideas which are useful.

- First divide the entire equation by any common factor of the coefficients, so as to obtain an equation of the form $ax^2+bx+c = 0$ where a, b and c have no common factors. For example, $2x^2 + 4x + 2 = 0$ can be written as $x^2 + 2x + 1 = 0$ by dividing by 2.
- Now, if $ax^2 + bx + c = (rx + s)(ux + v)$, then $sv = c$ and $ru = a$. Therefore, by finding all the factors of a and c, one can try all the combinations and see if there is one which gives the correct result for $b = su + rv$.
- Once writing the equation in the form $(rx + s)(ux + v) = 0$, it then follows that the two solutions are $x = -\frac{s}{r}$ and $x = -\frac{u}{v}$.

Worked Examples:

Example 1: Q: Solve the equation $x^2 + 3x - 4$.

A: Since $a = 1$, if this equation can be factorised, it must have the form

$$x^2 + 3x - 4 = (x + s)(x + v) = x^2 + (s + v)x + sv \tag{8.87}$$

Now, as $sv = -4$, we know that $s = -2$, $v = 2$ or $s = -1$, $v = 4$ or $s = 1$, $v = -4$ (excluding the options which just involve interchanging s and v, which makes no difference to the final answer).

Also $s + v = 3$, so $s = -1$ and $v = 4$ is the correct combination. Thus the quadratic equation can be written as

$$x^2 + 3x - 4 = (x - 1)(x + 4) = 0 \tag{8.88}$$

Therefore the solutions are $x = 1$ and $x = -4$. **Example 2:** Q: Find the roots of the quadratic function $f(x) = -2x^2 + 4x - 2$.

A: We must find the solutions to the equation $f(x) = -2x^2 + 4x - 2 = 0$. First we divide both sides of the equation be a factor of -2. This gives the equation

$$x^2 - 2x + 1 = 0 \tag{8.89}$$

Now, let us assume that

$$x^2 - 2x + 1 = (x + s)(x + v) = x^2 + (s + v)x + sv \tag{8.90}$$

Then $sv = 1$ and therefore either $s = v = 1$ or $s = v = -1$. Since $s + v = -2$, it follows that $s = v = -1$ and thus

$$x^2 - 2x + 1 = (x - 1)(x - 1) = (x - 1)^2 = 0 \tag{8.91}$$

The only solution is therefore $x = 1$. **Example 3:** Q: Solve the equation $2x^2 - 5x - 12 = 0$.

A: This equation has no common factors, but still has $a = 2$. Therefore, we must look for a factorisation in the form

$$2x^2 - 5x - 12 = (2x + s)(x + v) = 2x^2 + (s + 2v)x + sv \tag{8.92}$$

We see that $sv = -12$ and $s + 2v = -5$. All the options for s and v are considered below. **Note:** Since we now have the factor of $2x$ in the first

s	v	$s + 2v$
2	-6	-10
-2	6	10
3	-4	-5
-3	4	5
4	-3	-2
-4	3	2
6	-2	2
-6	2	-2

bracket, it does make a difference, in this case, whether we interchange the s and v values. For example, $s = 2$, $v = -6$ and $s = -6$, $v = 2$ give different solutions. We must therefore consider both options.

We can see that the combination $s = 3$ and $v = -4$ gives $s + 2v = -5$. Therefore one can check that

$$2x^2 - 5x + 12 = (2x + 3)(x - 4) = 0 \tag{8.93}$$

Therefore the solutions are $x = -\frac{3}{2}$ and $x = 4$.

8.2.5 Completing the Square

It is not always possible to factorize a quadratic function. We shall now derive a general formula, which gives the solutions to any quadratic equation.

Consider a general quadratic equation $ax^2 + bx + c = 0$.

Adding and subtracting $\frac{b^2}{4a}$ from the left-hand side does not change the equation. Thus

$$ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c = 0 \tag{8.94}$$

Taking out a factor of a from the 1st 3 terms gives

$$a(x^2 + \frac{b}{a} + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = 0 \tag{8.95}$$

$$\Rightarrow a(x^2 + \frac{b}{a} + (\frac{b}{2a})^2) = \frac{b^2}{4a} - c \tag{8.96}$$

We can now see that the term in brackets is the perfect square $(x + \frac{b}{2a})^2$ and therefore

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c \tag{8.97}$$

Now dividing by a and taking the square root of either side gives the expression

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \tag{8.98}$$

Finally, solving for x implies that

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{8.99}$$

and taking the square root of $4a^2$ to obtain $2a$ gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{8.100}$$

These are the solutions to the quadratic equation. Notice that there are two solutions in general, but these may not always exists (depending on the sign of the expression $b^2 - 4ac$ under the square root).

Worked Examples:

Example 1:

Q: Solve for the roots of the function $f(x) = 2x^2 + 3x - 7$.

A: One should first try to factorise this expression, but in this case it turns out that this is not possible. Therefore we must make use of the general formula as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (8.101)$$

$$= \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-7)}}{2(2)} \quad (8.102)$$

$$= \frac{-3 \pm \sqrt{56}}{4} \quad (8.103)$$

$$= \frac{-3 \pm 2\sqrt{14}}{4} \quad (8.104)$$

Therefore the two roots of the quadratic function are $x = \frac{-3+2\sqrt{14}}{4}$ and $\frac{-3-2\sqrt{14}}{4}$.

Example 2:

Q: Solve for the solutions to the quadratic equation $x^2 - 5x + 8$.

A: Again it is not possible to factorise this equation. The general formula shows that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (8.105)$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)} \quad (8.106)$$

$$= \frac{5 \pm \sqrt{-7}}{2} \quad (8.107)$$

$$(8.108)$$

Since the expression under the square root is negative these are not real solutions ($\sqrt{-7}$ is not a real number). Therefore there are no real solutions to the quadratic equation $x^2 - 5x + 8$. This means that the quadratic function $f(x) = x^2 - 5x + 8$ has no x-intercepts, but the entire function lies above the x-axis.

Note to self: maybe add quadratic example about distance, velocity and acceleration ... object falling under action of gravity (giving formula for distance as a function of time)?.

8.2.6 Theory of Quadratic Equations

What is the Discriminant of a Quadratic Equation?

Consider a general quadratic function of the form $f(x) = ax^2 + bx + c$. The discriminant is defined as $\Delta = b^2 - 4ac$. This is the expression under the square root in the formula for the roots of this function. We have already seen that whether the roots exist or not depends on whether this factor Δ is negative or positive.

The Nature of the Roots

Real Roots:

Consider $\Delta \geq 0$ for some quadratic function $f(x) = ax^2 + bx + c$. In this case there are solutions to the equation $f(x) = 0$ given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \tag{8.109}$$

since the square roots exists (the expression under the square root is non-negative.) These are the roots of the function $f(x)$.

There are various possibilities:

Equal Roots:

If $\Delta = 0$, then the roots are equal and, from the formula, these are given by

$$x = -\frac{b}{2a} \tag{8.110}$$

Unequal Roots:

There will be 2 unequal roots if $\Delta > 0$. The roots of $f(x)$ are **rational** if Δ is a perfect square (a number which is the square of a rational number), since, in this case, $\sqrt{\Delta}$ is rational. Otherwise, if Δ is not a perfect square, then the roots are **irrational**.

Imaginary Roots:

If $\Delta < 0$, then the solution to $f(x) = ax^2 + bx + c = 0$ contains the square root of a negative number and therefore there are no real solutions. We therefore say that the roots of $f(x)$ are imaginary (the function $f(x)$ does not intersect the x-axis).

Summary of Cases:

- Real Roots ($\Delta \geq 0$)
  - Equal Roots ($\Delta = 0$)
  - Unequal Roots ($\Delta > 0$)
    - Rational Roots (Δ a perfect square)
    - Irrational Roots (Δ not a perfect square)
- Imaginary Roots ($\Delta < 0$)

Note to self: maybe add pictures showing these cases graphically?

Worked Examples:

Example 1:

Q: Consider the function $f(x) = 2x^2 + 5x - 11$. Without solving the equation $f(x) = 0$, discuss the nature of the roots of $f(x)$.

A: We need to calculate and classify $\Delta = b^2 - 4ac$ according to the cases for the roots.

$$\Delta = (5)^2 - 4(2)(-11) = 25 + 88 = 113 \tag{8.111}$$

Now Δ is positive, so the roots are real and unequal. Also, since 113 is not a perfect square, the roots are irrational.

Example 2:

Q: Consider the quadratic function $f(x) = x^2 + bx + (2b - 5)$, where b is some constant. Classify the roots of this function as far as possible.

A: Let us calculate the discriminant

$$\Delta = b^2 - 4(1)(2b - 5) = b^2 - 8b + 20 \tag{8.112}$$

We shall now use a useful trick, which is to write the above expression as a perfect square plus a number.

$$\Delta = b^2 - 8b + 20 = (b^2 - 8b + 16) + 4 = (b - 4)^2 + 4 \tag{8.113}$$

Now $(b-4)^2 \geq 0$ because this is a perfect square. Therefore we know that $\Delta \geq 4 > 0$.

We can thus say that $f(x)$ has real unequal roots. We do not know whether Δ is a perfect square, since we do not know that value of the constant b, and therefore we cannot say whether the roots are rational or irrational.

8.3 Cubic Equations

`The syllabus requires:`

- `(grade 12) solve cubic equations using factor theorem ‘‘and other techniques’’ (NOTE: SH: i want to hit whoever wrote this syllabus)`

8.4 Exponential Equations

`The syllabus requires:`

- `solve exponential equations`
- `(grade 12) switch between log and exp form of an equation`

8.5 Trigonometric Equations

`The syllabus requires:`

- `solve trig equations (NOTE: this implies that this section comes after trig, which is also probably after geometry)`

8.6 Simultaneous Equations

`The syllabus requires:`

- `solve simultaneous equations algebraically and graphically`

8.7 Inequalities

`The syllabus requires:`

- `solve linear inequalities in 1 and 2 variables and illustrate graphically`
- `solve quadratic inequalities in 1 variable and illustrate graphically`

8.7.1 Linear Inequalities

Let us say that we are given a general inequality as follows:

$$ax+by+c \geq 0, \quad ax+by+c > 0, \quad ax+by+c \leq 0 \quad \text{or} \quad ax+by+c < 0 \tag{8.114}$$

Now there are many possible values of x and y, for which this may be true (these will depend on the values of the constants a, b and c). The set of all the (x,y) values which satisfy this inequality is called the solution set.

We shall now see how to draw the solution set on a graph. Let's consider the following example.

Worked Example 1:

Q: Find the solution set of the inequality $2x + y - 3 \geq 0$.

A: First we solve for y by writing the inequality as

$$y \geq -2x + 3 \tag{8.115}$$

Now the function $y = -2x+3$ is a straight line and the points (x,y), which satisfy the inequality are therefore all the points above the line. These can be drawn as

The shaded section on the graph, which shows the solution set, is called the feasible region.

Now sometimes x and y must satisfy more than one inequality. In this case, we consider each inequality separately and then the feasible region is where the feasible regions of each inequality overlap.

