Solving Polynomial Equations

Introduction

Linear and quadratic equations, dealt within Sections 3.1 and 3.2, are members of a class of equations, called **polynomial equations**. These have the general form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

in which x is a variable and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are given constants. Also n must be a positive integer and $a_n \neq 0$. Examples include $x^3 + 7x^2 + 3x - 2 = 0$, $\quad 5x^4 - 7x^2 = 0$ and $-x^6 + x^5 - x^4 = 0$. In this Section you will learn how to factorise some polynomial expressions and solve some polynomial equations.

Prerequisites Before starting this Section you should ...	• be able to solve linear and quadratic equations
Learning Outcomes On completion you should be able to ...	• recognise and solve some polynomial equations

1. Multiplying polynomials together

> ### Key Point 7
>
> A **polynomial expression** is one of the form
>
> $$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$
>
> where a_0, a_1, ..., a_n are known coefficients (numbers), $a_n \neq 0$, and x is a variable.
>
> n must be a positive integer.

For example $x^3 - 17x^2 + 54x - 8$ is a polynomial expression in x. The polynomial may be expressed in terms of a variable other than x. So, the following are also polynomial expressions:

$$t^3 - t^2 + t - 3 \qquad z^5 - 1 \qquad w^4 + 10w^2 - 12 \qquad s + 1$$

Note that only non-negative whole number powers of the variable (usually x) are allowed in a polynomial expression. In this Section you will learn how to factorise simple polynomial expressions and how to solve some polynomial equations. You will also learn the technique of **equating coefficients**. This process is very important when we need to perform calculations involving partial fractions which will be considered in Section 6.

The **degree** of a polynomial is the highest power to which the variable is raised. Thus $x^3 + 6x + 2$ has degree 3, $t^6 - 6t^4 + 2t$ has degree 6, and $5x + 2$ has degree 1.

Let us consider what happens when two polynomials are multiplied together. For example

$$(x + 1)(3x - 2)$$

is the product of two first degree polynomials. Expanding the brackets we obtain

$$(x + 1)(3x - 2) = 3x^2 + x - 2$$

which is a second degree polynomial.

In general we can regard a second degree polynomial, or quadratic, as the product of two first degree polynomials, provided that the quadratic can be factorised. Similarly

$$(x - 1)(x^2 + 3x - 7) = x^3 + 2x^2 - 10x + 7$$

is a third degree, or **cubic**, polynomial which is thus the product of a linear polynomial and a quadratic polynomial.

In general we can regard a cubic polynomial as the product of a linear polynomial and a quadratic polynomial or the product of three linear polynomials. This fact will be important in the following Section when we come to factorise cubics.

Key Point 8

A cubic expression can always be formulated as a linear expression times a quadratic expression.

Task

If $x^3 - 17x^2 + 54x - 8 = (x - 4) \times$ (a polynomial), state the degree of the undefined polynomial.

Your solution

Answer

second.

Task

(a) If $3x^2 + 13x + 4 = (x + 4) \times$ (a polynomial), state the degree of the undefined polynomial.

(b) What is the coefficient of x in this unknown polynomial ?

Your solution

(a)   (b)

Answer

(a) First.   (b) It must be 3 in order to generate the term $3x^2$ when the brackets are removed.

Task

If $2x^2 + 5x + 2 = (x + 2) \times$ (a polynomial), what must be the coefficient of x in this unknown polynomial ?

Your solution

Answer

It must be 2 in order to generate the term $2x^2$ when the brackets are removed.

Task

Two quadratic polynomials are multiplied together. What is the degree of the resulting polynomial?

Your solution

Answer

Fourth degree.

2. Factorising polynomials and equating coefficients

We will consider how we might find the solution to some simple polynomial equations. An important part of this process is being able to express a complicated polynomial into a product of simpler polynomials. This involves **factorisation**.

Factorisation of polynomial expressions can be achieved more easily if one or more of the factors is already known. This requires a knowledge of the technique of 'equating coefficients' which is illustrated in the following example.

Example 23

Factorise the expression $x^3 - 17x^2 + 54x - 8$ given that one of the factors is $(x - 4)$.

