Chapter 3

Continuity and Limits of Functions

The concept of continuity is an important first step in the analysis leading to differential and integral calculus. It is also an important analytical tool in its own right, with significant practical applications. Fortunately the main theorems are intuitive, though their proofs can be technically challenging. Nonetheless we prove most of the continuity theorems we state, while the remaining theorems we discuss and apply without proof, since they are intuitive and useful but their proofs require background material from at least a junior level real analysis or topology course.

Related to continuity is the concept of limit, which is vital for calculus since it puts calculus on the same rigorous footing as other mathematical disciplines such as algebra and geometry. In our more modern times it has further conceptual appeal, as it is often only possible to approximate the solution to some problem, even though our method of approximation may be arbitrarily close to the actual solution if given enough computing resources. However, the real value of limits lies in its use in confronting an interesting phenomenon. This is the fact that in mathematics we at times find our analysis (algebraic, geometric or otherwise) breaking down at exactly the value of some variable where we would like to compute something; that is, we are allowed to let that variable "approach" the desired value as closely as we would like but it cannot equal that value according to our classical, pre-calculus mathematics. The relatively modern mathematical tool called "limits" can often break through the analytic barrier at that value, in turn opening us to the extensive and spectacularly useful field we call calculus.

Many examples of continuity and limits in action seem straightforward enough, but without a sufficiently deep understanding it is all too easy for students to fall victim to common errors. For this reason we introduce the rather technical definition of continuity here, and develop a method to prove continuity in cases which may seem obvious. Some powerful continuity theorems follow, as do applications. We then employ limits for cases where continuity is "broken," and in numerous other contexts to make calculus possible, and as an analytical tool in its own right.

Understanding limits and continuity sufficiently to avoid numerous common mistakes requires a care and depth of thought which we attempt to foster in this chapter. After continuity, we use a "forms" approach to limits, those forms themselves being ultimately intuitive but nonetheless requiring students to study them carefully and extensively to achieve satisfactory proficiency.[1]

[1]Unfortunately limit computations can be deceiving in that perhaps the majority of problems one first encounters do not require a deep understanding in order to "guess" their correct answers. However the interesting (and more advanced) cases tend to lie outside of those which are easily guessed, and so computing the correct answers in such cases requires a much deeper understanding. We take the approach here that it is better to heavily analyze the simpler cases, so that the later cases are more easily learned.

3.1 Definition of Continuity at a Point

The function $f(x)$ is continuous at $x = a$ if and only if we can guarantee $f(x)$ to be close to the value $f(a)$ by restricting x to be close to a. To rephrase, we say $f(x)$ is continuous at $x = a$ if, given any positive tolerance $\varepsilon > 0$ we choose for $f(x)$ as an approximation for $f(a)$, we can then find a positive tolerance $\delta > 0$ for x as an approximation for a so that δ-tolerance in x allows at most ε tolerance in $f(x)$. The definition below is very technical, but through reflection and exposure to examples, one eventually sees that this is exactly what is required.

Definition 3.1.1 *The function $f(x)$ is continuous at the point $x = a$ if and only if*[2]

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right). \tag{3.1}$$

This is sometimes called the **epsilon-delta** (ε-δ) definition of continuity. (Note that all values are assumed real, i.e., $\varepsilon, \delta, x, a, f(x), f(a) \in \mathbb{R}$, and so for instance $(\forall x)$ is short for $(\forall x \in \mathbb{R})$.) Now let us examine the various parts of the definition.

$$
\begin{aligned}
|f(x) - f(a)| \quad &< \quad \varepsilon && (3.2)\\
|x - a| \quad &< \quad \delta && (3.3)\\
(\forall x)\,(|x - a| < \delta \quad &\longrightarrow \quad |f(x) - f(a)| < \varepsilon) && (3.4)\\
(\forall \varepsilon > 0) \quad & \qquad (\exists \delta > 0) && (3.5)
\end{aligned}
$$

(3.2): $f(x)$ will be *within* ε of $f(a)$. In other words, the function at x will be near in value to the value of the function at a. How near? Less than ε distance away.

(3.3): x is within δ of a. (Otherwise the implication holds true vacuously, but that case is useless. What is important is what occurs when $|x - a| < \delta$.)

(3.4): The condition that x be within δ of a forces $f(x)$ to be within ε of $f(a)$. In other words, allowing x to stray by less than δ from a keeps $f(x)$ within ε of $f(a)$. By controlling x by allowing it a tolerance of less than δ, we control $f(x)$ to have a tolerance of less than ε.

(3.5): Whatever positive value of ε we choose, we can find a δ which satisfies (3.4). In particular, *no matter how small* we choose $\varepsilon > 0$, we can find a positive δ so that (3.4) is satisfied.

For a final rephrasing, we have the statement that **we can control the tolerance ε in the output $f(x)$ as much as we would like, so long as $\varepsilon > 0$, by controlling the tolerance δ (which must also be positive**[3]**) in the input variable x.**

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[2] Many texts abbreviate statements like (3.1) as follows:

$$(\forall \varepsilon > 0)(\exists \delta > 0)\left(|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right),$$

the idea being that the $\forall x$ is understood when we make an unquantified (in x) statement like $|x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon$. For a similar example in English, consider the following two statements, usually deemed equivalent:

All Americans have trouble speaking English.

If X is an American, then X has trouble speaking English.

Most see both as false exactly when we can find one American (counterexample) who has no such trouble. In other words, the second statement is as much a "blanket" statement as the first, and is in fact equivalent to the first. We leave the $\forall x$ in (3.1) for precision and to aid in negating the statement, using rules from Section 1.4.

[3] Notice that $\delta = 0$ would be worthless for several reasons. First, the implication would be vacuously true and all functions would be continuous everywhere, since $|x - a| < 0$ would never be satisfied. Second, when—in reality—do we ever have a tolerance of zero in a measurement? Finally, as we explore the implications of continuity, we will see that having positive δ is central to the spirit of what follows, particularly with regards to limits; $\delta > 0$ allows for some "wiggle room" for x near $x = a$, and this wiggle room is crucial to the concept of continuity.

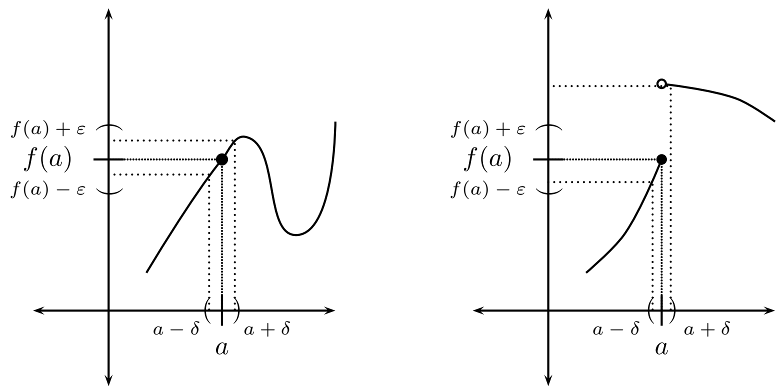


Figure 3.1: The first graph shows a function continuous at $x = a$, illustrating that we can force $f(x)$ to be within any fixed $\varepsilon > 0$ of $f(a)$ by keeping x to within some $\delta > 0$ (depending upon the ε) of a. On the other hand, in the second graph we see an $\varepsilon > 0$ for which no positive δ-tolerance in x can force $f(x)$ to be within ε-tolerance of $f(a)$, and so $f(x)$ is not continuous at $x = a$.

Example 3.1.1 *Show (by a proof!) that the function* $f(x) = 5x - 9$ *is continuous at the point* $x = 2$*, according to the definition* (3.1).

Solution: *First we notice that* $f(2) = 1$*, so we are trying to show the truth of the statement*

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)\,(|x-2| < \delta \longrightarrow |f(x)-1| < \varepsilon)\,.$$

Before completing this example, we insert here the following general strategy for all such proofs.

Strategy for Writing ε-δ Proofs

1. Use the statement $|f(x)-f(a)| < \varepsilon$ to see how it can be controlled by $|x-a|$, in particular if $|x-a|$ is a factor of $|f(x)-f(a)|$.
2. If necessary, assume *a priori*[4] that δ is smaller than some fixed positive number to control the other factors contained in $|f(x)-f(a)|$.
3. Find δ as a function of ε, i.e.,
$$\delta = \delta(\varepsilon) \tag{3.6}$$
with $\varepsilon \in (0,\infty)$ (technically, the *domain* of this function $\delta(\varepsilon)$) and $\delta \in (0,\infty)$, and such that the preliminary (or exploratory) analysis indicates that choice of δ satisfies the definition (3.1).
4. Verify that (3.1) holds with this choice of δ, and in doing so write the actual proof.

The first three steps are analysis, or "scratch-work" to determine the form of δ. The final step is the actual proof, though elements of it are often contained in the analysis/scratch-work. Let us apply this strategy to the problem at hand.

[4]Presumptive; before observations. When Step 2 is necessary, we make such suppositions not necessarily based upon observation, but to help focus our search for δ. If continuity is true, (as we will see) we will find that a legitimate δ is still available even with the restriction. In fact, if the limit definition holds for a value of $\delta > 0$, it holds for any smaller positive value δ, so this is not a fatal restriction at all. Note that if $0 < \delta_1 < \delta 2$, and $|x-a| < \delta_1$, then $|x-a| < \delta_2$ as well so we can always take a smaller value for δ in our proof. The upshot is that *a priori* restricting the size of $\delta > 0$ from the start never jeopardizes our ability to prove continuity.

Scratch-work: We want $|f(x) - f(a)| < \varepsilon$ to follow from our choice of δ. We work backwards from that statement, with $f(x) = 5x - 9$, $a = 2$, and $f(a) = f(2) = 1$.

$$\begin{array}{rrcll}
 & |f(x) - f(a)| & < & \varepsilon & \textit{(what we need)} \\
\iff & |f(x) - 1| & < & \varepsilon & \\
\iff & |5x - 9 - 1| & < & \varepsilon & \\
\iff & |5x - 10| & < & \varepsilon & \\
\iff & 5|x - 2| & < & \varepsilon & \\
\iff & |x - 2| & < & \frac{1}{5}\varepsilon & \textit{(how to get it).}
\end{array}$$

We can see from this that, if we take $\delta = \frac{1}{5}\varepsilon$, then $\delta > 0$ (since $\varepsilon > 0$ is assumed), and the bottom could be written $|x - 2| < \delta$. Then the implication could be read from that statement upwards to get $|f(x) - 1| < \varepsilon$. We summarize this in the proof.

Proof: *For any $\varepsilon > 0$ choose $\delta = \frac{1}{5}\varepsilon$. Then $\delta > 0$ exists and satisfies*

$$\begin{aligned}
|x - 2| < \delta \implies & |f(x) - f(2)| = |(5x - 9) - 1| \\
& = |5x - 10| = \underbrace{5|x - 2| < 5\delta}_{\textit{since } |x-2|<\delta} = 5 \cdot \frac{1}{5}\varepsilon = \varepsilon, \text{ q.e.d.}
\end{aligned}$$

Note that the final line of the proof does imply that $|f(x) - f(2)| < \varepsilon$, with intermediate calculations, some of which one may wish to omit with practice.

Example 3.1.2 *Show that $f(x) = 2x + 3$ is continuous at $x = -5$.*

Scratch-work: Here $a = -5$ and $f(a) = f(-5) = -7$. Hence we wish to find $\delta > 0$ so that

$$|x - (-5)| < \delta \implies |f(x) - (-7)| < \varepsilon,$$

i.e.,

$$|x + 5| < \delta \implies |f(x) + 7| < \varepsilon.$$

Again we work backwards from the conclusion we wish to justify.

$$\begin{array}{rrcl}
 & |f(x) + 7| & < & \varepsilon \\
\iff & |2x + 3 + 7| & < & \varepsilon \\
\iff & |2x + 10| & < & \varepsilon \\
\iff & 2|x + 5| & < & \varepsilon \\
\iff & |x + 5| & < & \frac{1}{2}\varepsilon.
\end{array}$$

This time we take $\delta = \frac{1}{2}\varepsilon$, and write the proof.

Proof: *For $\varepsilon > 0$, set $\delta = \frac{1}{2}\varepsilon$. Then $\delta > 0$ exists and satisfies*

$$|x+5| < \delta \implies |f(x)+7| = |2x+3+7| = |2x+10| = 2\underbrace{|x + 5|}_{<\delta} < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon, \text{ q.e.d.}$$

Proving continuity for first-degree polynomials is rather routine at any $x = a$. The strategy is the same for each, with the only complications coming from the signs of the values in question. For completeness we include one more such example.

Example 3.1.3 *Show that $f(x) = 9 - 4x$ is continuous at $x = 2$.*

Scratch-work: Here $a = 2$, $f(a) = f(2) = 1$. Now we must be a little more careful, and will make use of the fact that $|a \cdot b| = |a| \cdot |b|$.

$$\begin{array}{rrcl} & |f(x) - 1| & < & \varepsilon \\ \iff & |9 - 4x - 1| & < & \varepsilon \\ \iff & |-4x + 8| & < & \varepsilon \\ \iff & |(-4)(x - 2)| & < & \varepsilon \\ \iff & 4|x - 2| & < & \varepsilon \\ \iff & |x - 2| & < & \frac{1}{4}\varepsilon. \end{array}$$

Proof: *For $\varepsilon > 0$, choose $\delta = \frac{1}{4}\varepsilon$. Then $\delta > 0$ (exists) and*

$$\begin{aligned} |x - 2| < \delta \implies |f(x) - 1| &= |9 - 4x - 1| = |-4x + 8| \\ &= |(-4)(x - 2)| = 4|x - 2| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon, \text{ q.e.d.} \end{aligned}$$

The function which represents a line is the easiest to confirm continuity at every point. If $f(x) = mx + b$, where $m \neq 0$, it is clear from the geometric meaning of slope m that a variation (absolute value of "rise") of less than ε in height $f(x)$ can be achieved by allowing a variation (absolute value of "run") of less than $\frac{1}{|m|}\varepsilon$ in x. Thus $\delta = \frac{1}{|m|}\varepsilon$ is the largest δ which satisfies the definition of continuity for such a function $f(x)$. (See again our three "linear" examples above, and compare their slopes with our choices of δ.)

Example 3.1.4 *Show that $f(x) = x^2$ is continuous at $x = 0$.*

Scratch-work: Here $a = 0$ and $f(a) = f(0) = 0$. We therefore want to choose $\delta > 0$ such that

$$\begin{aligned} |x - 0| < \delta &\implies |f(x) - 0| < \varepsilon, \quad \textit{i.e.,} \\ |x| < \delta &\implies |x|^2 < \varepsilon. \end{aligned}$$

Again we begin with the inequality we would like to result, and see how we might get it.

$$|x|^2 < \varepsilon \iff |x| < \sqrt{\varepsilon}.$$

Here we do have $\iff$ because we are dealing with only positive quantities (recall that $\sqrt{\cdot}$ is an increasing function on $[0, \infty)$). Thus we have a good choice for δ, namely $\delta = \sqrt{\varepsilon}$.