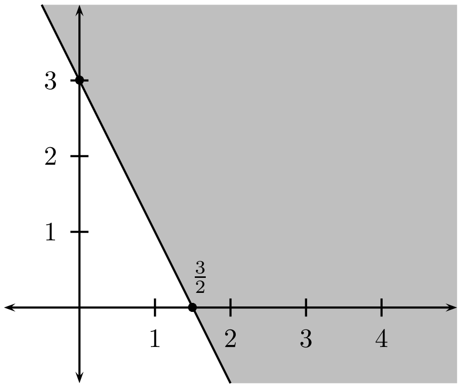


Figure 8.4: Graph of $y = -2x + 3$. Points satisfying $y \geq -2x + 3$ are shaded

Worked Example 2:

Q: Graphically represent the solution set for the following inequalities:

$$y \geq 1 \tag{8.116}$$
$$-2x + y - 5 < 0 \tag{8.117}$$
$$x + y - 10 \leq 0 \tag{8.118}$$

A: Solving for y gives

$$y \geq 1 \tag{8.119}$$
$$y < 2x + 5 \tag{8.120}$$
$$y \leq -x + 10 \tag{8.121}$$

Now we draw the solution set to each of the inequalities separately and find the region where these overlap as shown below. We draw the line $y = 2x + 5$ as a dashed line because the inequality is $<$ and not $\leq$ (the line is not included in the feasible region).

8.7.2 What is a Quadratic Inequality

A quadratic inequality is an inequality of the form $ax^2 + bx + c > 0$, $ax^2 + bx + c \geq 0$, $ax^2 + bx + c < 0$ or $ax^2 + bx + c \leq 0$.

8.7.3 Solving Quadratic Inequalities

Solving a quadratic inequality corresponds to working out in what region a quadratic function lies above or below the x-axis. Here are some examples showing how this is done.

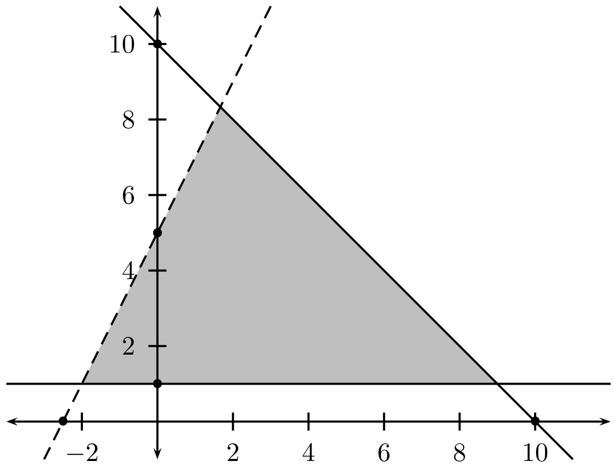


Figure 8.5: Graph of TODO

Worked Examples:

Example 1:

Q: Find all the solutions to the inequality $x^2 - 5x + 6 \geq 0$.

A: Consider the function $f(x) = x^2 - 5x + 6$. We need to find out where $f(x) \geq 0$; in other words, where the function $f(x)$ lies above/on the x-axis.

We shall first work out where $f(x)$ intersects the x-axis by solving the equation

$$x^2 - 5x + 6 = 0 \tag{8.122}$$

which can be factorised to give

$$(x - 3)(x - 2) = 0 \tag{8.123}$$

The x-intercepts are therefore $x_1 = 2$ and $x_2 = 3$.

We can see from figure 8.6 that $f(x)$ is above/on the x-axis when $x \geq 3$ or $x \leq 2$.

Therefore the solution to the quadratic inequality is $\{x : x \geq 3 \text{ or } x \leq 2\}$ or in interval notation $(-\infty,2] \cup [3,\infty)$.

Note: The x-intercepts are included in this solution, since the $f(x) \leq 0$ inequality includes the solution $f(x) = 0$.

Example 2:

Q: Solve the quadratic inequality $-x^2 - 3x + 5 > 0$.

A: Let $f(x) = -x^2 - 3x + 5$. The x-intercepts are solutions to the quadratic equation

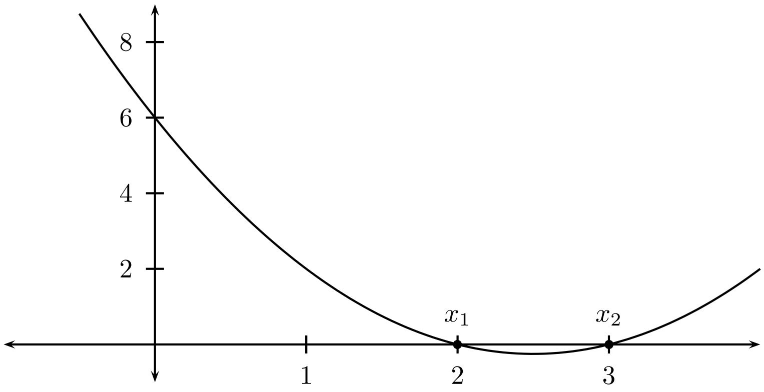


Figure 8.6: Graph of $f(x) = x^2 - 5x + 6$

$$\begin{aligned} -x^2 - 3x + 5 &= 0 \qquad (8.124) \\ \Rightarrow x^2 + 3x - 5 &= 0 \qquad (8.125) \end{aligned}$$

which has solutions (using the formula for the roots of a quadratic function (NOTE: reference that equation)) given by

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)} \qquad (8.126) \\ &= \frac{-3 \pm \sqrt{29}}{2} \qquad (8.127) \\ x_1 &= \frac{-3 - \sqrt{29}}{2} \qquad (8.128) \\ x_2 &= \frac{-3 + \sqrt{29}}{2} \qquad (8.129) \end{aligned}$$

The graph of $f(x)$ is shown in figure 8.7. The points x_1 and x_2 (where the function $f(x)$ cuts the x axis) are labelled.

Now $f(x) > 0$ (the function is above the x-axis) when $\frac{-3-\sqrt{29}}{2} < x < \frac{-3+\sqrt{29}}{2}$.

Therefore the solution to the inequality is $\{x : \frac{-3-\sqrt{29}}{2} < x < \frac{-3+\sqrt{29}}{2}\}$, or in interval notation $(\frac{-3-\sqrt{29}}{2}, \frac{-3+\sqrt{29}}{2})$.

Note: The x-intercepts are not included in the solution because the $>$ sign has been used and therefore $f(x) = 0$ does not define a solution to the inequality.

Example 3:

Q: Solve the inequality $4x^2 - 4x + 1 \leq 0$.

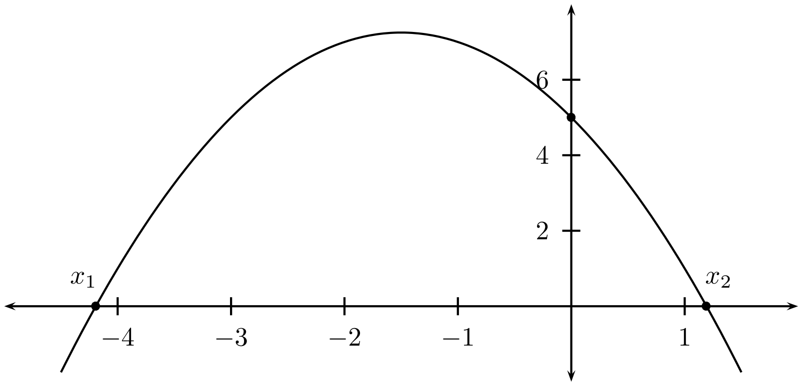


Figure 8.7: Graph of $f(x) = -x^2 - 3x + 5$

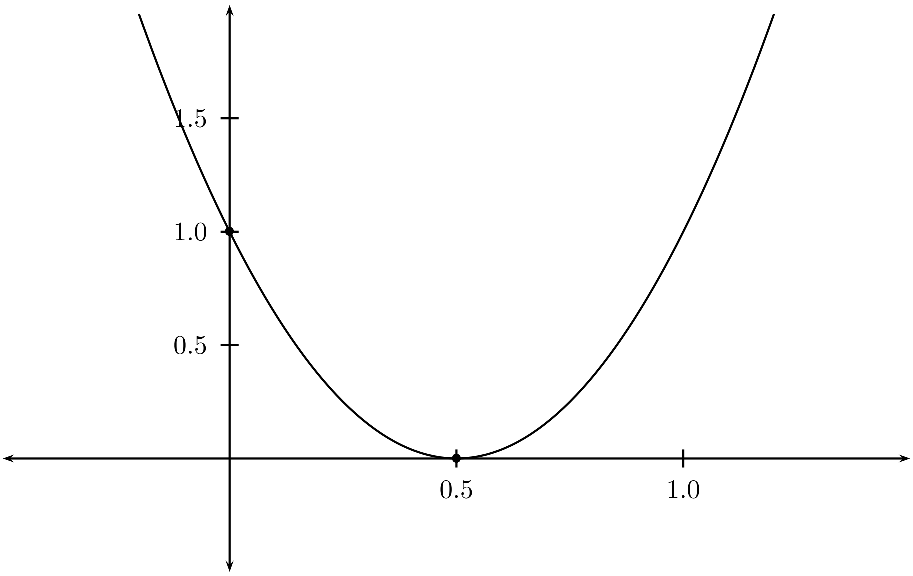


Figure 8.8: Graph of $f(x) = 4x^2 - 4x + 1$

A: Let $f(x) = 4x^2 - 4x + 1$. Factorising this quadratic function gives $f(x) = (2x - 1)^2$, which shows that $f(x) = 0$ only when $x = \frac{1}{2}$.

The function $f(x)$ lies below/on the x-axis only at the x-intercept. Therefore the only solution to the inequality is $x = \frac{1}{2}$.

8.8 Intersections

`The syllabus requires:`

- `find solutions of 2 lines and interpret the common solution as the intersection`

— find solutions of linear and quadratic, interpret the common solution(s) as intersections

Chapter 9

Working with Data

9.1 Statistics

The syllabus requires:

- organise univariate numerical data to determine measure of central tendency; mean, median, mode and when each is appropriate. measure of dispersion; ranger, percentiles, quantities, interquartile, semi-interquartile range
- represent data effectively, choosing from; bar and compound bar graphs, histograms, frequency polygons, pie charts, dot plots, line and broken line graphs, stem and leaf plots, box and whisker diagrams
- (grade 12) variance, standard deviation
- (grade 12) draw a suitable random sample from a populations, and understand the importance of sample size in predicting the mean and standard deviation
- (grade 12) identifies data which is normally distributed about a mean by investigating appropriate histograms and frequency polygons

9.2 Function Fitting

The syllabus requires:

- represent bivariate data as a scatter plot and suggest what function (linear, quadratic, exp) would best describe it
- tell the difference between symmetric and skewed data and make relevant deductions
- (grade 12) use appropriate technology to calculate linear regression line which best fits a given set of bivariate data

9.3 Probability

The syllabus requires:

- Venn diagrams to solve probability problems. must know

$$P(s) = 1qquad P(AorB) = P(A) + P(B) - P(AandB)$$

- identify dependent and independent events and calculate the prob of 2 independent events occurring by applying the product rule for independent events $P(AandB) = P(A).P(B)$
- identify mutually exclusive events. calculate prob of the events occuring by applying additive rule for mutually exclusive events $P(AorB) = P(A) + P(B)$
- identify complementary events $P(notA) = 1 - P(A)$
- use prob models for comparing experimental results with theory; need many trials to get comparable results... flipping a coin example
- comparing experimental results with each other
- potential sources of bias, error in measurement, potential uses and misuses of stats and charts (NOTE: if SA people could get some popular TV adverts as examples, that would be good)
- converts this theory into a project (NOTE: i don't know how much of this project stuff we should do in the book. possibly just ignore its existence)

