MISCELLANEOUS TOPICS IN FIRST YEAR MATHEMATICS

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Chapter 2

POLYNOMIALS

2.1. Introduction

We shall be considering polynomials with rational, real or complex coefficients. Accordingly, throughout this chapter, $\mathbb{F}$ denotes $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Definition. We denote by $\mathbb{F}[x]$ the set of all polynomials of the form

$$p(x) = p_k x^k + p_{k-1} x^{k-1} + \ldots + p_1 x + p_0, \quad \text{where } k \in \mathbb{N} \cup \{0\} \text{ and } p_0, \ldots, p_k \in \mathbb{F};$$

in other words, $\mathbb{F}[x]$ denotes the set of all polynomials in variable x and with coefficients in $\mathbb{F}$. Suppose further that $p_k \neq 0$. Then p_k is called the leading coefficient of the polynomial $p(x)$, $p_k x^k$ is called the leading term of the polynomial $p(x)$, and k is called the degree of the polynomial $p(x)$. In this case, we write $k = \deg p(x)$. Furthermore, if $p_k = 1$, then the polynomial $p(x)$ is called monic.

Example 2.1.1. The polynomial $3x^2 + 2$ is in $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Furthermore, it has degree 2 and leading coefficient 3.

Example 2.1.2. The polynomial $4x^2 + 3x - \sqrt{5}$ is in $\mathbb{R}[x]$ and $\mathbb{C}[x]$ but not $\mathbb{Q}[x]$. Furthermore, it has degree 2 and leading coefficient 4.

Example 2.1.3. The polynomial $x^3 + (3 + 2\mathrm{i})x - 3$ is in $\mathbb{C}[x]$ but not $\mathbb{Q}[x]$ or $\mathbb{R}[x]$. Furthermore, it has degree 3 and is monic.

Example 2.1.4. The constant polynomial 5 is in $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Furthermore, it has degree 0 and leading coefficient 5.

REMARK. We have defined the degree of any non-zero constant polynomial to be 0. Note, however, that we have not defined the degree of the constant polynomial 0. The reason for this will become clear from Proposition 2A.

DEFINITION. Suppose that

$$p(x) = p_k x^k + p_{k-1}x^{k-1} + \ldots + p_1 x + p_0 \qquad \text{and} \qquad q(x) = q_m x^m + q_{m-1}x^{m-1} + \ldots + q_1 x + q_0$$

are two polynomials in $\mathbb{F}[x]$. Then we write

$$p(x) + q(x) = (p_n + q_n)x^n + (p_{n-1} + q_{n-1})x^{n-1} + \ldots + (p_1 + q_1)x + (p_0 + q_0), \tag{1}$$

where $n = \max\{k, m\}$. Furthermore, we write

$$p(x)q(x) = r_{k+m}x^{k+m} + r_{k+m-1}x^{k+m-1} + \ldots + r_1 x + r_0, \tag{2}$$

where, for every $s = 0, 1, \ldots, k+m$,

$$r_s = \sum_{j=0}^{s} p_j q_{s-j}. \tag{3}$$

Here, we adopt the convention $p_j = 0$ for every $j > k$ and $q_j = 0$ for every $j > m$.

EXAMPLE 2.1.5. Suppose that $p(x) = 3x^2 + 2$ and $q(x) = x^3 + (3 + 2\mathrm{i})x - 3$. Note that $k = 2$, $p_0 = 2$, $p_1 = 0$ and $p_2 = 3$. Note also that $m = 3$, $q_0 = -3$, $q_1 = 3 + 2\mathrm{i}$, $q_2 = 0$ and $q_3 = 1$. If we adopt the convention, then $k + m = 5$ and $p_3 = p_4 = p_5 = q_4 = q_5 = 0$. Now

$$p(x) + q(x) = (0 + 1)x^3 + (3 + 0)x^2 + (0 + 3 + 2\mathrm{i})x + (2 - 3) = x^3 + 3x^2 + (3 + 2\mathrm{i})x - 1.$$

On the other hand,

$$\begin{aligned}
r_5 &= p_0q_5 + p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1 + p_5q_0 = p_2q_3 = 3,\\
r_4 &= p_0q_4 + p_1q_3 + p_2q_2 + p_3q_1 + p_4q_0 = p_1q_3 + p_2q_2 = 0 + 0 = 0,\\
r_3 &= p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0 = p_0q_3 + p_1q_2 + p_2q_1 = 2 + 0 + 3(3 + 2\mathrm{i}) = 11 + 6\mathrm{i},\\
r_2 &= p_0q_2 + p_1q_1 + p_2q_0 = 0 + 0 - 9 = -9,\\
r_1 &= p_0q_1 + p_1q_0 = 2(3 + 2\mathrm{i}) + 0 = 6 + 4\mathrm{i},\\
r_0 &= p_0q_0 = -6,
\end{aligned}$$

so that

$$p(x)q(x) = 3x^5 + (11 + 6\mathrm{i})x^3 - 9x^2 + (6 + 4\mathrm{i})x - 6.$$

Note that our technique for multiplication is really just a more formal version of the usual technique involving distribution, as

$$\begin{aligned}
p(x)q(x) &= (3x^2 + 2)(x^3 + (3 + 2\mathrm{i})x - 3)\\
&= (3x^2 + 2)x^3 + (3x^2 + 2)(3 + 2\mathrm{i})x - 3(3x^2 + 2)\\
&= (3x^5 + 2x^3) + (3(3 + 2\mathrm{i})x^3 + 2(3 + 2\mathrm{i})x) - (9x^2 + 6)\\
&= 3x^5 + (11 + 6\mathrm{i})x^3 - 9x^2 + (6 + 4\mathrm{i})x - 6.
\end{aligned}$$