Proof: *For $\varepsilon > 0$, set $\delta = \sqrt{\varepsilon}$. Then $\delta > 0$ exists and satisfies*

$$|x - 0| < \delta \implies |f(x) - f(0)| = |x^2| = |x|^2 < \delta^2 = \left(\sqrt{\varepsilon}\right)^2 = \varepsilon, \quad \text{q.e.d.}$$

It should be clear that we could easily modify this example to show that $f(x) = x^n$ is continuous at $x = 0$ for $n \in \mathbb{N}$. From there it is not hard to show $f(x) = x^{m/n}$ is also continuous at $x = 0$, as long as n is odd and $m, n \in \mathbb{Z}$.[5] Once we stray from $x = 0$, we begin to have more difficulties, as illustrated in the next example.

[5] It is false if n is even, assuming m/n is a reduced fraction. The trouble there is that $x^{m/n} = \left(\sqrt[n]{x}\right)^m$ is undefined for $x < 0$, so the second part of $(|x - 0| < \delta) \longrightarrow (|f(x) - f(0)| < \varepsilon)$ is false for such values of x.

Example 3.1.5 *Show that $f(x) = x^2$ is continuous at $x = 4$.*

Scratch-work: This has a complication that the previous problem did not. To see this we first attempt to proceed as in the previous Example 3.1.4. *Here $a = 4$ and $f(a) = 16$. We therefore want to choose $\delta > 0$ such that*

$$|x-4| < \delta \implies |f(x) - 16| < \varepsilon.$$

Working backwards as before we get

$$\begin{aligned} |f(x) - 16| < \varepsilon &\iff \quad |x^2 - 16| < \varepsilon \\ &\iff \quad |x+4| \cdot |x-4| < \varepsilon. \end{aligned}$$

We would like to be able to divide both sides by $|x+4|$, except that **it is not constant**. *Here Step 2 in our strategy comes into play. We will control the $|x+4|$ term by assuming a priori that (in all cases) $\delta \le 1$.*[6]

$$(|x-4| < \delta) \wedge (\delta \le 1) \implies |x-4| < 1 \implies -1 < x - 4 < 1 \implies 3 < x < 5.$$

Now we add 4 to this inequality to get

$$(|x-4| < \delta) \wedge (\delta \le 1) \implies 7 < x + 4 < 9.$$

With $x+4$ between 7 and 9, its absolute size is strictly bounded by the number with the largest absolute value, 9, i.e.,

$$(|x-4| < \delta) \wedge (\delta \le 1) \implies 7 < x + 4 < 9 \implies |x+4| < 9.$$

Continuing the scratch-work, we would get

$$(|x-4| < \delta) \wedge (\delta \le 1) \implies |f(x) - 16| = |x+4||x-4| \le 9|x-4| < 9\delta,$$

(where we only had $\le$ in our inequality $|x+4||x-4| \le 9|x-4|$ because of the case $x = 4$, where we have the equality $0 = 0$) and now this looks similar to the earlier examples. To achieve $|f(x) - 16| < \varepsilon$ it would be sufficient to have $9\delta \le \varepsilon$, which is to say $\delta \le \frac{1}{9}\varepsilon$.

Picking $\delta = \frac{1}{9}\varepsilon$ **is not quite enough**, *since we assumed $\delta \le 1$ (to get our estimate $|x+4| < 9$), and this would be false if $\varepsilon > 9$. To cover both of these requirements for δ for every given ε (as the definition requires), we choose $\delta = \min\left\{1, \frac{1}{9}\varepsilon\right\}$, i.e., the minimum of the two numbers, which will still be positive. Now we write the proof.*

Proof: *For $\varepsilon > 0$, choose $\delta = \min\left\{1, \frac{1}{9}\varepsilon\right\}$. Then $\delta > 0$ exists and satisfies*

$$|x-4| < \delta \implies |f(x) - 16| = |x+4| \cdot |x-4| < 9|x-4| < 9\delta \le 9 \cdot \frac{1}{9}\varepsilon = \varepsilon, \text{ q.e.d.}$$

An experienced reader of mathematical proofs would be able to make sense of the proof above on its own—perhaps with minimal writing to check one inequality—but for our purposes we note that much of the explanation of the proof can be found again in the scratchwork. A key observation is that since $\delta = \min\left\{1, \frac{1}{9}\varepsilon\right\}$, we have both $\delta \le 1$ and $\delta \le \frac{1}{9}\varepsilon$. Some "steps" in our proof rely on related implications of these in turn, namely $(|x-4| < \delta) \wedge (\delta \le 1) \implies |x+4| < 9$, which is believable on its face (as it is not hard to see that $|x-4| < 1 \implies x \in (3,5) \implies$

[6] Here we chose $\delta \le 1$, but we could have chosen any positive number for the maximum we allow δ to be. We just need to restrict δ (though keeping it positive) to control the other factors of $|f(x) - 16|$.

$x+4 \in (7,9) \implies |x+4| < 9$), and $\delta \le \frac{1}{9}\varepsilon \implies 9\delta \le 9 \cdot \frac{1}{9}\varepsilon = \varepsilon$. As in previous examples, it is useful that in the statement of our proof we can see the outline of the definition of continuity (3.1), page 169. Also important to note is that while we could have made a different choice for δ, the obvious possibilities would still require δ to be defined as a similar type of minimum (depending upon the *a priori* restriction of the form $\delta \le M$).

Example 3.1.6 *Show that $f(x) = -x^3$ is continuous at $x = -2$.*

Scratchwork: Here $a = -2$, and $f(a) = f(-2) = 8$, so we hope for every $\varepsilon > 0$ to find $\delta > 0$ such that $|x - (-2)| < \delta \implies |f(x) - 8| < \varepsilon$, i.e., $|x+2| < \delta \implies |f(x) - 8| < \varepsilon$. Working backwards as before, we see (using $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$) that

$$\begin{aligned} |f(x) - f(2)| < \varepsilon &\iff \left|-x^3 - 8\right| < \varepsilon \\ &\iff \left|(-1)(x^3+8)\right| < \varepsilon \\ &\iff 1 \cdot \left|x^3 + 8\right| < \varepsilon \\ &\iff \underbrace{|x+2|}_{<\delta} \cdot \left|x^2 - 2x + 4\right| < \varepsilon. \end{aligned}$$

Here we will make a rather crude estimate[7] *on the size of $|x^2 - 2x + 4|$, eventually using the triangle inequality* (2.35), page 98 *but to do so will require as before an a priori bound on δ. So for simplicity we will again assume in all cases that $\delta \le 1$ for what follows (and incorporate that restriction into our proof as well).*

$$\begin{aligned} (|x+2| < \delta) \wedge (\delta \le 1) &\implies |x+2| \le 1 \iff -1 < x+2 < 1 \iff -3 < x < -1 \\ &\implies |x| < 3, \end{aligned}$$

from which we can summarize (in the first line below) and extend using the triangle inequality (in the second) to get

$$\begin{aligned} (|x+2| < \delta) \wedge (\delta \le 1) &\implies |x| < 3 \\ &\implies |x^2 - 2x + 4| \le |x^2| + |-2x| + |4| = |x|^2 + 2|x| + 4 \\ &\qquad\qquad\qquad\qquad < 3^2 + 2(3) + 4 \\ &\qquad\qquad\qquad\qquad = 19 \end{aligned}$$

that is, $(|x+2| < \delta) \wedge (\delta \le 1) \implies |x^2 - 2x + 4| \le 19.$

So far, under the assumptions that $|x+2| < \delta$ and $\delta \le 1$ we have (slightly rearranged)

$$|f(x) - 8| = |x^2 - 2x + 4| \cdot |x+2| \le 19|x+2| < \underbrace{19\delta.}_{\text{want } \le \varepsilon}$$

If we also have $\delta \le \varepsilon/19$ we will have $|f(x) - 8| < 19\frac{\varepsilon}{19} \le \varepsilon$, and our proof would be complete. However we must remember that this was possible because we already assumed $\delta \le 1$, and so accomplish both of these equally important restrictions by setting $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$. The inequalities involved in the proof (after the "$\implies$") include, in order, the triangle inequality, one implied by $\delta \le 1$, $|x+2| < \delta$ and finally $\delta \le \frac{\varepsilon}{19}$.

[7] In mathematical analysis, the term *estimate* often refers to a bound on the size of a quantity. For instance, $|x| < 100$ means $-100 < x < 100$, giving a lower and upper bound for x. In common usage, the word "estimate" often refers instead to what mathematicians and other scientists would call "approximation."

Proof: *Let $\varepsilon > 0$, and set $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$. Then $\delta > 0$ exists and*

$$\begin{aligned}|x+2| < \delta \implies |f(x) - f(-2)| &= |x^2 - 2x + 4| \cdot |x+2| \le \left(|x|^2 + 2|x| + 4\right)|x+2| \\ &\le (3^2 + 2(3) + 4)|x+2| \\ &= 19|x+2| < 19\delta \le 19\frac{\varepsilon}{19} = \varepsilon, \qquad \text{q.e.d.}\end{aligned}$$

The next example requires a somewhat different bit of cleverness, and similar *a priori* restriction on δ.

Example 3.1.7 *Show that $f(x) = \dfrac{1}{x}$ is continuous at $x = 5$.*

Scratch-work*: Here $a = 5$ and $f(a) = f(5) = \frac{1}{5}$. Again we work backwards:*

$$\begin{aligned} & \left|f(x) - \frac{1}{5}\right| < \varepsilon \\ \iff & \left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon \\ \iff & \left|\frac{5-x}{5x}\right| < \varepsilon \\ \iff & \frac{1}{5} \cdot \frac{1}{|x|} \cdot |x-5| < \varepsilon.\end{aligned}$$

As before, the $|x-5|$ will be controlled by δ, but we need to also use δ to control the factor $\frac{1}{|x|}$. Again we will assume a priori that $\delta \le 1$.

$$|x-5| < \delta \implies |x-5| < 1 \iff -1 < x-5 < 1 \iff 4 < x < 6 \iff 4 < |x| < 6.$$

With these bounds on $|x|$, we also get bounds on $\frac{1}{|x|}$ (noting that if $z_1, z_2 > 0$ and $z_1 < z_2$ then $1/z_1 > 1/z_2$):

$$|x-5| < 1 \implies 4 < |x| < 6 \iff \frac{1}{6} < \frac{1}{|x|} < \frac{1}{4}.$$

With all these assumptions, then, we have (continuing from before and noting the $x = 5$ case)

$$\left|f(x) - \frac{1}{5}\right| = \frac{1}{5} \cdot \frac{1}{|x|} \cdot |x-5| \le \underbrace{\frac{1}{5} \cdot \frac{1}{4}|x-5|}_{<\frac{1}{20}\delta}.$$

This is less than ε if δ is no bigger than 20ε. Now the analysis above also assumed $\delta \le 1$, so we take

$$\delta = \min\{1, 20\varepsilon\}.$$

Now we state the proof.

Proof: *For $\varepsilon > 0$, choose $\delta = \min\{1, 20\varepsilon\}$. Then $\delta > 0$ exists and satisfies*

$$\begin{aligned}|x-5| < \delta \implies |f(x) - f(5)| &= \left|\frac{1}{x} - \frac{1}{5}\right| = \left|\frac{5-x}{5x}\right| \\ &= \frac{1}{5} \cdot \frac{1}{|x|} \cdot |x-5| < \frac{1}{5} \cdot \frac{1}{4} \cdot \delta \le \frac{1}{20} \cdot 20\varepsilon = \varepsilon, \qquad \text{q.e.d.}\end{aligned}$$

We should note here that if we had chosen $a = 0.5$, then we could not use 1 as the upper bound for δ, since the function is undefined at a point within 1 of 0.5 (namely at $x = 0$). For such an a we should instead assume *a priori* that $\delta < 0.25$, or a similar number to be sure to avoid any problems with the definition of the function for any values of x in which $|x - a| < \delta$. Indeed, we would wish to be sure that $f(x)$ is defined within the "wiggle room" allowed by δ in the continuity definition (3.1), page 169 at the start of this section.

Example 3.1.8 *Show that $f(x) = x^4$ is continuous at $x = -2$.*

Scratch-work: Here $a = -2$ and $f(a) = 16$. Again we will attempt to work backwards.

$$\begin{aligned} & |f(x) - f(-2)| < \varepsilon \\ \iff & |x^4 - 16| < \varepsilon \\ \iff & |x^2 + 4| \cdot |x - 2| \cdot |x + 2| < \varepsilon. \end{aligned}$$

Now our "$|x - a|$", namely $|x + 2|$ is controlled by δ, so we need to control the other two factors. Again let us assume that $\delta \leq 1$. Then

$$|x + 2| < \delta \implies |x + 2| < 1 \iff -1 < x + 2 < 1 \iff -3 < x < -1.$$

Note for later reference that $|x| < 3$.

For the $|x - 2|$ term we can subtract 2 from the above to get $-5 < x - 2 < -3$, giving $|x - 2| < 5$.[8] *For the $|x^2 + 4|$ term, we have $x^2 + 4 > 0$, so*

$$|x^2 + 4| = x^2 + 4 = |x|^2 + 4 < (3)^2 + 4 = 13.$$

(Note that this was because $|x| < 3$.) So far we have

$$\delta \leq 1 \implies |f(x) - f(-2)| = (x^2 + 4)|x - 2| \cdot |x + 2| \underbrace{< 13 \cdot 5 \cdot \delta}_{\text{want} < \varepsilon}.$$

Taking $\delta = \min\left\{\frac{\varepsilon}{65}, 1\right\}$ should give us a proof.

Proof: *Let $\varepsilon > 0$ and choose $\delta = \min\left\{\frac{\varepsilon}{65}, 1\right\}$. Then $\delta > 0$ and*

$$\begin{aligned} |x - (-2)| < \delta \implies |f(x) - f(-2)| &= |x^4 - 16| = (x^2 + 4)|x - 2| \cdot |x + 2| \\ &= (x^2 + 4)|x - 2| \cdot |x - (-2)| \\ &< 13 \cdot 5 \cdot \delta \\ &\leq 26 \cdot \frac{\varepsilon}{65} = \varepsilon, \qquad \text{q.e.d.} \end{aligned}$$

For our final example, we look at a case where we introduce factors in our computation to extract the $|x - a|$ factor.

[8] We could also use the triangle inequality to get $|x - 2| \leq |x| + |-2| < 3 + 2 = 5$. This happens to give the same bound, but in more complicated cases that might not happen. Either bound would then work. For an engineering application, one would likely prefer whatever gives the larger δ, which indicates less sensitivity to tolerance in x to achieve ε tolerance in $f(x)$.