9.4 Permutations and Tree Diagrams

The syllabus requires:

- tree diagrams and other methods of listing all options to generalise counting principle (successive choices)
- calculate the probability of compound events which are not independent
- assess the odds in a variety of games of chance, lotteries, raffles
- (grade 12) use investigate and solve problems involving the number of arrangements (permutations) of a number of discrete objects (when order matters) $m!$ (m different items), m items selected from n
- (grade 12) investigate and solve problems involving the number of possible solutions when order is not important (combinations) of m items from n where all are different or distinguishable
- (grade 12) uses permutations and combinations to correctly calculate the probability of specified events occurring
- determines the odds of various games of chance and the probability of events which depend on combinations and permutations

9.5 Finance

The syllabus requires:

- use simple and compound interest for relevant problems (hire purchase and inflation)
- effective and nominal interest
- understand fluctuating foreign exchange rates and their effect on local prices, travelling prices, imports and exports
- solve straight line (simple) depreciation and depreciation on a reducing budget (compound depreciation)
- (grade 12) apply geometric series to solve problems (future values of annuities, bond repayments, sinking fund contributions including the difference in time taken to pay when the monthly payment is changed)
- (grade 12) critically analyse investment and loan options and make informed decisions to the best options (pyramid and micro lenders schemes)

9.6 Worked Examples

TODO

9.7 Exercises

TODO

Part II

Old Maths

Chapter 10

Worked Examples

10.1 Exponential Numbers

(NOTE: All of these worked examples need to be updated to use the FHSST internal environments and also to use the new rules (i changed the originals). They are for the Exponential Numbers section)

***Worked Example 9* : Manipulating Exponential Numbers**

Question:
Simplify the expression $\frac{4^2.3^3}{6^3}$
A: Noting that $4 = 2 \times 2 = 2^2$ and $6 = 2.3$ it follows that

$$\begin{aligned} \tfrac{4^2.3^3}{6^3} &= \frac{(2^2)^2.3^2}{(2.3)^3} && (10.1)\\ &= \frac{2^4.3^3}{2^3.3^3} && (10.2)\\ &= 2^{4-3} && (10.3)\\ &= 2^1 && (10.4)\\ &= 2 && (10.5) \end{aligned}$$

Example 2:
Q: Simplify $(\frac{5}{2})^2.20$.
A: First note that $20 = 2 \times 2 \times 5 = 2^2.5$. Therefore

$$
\begin{aligned}
(\tfrac{5}{2})^2.20 &= \frac{5^2}{2^2}.2^2.5 && (10.6)\\
&= 5^{2+1} && (10.7)\\
&= 5^3 && (10.8)\\
&= 5 \times 5 \times 5 && (10.9)\\
&= 125 && (10.10)
\end{aligned}
$$

Example 3:
Q: Solve for the variable x in the equation $2^{x+1} = 2^x + 8$.
A:

$$
\begin{aligned}
& 2^{x+1} = 2^x + 8 && (10.11)\\
\Rightarrow\ & 2^x.2 - 2^x = 8 && (10.12)\\
\Rightarrow\ & 2^x(2-1) = 8 && (10.13)\\
\Rightarrow\ & 2^x = 8 = 2 \times 2 \times 2 = 2^3 && (10.14)\\
\Rightarrow\ & x = 3 && (10.15)
\end{aligned}
$$

It is also possible to talk about zero, negative and even fractional exponents. We shall assume that laws 1–5 are also true in these cases. This gives us 3 more laws.

$$a^0 = a^{1-1} = \frac{a^1}{a^1} = \frac{a}{a} = 1$$

Law 7

$$a^{-n} = \frac{1}{a^n} \qquad (10.16)$$

Since $-n = 0 - n$, it follows from laws 2 and 6 that

$$a^{-n} = a^{0-n} = \frac{a^0}{a^n} = \frac{1}{a^n} \qquad (10.17)$$

This defines what is meant by a negative exponent.

Note to self: add aside lay-out to following paragraph
Aside:

A fraction to the power of a negative exponent is the same as the inverse fraction to the power of the corresponding positive exponent. Therefore

$$(\frac{a}{b})^{-n} = \frac{1}{\frac{a}{b}} = \frac{a}{\frac{a^n}{b^n}} = (\frac{b}{a})^n \qquad (10.18)$$

Example 1:
Q: Simplify the expression $(\frac{1}{2})^{-3}$.
A:

$$(\frac{1}{2})^{-3} = \frac{1}{(\frac{1}{2})^3} = \frac{1}{\frac{1}{2^3}} = 2^3 = 8 \tag{10.19}$$

Example 2:
Q: Simplify $(27)^{\frac{1}{3}}.\sqrt{\frac{2}{9}}$.
A: Now $27 = 3 \times 3 \times 3 = 3^3$ and $9 = 3 \times 3 = 3^2$, so

$$(27)^{\frac{1}{3}}.\sqrt{\frac{2}{9}} = (3^3)^{\frac{1}{3}}.\sqrt{\frac{2}{3^2}} \tag{10.20}$$
$$= 3^{3.\frac{1}{3}}.(\frac{2}{3^2})^{\frac{1}{2}} \tag{10.21}$$
$$= \frac{3^1.2^{\frac{1}{2}}}{(3^2)^{\frac{1}{2}}} \tag{10.22}$$
$$= \frac{3.\sqrt{2}}{3} \tag{10.23}$$
$$= \sqrt{2} \tag{10.24}$$

Example 3:
Q: Simplify $(a^n + a^n)^{\frac{1}{2}}$
A:

$$(a^n + a^n)^{\frac{1}{2}} = (2a^n)^{\frac{1}{2}} \tag{10.25}$$
$$= (2)^{\frac{1}{2}}.(a^n)^{\frac{1}{2}} \tag{10.26}$$
$$= \sqrt{2}\ a^{n.\frac{1}{2}} \tag{10.27}$$
$$= \sqrt{2}\ a^{\frac{n}{2}} \tag{10.28}$$

Example 4:
Q: Find a solution to the equation $ax^{\frac{m}{n}} - b = 0$, where a and b are constants.
A:

$$ax^{\frac{m}{n}} - b = 0 \tag{10.29}$$
$$\Rightarrow \quad ax^{\frac{m}{n}} = b \tag{10.30}$$
$$\Rightarrow \quad x^{\frac{m}{n}} = \frac{b}{a} \tag{10.31}$$
$$\Rightarrow \quad x^m = (x^{\frac{m}{n}})^n = (\frac{b}{a})^n \tag{10.32}$$
$$\Rightarrow \quad x = (x^m)^{\frac{1}{m}} = ((\frac{b}{a})^n)^{\frac{1}{m}} \tag{10.33}$$
$$\Rightarrow \quad x = (\frac{b}{a})^{\frac{n}{m}} = \sqrt[m]{(\frac{b}{a})^n} \tag{10.34}$$

Therefore $x = \sqrt[m]{(\frac{b}{a})^n}$ is a solution to the equation $ax^{\frac{m}{n}} - b = 0$.

(NOTE: surds)

Worked Example:

Q: Simplify the expression $\frac{\sqrt{32}}{2}$.

A:

$$\begin{aligned}
\tfrac{\sqrt{32}}{2} &= \frac{\sqrt{32}}{\sqrt{4}} && (10.35)\\
&= \sqrt{\frac{32}{4}} && (10.36)\\
&= \sqrt{8} && (10.37)\\
&= \sqrt{4.2} && (10.38)\\
&= \sqrt{4}.\sqrt{2} && (10.39)\\
&= 2\sqrt{2} && (10.40)
\end{aligned}$$

Worked Example:

Q: Which of the numbers $\sqrt[3]{100}$ and $\sqrt{20}$ is bigger? (You may not use a calculator to answer this question.)

A: The two numbers must first be converted into like surds. Since we have a cube root and a square root, we must first find the lowest common multiple of 2 and 3 which is 6. We then convert each of the surds into 6^{th} roots as follows:

$$\sqrt[3]{100} = \sqrt[3]{\sqrt{100^2}} = \sqrt[3.2]{100^2} = \sqrt[6]{10000} \qquad (10.41)$$

$$\sqrt{20} = \sqrt[2]{\sqrt[3]{20^3}} = \sqrt[2.3]{20^3} = \sqrt[6]{8000} \qquad (10.42)$$

Now, since 100000 is bigger than 8000, it follows that $\sqrt[6]{10000}$ is bigger than $\sqrt[6]{8000}$. Therefore $\sqrt[3]{100}$ is bigger than $\sqrt{20}$.

Worked Examples:

Example 1:

Q: Rationalize the denominator of the fraction $\frac{1}{\sqrt{6}}$.

A: The denominator can be changed into the rational number 6 by multiplying the numerator and denominator by $\sqrt{6}$. Therefore

$$\frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6} \qquad (10.43)$$

Example 2:

Q: Rationalize the denominator of $\frac{\sqrt{2}}{(\sqrt{3}+\sqrt{8})}$.

A: Multiplying the numerator and denominator by $(\sqrt{3}-\sqrt{8})$ gives

$$\begin{aligned}
\tfrac{\sqrt{2}}{(\sqrt{3}+\sqrt{8})} &= \frac{\sqrt{2}}{(\sqrt{3}+\sqrt{8})} \times \frac{(\sqrt{3}-\sqrt{8})}{(\sqrt{3}-\sqrt{8})} && (10.44)\\
&= \frac{\sqrt{2}(\sqrt{3}-\sqrt{8})}{3-8} && (10.45)\\
&= \frac{\sqrt{2}\sqrt{3}-\sqrt{2}\sqrt{8}}{-5} && (10.46)\\
&= -\frac{\sqrt{6}-\sqrt{16}}{5} && (10.47)\\
&= \frac{4-\sqrt{6}}{5} && (10.48)
\end{aligned}$$

1.1.4 d) Equations

Here we shall solve some equations involving surds.