More formally, we have

$$\begin{aligned}
&p(x)q(x) = (p_2x^2 + p_1x + p_0)(q_3x^3 + q_2x^2 + q_1x + q_0)\\
&\ = (p_2x^2 + p_1x + p_0)q_3x^3 + (p_2x^2 + p_1x + p_0)q_2x^2 + (p_2x^2 + p_1x + p_0)q_1x + (p_2x^2 + p_1x + p_0)q_0\\
&\ = (p_2q_3x^5 + p_1q_3x^4 + p_0q_3x^3) + (p_2q_2x^4 + p_1q_2x^3 + p_0q_2x^2) + (p_2q_1x^3 + p_1q_1x^2 + p_0q_1x)\\
&\qquad + (p_2q_0x^2 + p_1q_0x + p_0q_0)\\
&\ = p_2q_3x^5 + (p_1q_3 + p_2q_2)x^4 + (p_0q_3 + p_1q_2 + p_2q_1)x^3 + (p_0q_2 + p_1q_1 + p_2q_0)x^2 + (p_0q_1 + p_1q_0)x + p_0q_0.
\end{aligned}$$

The following result follows immediately from the definitions. For purely technical reasons, we define $\deg 0 = -\infty$, where 0 represents the constant zero polynomial.

PROPOSITION 2A. *Suppose that*

$$p(x) = p_k x^k + p_{k-1}x^{k-1} + \ldots + p_1 x + p_0 \qquad \text{and} \qquad q(x) = q_m x^m + q_{m-1}x^{m-1} + \ldots + q_1 x + q_0$$

are two polynomials in $\mathbb{F}[x]$*, where both* p_k *and* q_m *are non-zero, so that* $\deg p(x) = k$ *and* $\deg q(x) = m$*. Then*
(a) $\deg p(x)q(x) = k + m$*; and*
(b) $\deg(p(x) + q(x)) \le \max\{k, m\}$*.*

PROOF. (a) Suppose first of all that $p(x)$ and $q(x)$ are both different from the zero polynomial 0. Then it follows from (2) and (3) that the leading term of the polynomial $p(x)q(x)$ is $r_{k+m}x^{k+m}$, where

$$r_{k+m} = p_0 q_{k+m} + \ldots + p_{k-1}q_{m+1} + p_k q_m + p_{k+1}q_{m-1} + \ldots + p_{k+m}q_0 = p_k q_m \ne 0.$$

Hence $\deg p(x)q(x) = k + m$. If $p(x)$ is the zero polynomial, then $p(x)q(x)$ is also the zero polynomial. Note now that $\deg p(x)q(x) = -\infty = -\infty + \deg q(x) = \deg p(x) + \deg q(x)$. A similar argument applies if $q(x)$ is the zero polynomial.

(b) Recall (1) and that $n = \max\{k, m\}$. If $p_n + q_n \ne 0$, then $\deg(p(x) + q(x)) = n = \max\{k, m\}$. If $p_n + q_n = 0$ and $p(x) + q(x)$ is non-zero, then there is a largest $j < n$ such that $p_j + q_j \ne 0$, so that $\deg(p(x) + q(x)) = j < n = \max\{k, m\}$. Finally, if $p_n + q_n = 0$ and $p(x) + q(x)$ is the zero polynomial 0, then $\deg(p(x) + q(x)) = -\infty < \max\{k, m\}$. $\bigcirc$

2.2. Polynomial Equations

An equation of the type $ax + b = 0$, where $a, b \in \mathbb{F}$ and $a \ne 0$, is called a linear polynomial equation, or simply a linear equation, and has unique solution

$$x = -\frac{b}{a}.$$

Occasionally a given linear equation may look a little more complicated. However, with the help of some simple algebra, one can reduce the given equation to the form given above.

An equation of the type

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{F}$ are constants and $a \ne 0$, is called a quadratic polynomial equation, or simply a quadratic equation. To solve such an equation, we observe first of all that

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0 \end{aligned}$$

precisely when

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{4}$$

Suppose first of all that $\mathbb{F} = \mathbb{R}$, so that we are considering quadratic equations with real coefficients. Then there are three cases:

(1) If $b^2 - 4ac > 0$, then (4) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct real solutions for the quadratic equation.

(2) If $b^2 - 4ac = 0$, then (4) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Indeed, this solution occurs twice, as we shall see later.

(3) If $b^2 - 4ac < 0$, then the right hand side of (4) is negative. It follows that (4) is never satisfied for any real number x, so that the quadratic equation has no real solution.

Suppose next that $\mathbb{F} = \mathbb{C}$, so that we are considering quadratic equations with complex coefficients. Then there are two cases:

(1) If $b^2 - 4ac \neq 0$, then (4) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct complex solutions for the quadratic equation.

(2) If $b^2 - 4ac = 0$, then (4) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Again, this solution occurs twice, as we shall see later.

For polynomial equations of degree greater than 2, we do not have general formulae for their solutions. However, we may occasionally be able to find some solutions by inspection. These may help us find other solutions. However, we need to understand better how the solutions of polynomial equations are related to factorization of polynomials. We therefore first study the general problem of attempting to divide a polynomial by another polynomial.

2.3. Division of Polynomials

Let us consider the question of division in $\mathbb{Z}$. It is possible to divide an integer b by a non-zero integer a to get a main term q and remainder r, where $0 \le r < |a|$. In other words, we can find $q, r \in \mathbb{Z}$ such that $b = aq + r$ and $0 \le r < |a|$. In fact, q and r are uniquely determined by a and b. Note that what governs the remainder r is the restriction $0 \le r < |a|$; in other words, the "size" of r.