Solution

Given that $x - 4$ is a factor we can write

$$x^3 - 17x^2 + 54x - 8 = (x - 4) \times \text{(a quadratic polynomial)}$$

The polynomial must be quadratic because the expression on the left is cubic and $x - 4$ is linear. Suppose we write this quadratic as $ax^2 + bx + c$ where a, b and c are unknown numbers which we need to find. Then

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(ax^2 + bx + c)$$

Removing the brackets on the right and collecting like terms together we have

$$x^3 - 17x^2 + 54x - 8 = ax^3 + (b - 4a)x^2 + (c - 4b)x - 4c$$

Solution (contd.)

Like terms are those which involve the same power of the variable (x).

Equating coefficients means that we compare the coefficients of each term on the left with the corresponding term on the right. Thus if we look at the x^3 terms on each side we see that $x^3 = ax^3$ which implies a must equal 1. Similarly by equating coefficients of x^2 we find $-17 = b - 4a$ With $a = 1$ we have $-17 = b - 4$ so b must equal -13. Finally, equating constant terms we find $-8 = -4c$ so that $c = 2$.

As a check we look at the coefficient of x to ensure it is the same on both sides. Now that we know $a = 1, b = -13, c = 2$ we can write the polynomial expression as

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2)$$

Exercises

Factorise into a quadratic and linear product the given polynomial expressions

1. $x^3 - 6x^2 + 11x - 6$, given that $x - 1$ is a factor

2. $x^3 - 7x - 6$, given that $x + 2$ is a factor

3. $2x^3 + 7x^2 + 7x + 2$, given that $x + 1$ is a factor

4. $3x^3 + 7x^2 - 22x - 8$, given that $x + 4$ is a factor

Answers

1. $(x - 1)(x^2 - 5x + 6)$, 2. $(x + 2)(x^2 - 2x - 3)$, 3. $(x + 1)(2x^2 + 5x + 2)$,
4. $(x + 4)(3x^2 - 5x - 2)$.

3. Polynomial equations

When a polynomial expression is equated to zero, a polynomial equation is obtained. Linear and quadratic equations, which you have already met, are particular types of polynomial equation.

Key Point 9

A **polynomial equation** has the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0 = 0$$

where a_0, a_1, ..., a_n are known coefficients, $a_n \neq 0$, and x represents an unknown whose value(s) are to be found.

Polynomial equations of low degree have special names. A polynomial equation of degree 1 is a linear equation and such equations have been solved in Section 3.1. Degree 2 polynomials are called quadratics; degree 3 polynomials are called cubics; degree 4 equations are called quartics and so on. The following are examples of polynomial equations:

$$5x^6 - 3x^4 + x^2 + 7 = 0, \qquad -7x^4 + x^2 + 9 = 0, \qquad t^3 - t + 5 = 0, \qquad w^7 - 3w - 1 = 0$$

Recall that the **degree** of the equation is the highest power of x occurring. The **solutions** or **roots** of the equation are those values of x which satisfy the equation.
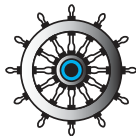

Key Point 10

A polynomial equation of degree n has n roots.

Some (possibly all) of the roots may be repeated.

Some (possibly all) of the roots may be complex.

Example 24
Verify that $x = -1$, $x = 1$ and $x = 0$ are solutions (roots) of the equation

$$x^3 - x = 0$$

Solution

We substitute each value in turn into $x^3 - x$.

$$(-1)^3 - (-1) = -1 + 1 = 0$$

so $x = -1$ is clearly a root.

It is easy to verify similarly that $x = 1$ and $x = 0$ are also solutions.

In the next subsection we will consider ways in which polynomial equations of higher degree than quadratic can be solved.

Exercises

Verify that the given values are solutions of the given equations.

1. $x^2 - 5x + 6 = 0$, $\qquad x = 3, x = 2$

2. $2t^3 + t^2 - t = 0$, $\qquad t = 0, t = -1, t = \frac{1}{2}$.