Worked Examples:

Example 1:

Q: Find a solution to the following equation

$$x + 3 - \sqrt{6x + 13} = 0 \qquad (10.49)$$

A: First the square root must be moved to the right-hand side of the equation, and everything else to the left.

$$x + 3 = \sqrt{6x + 13} \qquad (10.50)$$

Now we square both sides of the equation.

$$\begin{aligned}
& (x+3)^2 = 6x + 13 && (10.51)\\
\Rightarrow\ & x^2 + 6x + 9 = 6x + 13 && (10.52)\\
\Rightarrow\ & x^2 = 4 && (10.53)\\
\Rightarrow\ & x = 2 \ \text{ or } \ x = -2 && (10.54)
\end{aligned}$$

When we square both sides of the equation, it is possible that we introduce extra solutions, which may not actually satisfy the original equation. This is the reason one should **always check the answers** by substituting these into the original equation.

x = 2:

$$x+3-\sqrt{6x+13} = 2+3-\sqrt{6(2)+13} = 5-\sqrt{25} = 5-5 = 0 \quad (10.55)$$

<u>x = -2:</u>

$$x+3-\sqrt{6x+13} = -2+3-\sqrt{6(-2)+13} = -1-\sqrt{1} = 1-1 = 0 \quad (10.56)$$

Therefore, in this case, both the solutions $x = 2$ and $x = -2$ are solutions to the original equation

<u>Note to self: Maybe the following example should only be included after the chapter on quadratics? But need non-trivial example where not all solutions valid.</u>

Example 2:

Q: Solve the equation

$$x - \sqrt{x+7} = -1 \quad (10.57)$$

As before, we first move the surd to the right-hand side and the other terms to the left.

$$x + 1 = \sqrt{x+7} \quad (10.58)$$

Squaring both sides of the equation gives

$$\begin{aligned} & (x+1)^2 = x+7 && (10.59)\\ \Rightarrow\quad & x^2+2x+1 = x+7 && (10.60)\\ \Rightarrow\quad & x^2+x-6 = 0 && (10.61)\\ \Rightarrow\quad & (x+3)(x-2) = 0 && (10.62)\\ \Rightarrow\quad & x = -3 \ \text{ or } \ x = 2 && (10.63) \end{aligned}$$

Again we must check these answers:

<u>x = -3:</u>

$$x-\sqrt{x+7} = -3-\sqrt{-3+7} = -3-\sqrt{4} = -3-2 = -5 \neq -1 \quad (10.64)$$

<u>x = 2:</u>

$$x-\sqrt{x+7} = 2-\sqrt{2+7} = 2-\sqrt{9} = 2-3 = -1 \quad (10.65)$$

Therefore $x = -3$ does not satisfy the original equation, so $x = 2$ is the only solution.

Scientific Notation

Example 1:

The speed of light in a vacuum c is

$$c = 2.9979 \times 10^8 \ \frac{m}{s} \tag{10.66}$$
$$= 2.9979 \times 100000000 \ \frac{m}{s} \tag{10.67}$$
$$= 299790000 \ \frac{m}{s} \tag{10.68}$$

Example 2:

The approximate radius of the hydrogen atom (called the Bohr radius a_0) is

$$a_0 = 5.3 \times 10^{-11} \ m \tag{10.69}$$
$$= 5.3 \times \frac{1}{10^{11}} \ m \tag{10.70}$$
$$= 5.3 \times \frac{1}{10000000000} \ m \tag{10.71}$$
$$= 5.3 \times 0.00000000001 \ m \tag{10.72}$$
$$= 0.000000000053 \ m \tag{10.73}$$

Note to self: check the above number ... I think $a_0 = 0.53\mathring{A}$? ...

Example 3:

Q: The universe is known to be 13.7 billion years old. Convert this number into scientific notation.

A: The age of the universe is 13700000000 years (since 1 billion is a thousand million or 1000000000). The decimal point must move 10 places to the left to convert this number into 1.37. Since 13700000000 is much bigger than 1.37, the power 10^m must make the number 1.37 bigger and therefore m must be positive (so that 10^m is greater than 1). Therefore $m = 10$ (the decimal point moves 10 places).

Therefore $13700000000 = 1.37 \times 10^{10}$. The age of the universe is thus 1.37×10^{10} years.

Example 4:

Q: The charge on an electron is $e = 0.0000000000000000001602$ C. What is this constant in scientific notation.

A: The decimal point must be moved 19 places to the right to change this number into 1.602 (so $a = 1.602$). Now e is much smaller than 1 and thus the exponent m must be negative, so that 10^m is less than 1. Therefore $m = -19$.

Therefore $e = 1.602 \times 10^{-19}$ C.

Note: As a general rule, if the decimal point must move to the left then the exponent m is positive and if the decimal point moves to the right then m is negative. patterns

sequences

Worked Example 10 **: Calculating the nth Term of a Sequence**

Question: Check that the formula for the nth term of the sequence {2,6,10,14,18,...} is given by $2+4(n-1)$, and calculate the thousandth term.
Answer: The sequence we are given starts at two, and each term is equal to the previous term plus four. We can check whether the formula is valid by going through the first few terms, and seeing whether the terms in the sequence correspond to the terms given by the formula. If we substitute $n = 1$ into the formula, we should get the first term of the sequence, and this is indeed the case: $2 = 2 + 4(1 - 1)$. If we substitute $n = 2$, we get the second term, namely 6. We can continue in this way, substituting $n = 3$, $n = 4$, and so on - each time we get the expected value back. This means that the formula is indeed valid. To calculate the thousandth term, we must substitute 1000 into the formula. When we do so we get $2 + 4(1000 - 1) = 2 + 4(999) = 3998$.

Worked Example 11 **: Calculating the Sequence When Given the Formula**

Question: The formula for the nth term of a sequence is given by $2(2^{n-1})$.Write down the first four terms of the sequence, and describe the pattern that you observe.
Answer: This question is similar to the one in the previous example. To get the first term of the sequence, we must substitute 1 into the formula. Doing so gives us $2(2^{1-1}) = 2(2^0) = 2(1) = 2$. To find the second term we must substitute two into the sequence: $2(2^{2-1}) = 2(2) = 4$. For the third and fourth terms we substitute three and four respectively, and so we find that the first four terms of the sequence are 2,4,8,16,.... ¿From these four terms we can see that every term in the sequence is equal to the previous term multiplied by two.

***Worked Example 12* : Calculating the Sequence When Given the Formula**

Question: The first term of a sequence is 3, and the formula for the $(n+1)$th term is given by $a_{n+1} = a_n + 2n$. Write down the first four terms of the sequence.
Answer: We start by calculating the second term, since we already know the first. To do this we need to substitute $n = 1$ into the formula, since the formula is for the $(n+1)$th term, NOT for the nth term, as was previously the case. So, substituting $n = 1$, we get that $a_{1+1} = a_1 + 2(1) = 3+2 = 5$. To get the next term, we must substitute $n = 2$: $a_{2+1} = a_2 + 2(2) = 5 + 4 = 9$. Lastly, we calculate that $a_4 = 9 + 2(3) = 15$. So the first four terms of the sequence are 3,5,9,15,....

***Worked Example 13* : Checking That a Given Sequence is Arithmetic**

Question: Check that the sequence given by the formula $a_n = 3 + 4(n-1)$ is an arithmetic sequence, and find d for this sequence.
Answer: We must check to see that the difference between successive terms is a constant. There are two ways of doing this: we could write out the first few terms of the sequence and check that they are evenly spread - i.e. they differ by a constant amount, or we could do the calculation in general, using the formula directly. We will do the example using both these approaches alternately.

– First approach: It is easy to calculate the first few terms using the formula. They are given by 3,7,11,15,19,.... We can see that the difference between successive terms is always 4, since $7-3 = 11-7 = 15-11 = 19-15 = 4$, so the sequence is indeed an arithmetic sequence, and $d = 4$
– Second approach: We know that the formula for the nth term is given by $a_n = 3 + 4(n-1)$. From this it should be clear that the $(n+1)$th term is given by $a_{n+1} = 3 + 4((n+1)-1) = 3 + 4n$. If we work out the difference between successive terms, we get that $a_{n+1} - a_n = 3+4n-3-4(n-1) = 4n - 4n + 4 = 4 = d$, which is the same answer that we got using the previous method.

***Worked Example 14* : Calculating the Formula for the nth term of a Sequence**

Question: Find a formula for the nth term of the sequence $6, 17, 28, 39$,.... Which term in the sequence equals 688?

Answer: First we must check that the given sequence is in fact an arithmetical one, otherwise we can't use the formula. This is easily seen, since the difference between successive terms is 11. Now we must use (**??**). We know that $a_1 = 6$ and that $d = 11$. If we substitute these into the formula, we see that $a_n = 6 + 11(n-1)$. Lastly, we must find which term equals 688. Notice that in the first worked example we were given an n (namely 1000), and asked to calculate a_n. Now we are given an a_n and asked to calculate n. We can do this easily by rearranging the formula:

$$\begin{aligned} a_n = 688 &= 6 + 11(n-1) \\ 688 - 6 &= 11(n-1) \\ \frac{688-6}{11} &= n - 1 \\ n = \frac{688-6}{11} &+ 1 = 63 \end{aligned}$$

We conclude that the 63rd term will equal 688.

geometric

Worked Example 15 **: Checking That a Given Sequence is Geometric**

Question: Check that the sequence given by the formula $a_n = 2(3^{n-1})$ is a geometric sequence, and find r for this sequence.
Answer: We must check to see that the ratio between successive terms is a constant. As in example four, there are two ways of doing this: we could write out the first few terms of the sequence and check that successive terms differ by a common factor, or we could do the calculation in general, using the formula directly. We will do the example using both these approaches alternately.

- First approach: It is easy to calculate the first few terms using the formula. They are given by 2,6,18,54,162,.... We can see that the ratio between successive terms is always 3, since $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3$, so the sequence is indeed an arithmetic sequence, and $r = 3$
- Second approach: We know that the formula for the nth term is given by $a_n = 2(3^{n-1})$. From this it should be clear that the $(n+1)$th term is given by $a_{n+1} = 2(3^n)$. If we work out the ratio between successive terms, we get that $\frac{a_{n+1}}{a_n} = \frac{2(3^n)}{2(3^{n-1})} = 3^{n-n+1} = 3$, which is the same answer that we got using the previous method.

Worked Example 16 **: Calculating the Formula for the nth term of a Sequence**

Question: Find a formula for the nth term of the sequence 2,4,8,16,.... Which term in the sequence equals 8192?
Answer: First we must check that the given sequence is in fact a geometric one, otherwise we can't use the formula. This is easily seen, since the ratio between successive terms is 2. Now we must use (**??**). We know that $a_1 = 2$ and that $r = 2$. If we substitute these into the formula, we see that $a_n = 2(2^{n-1})$. Lastly, we must find which term equals 8192. We can do this easily by rearranging the formula:

$$\begin{array}{rclcl} a\_n & = & 688 & = & 6 + 11(n-1) \\ & & 688 - 6 & = & 11(n-1) \\ & & (688-6)/11 & = & n-1 \\ n & = & (688-6)/11 + 1 & = & 63 \end{array}$$

We conclude that the 13th term will equal 8192.

10.2 series

Worked Example 17 **: Calculating S_n**

Question: Calculate S_4 for the series $2 + 4 + 8 + 16 + 32 + ...$
Answer: Recall that S_4 is the sum of the first four terms of the series. This is given by $2 + 4 + 8 + 16 = 30$.

Worked Example 18 **: Calculating a Series**

Question: Calculate first five terms of the series which corresponds to the sequence $a_n = 2 + 2(n-1)$.
Answer: First we calculate the first five terms of the sequence. They are 2,4,6,8,10. To get the corresponding series, we simply need to put addition signs in between the terms: 2+4+6+8+10.

Worked Example 19 **: Checking S_n for a given series**

Question: For the series $2+4+6+8+...$, check that the formula for the sum of the first n terms is given by $S_n = \frac{n}{2}[4 + 2(n-1)]$
Answer: Let us start by writing out the first few terms of S_n. S_1 equals the first term of the series, so $S_1 = 2$. S_2 is the sum of the first two terms, so $S_2 = 2 + 4 = 6$. S_3 is the sum of the first three terms, namely 12. Continuing in this fashion, we can see that the first few terms of S_n are 2,6,12,20,28,....What we need to determine is whether these correspond to the given formula for S_n, and indeed they do. We must simply note that when we substitute 1 into $\frac{n}{2}[4 + 2(n-1)]$, we get 2, when we substitute 2, we get 6, when we substitute 3, we get 12, then 20, then 28,

and so on. This means that the formula does indeed give us the sum up to n terms.