If one is to propose a theory of division in $\mathbb{F}[x]$, then one needs to find some way to measure the "size" of polynomials. This role is played by the degree. Let us now see what we can do.

Example 2.3.1. Let us attempt to divide the polynomial $b(x) = x^4 + 3x^3 + 2x^2 - 4x + 4$ by the non-zero polynomial $a(x) = x^2 + 2x + 2$. Then we can perform long division is a way similar to long division for integers.

$$\begin{array}{r|rrrrr}
 & & & x^2+ & x- & 2 \\ \cline{2-6}
x^2+\ 2x+\ 2\ ) & x^4+ & 3x^3+ & 2x^2- & 4x+ & 4 \\
 & x^4+ & 2x^3+ & 2x^2 & & \\ \cline{2-6}
 & & x^3 & - & 4x+ & 4 \\
 & & x^3+ & 2x^2+ & 2x & \\ \cline{3-6}
 & & & -\,2x^2- & 6x+ & 4 \\
 & & & -\,2x^2- & 4x- & 4 \\ \cline{4-6}
 & & & - & 2x+ & 8
\end{array}$$

Let us explain this a bit more carefully. We attempt to divide the polynomial $x^4 + 3x^3 + 2x^2 - 4x + 4$ by the polynomial $x^2 + 2x + 2$. A factor of x^2 will lift the term x^2 in the "smaller" polynomial to the term x^4 in the "bigger" polynomial, so let us take this first step, and examine the consequences.

$$\begin{array}{r|rrrrr}
 & & & x^2 & & \\ \cline{2-6}
x^2+\ 2x+\ 2\ ) & x^4+ & 3x^3+ & 2x^2- & 4x+ & 4 \\
 & x^4+ & 2x^3+ & 2x^2 & & \\ \cline{2-6}
 & & x^3 & - & 4x+ & 4
\end{array}$$

We next attempt to divide the polynomial $x^3 - 4x + 4$ by the polynomial $x^2 + 2x + 2$. A factor of x will lift the term x^2 in the "smaller" polynomial to the term x^3 in the "bigger" polynomial, so let us take this second step, and examine the consequences.

$$\begin{array}{r|rrrrr}
 & & & x^2+ & x & \\ \cline{2-6}
x^2+\ 2x+\ 2\ ) & x^4+ & 3x^3+ & 2x^2- & 4x+ & 4 \\
 & x^4+ & 2x^3+ & 2x^2 & & \\ \cline{2-6}
 & & x^3 & - & 4x+ & 4 \\
 & & x^3+ & 2x^2+ & 2x & \\ \cline{3-6}
 & & & -\,2x^2- & 6x+ & 4
\end{array}$$

We then attempt to divide the polynomial $-2x^2 - 6x + 4$ by the polynomial $x^2 + 2x + 2$. A factor of -2 will reconcile the x^2 terms, so let us take this third step, and complete our task. If we now write $q(x) = x^2 + x - 2$ and $r(x) = -2x + 8$, then $b(x) = a(x)q(x) + r(x)$. Note that $\deg r(x) < \deg a(x)$. We can therefore think of $q(x)$ as the main term and $r(x)$ as the remainder. If we think of the degree as a measure of size, then the remainder $r(x)$ is clearly "smaller" than $a(x)$.

In general, we have the following important result.

PROPOSITION 2B. *Suppose that $a(x), b(x) \in \mathbb{F}[x]$, and that $a(x) \neq 0$. Then there exist unique polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that*
(a) $b(x) = a(x)q(x) + r(x)$; and
(b) either $r(x) = 0$ or $\deg r(x) < \deg a(x)$.

Proof. Consider all polynomials of the form $b(x) - a(x)Q(x)$, where $Q(x) \in \mathbb{F}[x]$. If there exists $q(x) \in \mathbb{F}[x]$ such that $b(x) - a(x)q(x) = 0$, our proof is complete. Suppose now that $b(x) - a(x)Q(x) \neq 0$ for any $Q(x) \in \mathbb{F}[x]$. Then among all polynomials of the form $b(x) - a(x)Q(x)$, where $Q(x) \in \mathbb{F}[x]$, there must be one with smallest degree. More precisely, $m = \min\{\deg(b(x) - a(x)Q(x)) : Q(x) \in \mathbb{F}[x]\}$ exists. Let $q(x) \in \mathbb{F}[x]$ satisfy $\deg(b(x) - a(x)q(x)) = m$, and let $r(x) = b(x) - a(x)q(x)$. Then $\deg r(x) < \deg a(x)$, for otherwise, writing $a(x) = a_n x^n + \ldots + a_1 x + a_0$ and $r(x) = r_m x^m + \ldots + r_1 x + r_0$, where $m \geq n$, we have

$$r(x) - (r_m a_n^{-1} x^{m-n})a(x) = b(x) - a(x)\left(q(x) + r_m a_n^{-1} x^{m-n}\right) \in \mathbb{F}[x].$$

Clearly $\deg(r(x) - (r_m a_n^{-1} x^{m-n})a(x)) < \deg r(x)$, contradicting the minimality of m. On the other hand, suppose that $q_1(x), q_2(x) \in \mathbb{F}[x]$ satisfy $\deg(b(x)-a(x)q_1(x)) = m$ and $\deg(b(x)-a(x)q_2(x)) = m$. Let $r_1(x) = b(x) - a(x)q_1(x)$ and $r_2(x) = b(x) - a(x)q_2(x)$. Then $r_1(x) - r_2(x) = a(x)(q_2(x) - q_1(x))$. If $q_1(x) \neq q_2(x)$, then $\deg(a(x)(q_2(x) - q_1(x))) \geq \deg a(x)$, while $\deg(r_1(x) - r_2(x)) < \deg a(x)$, a contradiction. It follows that $q(x)$, and hence $r(x)$, is unique. ○

2.4. Roots of Polynomials

In this section, we study the relationship between roots and factors of a polynomial. The first step in our investigation is given by the following special case of Proposition 2B.