4. Solving polynomial equations when one solution is known

In Section 3.2 we gave a formula which can be used to solve quadratic equations. Unfortunately when dealing with equations of higher degree no simple formulae exist. If one of the roots can be spotted or is known we can sometimes find the others by the method shown in the next Example.
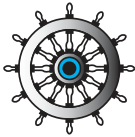
Example 25

Let the polynomial expression $x^3 - 17x^2 + 54x - 18$ be denoted by $P(x)$. Verify that $x = 4$ is a solution of the equation $P(x) = 0$. Hence find the other solutions.

Solution

We substitute $x = 4$ into the polynomial expression $P(x)$:

$$P(4) = 4^3 - 17(4^2) + 54(4) - 8 = 64 - 272 + 216 - 8 = 0$$

So, when $x = 4$ the left-hand side equals zero. Hence $x = 4$ is indeed a solution. Knowing that $x = 4$ is a root we can state that $(x-4)$ must be a factor of $P(x)$. Therefore $P(x)$ can be re-written as a product of a linear and a quadratic term:

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4) \times \text{(quadratic polynomial)}$$

The quadratic polynomial has already been found in a previous task so we deduce that the given equation can be written

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2) = 0$$

In this form we see that $\quad x - 4 = 0 \quad$ or $\quad x^2 - 13x + 2 = 0$

The first equation gives $x = 4$ which we already knew.

The second equation must be solved using one of the methods for solving quadratic equations given in Section 3.2. For example, using the formula we find

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{with } a = 1,\ b = -13,\ c = 2 \\
&= \frac{13 \pm \sqrt{(-13)^2 - 4.1.2}}{2} \\
&= \frac{13 \pm \sqrt{161}}{2} = \frac{13 \pm 12.6886}{2}
\end{aligned}
$$

So $x = 12.8443$ and $x = 0.1557$ are roots of $x^2 - 13x + 2$.

Hence the three solutions of $P(x) = 0$ are $x = 4$, $x = 12.8443$ and $x = 0.1557$, to 4 d.p.

Task

Solve the equation $x^3 + 8x^2 + 16x + 3 = 0$ given that $x = -3$ is a root.
Consider the equation $x^3 + 8x^2 + 16x + 3 = 0$.

Given that $x = -3$ is a root state a linear factor of the cubic:

Your solution

Answer

$x + 3$

The cubic can therefore be expressed as

$$x^3 + 8x^2 + 16x + 3 = (x + 3)(ax^2 + bx + c)$$

where a, b, and c are constants. These can be found by expanding the right-hand side.

Expand the right-hand side:

Your solution

Answer

$$x^3 + 8x^2 + 16x + 3 = ax^3 + (3a + b)x^2 + (3b + c)x + 3c$$

Equate coefficients of x^3 to find a:

Your solution

Answer

1

Equate constant terms to find c:

Answer

$3 = 3c$ so that $c = 1$

Your solution

Equate coefficients of x^2 to find b:

Your solution

Answer

$8 = 3a + b$ so $b = 5$

This enables us to write the equation as $(x + 3)(x^2 + 5x + 1) = 0$ so $x + 3 = 0$ or $x^2 + 5x + 1 = 0$. Now solve the quadratic and state all three roots:

Your solution

Answer

The quadratic equation can be solved using the formula to obtain $x = -4.7913$ and $x = -0.2087$. Thus the three roots of $x^3 + 8x^2 + 16x + 3$ are $x = -3$, $x = -4.7913$ and $x = -0.2087$.

Exercises

1. Verify that the given value is a solution of the equation and hence find all solutions:

(a) $x^3 + 7x^2 + 11x + 2 = 0$, $x = -2$ (b) $2x^3 + 11x^2 - 2x - 35 = 0$, $x = -5$