Example 3.1.9 *Show that* $f(x) = \sqrt{x}$ *is continuous at* $x = 4$.

Scratch-Work: As always we begin with $|f(x) - f(a)| < \varepsilon$ *and work backwards to find a case where this is implied by* $|x - a| < \delta$ *for a strategically-chosen* $\delta > 0$.

$$\begin{aligned} |f(x) - f(4)| < \varepsilon &\iff \left|\sqrt{x} - 2\right| < \varepsilon \\ &\iff \left|\frac{\sqrt{x}-2}{\sqrt{x}+2} \cdot \left(\sqrt{x}+2\right)\right| < \varepsilon \\ &\iff \frac{1}{\sqrt{x}+2}\underbrace{|x-4|}_{<\delta} < \varepsilon. \end{aligned}$$

Note that the fraction $1/\left(\sqrt{x}+2\right)$ *is (1) positive wherever it is defined, which is where* $x > 0$, *and therefore (2) maximized when the denominator is minimized, which will happen when the square root term is minimized, i.e., when* x *itself is minimized, but nonnegative, lest* $\sqrt{x}$ *is undefined. For this case we will let* $\delta \le 4$, *so that*

$$\begin{aligned} |x-4| < \delta \implies |x - 4| < 4 &\implies x \in (4-4, 4+4) = (0, 8) \\ &\implies \frac{1}{\sqrt{x}+2} \in \left(\frac{1}{\sqrt{8}+2}, \frac{1}{2}\right) \\ &\implies \frac{1}{\sqrt{x}+2} < \frac{1}{2}. \end{aligned}$$

Using the previous computation we can say that

$$(|x-4| < \delta) \wedge (\delta \le 4) \implies |f(x) - f(4)| = \frac{1}{\sqrt{x}+2}|x-4| \le \frac{1}{2}|x-4| < \frac{1}{2}\delta,$$

and so we can accomplish having this be less than ε *if* $\delta \le 2\varepsilon$. *We will therefore take* $\delta = \min\{4, 2\varepsilon\}$ *in our proof.*

Proof: *For* $\varepsilon > 0$, *let* $\delta = \min\{4, 2\varepsilon\}$, *so* $\delta > 0$ *exists and*

$$\begin{aligned} |x - 4| < \delta \implies |f(x) - f(4)| = \left|\sqrt{x} - 2\right| &= \frac{1}{\sqrt{x}+2}|x-4| \\ &\le \frac{1}{2}|x-4| < \frac{1}{2}\delta \le \frac{1}{2} \cdot 2\varepsilon = \varepsilon, \text{ q.e.d.} \end{aligned}$$

While this section discusses the technical definition of continuity at a point $x = a$, the reader is invited to consider where else this idea of continuity may apply. Anytime we wish to control the output of some process, the presence of continuity with respect to the input would mean that we can in principle guarantee there would be only small changes in output (of size up to ε, whatever we choose that to be) by controlling the changes in the input (keeping them within δ). For instance, if an audio amplifier's gain changes continuously with the position of its volume control, it is much easier to make minor changes in gain by carefully moving the volume control. If (as in the case with old or dirty internal contacts) the volume control's effect is not always continuous, it becomes much more difficult to control the gain. There are numerous other practical examples where continuity is desirable, for instance in the context of tolerances discussed previously.

However, the proofs above are technical (and the exercises somewhat difficult). Still, the reader should be encouraged that the study of subsequent sections will benefit to the extent time is spent studying this section, even if it is not mastered immediately. Furthermore, this section can and should be revisited after study of future sections, so it can be seen with the benefit of a knowledge of the larger context. Of course this is true of all sections in this or any other textbook whose subject is the least bit challenging.

Exercises[9]

1. Show that $f(x) = 9x - 11$ is continuous at $x = 2$.
2. Show that $f(x) = 9x - 11$ is continuous at $x = -2$.
3. Show that $f(x) = 3x + 1$ is continuous at $x = 5$.
4. Show that $f(x) = 6 - 2x$ is continuous at $x = -8$.
5. Show that if $m \neq 0$, then $f(x) = mx + b$ is continuous at every $x = a$. What if $m = 0$? (See Exercise 11 below.)
6. Show that $f(x) = x^3$ is continuous at $x = 0$.
7. Show that $f(x) = x^2$ is continuous at $x = 9$.
8. Show that $f(x) = x^2$ is continuous at $x = -3$.
9. Show that $f(x) = 5x^2 - 3$ is continuous at $x = 2$.
10. Show that $f(x) = \frac{1}{x^2}$ is continuous at $x = 5$.
11. Show that $f(x) = b$ (i.e., a line with slope zero) is continuous at every point $x = a$. (Hint: choosing any $\delta > 0$ will work for the continuity definition.)
12. Show that $f(x) = x^3$ is continuous at $x = 1$.
13. Show that $f(x) = x^3$ is continuous at $x = -3$.
14. Show that $f(x) = \sqrt{x}$ is continuous at $x = 9$.
15. Show that $f(x) = \sqrt{x}$ is continuous at any $a > 0$.
16. Show that $f(x) = \frac{1}{x}$ is continuous at any $a \neq 0$. (Letting $\delta \leq 1$ as in Example 3.1.7 works fine until $0 < |a| \leq 1$. For this more general case desired here, one approach is to assume a priori that $\delta \leq \frac{1}{2}|a|$, so that

$$|x - a| < \delta \implies |x - a| < \frac{1}{2}|a|$$
$$\implies |x| > \frac{1}{2}|a| \implies \frac{1}{|x|} < \frac{2}{|a|}.$$

There is some detail to showing the second implication. From this a δ can be chosen, using some minimum of two values, for each $\varepsilon > 0$.)

17. Consider $f(x) = \sqrt[3]{x}$.

    (a) Show that $f(x)$ is continuous at $x = 0$.

---

[9] In all fairness, it should be pointed out to the reader that some of the exercises here are likely to be quite difficult, particularly for beginning calculus students. This is because the proofs are very involved and use a large variety of methods. Furthermore, reading the proofs of the examples is very different from producing one's own proofs from scratch.

With these exercises, students are thus advised to adopt the following general approach:

(a) attempt as many problems as possible, looking back on earlier examples for ideas;

(b) move on to the rest of the text even if few problems are completed on the first attempt; and

(c) revisit this section and its problems from time to time to attempt complete proofs for the results which were not finished previously.

With further calculus experience, the ideas and techniques should become clearer, just as the inner workings of an automobile likely make more sense—and seem more important—as one gains experience from actually driving.

(b) Show that $f(x)$ is continuous at $x = 8$.

(c) Show that $f(x)$ is continuous at a for any $a > 0$.

(d) Show that $f(x)$ is continuous at any $a \neq 0$.

Conclude from a–d that $f(x)$ is continuous for all $x \in \mathbb{R}$.

18. Recall that $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1})$. Use this to show that $f(x) = \sqrt[n]{x}$ is continuous for all $x > 0$ if $n \in \mathbb{N}$ is even, and continuous for all $x \in \mathbb{R}$ if $n \in \mathbb{N}$ is odd. (For the latter you should do the case $x = 0$ separately.) Notice why the even case does not allow $x = 0$ or $x < 0$. <u>Hint</u>: the triangle inequality is needed, as is the fact that $\sqrt[n]{|x_1|} < \sqrt[n]{|x_2|}$ if $|x_1| < |x_2|$.

3.2 Continuity Theorems

Though fundamental and technically important, using the ε-δ definition to locate each point—or even a single point—where a function is continuous is an unwieldy approach. Fortunately there are many very general theorems which allow us to see where a function is continuous by simple inspection. The theorems below are of great use, collectively and individually, for doing just that. Here we list and prove these theorems, and later in the section we will show how to make use of them. The proofs are interesting and introduce some new techniques, but are not crucial for most of what we do. We include them here for completeness. *The reader is advised to first concentrate on the theorems.*

3.2.1 Basic Theorems

For each theorem below, it is useful to reflect upon some intuitions regarding what makes a function $f(x)$ continuous at a point $x = a$. While ultimately all continuity refers back to the technical definition found in (3.1), page 169, namely

$$f(x) \text{ is continuous at } x = a \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(|x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right),$$

a seemingly less precise interpretation can nonetheless be of some use, namely that if $f(x)$ is continuous at $x = a$, we expect that

> *if x is changed very little from the value $x = a$, then $f(x)$ will change very little from the value $f(a)$; sufficiently small changes away from $x = a$ in the value of the input of $f(x)$ will result in only small changes from $f(a)$ in the output of $f(x)$ very little from $f(a)$. (This is not true if $f(x)$ is not continuous at $x = a$.)*

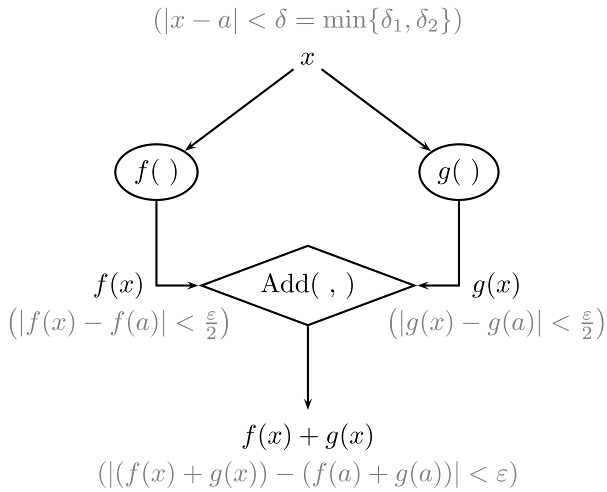
It is the ε-δ definition which makes this notion precise.

The theorems below extend this to many combinations of functions. For instance if $f(x)$ and $g(x)$ are both continuous at $x = a$, then so is $f(x) + g(x)$; if $f(x)$ and $g(x)$ change very little if x changes a small amount from the value $x = a$, it is natural to expect the sum $f(x) + g(x)$ to change little as well.

Theorem 3.2.1 *Suppose that $f(x)$ and $g(x)$ are continuous at $x = a$. Then so is $f(x) + g(x)$.*

> **<u>Proof</u>:** We are beginning under the assumption that $f(x)$ and $g(x)$ are continuous at $x = a$, i.e.,
>
> $$\begin{aligned}&(\forall \varepsilon_1 > 0)(\exists \delta_1 > 0)(\forall x)(|x-a| < \delta_1 \longrightarrow |f(x) - f(a)| < \varepsilon_1),\\ &(\forall \varepsilon_2 > 0)(\exists \delta_2 > 0)(\forall x)(|x-a| < \delta_2 \longrightarrow |g(x) - g(a)| < \varepsilon_2).\end{aligned}$$
>
> Define $h(x) = f(x) + g(x)$. We want to show that $h(x)$ is also continuous at $x = a$. For a given $\varepsilon > 0$, set both $\varepsilon_1, \varepsilon_2 = \varepsilon/2$. Next find corresponding δ_1, δ_2 satisfying the definitions above of continuity for $f(x)$ and $g(x)$ at $x = a$, respectively. Finally, pick $\delta = \min\{\delta_1, \delta_2\}$. Recalling the triangle inequality, $|\underline{A} + \underline{\underline{B}}| \le |\underline{A}| + |\underline{\underline{B}}|$, we have
>
> $$\begin{aligned}|x-a| < \delta \implies |h(x) - h(a)| &= |f(x) + g(x) - f(a) - g(a)|\\ &= |\underline{f(x) - f(a)} + \underline{\underline{g(x) - g(a)}}| \le |\underline{f(x) - f(a)}| + |\underline{\underline{g(x) - g(a)}}|\\ &< \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad \text{q.e.d.}\end{aligned}$$

It is somewhat interesting to diagram $f(x)+g(x)$ and note the flow of the tolerances which appear in the proof. We do this below, with tolerances in gray:



It is not difficult to see that this can be extended to include sums of more functions. Indeed, if $i(x)=f(x)+g(x)+h(x)$ where $f(x),g(x),h(x)$ are all continuous at $x=a$, then so is the sum $(f(x)+g(x))$ by the previous theorem, and then again so will be the sum $[f(x)+g(x)]+(h(x))=i(x)$ by that same theorem. (A proof "from scratch" can be done as well, following the same pattern but using $\varepsilon_1,\varepsilon_2,\varepsilon_3=\varepsilon/3$ and $\delta=\min\{\delta_1,\delta_2,\delta_3\}$, but it is faster to "bootstrap" later results from theorems which were already proved.)

The next theorem's proof is less difficult, and is left as an exercise (Exercise 23, page 193):

Theorem 3.2.2 *If $f(x)$ is continuous at $x=a$, then so is $Cf(x)$ for any constant C.*

The next theorem is the most difficult of these to prove, but will also be one of the most useful in practice.

Theorem 3.2.3 *Suppose that $f(x)$ and $g(x)$ are continuous at $x=a$. Then so is $f(x)g(x)$.*

Proof: Let $\varepsilon>0$, for the definition of continuity for the product $h(x)=f(x)g(x)$. Now we carefully construct a $\delta>0$ so that we can prove $h(x)$ continuous at $x=a$, i.e.,

$$|x-a|<\delta \implies |h(x)-h(a)|=|f(x)g(x)-f(a)g(a)|<\varepsilon \qquad \text{(to prove!)}.$$

As before we are assuming that $f(x),g(x)$ are continuous at $x=a$, i.e.,

$$(\forall\varepsilon_1>0)(\exists\delta_1>0)(\forall x)(|x-a|<\delta_1 \longrightarrow |f(x)-f(a)|<\varepsilon_1), \tag{3.7}$$
$$(\forall\varepsilon_2>0)(\exists\delta_2>0)(\forall x)(|x-a|<\delta_2 \longrightarrow |g(x)-g(a)|<\varepsilon_2). \tag{3.8}$$

First we note that what we need to control is $|f(x)g(x)-f(a)g(a)|$, which can be expanded and then bounded using the triangle inequality as follows:

$$\begin{aligned}
&|f(x)g(x) - f(a)g(a)| \\
&\quad = \frac{1}{2}|(f(x) - f(a))(g(x) + g(a)) + (f(x) + f(a))(g(x) - g(a))| \\
&\quad \leq \frac{1}{2}|(f(x) - f(a))(g(x) + g(a))| + \frac{1}{2}|(f(x) + f(a))(g(x) - g(a))| \\
&\quad = \frac{1}{2}|f(x) - f(a)| \cdot |g(x) + g(a)| + \frac{1}{2}|f(x) + f(a)| \cdot |g(x) - g(a)|.
\end{aligned}$$

It is enough that the final line in the above be less than ε.