PROPOSITION 2C. *Suppose that $b(x) \in \mathbb{F}[x]$, and that $\alpha \in \mathbb{F}$. Then there exists a unique polynomial $q(x) \in \mathbb{F}[x]$ such that $b(x) = (x - \alpha)q(x) + b(\alpha)$.*

PROOF. By Proposition 2B, there exist unique polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that

$$b(x) = (x - \alpha)q(x) + r(x),$$

where either $r(x) = 0$ or $\deg r(x) = 0$. It follows that $r(x)$ is a constant polynomial, so that

$$b(x) = (x - \alpha)q(x) + r,$$

where $r \in \mathbb{F}$. Substituting $x = \alpha$, we have $r = b(\alpha)$. ○

We can now establish the following nice relationship between roots and factors of a polynomial. First we need a precise definition of factors of a polynomial.

DEFINITION. Suppose that $a(x), b(x) \in \mathbb{F}[x]$. Then we say that $a(x)$ is a factor of $b(x)$, denoted by $a(x) \mid b(x)$, if there exists $c(x) \in \mathbb{F}[x]$ such that $b(x) = a(x)c(x)$.

EXAMPLE 2.4.1. In $\mathbb{R}[x]$, $x - \sqrt{2}$ is a factor of $x^2 - 2$, for $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$. However, in $\mathbb{Q}[x]$, $x^2 - 2$ has no factor of the form $x - \alpha$, where $\alpha \in \mathbb{Q}$.

EXAMPLE 2.4.2. In $\mathbb{C}[x]$, $x - 2\mathrm{i}$ is a factor of $x^2 + 4$, for $x^2 + 4 = (x - 2\mathrm{i})(x + 2\mathrm{i})$. However, in $\mathbb{R}[x]$, $x^2 + 4$ has no factor of the form $x - \alpha$, where $\alpha \in \mathbb{R}$.

PROPOSITION 2D. *Suppose that $b(x) \in \mathbb{F}[x]$. Then $\alpha \in \mathbb{F}$ is a root of $b(x)$ if and only if $x - \alpha$ is a factor of $b(x)$.*

PROOF. By Proposition 2C, there exists a unique polynomial $q(x) \in \mathbb{F}[x]$ such that

$$b(x) = (x - \alpha)q(x) + b(\alpha). \tag{5}$$

If α is a root of $b(x)$, then $b(\alpha) = 0$, so that $b(x) = (x - \alpha)q(x)$, whence $x - \alpha$ is a factor of $b(x)$. On the other hand, if $x - \alpha$ is a factor of $b(x)$, then it follows from (5) that $x - \alpha$ must be a factor of $b(\alpha) = b(x) - (x - \alpha)q(x)$. Clearly we must have $b(\alpha) = 0$. ○

Let us apply this result to study roots of quadratic and cubic polynomials. For simplicity, we study only the case when the roots are distinct. The results in fact hold without this restriction.

Suppose that $\alpha, \beta \in \mathbb{F}$ are the two distinct roots of the quadratic polynomial $ax^2 + bx + c$, where $a \neq 0$. Then the polynomial must have two factors $x - \alpha$ and $x - \beta$. We must therefore have

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta.$$

Equating coefficients, we have

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = \frac{c}{a}.$$

Suppose that $\alpha, \beta, \gamma \in \mathbb{F}$ are the three distinct roots of the cubic polynomial $ax^3 + bx^2 + cx + d$, where $a \neq 0$. Then the polynomial must have three factors $x - \alpha$, $x - \beta$ and $x - \gamma$. We must therefore have

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta + \alpha\gamma + \beta\gamma)x - a\alpha\beta\gamma.$$

Equating coefficients, we have

$$\alpha + \beta + \gamma = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \qquad \text{and} \qquad \alpha\beta\gamma = -\frac{d}{a}.$$

2.5. Fundamental Theorem of Algebra

Consider the polynomial $x^2 - 2 \in \mathbb{Q}[x]$. This has no roots in $\mathbb{Q}$. To see this, suppose on the contrary that there exists $x \in \mathbb{Q}$ such that $x^2 = 2$. In other words, suppose that there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $(p/q)^2 = 2$. Without loss of generality, we can assume that p and q are relatively prime. Clearly, $p^2 = 2q^2$, so that p must be even. Write $p = 2r$, where $r \in \mathbb{Z}$. Then $4r^2 = 2q^2$, so that $q^2 = 2r^2$, whence q is even. It follows that both p and q are even, giving a contradiction. On the other hand, clearly $\sqrt{2} \in \mathbb{R}$ is a root of $x^2 - 2$. It follows from Proposition 2D that if we study the polynomial $x^2 - 2$ in $\mathbb{R}[x]$, then it has a factor $x - \sqrt{2}$. Furthermore, on dividing, we have $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$.

Next, consider the polynomial $x^2 + 4 \in \mathbb{R}[x]$. This has no roots in $\mathbb{Q}$. To see this, note simply that for every $x \in \mathbb{R}$, we have $x^2 \geq 0$, and so $x^2 \neq -4$. On the other hand, clearly $2\mathrm{i} \in \mathbb{C}$ is a root of $x^2 + 4$. It follows from Proposition 2D that if we study the polynomial $x^2 + 4$ in $\mathbb{C}[x]$, then it has a factor $x - 2\mathrm{i}$. Furthermore, on dividing, we have $x^2 + 4 = (x - 2\mathrm{i})(x + 2\mathrm{i})$.