Worked Example 20 **: Calculating the Sequence that Corresponds to a Given S_n**

Question: For a certain series, S_n is given by the formula $S_n = 2n^2 - 4n$. Find the sequence which corresponds to this series.
Answer: Let us start by writing out the first few terms of S_n: -2,0,6,16,30,.... Think carefully about what this means - $S_1 = -2$ means that the first term of the series is -2, $S_2 = 0$ means that the sum of the first two terms is 0. Therefore the second term must be 2, since when we add 2 to -2 we get 0. $S_3 = 6$ means that the sum of the first three terms of the series is 6, so, by similar reasoning, the third term must be 6. $S_4 = 16$ means that the fourth term must be 10. Now one should start to see the pattern - the first few terms of the sequence which corresponds to this series are -2,2,6,10,..., so we are dealing with an arithmetical sequence that has a common difference of 4.

Worked Example 21 **: Calculating S_n for a Given Series**

Question: Calculate the value of the series $1+4+7+10+13+ ... +46$.
Answer: We wish to find S_{16} (since 46 is the 16th term of the series). Of course we can do this with a calculator, but there is a much quicker way.

$$\begin{aligned}
S_{16} &= 1 + 4 + ... + 43 + 46 \\
S_{16} &= 46 + 43 + ... + 4 + 1 \\
2S_{16} &= 47 + 47 + ... + 47 + 47 \\
2S_{16} &= 16 \times 47 = 752 \\
S_{16} &= \tfrac{752}{2} = 376
\end{aligned}$$

Worked Example 22 **: Using the Formula for S_n**

Question: Find the sum of all the integers between 1 and 100, i.e. find 1+2+3+...+99+100.
Answer: Since we are dealing with an arithmetic series,we can use the formula for S_n that we have derived. In order to use it we need to know which values to put in for a_1, d, and n. Since the series starts at 1, we know that $a_1 = 1$. It should

also be clear that $d = 1$ (since the common difference between successive terms is 1), and that $n = 100$ (since we are summing up to the hundredth term). Now it is just a question of plugging these values into the formula.

$$S_{100} = \tfrac{n}{2}[2a_1 + (n-1)d]$$
$$S_{100} = \tfrac{100}{2}[2 + (100-1)]$$
$$S_{100} = 50(101) = 5050$$

Worked Example 23 **: Calculating S_n for a Geometric Series in General**

Question: Calculate a formula for S_n for any geometric series, given r and a_1.
Answer: This example is slightly different from the case for arithmetic series. Try writing out the calculation for yourself to make sure that you understand all the steps.

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + ... + a_1 r^{n-2} + a_1 r^{n-1} \\ r \times S_n &= a_1 r + a_1 r^2 + ... + a_1 r^{n-2} + a_1 r^{n-1} + a_1 r^n \\ rS_n - S_n &= -a_1 + 0 + 0 ... + 0 + a_1 r^n \\ S_n(r-1) &= a_1 r^n - a_1 \end{aligned}$$

$$S_n = \frac{a_1(r^n - 1)}{r - 1} \tag{10.74}$$

This formula is only valid when $r \neq 1$, otherwise we would have 0 in the denominator. When $r = 1$, $S_n = a_1 + a_1 + ... + a_1 = na_1$.

Worked Example 24 **: Using the Formula for S_n**

Question: What is $2 + 4 + 8 + 16 + ... + 32768$?
Answer: We are dealing with a geometric series, so we have to use equation (10.74). In order to use it we need to know which values to put in for a_1, r, and n. Since the series starts at 2, we know that $a_1 = 2$. It should also be clear that $d = 2$ (since the common ratio between successive terms is 2). It is not so clear what n should be, but we can work it out using the equation for the nth term of a geometric series.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ 32768 &= 2(2^{n-1}) = 2^n \\ \log 32768 &= n \log 2 \\ n &= \frac{\log 32768}{\log 2} = 15 \end{aligned}$$

Now that we have values for a_1,n, and r, we can use the formula to work out S_{15}.

$$S_n = \frac{a_1(r^n-1)}{r-1}$$
$$S_{15} = \frac{2(2^{15}-1)}{2-1}$$
$$S_{15} = 2(2^{15}-1) = 65534$$

Worked Example 25 **: Sigma Notation**

Question: Calculate the values of the following expressions:

- $\sum_{k=2}^{7} k+1$
- $\sum_{t=2}^{4} t^2$
- $\sum_{t=2}^{4} 2^t$
- $\sum_{k=1}^{4} 3$

Answer:

- We have to sum the expression $k+1$ from $k=2$ until $k=7$: $(1+1)+(2+1)+(3+1)+(4+1)+(5+1)+(6+1)+(7+1) = 35$.
- $\sum_{t=2}^{4} t^2 = 2^2+3^2+4^2 = 4+9+16 = 29$
- $\sum_{t=2}^{4} 2^t = 2^2+2^3+2^4 = 4+8+16 = 28$
- $\sum_{k=1}^{4} 3 = 3+3+3+3 = 12$

Worked Example 26 **: Converging or diverging**

Question: Which of the following infinite series do you think will converge, and which do you think will diverge:

- $\sum_{k=1}^{\infty} k$
- $\sum_{k=1}^{\infty} \frac{1}{k}$
- $\sum_{k=1}^{\infty} \frac{1}{k^2}$
- $\sum_{k=1}^{\infty} (-1)^(k+1)$

Answer: We don't really have any systematic way of working this out yet, but we can easily guess by using our calculators.

- Working out the first few terms of the sum, we get $S_1 = 1$, $S_2 = 1+2 = 3$, $S_3 = 1+2+3 = 6$, $S_4 = 1+2+3+4 = 10$, $S_5 = 1+2+3+4+5 = 15$. Clearly this is getting larger and larger, and it would be reasonable to guess that if we kept on adding terms, we would not get a finite number. So this series diverges.
- As before, we can work out the first few terms by hand: $S_1 = 1$, $S_2 = 1+\frac{1}{2} = 1\frac{1}{2}$, $S_3 = 1+\frac{1}{2}+\frac{1}{3} = 1\frac{5}{6}$, $S_4 = 2\frac{1}{12}$, $S_5 = 2\frac{17}{60}$, and, using a calculator, $S_{20} = 3.6$. This is still not absolutely clear, so now we can write a computer program to work out even higher values of S_n. It turns out that

$S_{100} = 5.19$, $S_{1000} = 7.49$, and $S_{1000000} = 14.4$. This series is in fact divergent. Even though it grows at a very slow rate, it never stops getting larger when we add more terms.

– As we mentioned in the text above, this series never gets larger than 1, no matter how many terms we care to add. In fact, the more terms we add, the closer the series gets to 1 (try it on your calculator), so we say that the series converges to 1.

– This example seems a bit strange at first. Let us try to write out a few terms of S_n to give us an idea of what is happening: $S_1 = 1$, $S_2 = 1-1 = 0$, $S_3 = 1-1+1 = 1$, $S_4 = 0$, $S_5 = 1$. It seems that the values we get are oscillating between 0 and 1. Remember we said that a series converges if we get closer and closer to some number when we add more terms - and that is clearly not what is happening here. That means that this series is divergent, even though it never gets larger than 1.

***Worked Example 27* : Using the formula for S_∞**

Question: Calculate the sum of the infinite series $\sum_{k=1}^{\infty}(\frac{1}{2})^{k-1}$
Answer: We are asked to determine the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ The common ratio is clearly $\frac{1}{2}$, which is between -1 and 1, so we can use the formula. The formula gives $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

10.3 functions

Q: State the domain and range of the function $y = x^2 - 4$ in set-builder notation and interval notation.

A: There is nothing to stop x from taking on the value of any real number, but, since x^2 cannot be negative, we see that $y \geq -4$. Thus the domain of the function is

$$\{x : x \in \mathbb{R}\} \quad \text{or} \quad (-\infty,\infty) \tag{10.75}$$

and the range is given by

$$\{y : y \geq 4 \text{ and } y \in \mathbb{R}\} \quad \text{or} \quad [-4,\infty) \tag{10.76}$$

<u>Worked Example:</u>

Q: Plot a graph of the function $f(x) = -x + 1$.

A: The x-intercept is

$$-x + 1 = 0 \tag{10.77}$$
$$\Rightarrow x = 1 \tag{10.78}$$

The y-intercept is

$$y = -(0) + 1 = 1 \tag{10.79}$$

Therefore the graph is as follows:

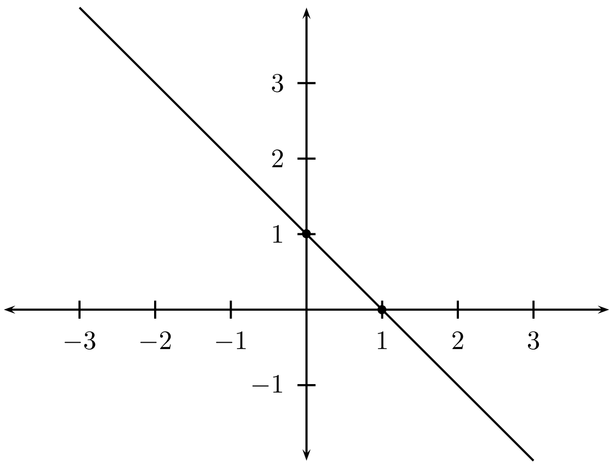


Figure 10.1: Graph of $f(x) = 1 - x$

Now we have seen that, in general, the constant b is the y-intercept, but what is a? The bigger a the faster the y-values change when the x values change. Therefore a is called the slope and shows how steep the straight line is.

Worked Example:

Example 1:

Q: We are given a straight line graph $f(x) = ax + b$ and two different points (x_1,y_1) and (x_2,y_2), where $x_1 \neq x_2$. What is the slope of the line?

A: Now we need to calculate the slope a and we know that $y_1 = ax_1 + b$ and $y_2 = ax_2 + b$, since (x_1,y_1) and (x_2,y_2) are points on the straight line. This means that

$$y_2 - y_1 = (ax_1 + b) - (ax_2 + b) \tag{10.80}$$
$$= ax_2 - ax_1 + b - b \tag{10.81}$$
$$= a(x_2 - x_1) \tag{10.82}$$

Therefore the slope describes the change in y (sometimes called $\Delta y = y_2 - y_1$) divided by the change in x ($\Delta x = x_2 - x_1$) between any two different points on the line, i.e.