We will choose $\varepsilon_1, \varepsilon_2$ based on the choice of ε. Let us first assume *a priori* that any $\varepsilon_1, \varepsilon_2 \leq 1$ in (3.7) and (3.8) and so with $|x - a| < \min\{\delta_1, \delta_2\}$ we get by the triangle inequality

$$|f(x) - f(a)| < 1 \implies |f(x)| = |f(a) + (f(x) - f(a))| < |f(a)| + 1, \tag{3.9}$$

$$|g(x) - g(a)| < 1 \implies |g(x)| = |g(a) + (g(x) - g(a))| < |g(a)| + 1. \tag{3.10}$$

Now define L and M as follow:

$$L = |f(a)| + 1 > 0, \tag{3.11}$$

$$M = |g(a)| + 1 > 0. \tag{3.12}$$

From (3.9), (3.10) and (3.11), (3.12) we get $|x - a| < \min\{\delta_1, \delta_1\} \implies$

$$|f(x)| < L + 1, \qquad |f(a)| < L, \tag{3.13}$$

$$|g(x)| < M + 1, \qquad |g(a)| < M. \tag{3.14}$$

Now we prove the statement of the continuity of $f(x)g(x)$ at $x = a$. For any $\varepsilon > 0$ define

$$\varepsilon_1 = \min\left\{1,\ \varepsilon,\ \frac{\varepsilon}{2M+1}\right\} = \min\left\{1,\ \frac{\varepsilon}{2M+1}\right\}, \tag{3.15}$$

$$\varepsilon_2 = \min\left\{1,\ \varepsilon,\ \frac{\varepsilon}{2L+1}\right\} = \min\left\{1,\ \frac{\varepsilon}{2L+1}\right\}, \tag{3.16}$$

and the respective δ_1, δ_2 from these $\varepsilon_1, \varepsilon_2$ as appear in the continuity conditions on $f(x)$ and $g(x)$ at $x = a$, namely (3.7) and (3.8). Note that we used the fact that $\varepsilon = \varepsilon/1 < \varepsilon/(2M+1)$, and similarly $\varepsilon < \varepsilon/(2L+1)$, in (3.15) and (3.16) respectively. Finally, choose $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ exists (because $\delta_1, \delta_2 > 0$) and

$$\begin{aligned}
&|x - a| < \delta \implies \\
&\quad |f(x)g(x) - f(a)g(a)| \\
&\quad \leq \frac{1}{2}|f(x) - f(a)| \cdot |g(x) + g(a)| + \frac{1}{2}|f(x) + f(a)| \cdot |g(x) - g(a)| \\
&\quad < \frac{1}{2}\varepsilon_1 \cdot (|g(x)| + |g(a)|) + \frac{1}{2}(|f(x)| + |f(a)|)\varepsilon_2 \\
&\quad \leq \frac{1}{2}\varepsilon_1(2M + 1) + \frac{1}{2}\varepsilon_2(2M + 1) \\
&\quad \leq \frac{1}{2} \cdot \frac{\varepsilon}{2M + 1} \cdot (2M + 1) + \frac{1}{2} \cdot \frac{\varepsilon}{2L + 1} \cdot (2L + 1) \\
&\quad = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad \text{q.e.d.}
\end{aligned}$$

We could diagram $f(x)g(x)$ as we did with $f(x)+g(x)$, though the tolerances are more complicated. In fact the above proof is more difficult than most in this text, and requires careful examination to understand completely, or to be able to reproduce it. Contained within it lies a common method for controlling $|f(x)g(x)-f(a)g(a)|$ by controlling $|x-a|$ which is worth tracing. In particular we used the identity

$$f(x)g(x)-f(a)g(a)=\frac{1}{2}(f(x)-f(a))(g(x)+g(a))+\frac{1}{2}(f(x)+f(a))(g(x)-g(a)).$$

All three continuity theorems so far can be described intuitively as saying, respectively (for functions continuous at $x=a$):

- **a sum of functions which are continuous at a will also be continuous at a;**
- **a constant multiple of a function which is continuous at a will also be continuous at a;**
- **a product of functions which are continuous at a will also be continuous at a.**

Rephrased, if we can control how far functions stray from their values at $x=a$—by controlling how far x strays from a—then we can control how far their sums, products, and constant multiples of the functions stray from their respective values at $x=a$ as well. For the next theorem we first need the following:

Lemma 3.2.1 $(\forall a\in\mathbb{R})[f(x)=x$ *is continuous at* $x=a]$.

This is fairly trivial to prove, because we would just set $\delta=\varepsilon$ in the definition of continuity. Details are left to the exercises.

Theorem 3.2.4 *Polynomial functions are continuous at every $a\in\mathbb{R}$.*

Hence the functions $f(x)=x^2+1$, $g(x)=55x^{39}+101x-10,000,000$, and $h(x)=(9-23x)^{15}$ are all continuous. Certainly this theorem gives a welcome relief from trying to prove these functions are continuous using ε-δ. The proof is really quite simple (and may even sound a bit flippant).

Proof: Any polynomial can be written

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.$$

Now each x^k can be written as a product of k factors of the continuous function $g(x)=x$, so the x^k are all continuous for every $x=a$, as are constant multiples of these, i.e., the a_kx^k terms. Of course $h(x)=a_0$ is a constant function and is therefore continuous (see Exercise 11, page 178), so a polynomial is just the sum of continuous functions, and is therefore continuous.

The next theorem is also very useful, and deals with compositions of functions. Compared with, say, Theorem 3.2.3 this is surprisingly simple to prove.

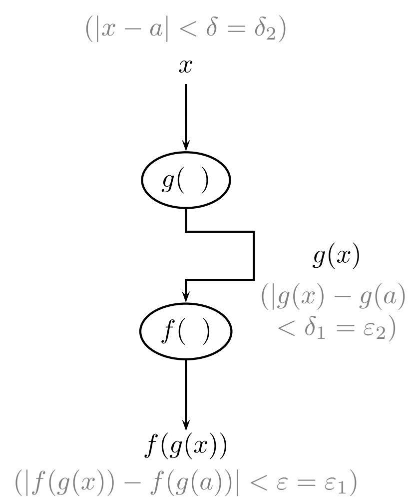
Theorem 3.2.5 *Suppose that $f(x)$ is continuous at $x=L=g(a)$, and that $g(x)$ is continuous at $x=a$. Then $f(g(x))$ is continuous at $x=a$.*

Put concisely, the composition of continuous functions is continuous.

Proof: Let $\varepsilon > 0$. We need to construct a $\delta > 0$ such that
$$|x - a| < \delta \implies |f(g(x)) - f(g(a))| < \varepsilon.$$
By our continuity assumptions, we know that
$$(\forall \varepsilon_1 > 0)(\exists \delta_1 > 0)(\forall x)(|x - L| < \delta_1 \longrightarrow |f(x) - f(L)| < \varepsilon_1),$$
$$(\forall \varepsilon_2 > 0)(\exists \delta_2 > 0)(\forall x)(|x - a| < \delta_2 \longrightarrow |g(x) - g(a)| < \varepsilon_2).$$
So for this ε, choose $\varepsilon_1 = \varepsilon$, which gives a $\delta_1 > 0$ so that
$$|x - L| < \delta_1 \implies |f(x) - f(L)| < \varepsilon.$$
Next set $\varepsilon_2 = \delta_1 > 0$. This gives a $\delta_2 > 0$ so that
$$|x - a| < \delta_2 \implies |g(x) - g(a)| < \varepsilon_2 = \delta_1.$$
Finally, let $\delta = \delta_2$, corresponding to ε_2 in the continuity requirement for g. This gives $\delta > 0$ and the following (note the only necessary directions are $\implies$ but we note the steps for which we actually have $\iff$):
$$\begin{aligned}
|x - a| < \delta &\iff |x - a| < \delta_2 \\
&\implies |g(x) - g(a)| < \varepsilon_2 && \text{(since input } x \text{ of } g \text{ is } \delta_2\text{-close to } a) \\
&\iff |g(x) - L| < \delta_1 && \text{(since } \varepsilon_2 = \delta_1) \\
&\implies |f(g(x)) - f(L)| < \varepsilon_1 && \text{(since input } g(x) \text{ of } f \text{ is } \delta_1\text{-close to } L) \\
&\iff |f(g(x)) - f(g(a))| < \varepsilon, && \text{q.e.d. (since } \varepsilon_1 = \varepsilon)
\end{aligned}$$

Intuitively, if $g(x)$ is changing continuously at $x = a$, and $f(\ )$ changes continuously at its input value $g(a)$, it seems reasonable that $f(g(x))$ should change continuously at $x = a$. Even less precisely, with the assumed continuity conditions of $f(\ )$ and $g(\ )$ we might expect that if $x \approx a$ then $g(x) \approx g(a)$, and then $f(g(x)) \approx f(g(a))$.

Now we have a proof, constructed as usual backwards from the function's flow, illustrated on the right. First we choose $\varepsilon > 0$ tolerance for the final output. With $\varepsilon_1 = \varepsilon > 0$ we have $\delta_1 > 0$ tolerance for the input of $f(\ )$ (near the value $g(a)$) guaranteeing ε tolerance in the output of $f(\ )$. Letting $\varepsilon_2 = \delta_1 > 0$, we have $\delta_2 > 0$ tolerance in x near $x = a$ allowing for at most ε_2 tolerance in the output of $g(x)$ to be near $g(a)$, guaranteeing ε-tolerance when this is fed to $f(\ )$.



We next work towards a simple theorem regarding quotients, which we argue deductively towards rather than stating and proving the steps in turn.[10]

First recall from Exercise 16, page 178 that we have the following (stated here without proof):

[10] When a thread contains theorems which are each difficult to prove, or contain some tangential clever technique, it is common for the discussion to proceed with a structure reading theorem-proof, theorem-proof, etc. When theorems flow from each other with minimal argument needed, the style often changes to short argument-result, short argument-result, etc.

Theorem 3.2.6

$$f(x) = \frac{1}{x} \qquad \textit{is continuous for all } a \neq 0. \tag{3.17}$$

Continuing an argument which leads to a theorem on quotients, next we suppose that $g(x)$ is continuous at $x = a$, and that $g(a) \neq 0$. Then (3.17) and the previous theorem (Theorem 3.2.5 above) with $f(x) = \frac{1}{x}$ conspire to give us that $f(g(x)) = 1/g(x)$ is thereby continuous at $x = a$:

Theorem 3.2.7 $(g(x)$ *continuous at* $x = a) \wedge (g(a) \neq 0) \implies \dfrac{1}{g(x)}$ *continuous at* $x = a$.

For arbitrary functions $f(x)$ and $g(x)$ which are continuous at $x = a$, if $g(a) \neq 0$ we can always write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$, a product of two functions now known to be continuous at $x = a$, and so by our Theorem 3.2.3, page 181 that product is also continuous and we get another theorem:

Theorem 3.2.8 *If* $f(x)$ *and* $g(x)$ *are continuous at* $x = a$*, and* $g(a) \neq 0$*, then* $f(x)/g(x)$ *is also continuous at* $x = a$*.*

This gives us a quick result on rational functions:

Theorem 3.2.9 *If* $f(x) = \dfrac{p(x)}{q(x)}$*, where* p *and* q *are polynomials, then* $f(x)$ *is continuous at every* $a \in \mathbb{R}$ *except where* $q(a) = 0$*, at which points* $f(x)$ *is undefined and therefore discontinuous.*

In other words, *rational functions are continuous where defined.* (The proofs of both of these are contained in the discussion above.) Next we mention the following reasonable definition:

Definition 3.2.1 *If* $f(x)$ *is not continuous at some point* c*, then* $f(x)$ *is called* **discontinuous** *at* $x = c$*, and* c *is called a* **point of discontinuity** *of* $f(x)$*.*[11]

It is sometimes easier to list those points at which a function is discontinuous than the set of points at which it is continuous. Of course continuity at a point, and discontinuity at a point, are negations of each other.

Example 3.2.1 *Find where the function* $f(x) = \dfrac{3x-5}{x^2-1}$ *is continuous.*

<u>*Solution:*</u> *As a rational function,* $f(x)$ *is continuous* **except** *where* $x^2 - 1 = 0$*, i.e., except where* $x^2 = 1$*, i.e., except where* $x = -1, 1$*. We can effectively describe where* $f(x)$ *is continuous in the following ways (all of which are equivalent):*

a. $f(x)$ *is continuous except at* $x = -1, 1$*;*

b. $f(x)$ *is continuous for* $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$*;*

c. $f(x)$ *is continuous for all* $x \neq \pm 1$*.*

It is important that we not simplify $f(x)$ in any way before describing where it is continuous, lest problem points in the definition of $f(x)$ be glossed over. For instance, consider the following:

[11] The word "point" sometimes refers to (among other things) a value (point) on the real line (or an element of $\mathbb{R}$), and other times refers to a point in the xy-plane. For instance, one may describe $f(x) = x^2$ "at the point $x = 3$" or "at the point $(3, 9)$." In most cases it is an imprecision in the language which is cleared up by an understanding of the context, though occasionally two authors will have different but strong opinions on how to make the language precise.

Example 3.2.2 *Find where* $f(x) = \dfrac{x^2 - 25}{x - 5}$ *is continuous.*

Solution: Clearly $f(x)$ *is not defined at* $x = 5$*, and therefore cannot possibly be continuous there. (The way in which* $f(a)$ *appears in the definition of continuity requires that it exist.) Simplifying is desirable, but we must note that* $x = 5$ *is not in the domain. To be clear on this we can write*

$$f(x) = \frac{x^2 - 25}{x - 5} = \frac{(x+5)(x-5)}{x-5} = x + 5, \qquad x \neq 5.$$

(Note that the cancellation step indeed required that $x - 5 \neq 0$*, i.e.,* $x \neq 5$ *because there we would essentially be dividing numerator and denominator by zero—not a valid arithmetic operation—when cancelling the* $x - 5$ *factors). Such a function is defined and continuous for* $x \in (-\infty, 5) \cup (5, \infty)$*, i.e., for* $x \neq 5$*.*

The proof of the next theorem was the subject of Exercise 18, page 179.

Theorem 3.2.10 *Suppose that* $f(x) = \sqrt[n]{x} = x^{1/n}$*, with* $n \in \mathbb{N}$*.*

(i) If n *is odd, then* $f(x)$ *is continuous at each* $x \in \mathbb{R}$*.*

(ii) If n *is even, then* $f(x)$ *is continuous at each* $x > 0$*, and discontinuous otherwise.*

Thus $f(x) = \sqrt[3]{x}$ is continuous for all $x \in \mathbb{R}$, where $g(x) = \sqrt{x} = \sqrt[2]{x}$ is only continuous for $x > 0$. Though $g(x)$ is defined at $x = 0$, but there is no room to the left of zero, so any interval $(0 - \delta, 0 + \delta)$, with $\delta > 0$, will always have points outside the domain of g (namely $x \in (0 - \delta, 0)$). Thus with even roots we need to look at more than where they are defined to determine continuity. If we consider only $x \geq 0$, then for any $\varepsilon > 0$ we can find a $\delta > 0$ so that we can at least say $(|x - 0| < \delta) \wedge (x \geq 0) \implies |g(x) - g(0)| < \varepsilon$, and this is occasionally interesting, and so we have use for so-called one-sided continuity, the types of which are defined below.