We may now reasonably ask the following question. Are there non-constant polynomials $b(x) \in \mathbb{C}[x]$ with no roots in $\mathbb{C}$? If so, how do we extend $\mathbb{C}$ in order to find a root of this polynomial?

The answer to these questions is given by the the following result.

FUNDAMENTAL THEOREM OF ALGEBRA. *Suppose that $b(x) \in \mathbb{C}[x]$ is a monic polynomial of degree $k \geq 1$. Then there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, not necessarily distinct, such that*

$$b(x) = (x - \alpha_1) \ldots (x - \alpha_k).$$

Furthermore, any $\beta \in \mathbb{C}$ different from $\alpha_1, \ldots, \alpha_k$ cannot be a root of $b(x)$.

We omit the rather complicated proof.

2.6. Roots of Real Polynomials

In this section, we study polynomials with coefficients in $\mathbb{R}$, and think of their roots as elements of $\mathbb{C}$, as justified by the Fundamental theorem of algebra. Our first result shows that non-real roots come in pairs.

PROPOSITION 2E. *Suppose that $b(x) \in \mathbb{R}[x]$. Suppose further that $\alpha \in \mathbb{C}$ is a root of $b(x)$. Then $\overline{\alpha}$, the complex conjugate of α, is also a root of $b(x)$.*

PROOF. Write

$$b(x) = b_k x^k + \ldots + b_1 x + b_0, \quad \text{where } b_0, \ldots, b_k \in \mathbb{R}.$$

Since $\alpha \in \mathbb{C}$ is a root, we must have $b(\alpha) = 0$. Hence

$$0 = \overline{b(\alpha)} = \overline{b_k \alpha^k + \ldots + b_1 \alpha + b_0} = \overline{b_k}\,\overline{\alpha}^k + \ldots + \overline{b_1}\,\overline{\alpha} + \overline{b_0} = b_k \overline{\alpha}^k + \ldots + b_1 \overline{\alpha} + b_0 = b(\overline{\alpha}).$$

Hence $\overline{\alpha}$ is a root of $b(x)$. ◯

A simple consequence is the following result.

PROPOSITION 2F. *Every polynomial $b(x) \in \mathbb{R}[x]$ of odd degree has a root in $\mathbb{R}$.*

PROOF. Consider $b(x)$ as a polynomial in $\mathbb{C}[x]$. Then all the roots are given by the Fundamental theorem of algebra. Suppose on the contrary that none of these is real. Then by Proposition 2E, the roots occur as conjugate pairs. It follows that there must be an even number of roots. This contradicts the Fundamental theorem of algebra. ◯

EXAMPLE 2.6.1. Consider the polynomial $b(x) = x^5 - 7x^4 + 21x^3 - 33x^2 + 28x - 10$. We are given that $x = 1 + \mathrm{i}$ is a root of the equation $b(x) = 0$. Then by Proposition 2E, $x = 1 - \mathrm{i}$ is also a root of this equation. It follows from Proposition 2D that $x - 1 - \mathrm{i}$ and $x - 1 + \mathrm{i}$ are both factors of $b(x)$. Note that $(x - 1 - \mathrm{i})(x - 1 + \mathrm{i}) = x^2 - 2x + 2$. Long division gives $b(x) = (x^2 - 2x + 2)(x^3 - 5x^2 + 9x - 5)$, and so the other roots of $b(x)$ must be roots of $x^3 - 5x^2 + 9x - 5$. This is a real polynomial of odd degree, so it follows from Proposition 2F that it must have a real root. By inspection, $x = 1$ is a solution of the equation $x^3 - 5x^2 + 9x - 5 = 0$, so it follows from Proposition 2D that $x - 1$ is a factor of $x^3 - 5x^2 + 9x - 5$. Long division gives $x^3 - 5x^2 + 9x - 5 = (x - 1)(x^2 - 4x + 5)$. The last two roots of $b(x)$ must then be roots of $x^2 - 4x + 5$. The quadratic equation can easily be solved to give roots $x = 2 \pm \mathrm{i}$.

2.7. Rational Functions

A rational function is the quotient of two polynomials; in other words, an expression of the form

$$\frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials in $\mathbb{R}[x]$.

If the degree of $p(x)$ is not smaller than the degree of $q(x)$, then in view of Proposition 2B (with slightly different notation), we can always find real polynomials $a(x)$ and $r(x)$ such that $p(x) = q(x)a(x) + r(x)$, where $r(x) = 0$ or has degree smaller than the degree of $q(x)$, so that

$$\frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)}.$$

Example 2.7.1. Consider the rational function

$$\frac{x^5+2x^4+4x^3+x+1}{x^2+x+1}.$$

Long division gives $x^5+2x^4+4x^3+x+1=(x^3+x^2+2x-3)(x^2+x+1)+(2x+4)$, so that

$$\frac{x^5+2x^4+4x^3+x+1}{x^2+x+1}=(x^3+x^2+2x-3)+\frac{2x+4}{x^2+x+1}.$$

We can therefore restrict our attention to the case when the polynomial $p(x)$ is of lower degree than the polynomial $q(x)$. We wish to express the given rational function as a sum of simpler rational functions. To do this, we study the technique of partial fractions.

The first step is to factorize the polynomial $q(x)$ into a product of irreducible factors. We shall use the fundamental result in algebra that a real polynomial $q(x)$ can be factorized into a product of irreducible linear factors and quadratic factors with real coefficients.

Example 2.7.2. Suppose that $q(x)=x^4-4x^3+5x^2-4x+4$. Then $q(x)$ can be factorized into a product of irreducible linear factors in the form $(x-2)^2(x^2+1)$.