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \tag{10.83}$$

We have used the fact that $x_1 \neq x_2$ (i.e. $x_2 - x_1 \neq 0$) because one cannot divide by zero. (The case of $x_1 = x_2$ but $y_1 \neq y_2$ gives an infinite slope and this describes a vertical line at constant x.)(NOTE: This section needs to work in the relevance of figure 10.2 more)

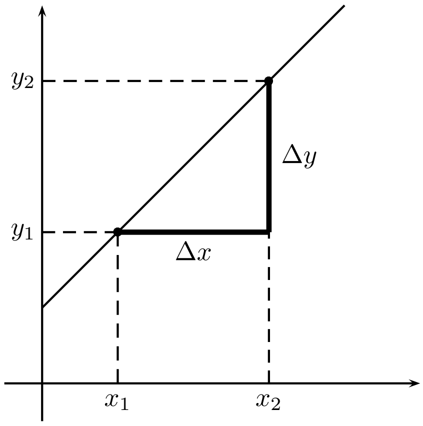


Figure 10.2: Graph showing Δx and Δy for a line of the form $y = ax + b$

Note: A straight line with a positive slope ($a > 0$) increases from left to right and a straight line with a negative slope ($a < 0$) increases from right to left.

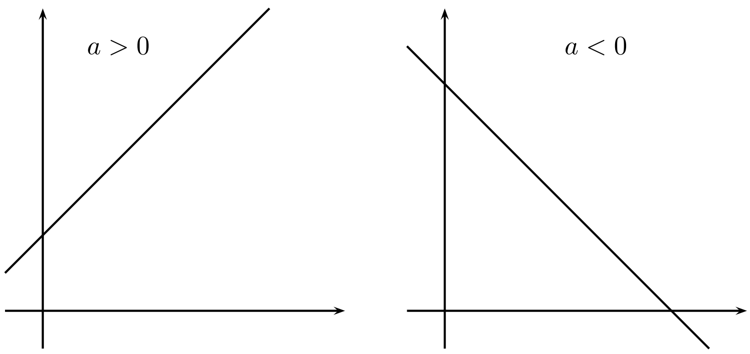


Figure 10.3: A straight line with positive slope ($a > 0$) and a straight line with negative slope ($a < 0$)

Example 2:

Q: Consider the straight line shown in the graph below:

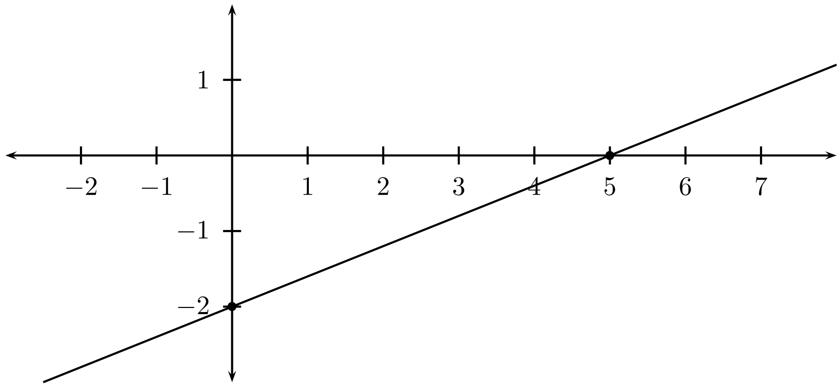


Figure 10.4: Graph of straight line with $y_0 = -2$ and $x_0 = 5$

Find the equation of the straight line describing this graph.

A: We need to find $f(x) = ax + b$, so we need the a and b values for this line.

We see that the y-intercept is $y = -2$ so $b = -2$. Since we have two points (the y-intercept (0,-2) and the x-intercept (5,0)), we can find the slope using the equation from the previous example. Therefore

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-2)}{5 - 0} = \frac{2}{5} \tag{10.84}$$

So the equation of the straight line is $f(x) = \frac{2}{5}x - 2$.

Now, for a general parabola of the form $f(x) = ax^2 + c$ where a is positive, the term ax^2 is always positive so the function is at a minimum when $x = 0$. Therefore the arms of the parabola point upwards. Otherwise, if a is negative, then ax^2 is always negative and thus the function is maximum at $x = 0$. This means that the arms of the parabola must point downwards.

Worked Examples:

Example 1:

Q: Find the intercepts and thus plot a graph of the function $f(x) = -x^2 + 4$.

A: The y-intercept is

$$y = -(0)^2 + 4 = 4 \tag{10.85}$$

and the x-intercepts are

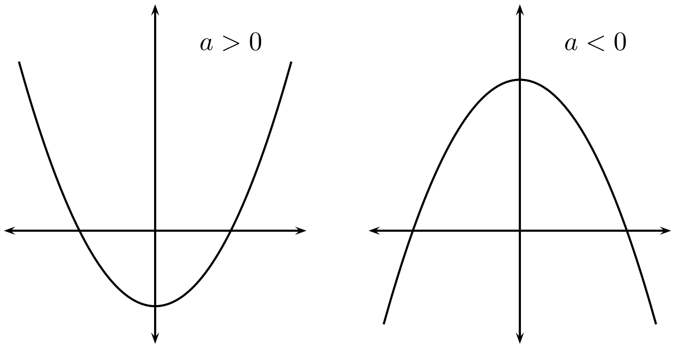


Figure 10.5: A parabola of the form $y = ax^2 + c$ with positive curvature ($a > 0$) and a parabola with negative curvature ($a < 0$)(NOTE: is curvature the right word to use here?)

$$-x^2 + 4 = 0 \tag{10.86}$$
$$\Rightarrow x^2 = 4 \tag{10.87}$$
$$\Rightarrow x = \pm 2 \tag{10.88}$$

Since $a = -1$ is negative, we know that the arms of the parabola must point downwards. We can now use the three intercepts to draw the graph of this function.

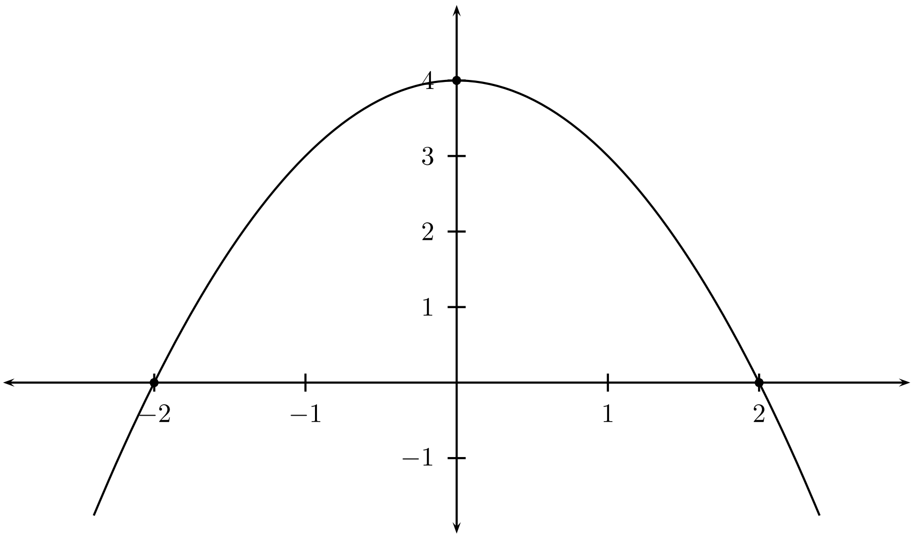


Figure 10.6: Graph of $f(x) = -x^2 + 4$

Example 2:

Q: Plot a graph of the parabola $f(x) = x^2 + 1$.

A: The y-intercept is

$$y = (0)^2 + 1 = 1 \tag{10.89}$$

and the x-intercepts are

$$x^2 + 1 = 0 \tag{10.90}$$
$$\Rightarrow x^2 = -1 \tag{10.91}$$
$$\tag{10.92}$$

This is not possible if x is a real number, so there are no x-intercepts. The parabola must therefore be entirely above the x-axis. This agrees with the fact that we know the arms of the parabola point upwards because $a = 1$ is positive. Thus the graph is as follows:

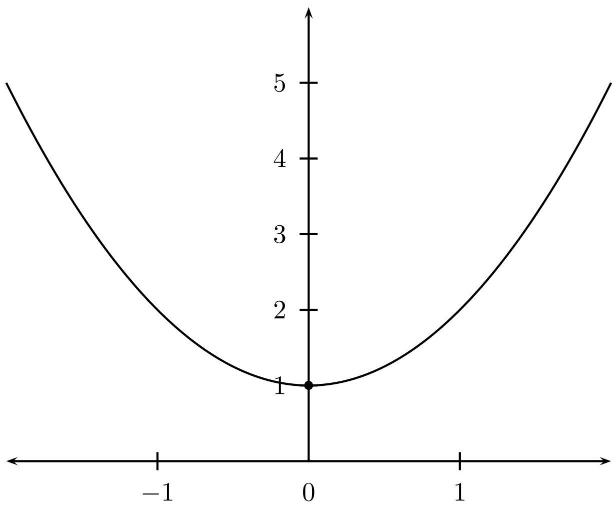


Figure 10.7: Graph of the parabola $f(x) = x^2 + 1$

Example 2:

Q: Plot a graph of the hyperbola $xy = -1$.

A: A table of x and y values is as follows:

x :	-4	-2	-1	$\frac{1}{2}$	-$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y = \frac{1}{x}$:	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	-4	-2	-1	-$\frac{1}{2}$	-$\frac{1}{4}$

This gives the graph shown below.

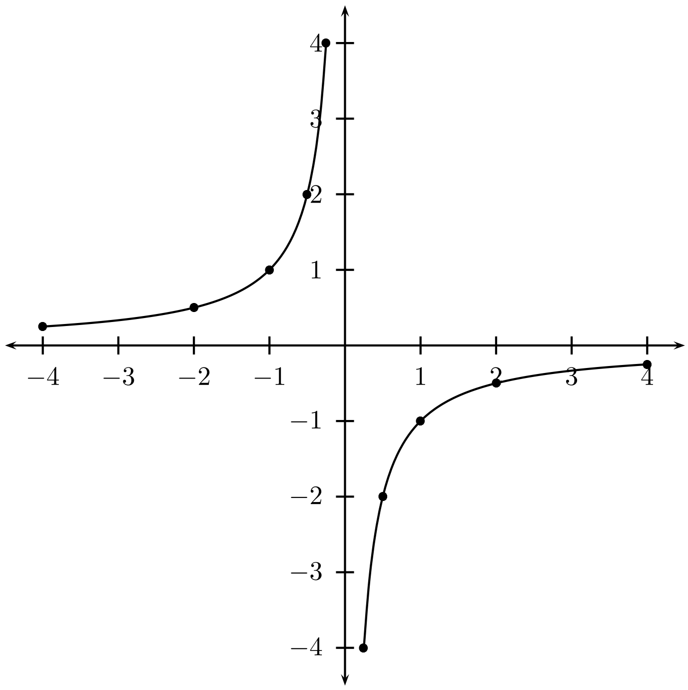


Figure 10.8: Graph of the hyperbola $xy = -1$

We can see that a hyperbola has no x or y-intercepts. However, there are two general forms for hyperbolae, depending on whether a is positive or negative.

Both types of hyperbola are symmetric about the lines $y = x$ and $y = -x$ (in other words, the part of the hyperbola on one side of the line is just the reflection of the part on the other side).