3.2.2 One-Sided Continuity

At this point it is useful to introduce the following concepts.

Definition 3.2.2 *We call* $f(x)$ **left-continuous** *at* $x = a$ *if and only if*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(x \in (a - \delta, a] \longrightarrow |f(x) - f(a)| < \varepsilon). \tag{3.18}$$

Similarly, we call $f(x)$ **right-continuous** *at* $x = a$ *if and only if*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(x \in [a, a + \delta) \longrightarrow |f(x) - f(a)| < \varepsilon). \tag{3.19}$$

Left-continuity at $x = a$ only considers points at or to the left of a $(x \leq a)$, while right-continuity considers those at or to the right of a $(x \geq a)$. In the definition of continuity (3.1), page 169 both sides of $x = a$ are considered. In fact that original definition of continuity can be rewritten

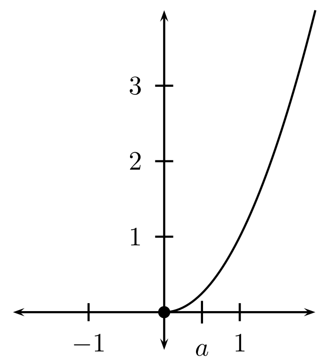
$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(x \in (a - \delta, a] \cup [a, a + \delta) \longrightarrow |f(x) - f(a)| < \varepsilon), \tag{3.20}$$

from which we can see how left-continuity (3.18) and right-continuity (3.19) are related to continuity. Thus these one-sided continuity conditions are effectively two halves of the continuity requirement as written in (3.20).

One aspect of this theory which has not been mentioned explicitly as yet is that either type of continuity (the original two-sided continuity, left-continuity and right-continuity) at $x = a$ require that $f(a)$ must be defined, in order that $|f(x) - f(a)|$ be true in the implications within the definitions. A function $f(x)$ can have none of these continuity properties at $x = a$ if $f(a)$ is undefined. This will be an issue in identifying where some functions are continuous.

Example 3.2.3 *Consider the function $f(x)$ defined only on $x \in [0,\infty)$, but on that interval given by $f(x) = x^2$. In other words,* $f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 0, \\ \text{undefined} & \text{if} \quad x < 0. \end{cases}$

- *Then $f(x)$ is continuous at $x = a$ for any $a > 0$.*
  - *Intuitively, we can see from the graph that we have "wiggle room" in x as long as $a > 0$ for which $|x - a| < a$ still has $f(x) = x^2$, which is clearly a continuous function; we can be guaranteed tolerances as small as we like in $f(x)$ being close to $f(a)$ by forcing x to be close to, but not necessarily equal to, a. This is not the case if $a = 0$, as we would have no "wiggle room" to the left of $x = 0$.*



  - *More technically this is because in the definition of continuity we can assume* a priori *that $\delta \leq a$, since $a > 0$, and then $|x-a| < \delta \implies f(a)$ is defined, and so we can borrow continuity results from our earlier theorem on the continuity of polynomials. To do so properly, we should consider the polynomial function $g(x) = x^2$ defined on all of $\mathbb{R}$. Now $g(x)$ is continuous for any $x = a$, with $a \in \mathbb{R}$. Thus for any $a \in \mathbb{R}$, if we let $\varepsilon > 0$ there exists $\delta_g > 0$ ("the Delta for g") such that $|x - a| < \delta_g \implies |g(x) - g(a)| < \varepsilon$. Now for our $f(x)$ defined above, and any $a > 0$ we can take $\delta = \min\{\delta_g, a\}$, so $\delta > 0$ exists and*

$$|x-a| < \delta \implies \begin{bmatrix} (|x - a| < \delta \leq \delta_g) \\ \wedge \\ (x > 0) \end{bmatrix} \implies |\underbrace{f(x) - f(a)}_{\text{exists}}| = \underbrace{|g(x) - g(a)|}_{\text{since } x>0} \quad \underbrace{< \varepsilon.}_{\text{since } \delta \leq \delta_g}$$

- *$f(x)$ is both left-continuous and right-continuous (as well as continuous in our original two-sided sense) at each $a > 0$. That should be clear because continuity is a stronger condition than the one-sided continuities. (See* Theorem 3.2.11 *below.)*

- *$f(x)$ is right-continuous (but neither continuous nor left-continuous) at $x = 0$. We only need to allow tolerance ("wiggle room") allowing for x to vary to the right of $x = 0$ (see graph). In fact, $\delta = \sqrt{\varepsilon}$ proves right-continuity: Let $\varepsilon > 0$, and choose $\delta = \sqrt{\varepsilon}$, so $\delta > 0$ exists and*

$$x \in [0, \delta) \implies |f(x) - f(0)| = x^2 < \delta^2 = \varepsilon, \text{ q.e.d.}$$

- *$f(x)$ has no continuity properties for $x < 0$, because the function is not defined there.*

Continuity is, in fact, a "local" phenomenon; what matters for $x = a$ is the behavior of the function "near $x = a$." This is because we can always restrict $\delta > 0$ to be as small as we like *a priori*, in essence ignoring what occurs outside of $(a - \delta, a + \delta)$ no matter how small we make δ, while keeping it positive. Even so restricted, we are still allowing for (and in fact requiring) a *continuum* containing a, extending from a in one or both directions, depending upon if we are considering one-sided continuity or the original, two-sided continuity.

Of course the different types of continuities are related. We leave for an exercise the following theorem:

Theorem 3.2.11 *$f(x)$ is continuous at $x = a$ if and only if $f(x)$ is both left-continuous and right-continuous at $x = a$.*

There are occasions where we check continuity by checking both one-sided continuity conditions, and other settings where we require only one-sided continuity but have that easily because we have two-sided continuity, so the theorem above has its uses.

Example 3.2.4 *Consider the function* $f(x) = \begin{cases} (x-2)^2 - 1 & \text{if} \quad x \ge 2, \\ 1 - x & \text{if} \quad x < 2. \end{cases}$

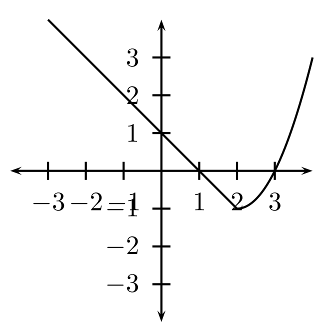
- *At $x = 0$ this function is continuous, because it is essentially $f(x) = 1 - x$ for $x \in (-\infty, 2)$, and $x = 0$ is safely inside this interval (in and surrounded by a continuum) in which $f(x)$ is simply the polynomial $1 - x$. If we wished to prove continuity at $x = 0$ from the definition, we can always choose $\delta \le 2$ so that when we write $|x - 0| < \delta$ we know we are within an interval in which $f(x)$ is a fixed polynomial and therefore continuous. (In fact $\delta = \min\{2, \varepsilon\}$ would work in the continuity definition.)*
- *At $x = 3$, this function is again continuous, because we are safely inside of $[2, \infty)$, and we could if necessary take $\delta \le 1$ so that $|x - 3| < \delta \implies x \in [2, \infty) \implies f(x) = (x-2)^2 - 1$, i.e., $f(x) = x^2 - 4x + 4 - 1 = x^2 - 4x + 3$, ultimately a polynomial, albeit one which requires a more careful choice of $\delta > 0$ for a given $\varepsilon > 0$ in the definition of continuity.*
- *At $x = 2$, we have to be more careful. Note that $f(2) = (2-2)^2 - 1 = -1$, which coincides with $1 - x$ where $x = 2$ as well. This means we could have used $x \le 2$ as the condition for which $f(x) = 1 - x$, because it did not matter which formula we applied at $x = 2$. Thus*

  (i) $x \in [2, \infty) \implies f(x) = (x-2)^2 - 1$,

  (ii) $x \in (-\infty, 2] \implies f(x) = 1 - x$.

  From (i) we know that $f(x)$, being a polynomial on $[2, \infty)$, is right-continuous at $x = 2$; and from (ii) we know that $f(x)$, being a polynomial on $(-\infty, 2]$, is left-continuous at $x = 2$. Since $f(x)$ is both left-continuous and right-continuous at $x = 2$, we conclude that $f(x)$ is continuous (in the original, two-sided sense) at $x = 2$.

In essence, if a function is defined piecewise and the defining formulas agree with $f(a)$ in their outputs at a point a on the boundary of the formulas' respective intervals, and if the defining formulas would themselves define appropriately continuous functions (left-continuous at a in the left-piece formula, and right-continuous at a in the right-piece formula), then $f(x)$ will be continuous at $x = a$. In the above example, we could simply say that the two formulas to the left and right of $x = 2$ would both yield $f(2)$ if we substituted $x = 2$ into them. Visually, the two pieces should "meet" at $(2, f(2))$, as in the graph on the right of the function in the above example.



While it is useful to consider the graphs when determining continuity, so far we have stressed the *analysis* without resorting to graphs. We will utilize graphs more in the next sections, but for now we will attempt to develop a "number sense" and "function sense" in a manner somewhat independent from the graphs. For instance, if we return again to the even roots $f(x) = \sqrt[n]{x}$, (where n is even) we see that, though these functions are not continuous at $x = 0$, they are right-continuous at $x = 0$. When dealing with functions which contain radicals, for continuity we often need only see where the functions are defined, and where we have some "wiggle room" on both sides of a particular $x = a$. Odd roots are defined and (two-sided) continuous for any real input so they do not in themselves limit where a function is continuous. Consider the following:

Example 3.2.5 *For each function, find where it is continuous.*

$$f(x) = \frac{1}{\sqrt[3]{x-1}} \qquad\qquad i(x) = \sqrt{x^2+1}$$
$$g(x) = \sqrt{2x+1} \qquad\qquad j(x) = \frac{1}{\sqrt{x^2-1}},$$
$$h(x) = \sqrt{20-4x} \qquad\qquad k(x) = \sqrt[3]{x^9 - 27x^4 + x - 11}.$$

Solution: We take each of these in turn, making use of our previous continuity theorems on sums, products, compositions and quotients of functions, as well as results on polynomials and roots.

- *The function $f(x)$ is a ratio of functions which are continuous everywhere, so we need only check where the denominator is zero. Thus $f(x)$ is discontinuous only at $x = 1$. (In other words, $f(x)$ is continuous on $(-\infty, 1) \cup (1, \infty)$, but we will be content in such a case to simply state where it is discontinuous.)*

- *The function $g(x)$ is definitely continuous where $2x + 1 > 0$, i.e., $2x > -1$, i.e., $x > -1/2$. It is not continuous at $x = -1/2$ because it $g(x)$ is undefined to the left of $x = -1/2$ as a quick check will show. ($g(x)$ is right-continuous at $x = -1/2$, but not continuous there.) Conclude $g(x)$ is continuous for $x > -1/2$.*

- *The function $h(x)$ is definitely continuous for $20-4x > 0$, i.e., $-4x > -20$, i.e., $x < 5$. It is not continuous at $x = 5$ since it is undefined to the right of that point. (It is left-continuous at $x = 5$, but again, that was not the question.) Conclude $h(x)$ is continuous where $x < 5$.*

- *The function $i(x)$ is continuous everywhere, since $x^2 + 1$ is continuous everywhere and for all $x \in \mathbb{R}$ we also have $x^2 + 1 > 0$. (In fact $x^2 + 1 \geq 1 > 0$.)*

- *The functions $j(x)$ is definitely discontinuous at $x = \pm 1$ because the denominator is zero there. With that consideration, and that of the square root disallowing its input being negative, for continuiut we need $x^2 - 1 > 0$, i.e., $x^2 > 1$, which occurs when $x > 1$ or $x < -1$.*

- *The function $k(x)$ is continuous everywhere, being an odd root of a (continuous) polynomial.*

Example 3.2.6 *Find where the function $f(x) = \dfrac{\sqrt{9-x^2}}{x^2-1}$ is continuous.*

Solution: For denominator considerations, we need only that $x \neq \pm 1$. The numerator is definitely continuous for $9-x^2 > 0$, i.e., $9 > x^2$, i.e., $-3 < x < 3$. It is not continuous at $x = \pm 3$ since both are on the edge of the domain (no "wiggle room"). Putting this together, we see $f(x)$ is continuous for $x \in (-3, 3) - \{-1, 1\} = (-3, -1) \cup (-1, 1) \cup (1, 3)$.

To be clear, we next look at a function which is the reciprocal of the previous function to illustrate the differences.

Example 3.2.7 *Find where the function* $f(x) = \dfrac{x^2-1}{\sqrt{9-x^2}}$ *is continuous.*

<u>Solution</u>: For this function, the numerator is continuous for all $x \in \mathbb{R}$, so there is no restriction on x implied by the form of the numerator. (Numerators can be equal to zero, while denominators cannot, for a quotient to be defined.) Clearly $x \neq -3, 3$ due to the denominator, lest we attempt to divide by zero. Requiring the input of the square root to be nonnegative forces $-3 \leq x \leq 3$, but we already disallowed $x = \pm 3$ due to the restriction against dividing by zero, though an alternative explanation here would be the lack of "wiggle room" either values have, -3 to the left and 3 to the right, avoiding again have a negative input for the square root function $\sqrt{(\ )}$. Our conclusion is that this $f(x)$ is continuous for $x \in (-3, 3)$.

A useful corollary to Theorem 3.2.10 is the fact that

$$f(x) = |x| = \sqrt{x^2}$$

is continuous for all $x \in \mathbb{R}$. To see this, note that clearly $|x|$ is continuous for $x \neq 0$, for which $x^2 > 0$. But there is "wiggle room" at $x = 0$ as well. In fact it is easier to see that $|x|$ is both left- and right-continuous at $x = 0$ if we write

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x \leq 0. \end{cases} \tag{3.21}$$

As a matter of form, we usually write definitions such as (3.21) with the "if" conditions being mutually exclusive (non-overlapping, as if we were instructing a computer), but we can see that at $x = 0$ both formulas give the same result. We can then see that on $x \in [0, \infty)$ we can write $|x| = x$, which is obviously right-continuous at $x = 0$. Similarly, on $x \in (-\infty, 0]$ we can write $|x| = -x$, which is obviously left-continuous at $x = 0$. Since $|x|$ as a function is both left-continuous and right-continuous at $x = 0$, it is continuous there.