Suppose that a linear factor $(ax+b)$ occurs n times in the factorization of $q(x)$. Then we write down a decomposition

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n},$$

where the constants $A_1,\ldots,A_n$ will be determined later. Suppose that a quadratic factor (ax^2+bx+c) occurs n times in the factorization of $q(x)$. Then we write down a decomposition

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_nx+B_n}{(ax^2+bx+c)^n},$$

where the constants $A_1,\ldots,A_n$ and $B_1,\ldots,B_n$ will be determined later. We proceed to add all the decompositions and equate their sum to

$$\frac{p(x)}{q(x)},$$

and then calculate all the constants by equating coefficients.

Example 2.7.3. Suppose that

$$\frac{p(x)}{q(x)}=\frac{3x+4}{x^2+x-6}.$$

We note that $x^2+x-6=(x+3)(x-2)$, so we write

$$\frac{p(x)}{q(x)}=\frac{3x+4}{(x+3)(x-2)}=\frac{c_1}{x+3}+\frac{c_2}{x-2}.$$

But then

$$\frac{c_1}{x+3}+\frac{c_2}{x-2}=\frac{c_1(x-2)+c_2(x+3)}{(x+3)(x-2)},$$

so that

$$c_1(x-2)+c_2(x+3)=3x+4.$$

Note now that $c_1(x-2)+c_2(x+3)=(c_1+c_2)x+(3c_2-2c_1)$. Equating coefficients gives rise to a system of linear equations $c_1+c_2=3$ and $3c_2-2c_1=4$, with solution $c_1=1$ and $c_2=2$. Another technique is to substitute particular values of x so as to eschew various terms. Putting $x=-3$ gives $-5c_1=-5$, so that $c_1=1$. Putting $x=2$ gives $5c_2=10$, so that $c_2=2$. Whichever method we use, we have

$$\frac{3x+4}{x^2+x-6}=\frac{1}{x+3}+\frac{2}{x-2}.$$

Example 2.7.4. Suppose that

$$\frac{p(x)}{q(x)}=\frac{x^2+x-3}{x^3-2x^2-x+2}.$$

We note that $x^3-2x^2-x+2=(x-2)(x+1)(x-1)$ – we easily note a root $x=1$, so the problem becomes a quadratic one. Accordingly we write

$$\frac{p(x)}{q(x)}=\frac{x^2+x-3}{(x-2)(x+1)(x-1)}=\frac{c_1}{x-2}+\frac{c_2}{x+1}+\frac{c_3}{x-1}.$$

But then

$$\frac{c_1}{x-2}+\frac{c_2}{x+1}+\frac{c_3}{x-1}=\frac{c_1(x+1)(x-1)+c_2(x-2)(x-1)+c_3(x-2)(x+1)}{(x-2)(x+1)(x-1)},$$

so that

$$c_1(x+1)(x-1)+c_2(x-2)(x-1)+c_3(x-2)(x+1)=x^2+x-3. \tag{6}$$

Note that the left hand side of (6) is equal to $(c_1+c_2+c_3)x^2-(3c_2+c_3)x-(c_1-2c_2+2c_3)$. Equating coefficients gives rise to a system of linear equations $c_1+c_2+c_3=1$, $3c_2+c_3=-1$ and $c_1-2c_2+2c_3=3$, with solution $c_1=1$, $c_2=-\frac{1}{2}$ and $c_3=\frac{1}{2}$. Alternatively, substituting $x=2,-1,1$ into equation (6), we obtain respectively $3c_1=3$, $6c_2=-3$ and $-2c_3=-1$, so that $c_1=1$, $c_2=-\frac{1}{2}$ and $c_3=\frac{1}{2}$ again. Hence

$$\frac{x^2+x-3}{x^3-2x^2-x+2}=\frac{1}{x-2}-\frac{1}{2(x+1)}+\frac{1}{2(x-1)}.$$

Example 2.7.5. Suppose that

$$\frac{p(x)}{q(x)}=\frac{2x^3-11x^2+17x-16}{x^4-4x^3+5x^2-4x+4}.$$

We note that $x^4-4x^3+5x^2-4x+4=(x-2)^2(x^2+1)$ – this is by no means obvious, and involves some trial and error. Accordingly we write

$$\frac{p(x)}{q(x)}=\frac{2x^3-11x^2+17x-16}{(x-2)^2(x^2+1)}=\frac{c_1}{x-2}+\frac{c_2}{(x-2)^2}+\frac{c_3x+c_4}{x^2+1}.$$

But then

$$\frac{c_1}{x-2}+\frac{c_2}{(x-2)^2}+\frac{c_3x+c_4}{x^2+1}=\frac{c_1(x-2)(x^2+1)+c_2(x^2+1)+(c_3x+c_4)(x-2)^2}{(x-2)^2(x^2+1)},$$

so that

$$c_1(x-2)(x^2+1)+c_2(x^2+1)+(c_3x+c_4)(x-2)^2 = 2x^3-11x^2+17x-16.$$

Note now that

$$\begin{aligned}
&c_1(x-2)(x^2+1)+c_2(x^2+1)+(c_3x+c_4)(x-2)^2 \\
&\quad = c_1(x^3-2x^2+x-2)+c_2(x^2+1)+c_3(x^3-4x^2+4x)+c_4(x^2-4x+4) \\
&\quad = (c_1+c_3)x^3+(-2c_1+c_2-4c_3+c_4)x^2+(c_1+4c_3-4c_4)x+(-2c_1+c_2+4c_4).
\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}
c_1 \qquad + c_3 \qquad &= 2, \\
-2c_1+c_2-4c_3+\ c_4 &= -11, \\
c_1 \qquad +4c_3-4c_4 &= 17, \\
-2c_1+c_2 \qquad +4c_4 &= -16.
\end{aligned}$$

This system has solution $c_1=1$, $c_2=-2$, $c_3=1$ and $c_4=-3$. Hence

$$\frac{2x^3-11x^2+17x-16}{x^4-4x^3+5x^2-4x+4} = \frac{1}{x-2} - \frac{2}{(x-2)^2} + \frac{x-3}{x^2+1}.$$

Remark. Note that the method reduces to solving a system of linear equations. The solution of such systems is discussed in detail in linear algebra.