In the first case ($a > 0$), the hyperbola intersects line $y = x$ at two points. At these intersections

$$xy = a \quad \text{and} \quad y = x \tag{10.93}$$

which implies that

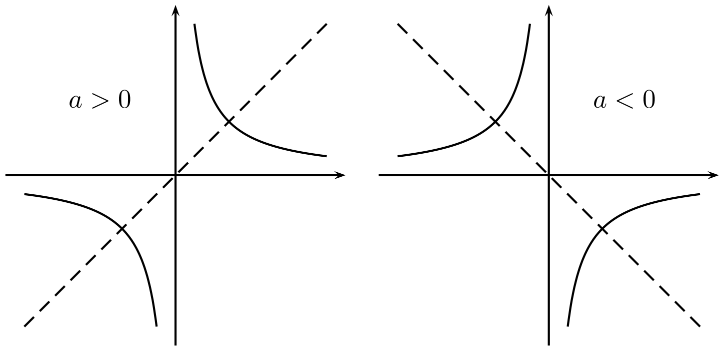


Figure 10.9: A hyperbola of the form $y = \frac{a}{x}$ with positive coefficient ($a > 0$) and straight line $y = x$. A hyperbola with negative curvature ($a < 0$) and straight line $y = -x$.

$$x(x) = x^2 = a \quad (10.94)$$
$$\Rightarrow \quad x = \pm\sqrt{a} \quad (10.95)$$

and, as $y = x$, it follows that the points of intersection are $(-\sqrt{a}, -\sqrt{a})$ and $(\sqrt{a}, \sqrt{a})$.

In the second case ($a < 0$), the hyperbola intersects the line $y = -x$. The two intersection points occur when

$$xy = a \quad \text{and} \quad y = -x \quad (10.96)$$

and therefore

$$x(-x) = -x^2 = a \quad (10.97)$$
$$\Rightarrow \quad x^2 = -a \quad (10.98)$$
$$\Rightarrow \quad x = \pm\sqrt{-a} \quad (10.99)$$

and, since $y = -x$, the two intersection points are $(-\sqrt{a}, \sqrt{a})$ and $(\sqrt{a}, -\sqrt{a})$.

Worked Examples:

Example 1:

Q: Draw a graph of the hyperbola $xy = 9$.

A: First we note that, since $a = 9$ is positive, the hyperbola must be in the top right and bottom left quadrants. The points at which it intersects the line $y = x$ are (-3,-3) and (3,3). It is also clear that the points (-1,-9), (-9,-1), (1,9) and (9,1) are part of the hyperbola. Thus the graph is

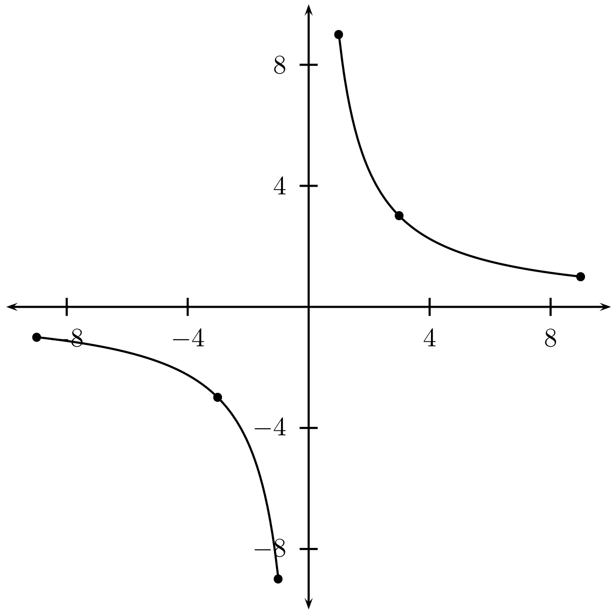


Figure 10.10: Graph of the hyperbola $xy = 9$

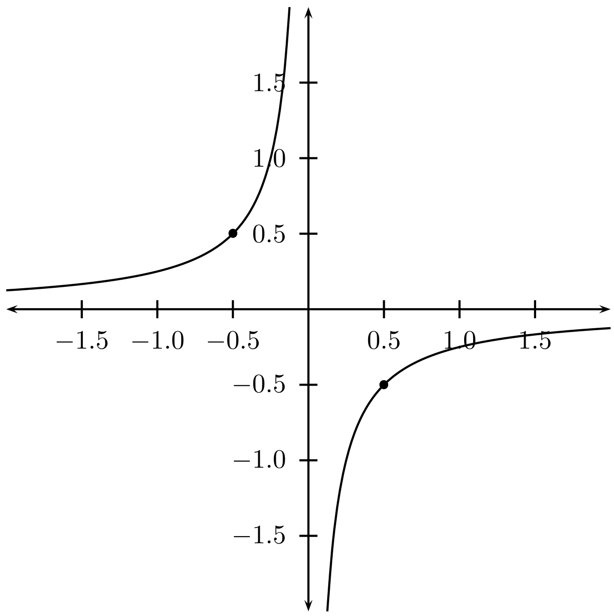


Figure 10.11: Graph of a hyperbolic function turning at the points $(-\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},-\frac{1}{2})$

Example 2:

Q: The graph of a hyperbolic function is shown below.

What is the function defining this hyperbola?

A: Since the hyperbola is in the top left and bottom right quadrants, we known that $a < 0$. Now the point $(-\frac{1}{2}, \frac{1}{2})$ lies on the hyperbola, so

$$a = xy = (-\frac{1}{2})(\frac{1}{2}) = -\frac{1}{4} \tag{10.100}$$

which is negative as we originally worked out.

Therefore the hyperbolic function is $f(x) = -\frac{1}{4x}$.

Worked Example:

Q: Plot graph of the relation $x^2 + y^2 = 4$.

A: This is the equation of a circle centered at the origin with radius 2. The graph is as follows:

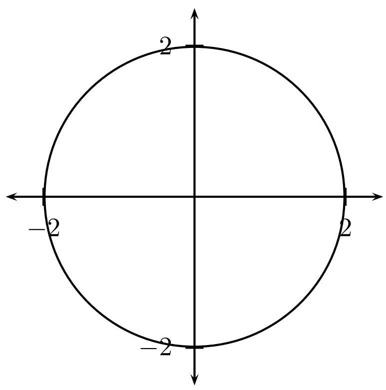


Figure 10.12: Graph of the circle $x^2 + y^2 = 4$

Worked Example:

Q: Draw the semi-circles described by the equations

$$y = \sqrt{9 - x^2} \quad \text{and} \quad y = -\sqrt{9 - x^2} \tag{10.101}$$

Also give the domains and ranges of these semi-circles.

A: These equations describe semi-circles with radius 3. The first semi-circle lies above the x-axis (since the positive square root is being considered, the y values are all positive) and the second semi-circle is below the x-axis (the y values are all negative because the equation involves the negative root). The graphs of these semi-circles are thus as follows:

From the above graphs we can see that, for the semi-circle $y = \sqrt{9 - x^2}$, the domain is $[-3,3]$ and the range is $[0,3]$ and, for the semi-circle $y = -\sqrt{9 - x^2}$, the domain is $[-3,3]$ and the range is [-3,0].

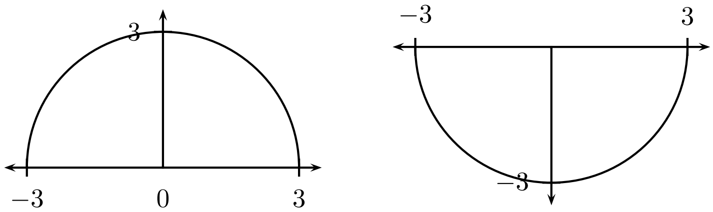


Figure 10.13: Semi-circles of radius 3, centered at the origin. The equation for the left semi-circle is $y = \sqrt{9 - x^2}$ whereas the right semi-circle is governed by $y = -\sqrt{9 - x^2}$

The equation $x^2 + y^2 = r^2$ can also be used to solve for x which gives

$$x = \pm\sqrt{r^2 - y^2} \tag{10.102}$$

Again the positive and negative roots each describe a semi-circle, but in this case, on either side of the y-axis. Therefore the equations for two other types of semi-circles are

$$x = \sqrt{r^2 - y^2} \quad \text{and} \quad x = -\sqrt{r^2 - y^2} \tag{10.103}$$

The domains of these semi-circles are $[0,r]$ (in the first case) and $[-r,0]$ (in the second case). In both cases, the range is $[-r,r]$.

Note: Again $y = -r$ and $y = r$ corresponds to the same $x = 0$ value. Thus these semi-circles are not functions.

Worked Example:

Q: Plot the graphs of the following relations

$$x = \sqrt{1-y^2} \quad \text{and} \quad x = -\sqrt{1-y^2} \tag{10.104}$$

State the domains and ranges of these relations.

A: The above equations describe semi-circles of radius 1 on either side of the y axis (in the first case x is always positive and in the second x is always negative). The graphs of these relations are as follows:

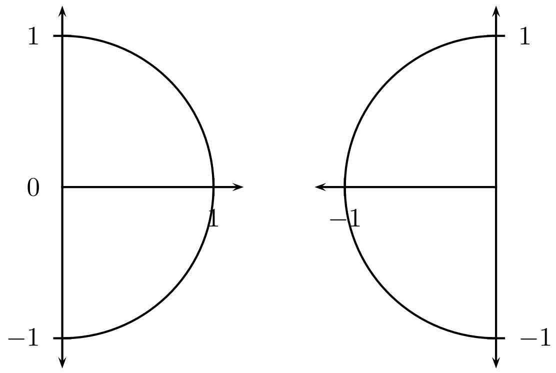


Figure 10.14: Semi-circles of radius 1 on either side of the y axis. In the first case x is always positive as $x = \sqrt{1-y^2}$ and in the second case x is always negative as $x = -\sqrt{1-y^2}$. (NOTE: This entire chapter needs a rethink on how it references graphs... instead of non-descriptive sentences like "in the first case", we need to use \ref a lot more)

For the first semi-circle, the domain is [0,1] and the range is [−1,1] and in the case of the second semi-circle, the domain is [−1,0] and the range is [−1,1].

Worked Example:

Q: Plot a graph of the function $f(x) = |x+2| - 5$.

A: Let us first work out a table of x and $f(x)$ values as follows:

x:	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$f(x)$	1	0	-1	-2	-3	-4	-5	-4	-3	-2	-1	0	1	2	3	4	5

The graph of the function $f(x)$ is thus shown below.

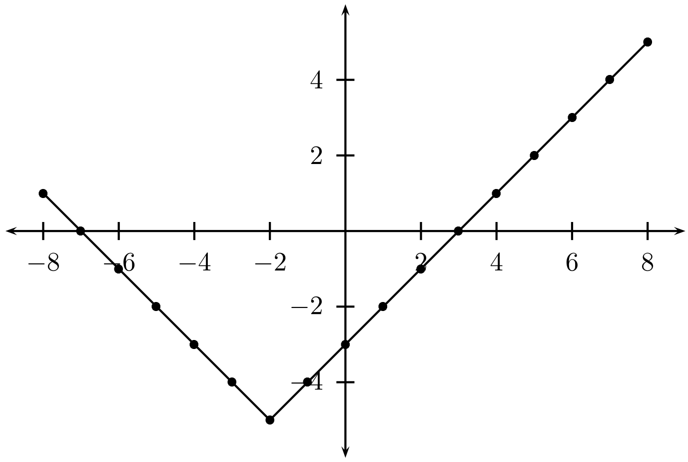


Figure 10.15: Graph of the function $f(x) = |x + 2| - 5$

Worked Examples:

Example 1:

Q: Consider the function $y = 2|x - 1| - 4$. Plot a graph of this absolute value function, showing all the intercepts, the turning point and the axis of symmetry.