Furthermore, at any $x \neq 0$ we are "safely" within either branch: where $x > 0 \implies |x| = x$ which is continuous, or where $x < 0 \implies |x| = -x$ which is also continuous. From these three cases ($x = 0$, $x > 0$, $x < 0$) we can see that $|x|$ is continuous at each $x \in \mathbb{R}$, as we state below:

Theorem 3.2.12 *The function $f(x) = |x|$ is continuous for all $x \in \mathbb{R}$.*

By our theorem on function compositions (Theorem 3.2.5 on page 183), we have that the absolute value of a continuous function is also continuous:

Theorem 3.2.13 *If $g(x)$ is continuous at $x = a$, then $f(x) = |g(x)|$ is also continuous at $x = a$.*

Example 3.2.8 *$f(x) = \sqrt{x^2 - 2x + 1}$ is continuous on $\mathbb{R}$ since $f(x) = \sqrt{(x-1)^2} = |x - 1|$ is the absolute value of a continuous function and is therefore continuous.*

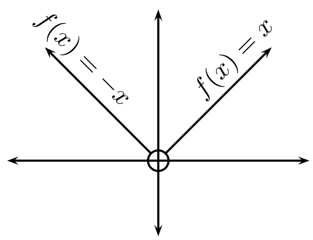
3.2.3 Essential versus Removable Discontinuities

Many functions which we encounter will have a discontinuity at a particular point, but where the discontinuity will be *removable*. This means that if we were to redefine the function at that point to have a well chosen output value, it would cease to be discontinuous there.

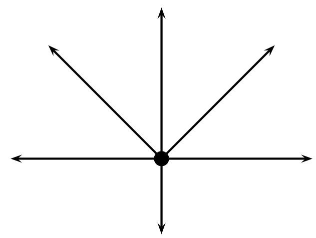
Example 3.2.9 *Consider the function $f(x) = \frac{x^2}{|x|}$. Clearly this function is undefined at $x = 0$, because we cannot divide by zero. Note that this function can instead be defined piece-wise:*

$$f(x) = \begin{cases} x^2/x & \text{if} \quad x > 0 \\ x^2/(-x) & \text{if} \quad x < 0 \end{cases} \quad = \quad \begin{cases} x & \text{if} \quad x > 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

In other words, $f(x) = |x|$ for $x \neq 0$. We say that $x = 0$ is a **removable discontinuity** *because if we were to simply redefine f so that $f(0) = 0$, we would have a continuous function at $x = 0$.*



$y = f(x)$

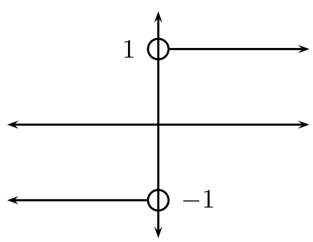


$y = f(x)$ *(redefined at $x = 0$)*

Clearly the only value we could use to redefine the output of $f(x)$ at $x = 0$ is $f(x) = 0$.

Example 3.2.10 *Consider the function $f(x) = \dfrac{x}{|x|}$. This function is also undefined at $x = 0$. Defined piece-wise, we have the easily graphed function*

$$f(x) = \begin{cases} x/x & \text{if} \quad x > 0 \\ x/(-x) & \text{if} \quad x < 0 \end{cases} \quad = \quad \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0. \end{cases}$$



This is graphed on the left. In order for the function to be right-continuous at $x = 0$ we would require $f(0) = 1$, but for left-continuity we would require $f(0) = -1$. Clearly we cannot have both (or we would not have a function!) and so the discontinuity at $x = 0$ is called **essential.**[12]

It is usually not difficult spot removable and essential discontinuities from the graph of the function: if we can "fill in a hole" to produce a function which is continuous at the point, then the discontinuity is removable; otherwise it is essential. However, one goal of the analysis is to be able to make such determinations based upon formulas for the functions, rather than based upon a requirement that we always produce a graph first.[13] At this point we are ready for our definitions:

[12]Another common term for essential discontinuities is **nonremovable**. Both terms have their respective advantages.

[13]There are approaches for learning calculus based upon exploring most problems with graphical calculators or software first—in essence, approaching calculus first as a visual exercise—though it does not take long for one to encounter functions complicated enough that a reliance on electronically-produced graphs is cumbersome or misleading. The calculator-based calculus instruction is especially popular in high school calculus courses, where there is an assumption that algebraic skills may be lacking, and that students will be more interested when they have a visual reference. One drawback is that the user can be fooled by limitations of a calculator display's resolution, or the analytical shortcomings of today's common graphing calculators.

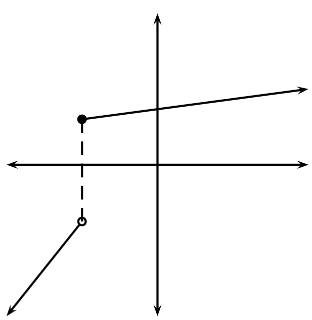
Definition 3.2.3 *Given a function $f(x)$, and a discontinuity $x = a$ of $f(x)$, then*

- *we call $x = a$ a* **removable discontinuity** *if there exists $y_0 \in \mathbb{R}$ such that, if $f(x)$ were redefined at $x = a$ so that $f(a) = y_0$, then $f(x)$ would be continuous at $x = a$;*
- *we call $x = a$ an* **essential (or nonremovable) discontinuity** *if no such y_0 exists. That is, no matter how we (re)define $f(x)$ at $x = a$, i.e., regardless of any redefinition $f(a) = y_0$, the function $f(x)$ is still discontinuous at $x = a$.*

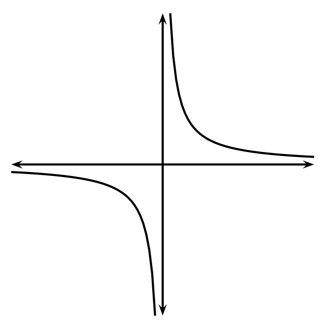
It is assumed in the definition that $f(x)$ is only redefined at $x = a$, and so the output of $f(x)$ will be the same as before for all $x \neq a$.

Note that if $f(x)$ has a removable discontinuity at $x = a$, then the redefinition is unique, that is there exists exactly one y_0 such that the redefinition $f(a) = y_0$ removes the discontinuity of $f(x)$ at $x = a$. That may seem intuitive from various graphs of removable discontinuities, but we will find tools later (namely limit theorems in later sections) which make a proof of this fact easier.

Examples of essential discontinuities include *jump discontinuities* and *vertical asymptotes*:



Jump Discontinuity



Vertical Asymptote

With a jump discontinuity, the function can be redefined as either left-continuous or right-continuous, but not both. In the case above, if we filled in the "hole" we would no longer have a function, since we would have two outputs (y-values) for that particular input a.

Vertical asymptotes of any kind are always essential discontinuities.

Some nonremovable discontinuities of we encounter are not terribly interesting. For instance, $x = -1$ is not a removable discontinuity of $f(x) = \sqrt{x}$, because it is well outside of the domain of $\sqrt{x}$. For that matter, $x = 0$ is an essential discontinuity of $f(x) = \sqrt{x}$ by our definition, because if is on the boundary of the domain of $\sqrt{x}$ (and no redefinition at $x = 0$ will change that it is a discontinuity). However $\sqrt{x}$ is right-continuous there. We could define left-removable or right-removable discontinuities, but we will not bother to do so here. Where applicable, we will later have other language for such phenomena.

Exercises

For 1–16, find all x for which the function is continuous.

1. $f(x) = \dfrac{x^2-4}{x^2+4}$.
2. $f(x) = \dfrac{1+x}{1-x^2}$.
3. $f(x) = \sqrt{3x-7}$.
4. $f(x) = \sqrt{9-3x}$.
5. $f(x) = \sqrt{16-x^2}$.
6. $f(x) = \sqrt{16+x^2}$.
7. $f(x) = \sqrt{x^2+6x+9}$.
8. $f(x) = \dfrac{1}{\sqrt{x^2+6x+9}}$.
9. $f(x) = \sqrt[3]{25-x^2}$.
10. $f(x) = \dfrac{x^2-1}{\sqrt[3]{25-x^2}}$.
11. $f(x) = \left|x^2-7x+12\right|$.
12. $f(x) = \sqrt{|x|}$.
13. $f(x) = \sqrt{\dfrac{1-x}{1+x^2}}$.
14. $f(x) = \sqrt[4]{\dfrac{1-x}{1+x^2}}$.
15. $f(x) = \sqrt[5]{\dfrac{1-x}{1+x^2}}$.
16. $f(x) = \dfrac{1}{\sqrt[3]{x^2-1}}$.

For Exercises 17–20, determine if the functions are continuous at the indicated points, and if not, whether the discontinuity is essential or removable.

17. $f(x) = \begin{cases} -6 & \text{if} \quad x<0 \\ 0 & \text{if} \quad x=0 \\ 6 & \text{if} \quad x>0, \end{cases}$ at $x=0$.
18. $f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if} \quad x \neq -1 \\ -2 & \text{if} \quad x=-1, \end{cases}$ at $x=-1$.
19. $f(x) = \begin{cases} -x^2 & \text{if} \quad x<0 \\ 1-\sqrt{x} & \text{if} \quad x \geq 0, \end{cases}$ at $x=0$.
20. $f(x) = \begin{cases} x^3+2 & \text{if} \quad x<2 \\ 8 & \text{if} \quad x=2 \\ 5x & \text{if} \quad x>2, \end{cases}$ at $x=2$.
21. For what value of A is the following function continuous on all of $\mathbb{R}$?
$$f(x) = \begin{cases} 2x^3 & \text{if} \quad x<1 \\ Ax-3 & \text{if} \quad x \geq 1. \end{cases}$$
22. Using ε-δ and the definition of right-continuity, show that $f(x) = \sqrt{x}$ is right-continuous at $x=0$.
23. Prove Theorem 3.2.2. Here is the general strategy:
    (a) For $C=0$, refer to Exercise 11, page 178.
    (b) If $C \neq 0$, let $\varepsilon_1 = \frac{\varepsilon}{|C|}$, and find the δ_1 corresponding to this ε_1 satisfying the definition of continuity for $f(x)$. Then take $\delta = \delta_1$.
24. Prove Lemma 3.2.1, page 183, i.e., that $f(x) = x$ is continuous at each $a \in \mathbb{R}$.
25. Prove Equation (3.17).
26. Prove Theorem 3.2.11. Hint: Let $\delta = \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 come from the one-sided continuity conditions when proving the "if," and let $\delta_1, \delta_2 = \delta$ for the "only if."

3.3 Continuity on Intervals

"A function $\phi(x)$ is said to be continuous between any limiting values of x, such as a and b, when to each value of x between those limits there corresponds a finite value of the function, and when an indefinitely small change in the value of x produces only an indefinitely small change in the function. In such cases the function in its passage from any one value to any other between the limits receives every intermediate value, and does not become infinite. This continuity can be readily illustrated by taking $\phi(x)$ as the ordinate of a curve, whose equation may then be written $y = \phi(x)$."

—*Encyclopædia Britannica: A Dictionary of Arts, Sciences, and General Literature*, Volume XIII, Ninth Edition, 1881, page 13 in the article "Infinitesimal Calculus," pages 5–72.

3.3.1 Continuity on Intervals: Main Definitions and Theorems

The significance of continuity is perhaps best understood when applied to whole intervals (a, b), or $[a, b]$, etc., rather than single points. Below we will define what it means for $f(x)$ to be continuous on (a, b), and continuous on $[a, b]$. (Other cases like $[a, b)$ would be defined in ways which would be obvious extensions once we have the (a, b) and $[a, b]$ cases.) Continuity on open intervals is rather trivial to define, but nonetheless has interesting consequences. In practice we will focus more on continuity on finite closed intervals $[a, b]$, properly defined, with $a < b$. But first we look at the open intervals.

Definition 3.3.1 *A function $f(x)$ is said to be* **continuous on the open interval** (a, b) *if and only if $f(x)$ is continuous at each value $x_0 \in (a, b)$.*[14]

What is interesting about continuity on (a, b) is it implies $f((a, b)) =$ is an interval of some kind (possibly infinite, or even a single point), and that the curve $y = f(x)$ is a *connected* graph for $a < x < b$.[15]

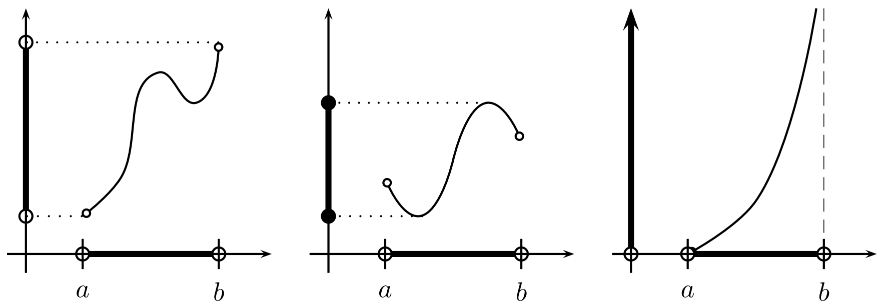
Theorem 3.3.1 *If $f(x)$ is continuous on (a, b), then $f((a, b))$ is an interval and the graph of $y = f(x)$, $a < x < b$ is a connected curve.*

Unfortunately the proof is a few steps beyond the scope of this textbook and is left, for the interested reader, to a course in advanced calculus, real analysis or topology.[16] Still, with the previous continuity sections behind us we should at least hear the ring of truth in the notion that the graph of $f(x)$, with domain restricted to $x \in (a, b)$, would be connected if f is continuous on (a, b). Put another way, we could draw the graph without lifting our pen from the paper. This is consistent with the elegant quote given at the top of this section, particularly the words, "... when an indefinitely small change in the value of x produces only an indefinitely small change

[14] Of course each x_0 may require a different δ for a given ε in the original definition of continuity given in Section 3.1, but the above definition only requires that each individual $x_0 \in (a, b)$ is a point at which $f(x)$ satisfies the ε-δ. To summarize, $(\forall x_0 \in (a, b))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(|x - x_0| < \delta \longrightarrow |f(x) - f(x_0)| < \varepsilon)$.

[15] Notationally, if S is a subset of the domain of $f(x)$, then $f(S) = \{y \,|\, (\exists x \in S)(y = f(x))\}$. In other words, $f(S)$ is the set of all possible outputs of $f(\ )$ if the inputs are taken from S. For instance, if $f(x) = 2x + 3$, then $f((0, 1)) = (3, 5)$, since $x \in (0, 1) \implies y \in (3, 5)$, as can easily be seen by graphing $y = 2x + 3$ on the interval $x \in (0, 1)$. Similarly, if $f(x) = \sin x$ then $f(\mathbb{R}) = [-1, 1]$.

[16] The preliminaries required to prove these are not terribly difficult, but would require a distracting amount of effort here.



$f((a,b))$ a finite, open interval

$f((a,b))$ a finite, closed interval.

$f((a,b)) = (0,\infty)$, an infinite, open interval.

Figure 3.2: Some examples of functions continuous on open intervals (a,b), with images $f((a,b))$ drawn on the vertical axis. The image will always be an interval of some kind—finite, infinite, open, closed, half-open or a "single-point" interval $[c,c]$ if f is a constant function.

in the function"; our pens would not "jump" off the page to a radically different height as we move slowly along the curve by increasing x through the interval (a,b). As a consequence it is then reasonable that the range will be a (connected) interval as the theorem also states.