2. Verify that $x = 1$ and $x = 2$ are solutions of $x^4 + 4x^3 - 17x^2 + 8x + 4$ and hence find all solutions.

Answers

1(a) $-2, -0.2087, -4.7913$ 1(b) $-5, -2.1375, 1.6375$

2. $1, 2, -0.2984, -6.7016$

5. Solving polynomial equations graphically

Polynomial equations, particularly of high degree, are difficult to solve unless they take a particularly simple form. A useful guide to the approximate values of the solutions can be obtained by sketching the polynomial, and discovering where the curve crosses the x-axis. The real roots of the polynomial equation $P(x) = 0$ are given by the values of the intercepts of the function $y = P(x)$ with the x-axis because on the x-axis $y = P(x)$, is zero. Computer software packages and graphics calculators exist which can be used for plotting graphs and hence for solving polynomial equations approximately. Suppose the graph of $y = P(x)$ is plotted and takes a form similar to that shown in Figure 6.
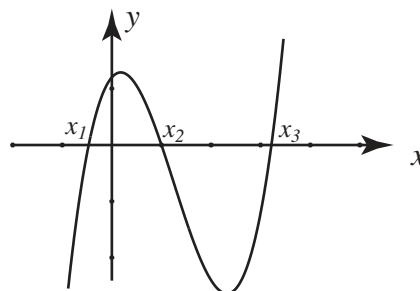
Figure 6: A polynomial function which cuts the x axis at points x_1, x_2 and x_3.

The graph intersects the x axis at $x = x_1$, $x = x_2$ and $x = x_3$ and so the equation $P(x) = 0$ has three roots x_1, x_2 and x_3, because $P(x_1) = 0$, $P(x_2) = 0$ and $P(x_3) = 0$.

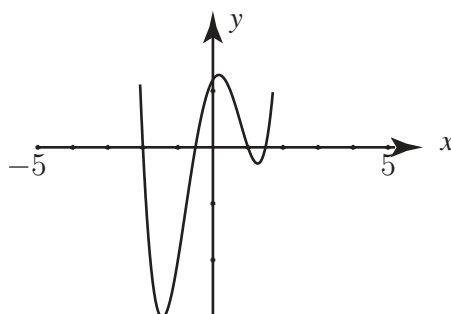### Example 26

Plot a graph of the function $y = 4x^4 - 15x^2 + 5x + 6$ and hence approximately solve the equation $4x^4 - 15x^2 + 5x + 6 = 0$.

Solution

The graph has been plotted here with the aid of a computer graph plotting package and is shown in Figure 7. By hand, a less accurate result would be produced, of course.

Figure 7: Graph of $y = 4x^4 - 15x^2 + 5x + 6$

The solutions of the equation are found by looking for where the graph crosses the horizontal axis. Careful examination shows the solutions are at or close to $x = 1$, $x = 1.5$, $x = -0.5$, $x = -2$.

An important feature of the graph of a polynomial is that it is **continuous**. There are never any gaps or jumps in the curve. Polynomial curves never turn back on themselves in the horizontal direction, (unlike a circle). By studying the graph in Figure 6 you will see that if we choose any two values of x, say a and b, such that $y(a)$ and $y(b)$ have opposite signs, then at least one root lies between $x = a$ and $x = b$.

Exercises

1. Factorise $x^3 - x^2 - 65x - 63$ given that $(x + 7)$ is a factor.

2. Show that $x = -1$ is a root of $x^3 + 11x^2 + 31x + 21 = 0$ and locate the other roots algebraically.

3. Show that $x = 2$ is a root of $x^3 - 3x - 2 = 0$ and locate the other roots.

4. Solve the equation $x^4 - 2x^2 + 1 = 0$.

5. Factorise $x^4 - 7x^3 + 3x^2 + 31x + 20$ given that $(x + 1)$ is a factor.

6. Given that two of the roots of $x^4 + 3x^3 - 7x^2 - 27x - 18 = 0$ have the same modulus but different sign, solve the equation.

   (Hint - let two of the roots be α and $-\alpha$ and use the technique of equating coefficients).

7. Consider the polynomial $P(x) = 5x^3 - 47x^2 + 84x$. By evaluating $P(2)$ and $P(3)$ show that at least one root of $P(x) = 0$ lies between $x = 2$ and $x = 3$.

8. Without solving the equation or using a graphical calculator, show that $x^4 + 4x - 1 = 0$ has a root between $x = 0$ and $x = 1$.

Answers

1. $(x + 7)(x + 1)(x - 9)$

2. $x = -1, -3, -7$

3. $x = 2, -1$ (repeated)

4. $x = -1, 1$ (each root repeated)

5. $(x + 1)^2 (x - 4)(x - 5)$

6. $(x + 3)(x - 3)(x + 1)(x + 2)$