A: We can see that, since $b = 1$ and $c = -4$, the turning point is $(1, -4)$ and the axis of symmetry is $x = 1$. Also, because $a = 2$ is positive, the absolute value function will be V-shaped.

Now let us find the y-intercept. At $x = 0$

$$y = 2|x - 1| - 4 = 2|(0) - 1| - 4 = 2| - 1| - 4 = 2(1) - 4 = -2 \quad (10.105)$$

Since the turning point is below the x-axis and the graph points upwards, we suspect that this function does have x-intercepts. Let us try to find these intercepts, which are the x-intercepts of the two straight lines $y = 2(x - 1) - 4$ and $y = 2(-x + 1) - 4$.

$$\begin{aligned} & y = 2(x-1) - 4 = 0 && (10.106)\\ \Rightarrow\ & 2x - 6 = 0 && (10.107)\\ \Rightarrow\ & 2x = 6 && (10.108)\\ \Rightarrow\ & x = 3 && (10.109)\\ & && (10.110) \end{aligned}$$

and

$$
\begin{aligned}
& y = 2(-x+1) - 4 = 0 && (10.111)\\
\Rightarrow\ & -2x - 2 = 0 && (10.112)\\
\Rightarrow\ & -2x = 2 && (10.113)\\
\Rightarrow\ & x = -1 && (10.114)\\
& && (10.115)
\end{aligned}
$$

Therefore the graph of the absolute values function is as follows:

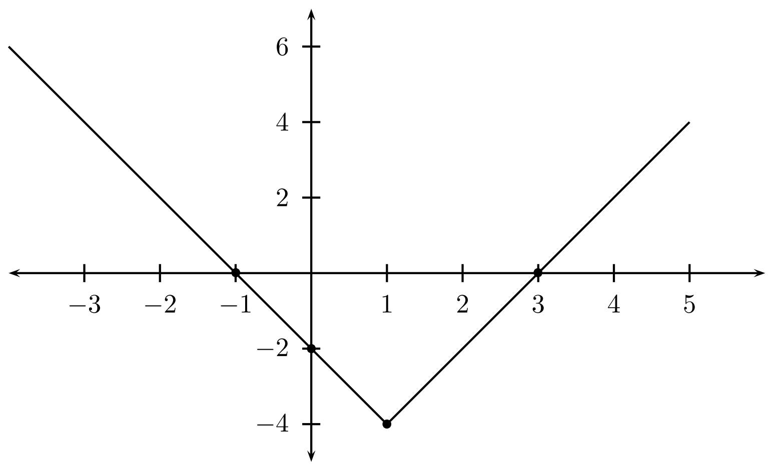


Figure 10.16: Graph of the absolute values function $y = 2|x-1| - 4$

Example 2:

Q: Plot a graph of the function $f(x) = -|x+3|-1$ showing the intercepts, turning point and axis of symmetry.

A: The turning point is of this function is $(-3, -1)$ and the axis of symmetry is given by $x = -3$. As $a = -1$ is negative the graph is shaped like an upside down V.

At $x = 0$, the y-intercept is

$$f(0) = -|(0)+3| - 1 = -|3| - 1 = -3 - 1 = -4 \qquad (10.116)$$

Now this function points downwards and obtains its maximum at the turning point $(-3, -1)$. This point is below the x-axis (since $y = -1$ is negative). Since the function is never greater than -1, it cannot become positive and therefore cannot cross the x-axis. Thus this absolute value function has no x-intercepts.

The graph of the function is shown below.

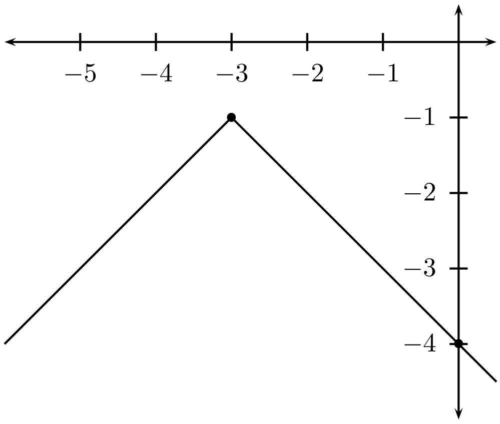


Figure 10.17: Graph of the function $f(x) = -|x+3| - 1$

Worked Example:

Q: What the does the relation $x^2 - 2x + y^2 + 4y = -4$ describe? Plot a graph of this relation.

A: We can see that it is not going to be easy to solve to x and y, but let us try another trick, which is to write the relation as the sum of perfect squares. We know that

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x-1)^2 - 1 \tag{10.117}$$

and also that

$$y^2 + 4y = (y^2 + 4y + 4) - 4 = (y+2)^2 - 4 \tag{10.118}$$

Therefore the relation can be written as follows:

$$(x-1)^2 - 1 + (y+2)^2 - 4 = -4 \tag{10.119}$$

and thus

$$(x-1)^2 + (y+2)^2 = 1 \tag{10.120}$$

But this is just the equation for a circle centered at the origin with radius 1 ($x^2 + y^2 = 1$), where x and y have been replaced by $x-1$ and $y+2$. So we can see that this is a circle which has been moved 1 to the right and 2 downwards. Therefore this relation describes a circle centered at the point (1,-2) with radius 1 (as shown in the diagram below).

Note: In general, the equation for a circle of radius r centered at the point (a,b) is

$$(x-a)^2 + (y-b)^2 = r^2 \tag{10.121}$$

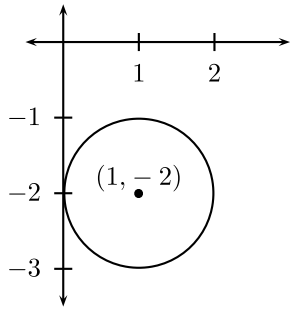


Figure 10.18: Graph of the circle $(x-1)^2 + (y+2)^2 = 1$

Worked Example:

Q: Show that the equation for an ellipse centered at the origin can be derived from that of a circle centered at the origin by performing suitable transformations. Draw a graph of the resulting ellipse.

A: We shall start with the equation of a circle centered at the origin, which is given by $x^2 + y^2 = r^2$, and replace x by $\frac{r}{a}x$ and y by $\frac{r}{b}y$.

We can see that the equation for the circle becomes

$$(\frac{r}{a}x)^2 + (\frac{r}{b}y)^2 = r^2 \tag{10.122}$$

$$\Rightarrow \quad r^2\frac{x^2}{a^2} + r^2\frac{y^2}{b^2} = r^2 \tag{10.123}$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{10.124}$$

which is the equation for an ellipse centred at the origin with y-intercepts $(0,-b)$ and $(0,b)$ and x-intercepts $(-a,0)$ and $(a,0)$.

The graph of this ellipse is as follows:

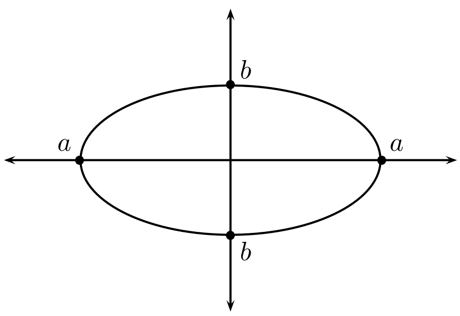


Figure 10.19: Graph of an ellipse centered at the origin with y-intercepts $(0,-b)$ and $(0,b)$ and x-intercepts $(-a,0)$ and $(a,0)$

10.3.1 Worked Examples:

Example 1:

Q: Consider the parabolic function $f(x) = x^2$. What is the equation for the function which can be obtained by stretching $f(x)$ vertically by a factor of a, then shifting this function to the right by p and upwards by q?

A: We start with the function $y = x^2$ and must be very careful to apply these changes in the right order. First we stretch $f(x)$ vertically by a factor of a. This means that we must change y to $\frac{y}{a}$, which gives

$$\frac{y}{a} = x^2 \tag{10.125}$$
$$\Rightarrow \quad y = ax^2 \tag{10.126}$$

Now we shift $y = ax^2$ to the right by p. In other words, we change x to $x - p$, which gives

$$y = a(x-p)^2 \tag{10.127}$$

Finally we shift $y = a(x-p)^2$ upwards by q. Therefore we must replace y by $y - q$. This means that

$$y - q = a(x-p)^2 \tag{10.128}$$
$$\Rightarrow \quad y = a(x-p)^2 + q \tag{10.129}$$

which is one way of writing the general formula for a quadratic function (see section 1.4).

Example 2:

Q: Let $f(x) = x + 6$. What is the function which is the result of reflecting $f(x)$ about the y-axis and then shrinking this function by a factor of 2 horizontally and then vertically? Plot graphs of the initial and final functions. There are other ways of getting this final function from $f(x)$ by performing only two changes. Give an example of such a method.

A: We start with the initial function $y = x + 6$. If we reflect this about the x-axis, we must change x to $-x$ and this gives

$$y = -x + 6 \tag{10.130}$$

Now we must shrink this function by a factor of 2 horizontally. Thus changing x to $2x$ result in the equation

$$y = -2x + 6 \tag{10.131}$$

If we now shrink this by a factor of 2 vertically (changing y to $2y$) we get

$$2y = -2x + 6 \quad (10.132)$$
$$\Rightarrow \quad y = -x + 3 \quad (10.133)$$

The graphs of the initial function $y = x + 6$ and the final function $y = -x + 3$ are shown below.

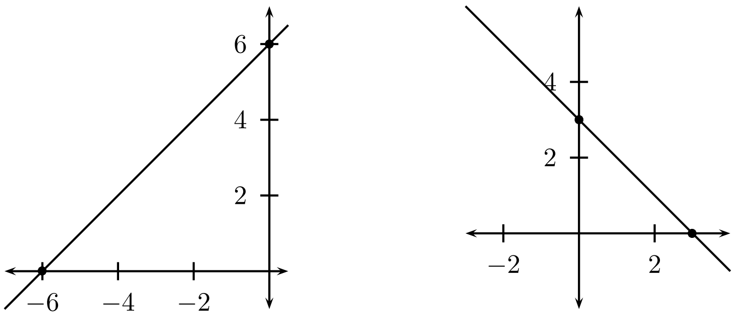


Figure 10.20: Graph of the initial function $y = x+6$ (left) and the final function $y = -x + 3$ (right)

Finally we must look for two transformations which give this same result. We suspect that there must be a reflection involved, because the final function involves $-x$ and the initial function contains x (also the above graph shows that the straight lines are perpendicular). Therefore, as before, we start by reflecting the function $f(x)$ about the y-axis. This gives

$$y = -x + 6 \quad (10.134)$$

Now we see that to get the final equation $y = -x + 3$ we must also add -3 to this equation. This is the same as replacing x by $x + 3$, so let us try shifting the equation by 3 to the left. This gives the equation

$$y = -(x + 3) + 6 \quad (10.135)$$
$$\Rightarrow \quad y = -x + 3 \quad (10.136)$$

Therefore, two changes which give the same final function are a reflection about the y-axis followed by a shift by 3 to left.

Appendix A

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