Figure 3.2 shows sample cases for continuity on open intervals, illustrating that the image is always an interval. Now we turn our attention to continuity on closed intervals $[a,b]$:

Definition 3.3.2 *A function* $f:[a,b]\longrightarrow\mathbb{R}$*, with* $a<b$ *is called* **continuous on the (finite) closed interval** $[a,b]$ *if and only if each of the following are true:*

1. $f(x)$ *is continuous on the open interval* (a,b)*, i.e., continuous at each* $x\in(a,b)$*, and*
2. $f(x)$ *is right-continuous at* $x=a$*, and*
3. $f(x)$ *is left-continuous at* $x=b$*.*

In other words, if we are on the *interior* of $[a,b]$, i.e., on (a,b), then we expect our original two-sided continuity at every point therein, but then we require only right-continuity at a, and left-continuity at b. In this way we ignore the behavior of $f(\ )$ outside of inputs from strictly from $[a,b]$.

Note that we can also define continuity on $(a,b]$ as requiring our original (two-sided) continuity on (a,b) and left-sided continuity at b. Similarly for continuity on $[a,b)$.

Two very important results which follow from continuity on closed intervals $[a,b]$ are the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT). Both are wrapped up nicely in the following analog to Theorem 3.3.1, the difference being that the previous theorem required continuity on (a,b), while this theorem requires continuity on $[a,b]$. That minor difference gives us a much stronger theorem (though again we omit the proof):

Theorem 3.3.2 *If* $f(x)$ *is continuous on* $[a,b]$*, where* $a<b$*, then* $f([a,b])$ *is a closed interval of the form* $[c,d]$ *(with the possibility that* $c=d$ *in the case* $f(x)$ *is constant on* $[a,b]$*).*

In other words, the continuous image of a closed and bounded interval $[a,b]$ is also a closed and bounded interval $[c,d]$. If we again think about graphing such a function we can see that

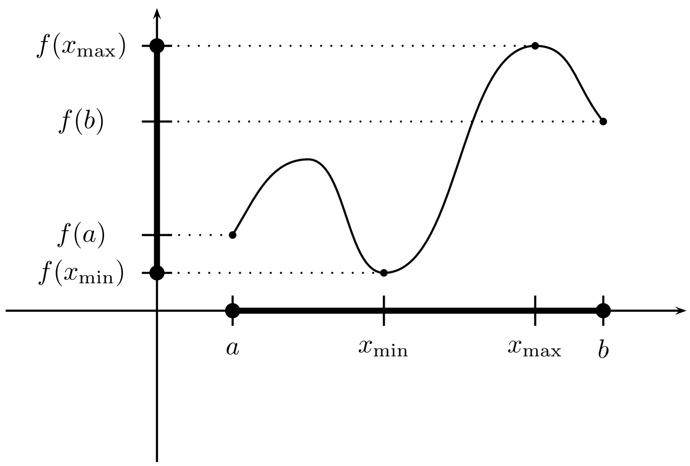


Figure 3.3: A typical graph of a function which is continuous on a finite, closed interval $[a, b]$. The image $f([a, b])$ is also a finite, closed interval (Theorem 3.3.2), containing a maximum value $f(x_{\max})$ and a minimum value $f(x_{\min})$, which are achieved at some $x_{\max}, x_{\min} \in [a, b]$ (Extreme Value Theorem, Corollary 3.3.1). Also note that all values between $f(a)$ and $f(b)$ (and more, for this particular graph) are in the image $f([a, b])$, and thus achieved for some x-values between a and b (Intermediate Value Theorem, Corollary 3.3.2).

this is also believable. After all, we would have to "pin down" the first point $(a, f(a))$, draw continuously until we end by "pinning down" the last point $(b, f(b))$. In doing so we would somewhere draw the highest and lowest points of our curve, and these heights would be d and c, respectively, from our theorem. Albeit we may hit those high and low vertical levels repeatedly, and maybe not at $x = a$ or $x = b$, but they will be achieved nonetheless.

The proof of Theorem 3.3.2 is also beyond the scope of this text, but the EVT and IVT follow very quickly from the theorem. We list these very important facts as corollaries.

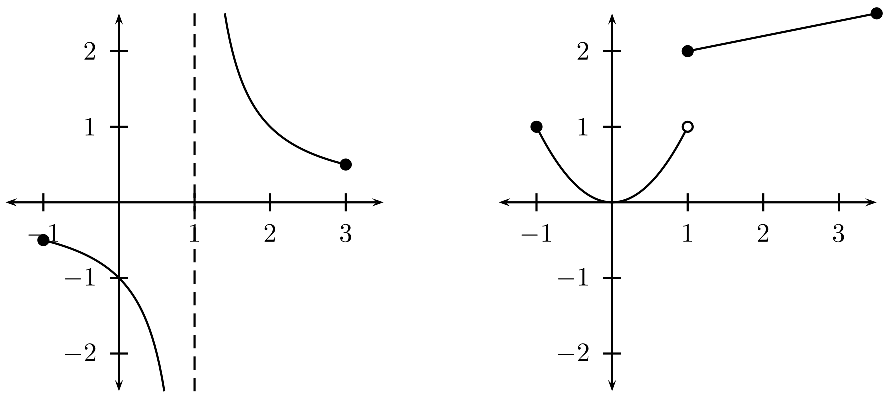
Corollary 3.3.1 (Extreme Value Theorem) *If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, then $f(x)$ achieves its maximum and minimum heights at some $x_{\max}, x_{\min} \in [a, b]$. In other words,*

$$\left(\exists x_{\min}, x_{\max} \in [a, b] \right) \left(f([a, b]) = [f(x_{\min}), f(x_{\max})] \right).$$

Corollary 3.3.2 (Intermediate Value Theorem) *If $f(x)$ is continuous on $[a, b]$, then $f(x)$ achieves every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a), f(b)$, then there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$.*

(Note that the statement of IVT is contained in the second sentence of the quote from *Britannica* on page 194, and the EVT is also hinted at in the quote.) A sample function which demonstrates the theorem and two corollaries is given in Figure 3.3.

It is crucial that the function in question be continuous, and that the domain in question be closed and bounded, i.e., of the form $[a, b]$ to guarantee that the image is another closed, bounded interval. The closed criterion is clearly necessary as shown in the the first and third graphs in Figure 3.2, page 195. In each graph in that figure, a function $f(x)$ defined on an *open* interval (a, b) is graphed: in the first graph, the image is an open interval and so no maximum



$$f(x) = \frac{1}{x-1}$$
$$x \in [-1,3] - \{1\}$$

$$f(x) = \begin{cases} x^2, & -1 \le x < 1 \\ 2 + 0.2(x-1), & 1 \le x \le 3.5 \end{cases}$$

Figure 3.4: Theorems 3.3.1 and 3.3.2—and their corollaries EVT and IVT—require continuity of f, without which the image can be disconnected or unbounded. In the first graph, there is a vertical asymptote at $x = 1$, which is a point of discontinuity. The image $(-\infty, -1/2] \cup [1/2, \infty)$ is disconnected (so no IVT conclusion) and unbounded (so no EVT conclusion). In the second case, there is a "jump" discontinuity at $x = 1$, and the image $[0,1] \cup [2, 2.5]$ is also disconnected (no IVT conclusion), though bounded. It happens that the second graph does have maximum and minimum values $f(0) = 0$ and $f(3.5) = 2.5$ though this was not guaranteed because $f(x)$ is not continuous on all of $[-1, 3.5]$. These examples do not violate the corollaries IVT and EVT since both corollaries claim the truth of tautologies of the form $P \to Q$, which is equivalent to $(\sim P) \vee Q$, and here we have $\sim P$ in both cases, making them true to the theorems and corollaries vacuously (in form $P \to Q$) or trivially (using the form $(\sim P) \vee Q$).

or minimum is actually achieved; in the third graph, the image is unbounded from above so no maximum is achieved, and no minimum is achieved either. It may happen that $f((a,b))$ is a closed and bounded interval, as in the second graph in that figure, but it clearly (from the other two graphs in that figure) is not guaranteed. Continuity is also required in these theorems, as we see in Figure 3.4.

Note that the first function in the figure is continuous on $[-1, 1)$ because it is continuous on $(-1, 1)$ and right-continuous at -1. Similarly it is continuous on $(1, 3]$. The second function is continuous on $[-1, 1)$ and $[1, 3.5]$. That is **not** to say it is continous at each $x \in [1, 3.5]$, but rather that the "piece" drawn on that interval is a continuous "piece," in the sense of Definition 3.3.2, page 195 and the discussion following that.

3.3.2 Simple Applications of the Intermediate Value Theorem

We will return to the extreme value theorem and its applications later in the text. Here we will instead look at the IVP and its usefulness in algebra. The following simple theorem can often be useful when we look at continuity considerations:

Theorem 3.3.3 *If I and J are intervals of any kind except for single points, with $I \subseteq J$, and $f : J \longrightarrow \mathbb{R}$ is continuous on J, then $f : I \longrightarrow \mathbb{R}$ is continuous on I.*

In other words when a function is known to be continuous on an interval, its restriction to a subinterval is also continuous on that subinterval. The proof is a matter of chasing down the definitions of continuity on the various intervals, and checking each of the cases. The following example shows a quick application.

Example 3.3.1 *Show that the equation $x^5 + 7 = x^2$ has a solution in $\mathbb{R}$.*
Solution: First notice that

$$x^5 + 7 = x^2 \iff x^5 - x^2 + 7 = 0.$$

Next define $f(x) = x^5 - x^2 + 7$, which is a polynomial and thus continuous on all of $\mathbb{R}$ (think of $\mathbb{R}$ as the open interval $(-\infty, \infty)$). Now

$$x^5 + 7 = x^2 \iff x^5 - x^2 + 7 = 0 \iff f(x) = 0,$$

so solving the original equation is equivalent to finding x so that $f(x) = 0$. Next notice that

$$\begin{aligned} f(-2) &= (-2)^5 - (-2)^2 + 7 = -29, \qquad \text{and} \\ f(1) &= 1 - 1 + 7 = 7. \end{aligned}$$

Because f is continuous in $\mathbb{R}$, it is also continuous on $[-2, 1] \subseteq \mathbb{R}$. Since $f(-2) = -29$ while $f(1) = 7$, there exists some x_0 between -2 and 1 such that $f(x_0) = 0$ (by IVT), and this x_0 therefore solves the original equation, q.e.d.

The technique in Example 3.3.1 is standard. It is similar to the usual algebraic trick where we try to solve an equation of the form $g(x) = h(x)$ by instead determining where $g(x) - h(x) = 0$. It is often convenient to define $f(x) = g(x) - h(x)$ and solve (or, as here, simply detect the presence of a solution of) the logically equivalent equation $f(x) = 0$. As we saw in the algebra sections of Chapter 2, solving $f(x) = 0$ is usually simpler than finding where $g(x) = h(x)$.

We can apply the IVT more than once to a single function, as in the following example:

Example 3.3.2 *Show that the equation $x^3 - 8x^2 + 15x = -1$ has at least three solutions.*
Solution: This is equivalent to the equation $x^3 - 8x^2 + 15x + 1 = 0$ having at least three solutions. Defining $f(x) = x^3 - 8x^2 + 15x + 1$, we thus want to prove that $f(x) = 0$ occurs at least three times.

First we notice that $f(x)$ is a polynomial, and therefore continuous on $\mathbb{R}$. Next we see that

$$\begin{aligned} f(-1) &= -23, \\ f(0) &= 1, \\ f(4) &= -3, \\ f(5) &= 1. \end{aligned}$$

We see that $f(x)$ must be zero for some $x \in (-1, 0)$, another $x \in (0, 4)$, and yet another $x \in (4, 5)$, ultimately therefore proving that there must be at least three solutions of $x^3 - 8x^2 + 15x = -1$, q.e.d.

3.3.3 Polynomial Inequalities

A very important consequence of the Intermediate Value Theorem is that, for a continuous function to change sign (positive to negative or vice versa) on an interval, its values must pass through zero. We can exploit this to solve polynomial and rational inequalities. The application of the IVT to polynomial inequalities is particularly straightforward, as the next two examples illustrate.

Example 3.3.3 *Solve the inequality $x^3 - 9x^2 + 20x \leq 0$.*

Solution: Define $f(x) = x^3 - 9x^2 + 20x$, which is a polynomial and therefore continuous on all of $\mathbb{R}$. We wish to see where $f(x) \leq 0$. First we will see where $f(x) = 0$, to detect the possible points at which $f(x)$ changes signs.

$$\begin{aligned} & f(x) = 0 \\ \iff & x^3 - 9x^2 + 20x = 0 \\ \iff & x(x^2 - 9x + 20) = 0 \\ \iff & x(x-4)(x-5) = 0 \\ \iff & x = 0, 4, 5. \end{aligned}$$

These points $x = 0, 4, 5$ are the only places where $f(x)$ can change signs as x passes through values along the x-axis continuum, so $f(x)$ will **not** *change signs anywhere* **within** *the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ or $(5, \infty)$. We can use this fact and a simple test, for example, to find the sign of $f(x)$ on all of $(-\infty, 0)$: if we know the sign of $f(x)$ at, say, $x = -10$, then we know it for the whole interval $x \in (-\infty, 0)$ because whatever is the sign at $x = -10$ will be the sign for that whole interval (because if it had two signs in that interval then continuity would guarantee there would be another point where $f(x) = 0$ in that interval, and we know that the only points where that happens, $x = 0, 4, 5$ are not in that interval). Furthermore, if we would like to find the sign of $f(x)$ on the interval $(-\infty, 0)$, it is enough to know the signs of all the factors of $f(x) = x(x-4)(x-5)$, since if an odd number of the factors are negative then f is negative there, whereas if an even number of factors are negative then they "cancel" to give f a positive sign.*

A nice visual device for determining where $f(x)$ is positive, and where it is negative, is a **sign chart***, as illustrated below. For this particular function the signs are shown on their respective intervals, with the points $x = 0, 4, 5$ as boundary points. In the style of sign chart is given below, we use $\oplus$ to represent a positive quantity, and $\ominus$ to represent a negative quantity. We also look at the signs of the factors to determine the sign of the function:*

Function:			$f(x) = x(x-4)(x-5)$				
Test $x =$	-10		2		4.5		10
Sign Factors:	$\ominus\ominus\ominus$		$\oplus\ominus\ominus$		$\oplus\oplus\ominus$		$\oplus\oplus\oplus$
Sign $f(x)$:	$\ominus$	0	$\oplus$	4	$\ominus$	5	$\oplus$

From the chart we see that $f(x) < 0$ on the first and third intervals, i.e., for $x \in (-\infty, 0) \cup (4, 5)$. Since $f(x) < 0$ is equivalent to our original inequality, this is also the solution of that inequality.