Example 2.7.6. Consider the rational function

$$\frac{x^6-2}{x^4+x^2}.$$

Using long division, we conclude that

$$\frac{x^6-2}{x^4+x^2} = x^2-1+\frac{x^2-2}{x^4+x^2},$$

so we concentrate our discussion on the rational function

$$\frac{p(x)}{q(x)} = \frac{x^2-2}{x^4+x^2}.$$

We note easily that $x^4+x^2=x^2(x^2+1)$, so we write

$$\frac{p(x)}{q(x)} = \frac{x^2-2}{x^2(x^2+1)} = \frac{c_1}{x}+\frac{c_2}{x^2}+\frac{c_3x+c_4}{x^2+1} = \frac{c_1x(x^2+1)+c_2(x^2+1)+(c_3x+c_4)x^2}{x^2(x^2+1)}.$$

It follows that

$$c_1x(x^2+1)+c_2(x^2+1)+(c_3x+c_4)x^2 = x^2-2.$$

Equating coefficients, we have

$$\begin{aligned}
c_1 \qquad + c_3 \qquad &= 0, \\
c_2 \qquad + c_4 &= 1, \\
c_1 \qquad\qquad\qquad &= 0, \\
c_2 \qquad\qquad &= -2.
\end{aligned}$$

This system has solution $c_1 = 0$, $c_2 = -2$, $c_3 = 0$ and $c_4 = 3$. Hence

$$\frac{x^2-2}{x^4+x^2} = -\frac{2}{x^2} + \frac{3}{x^2+1},$$

and so

$$\frac{x^6-2}{x^4+x^2} = x^2 - 1 - \frac{2}{x^2} + \frac{3}{x^2+1}.$$

2.8. Greatest Common Divisor

PROPOSITION 2G. *Suppose that $a(x), b(x) \in \mathbb{F}[x]$, not both zero. Then there exists a unique monic polynomial $d(x) \in \mathbb{F}[x]$ such that*
(a) $d(x) \mid a(x)$ and $d(x) \mid b(x)$;
(b) if $c(x) \in \mathbb{F}[x]$ and $c(x) \mid a(x)$ and $c(x) \mid b(x)$, then $c(x) \mid d(x)$. Furthermore,
(c) there exist polynomials $s(x), t(x) \in \mathbb{F}[x]$ such that $d(x) = a(x)s(x) + b(x)t(x)$.

PROOF. Consider the set

$$I = \{a(x)u(x) + b(x)v(x) : u(x), v(x) \in \mathbb{F}[x]\}.$$

Clearly I contains some non-zero polynomial in $\mathbb{F}[x]$. Let $d(x) \in I$ be a non-zero polynomial of smallest degree. We may assume without loss of generality that $d(x)$ is monic. Since $d(x) \in I$, there exist $s(x), t(x) \in \mathbb{F}[x]$ such that

$$d(x) = a(x)s(x) + b(x)t(x).$$

This gives (c). Suppose that $c(x) \in \mathbb{F}[x]$ and $c(x) \mid a(x)$ and $c(x) \mid b(x)$. Then there exist $f(x), g(x) \in \mathbb{F}[x]$ such that $a(x) = c(x)f(x)$ and $b(x) = c(x)g(x)$. It follows that

$$d(x) = c(x)(f(x)s(x) + g(x)t(x)),$$

so that $c(x) \mid d(x)$. This gives (b). On the other hand, the uniqueness of $d(x)$ follows from (a) and (b), since if $d_1(x)$ and $d_2(x)$ both satisfy the requirements of $d(x)$, then we must have $d_1(x) \mid d_2(x)$ and $d_2(x) \mid d_1(x)$. Since both $d_1(x)$ and $d_2(x)$ are monic, it follows that $d_1(x) = d_2(x)$. It remains to prove (a). Suppose on the contrary that $d(x) \nmid a(x)$. Then it follows from Proposition 2B that there exist unique polynomials $q(x), r(x) \in \mathbb{F}[x]$ such that $a(x) = d(x)q(x) + r(x)$, where $r(x) \neq 0$ and $\deg r(x) < \deg d(x)$. Note now that

$$r(x) = a(x) - d(x)q(x) = a(x) - a(x)s(x)q(x) - b(x)t(x)q(x) = a(x)(1 - s(x)q(x)) - b(x)t(x)q(x) \in I,$$

contradicting the assumption that $d(x) \in I$ is of smallest degree. Hence $d(x) \mid a(x)$. A similar argument gives $d(x) \mid b(x)$. ◯

DEFINITION. Suppose that $a(x), b(x) \in \mathbb{F}[x]$, not both zero. Then the unique monic polynomial $d(x) \in \mathbb{F}[x]$ in Proposition 2G is called the greatest common divisor of the polynomials $a(x)$ and $b(x)$. We write $d(x) = (a(x), b(x))$.

Proposition 2G and its proof give us little help in finding the greatest common divisor of the polynomials $a(x)$ and $b(x)$. A much easier way is given by the following result.

PROPOSITION 2H. (EUCLID'S ALGORITHM) *Suppose that $a(x), b(x) \in \mathbb{F}[x]$ are non-zero, with $\deg a(x) \geq \deg b(x)$. Suppose further that $q_1(x), \ldots, q_{n+1}(x) \in \mathbb{F}[x]$ and $r_1(x), \ldots, r_n(x) \in \mathbb{F}[x]$ are polynomials such that $0 \leq \deg r_n(x) < \ldots < \deg r_1(x) < \deg b(x)$ and*

$$\begin{aligned}
a(x) &= b(x)q_1(x) + r_1(x), \\
b(x) &= r_1(x)q_2(x) + r_2(x), \\
r_1(x) &= r_2(x)q_3(x) + r_3(x), \\
&\vdots \\
r_{n-2}(x) &= r_{n-1}(x)q_n(x) + r_n(x), \\
r_{n-1}(x) &= r_n(x)q_{n+1}(x).
\end{aligned}$$

Then the greatest common divisor $(a(x), b(x))$ is a constant multiple of $r_n(x)$.

Proof. We shall first of all prove that

$$(a(x), b(x)) = (b(x), r_1(x)). \tag{7}$$

Note that $(a(x), b(x)) \mid b(x)$ and $(a(x), b(x)) \mid (a(x) - b(x)q_1(x)) = r_1(x)$, so that

$$(a(x), b(x)) \mid (b(x), r_1(x)).$$

On the other hand, $(b(x), r_1(x)) \mid b(x)$ and $(b(x), r_1(x)) \mid (b(x)q_1(x) + r_1(x)) = a(x)$, so that

$$(b(x), r_1(x)) \mid (a(x), b(x)).$$

Since both $(a(x), b(x))$ and $(b(x), r_1(x))$ are monic, (7) follows. Similarly,

$$(b(x), r_1(x)) = (r_1(x), r_2(x)) = (r_2(x), r_3(x)) = \ldots = (r_{n-1}(x), r_n(x)). \tag{8}$$

Note now that

$$(r_{n-1}(x), r_n(x)) = (r_n(x)q_{n+1}(x), r_n(x)) = cr_n(x) \quad \text{for some } c \in \mathbb{F} \setminus \{0\}. \tag{9}$$

The result follows on combining (7)–(9). ○

Example 2.8.1. Let $a(x) = 2x^5 - 3x^4 + x^3 - 2x^2 + x + 1$ and $b(x) = x^3 + x^2 - x - 1$. Then

$$\begin{aligned}
2x^5 - 3x^4 + x^3 - 2x^2 + x + 1 &= (x^3 + x^2 - x - 1)(2x^2 - 5x + 8) + (-13x^2 + 4x + 9), \\
x^3 + x^2 - x - 1 &= (-13x^2 + 4x + 9)\left(-\frac{1}{13}x - \frac{17}{169}\right) + \left(\frac{16}{169}x - \frac{16}{169}\right), \\
-13x^2 + 4x + 9 &= \left(\frac{16}{169}x - \frac{16}{169}\right)\left(-\frac{2197}{16}x - \frac{1521}{16}\right).
\end{aligned}$$

It follows that $(a(x), b(x))$ is a constant multiple of

$$\frac{16}{169}x - \frac{16}{169}.$$

Since $(a(x), b(x))$ is monic, we must have

$$(a(x), b(x)) = \frac{169}{16}\left(\frac{16}{169}x - \frac{16}{169}\right) = x - 1.$$

Problems for Chapter 2

1. For each of the following, find $q(x)$ and $r(x)$ such that $b(x) = a(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg a(x)$:
   a) $a(x) = x^2 + 3x + 1$, $b(x) = x^4 + x^2 + 5x + 2$
   b) $a(x) = x^4 + x^2 + 5x + 2$, $b(x) = x^2 + 3x + 1$
   c) $a(x) = 2x^2$, $b(x) = 3x^3 + x + 1$

2. For each of the following real polynomials, try to guess one integer root, and then use it to find all the roots of the polynomial:
   a) $x^3 - x^2 + 3x + 5$ b) $x^3 + 8x^2 + 22x + 20$ b) $x^3 + 5x^2 + 7x - 13$

3. Find a real polynomial $p(x)$ that satisfies all of the following properties, and explain carefully whether the polynomial that you have found is the only one that satisfies all these properties:
   - $p(x)$ has $x = 1$, $x = 3 + \mathrm{i}$ and $x = 1 - 2\mathrm{i}$ as three of its roots;
   - $p(x)$ has degree 5; and
   - the coefficient of the x^5 term in $p(x)$ is equal to 3.

4. Apply partial fraction technique to each of the following rational functions:
   a) $\dfrac{1}{x^2 + 4x - 5}$ b) $\dfrac{x^2}{x^3 + 3x^2 + 3x + 1}$ c) $\dfrac{x^2 + 3x - 1}{x^4 + x^3 + x^2 + x}$ d) $\dfrac{1}{x^2 - 5x - 36}$
   e) $\dfrac{2x + 3}{x^2 + 3x - 4}$ f) $\dfrac{1}{x^2 - 4x + 3}$ g) $\dfrac{x - 4}{(x^2 + 4)(x + 1)}$ h) $\dfrac{x^2 + 1}{(x + 1)^4}$

5. For each of the following, find the greatest common divisor $(a(x), b(x))$:
   a) $a(x) = x^5 + 3x^3 - 2x^2 + x - 3$, $b(x) = 4x^3 - 2x^2 + 7x - 9$
   b) $a(x) = 2x^7 + 4x$, $b(x) = x^{10} + 7x^5$
   c) $a(x) = x^3 + 6x^2$, $b(x) = x^2 + 2x + 2$