The information at hand will not give us a complete picture of the graph of $f(x)$, but it is instructive to see what the graph looks like, and how an accurate enough (for our purposes here) picture can be easily imagined from the sign chart. For this reason the graph is given in Figure 3.5.

The logic which was used in constructing the sign chart bears repeating. Since f is continuous, the only way it can change sign is to pass through zero (by IVT, see also Figure 3.5), so we chart all the x-values for which $f(x) = 0$. These mark boundaries of subintervals of $\mathbb{R}$ on which f does not change sign. For each such interval, knowing the sign of $f(x)$ at any value in the interval gives us the sign for the whole interval (since, again, it cannot change sign in the interval without there being another zero in the interval, and all such points are accounted for). If $f(x)$ happens to be factored, we only need to check the signs of each factor to see if the negative factors "cancel" completely to leave a positive function, or if we have an odd number of negative factors to make f negative on the interval in question.

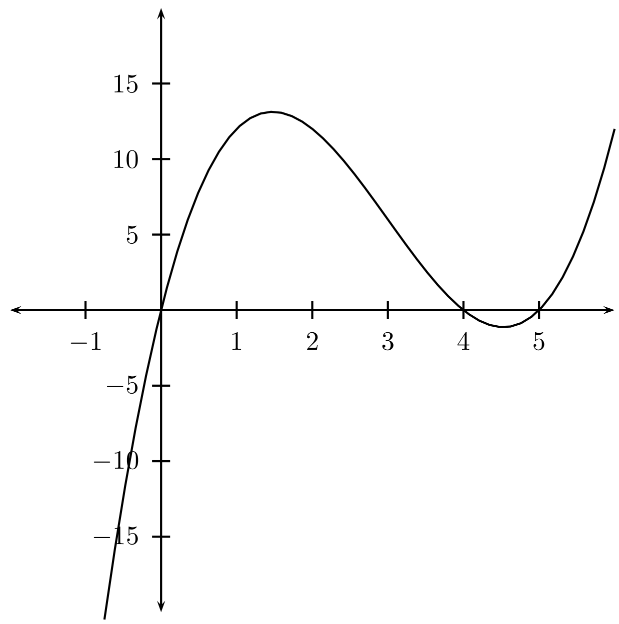


Figure 3.5: Actual graph of $f(x) = x(x-4)(x-5)$, showing where $f(x) > 0$, where $f(x) < 0$, and how the transitions between these occur: in this case by the function's height passing through zero. Compare to the sign chart in Example 3.3.3.

Example 3.3.3 was relatively straightforward. There can be complications, and we have to be careful to answer the given question. For instance, we do not always have *strict inequalities* $<, >$, but may have *inclusive inequalities* $\le, \ge$.

Example 3.3.4 *Solve* $x^2 \ge x + 1$.

Solution: First we subtract, and then define $f(x) = x^2 - x - 1$, *so that*

$$x^2 \ge x + 1 \iff x^2 - x - 1 \ge 0 \iff f(x) \ge 0.$$

Now solving $f(x) = 0$ *requires the quadratic formula or completing the square. We will opt for the former. Recall first that* $f(x) = 0 \iff x^2 - x - 1 = 0$.

$$f(x) = 0 \iff x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5} \approx -0.61803,\ 1.61803.$$

We will always use the exact values, but the approximate ones are also useful since we need to know where to find our test points.[17]

[17] We could factor $f(x)$ based upon the solutions to $f(x) = 0$, namely $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$:

$$f(x) = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$

Such an approach is perhaps more sophisticated than our method in Example 3.3.4, where we did not bother to factor $f(x)$, but is often unwieldy and requires more subtlety than necessary to solve the inequality.

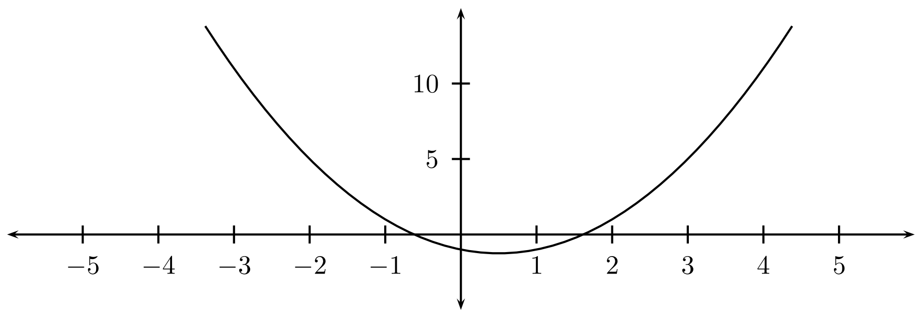


Figure 3.6: Actual graph of $f(x) = x^2 - x - 1$. The function is zero at $\frac{1}{2} \pm \frac{1}{2}\sqrt{5} \approx -0.61803,\ 1.61803$. Compare to the sign chart in Example 3.3.4.

Function:			$f(x) = x^2 - x - 1$		
Test $x =$	-10		0		10
Test $f(x) =$	109		-1		89
Sign $f(x)$*:*	$\oplus$	$\frac{1-\sqrt{5}}{2}$	$\ominus$	$\frac{1+\sqrt{5}}{2}$	$\oplus$

Recall that we are searching for all points for which $f(x) \geq 0$. These include the cases where $f(x) = 0$ as well as where $f(x) > 0$. Therefore we include the endpoints when we report the solution: $x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$.

The function $f(x) = x^2 - x - 1$ is graphed in Figure 3.6, page 201. *Compare the graph with the sign chart above.*

In the first two examples, we had the function $f(x)$ switch signs at every point where $f(x) = 0$. This is not always the case. It is possible for the graph to touch the axis, and retreat back to the same side. Consider the following example.

Example 3.3.5 *Solve the inequality $x^3 - 6x^2 + 9x > 0$.*

<u>Solution</u>: This is already a question about sign, so we will simply take $f(x) = x^3 - 6x^2 + 9x$ and solve $f(x) > 0$. Now

$$f(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2.$$

We see that $f(x) = 0$ at $x = 0, 3$. This gives us the following sign chart:

Function:			$f(x) = x(x-3)^2$		
Test $x =$	-1		2		4
Sign Factors:	$\ominus\ominus^2$		$\oplus\ominus^2$		$\oplus\oplus^2$
Sign $f(x)$:	$\ominus$	0	$\oplus$	3	$\oplus$

We see from the sign chart that $f(x) > 0$ for $x \in (0,3) \cup (3, \infty)$.

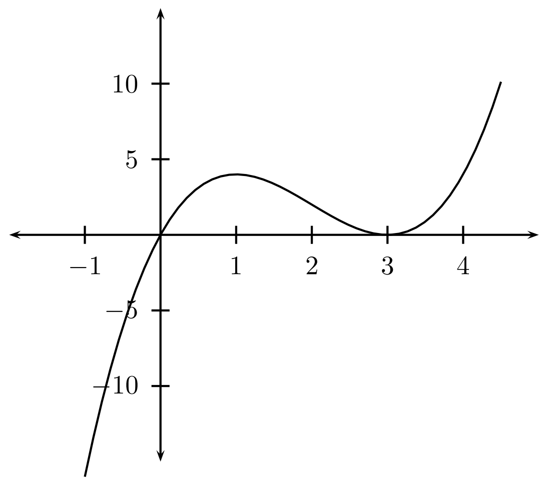


Figure 3.7: Graph of $f(x) = x(x-3)^2$ as in Example 3.3.5, illustrating that a function's height can be zero without the function changing signs there.

We can get an idea how a function might look given a sign chart as in Example 3.3.5, and indeed we see the expected behavior in the graph of the function, given in Figure 3.7.

An interesting phenomenon regarding *zeroes* of polynomials becomes apparent in constructing these sign charts: if $(x-a)$ appears as a factor in a polynomial $f(x)$ to an odd power, the polynomial changes signs at a, while if $(x-a)$ appears to an even power, the polynomial does not change signs at $x = a$. Some terminology helps to express this.

Definition 3.3.3 *If there is a polynomial $g(x)$ and a positive integer $k \in \mathbb{N}$ so that $f(x)$ can be written*

$$f(x) = (x-a)^k g(x),$$

but $x - a$ is not a factor of $g(x)$—in other words $(x-a)^k$ is a factor of $f(x)$ but $(x-a)^{k+1}$ is not—then $x = a$ is called a **zero***, or* **root***,* **of multiplicity** *or* **degree** *k of $f(x)$.*

Then we can observe the following for a polynomial $f(x)$:

1. if $x = a$ is a zero of odd multiplicity, then $f(x)$ changes sign at $x = a$;
2. if $x = a$ is a zero of even multiplicity, then $f(x)$ does not change sign at $x = a$.

In either case $f(a) = 0$ and so $x = a$ is an x-intercept of the function.

In Example 3.3.5 we had $x = 3$ was a zero of multiplicity 2, which is even and so $f(x)$ did not change sign (in the sense of changing from positive to negative or vice versa) as x passed through the value 3, while $x = 0$ was a zero of multiplicity 1, which is odd and so the function did change sign there. If one knew the degrees of each zero, and the sign on one interval, one could construct the sign chart from that information alone, depending upon whether the function changes sign while x passes from one interval to the next (left or right, all bordered by the zeroes of f), or does not.

It is sometimes the case that, even when $f(x)$ is factored, some of the factors might never be zero. In such a case that factor will never change signs either. One common such type of factor is of the form $x^{2k} + a$, where $k \in \mathbb{N}$ and $a > 0$. Such a factor is always positive regardless of $x \in \mathbb{R}$. Another type is a factor of the form $ax^2 + bx + c$ where $b^2 - 4ac < 0$, in which case

$ax^2 + bx + c = 0$ has no real solutions. By the Intermediate Value Theorem, if we determine a factor has no real zeros, we know it will not change signs. The following examples illustrates one case.

Example 3.3.6 *Solve the inequality $x^5 \le 25x$.*

<u>Solution</u>: *As before, we construct $f(x) = x^5 - 25x$, so that*

$$x^5 \le 25x \iff x^5 - 25x \le 0 \iff f(x) \le 0.$$

Next we factor[18] $f(x) = x(x^4 - 25) = x(x^2 + 5)(x^2 - 5)$. *Now the first factor is zero for $x = 0$, the second factor is never zero, and the third is zero for $x = \pm\sqrt{5} \approx \pm 2.23607$. Hence we get the following sign chart:*

Function:	$f(x) = x(x^2+5)(x^2-5)$						
Test $x =$	-4		-1		1		4
Sign Factors:	$\ominus\oplus\oplus$		$\ominus\oplus\ominus$		$\oplus\oplus\ominus$		$\oplus\oplus\oplus$
Sign $f(x)$:	$\ominus$	$-\sqrt{5}$	$\oplus$	0	$\ominus$	$\sqrt{5}$	$\oplus$

We see that $f(x) \le 0$ for $x \in \left(\infty, -\sqrt{5}\,\right] \cup \left[0, \sqrt{5}\,\right]$.

3.3.4 Rational Inequalities

When applying IVT we have to be careful that we actually do have continuity. For instance, if we define $f(x) = 1/(x-1)$, we see that $f(-1) = -1/2$, while $f(3) = 1/2$. However, there are no points between -1 and 3 at which $f(x) = 0$. The reason the IVT did not apply is that $f(x) = 1/(x-1)$ is not continuous in $[-1, 3]$, since it is discontinuous at $x = 1$. If we allow for discontinuities, then a function's output can "jump" past a particular value, and the image be disconnected (i.e., not an interval). See the first graph in Figure 3.4, page 197 for an illustration of this phenomenon with this particular function. (The second graph also shows how discontinuity allows "jumping.")

Now we can still use the IVT to solve rational inequalities. We simply need to analyze them and the IVT further. For instance, we can use the following corollary to that theorem:

Corollary 3.3.3 *Suppose that $f(a) = A$ and $f(b) = B$, where $a < b$, and suppose further that C is between A and B. Then at least one of the following must hold:*

(i) *there exists c between a and b so that $f(c) = C$; or*

(ii) *$f(x)$ has at least one discontinuity on $[a, b]$.*

In other words, for a function to pass from the height A to the height B, it must either pass through every height in between, or must be discontinuous, as proved below (see "q.e.d.").

[18]For this example we could continue factoring $f(x) = x(x^2+5)(x^2-5) = x(x^2+5)\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right)$. However here we will work with the partially factored form, as this will be sufficient. It is really a matter of personal taste.

Proof: We can use some symbolic logic to prove this. The idea is that

$$f \text{ continuous on } [a,b] \implies \text{(IVT)} \implies \text{(i)}.$$

Thus f continuous on $[a,b] \implies$ (i), meaning that f continuous on $[a,b] \longrightarrow$ (i) is a tautology. Finally, this is equivalent to the statement of the corollary:

$$\sim (f \text{ continuous on } [a,b]) \ \vee \ \text{(i)}, \qquad \text{q.e.d.}$$

This proof is, of course, true and precise, but the corollary should also be intuitive: as we move x along the interval $[a,b]$, to pass continuously from one height to another we pass through all intermediate heights; if we are not going to pass through all intermediate heights, we have to somehow "jump" over any missed heights, requiring a discontinuity.

One implication for rational, or any other function is immediate. *For a function's output to change signs as we vary its input, that output must pass through zero or be discontinuous.* Thus, when we produce a sign chart for a function f, we include as boundary points all those points where f is zero or discontinuous. For rational functions this means we look at where the numerator is zero, and where the denominator is zero, respectively.

Example 3.3.7 *Solve* $\dfrac{x}{x^2-1} \le 0$.

Solution: Defining $f(x) = \dfrac{x}{x^2-1} = \dfrac{x}{(x+1)(x-1)}$, *we see that* f *is zero at* $x=0$, *and discontinuous at* $x = \pm 1$. *We now use all three of these points to construct the sign chart.*

Function:	$f(x) = \dfrac{x}{(x+1)(x-1)}$						
Test $x =$	-2		-0.5		0.5		2
Sign Factors:	$\dfrac{\ominus}{\ominus\ominus}$		$\dfrac{\ominus}{\oplus\ominus}$		$\dfrac{\oplus}{\oplus\ominus}$		$\dfrac{\oplus}{\oplus\oplus}$
Sign $f(x)$*:*	$\ominus$	-1	$\oplus$	0	$\ominus$	1	$\oplus$

Since we are interested in the points where $f(x) \le 0$, *we need the open intervals on which* $f(x) < 0$, *i.e., the first and third intervals, and all points where* $f(x) = 0$, *i.e.,* $x = 0$. *Collecting all these we get* $x \in (-\infty, -1) \cup [0, 1)$.

We include a graph of $f(x) = \frac{x}{(x+1)(x-1)}$ in Figure 3.8 to illustrate how f changes signs by passing through, or leaping over zero. There are other types of discontinuities besides vertical asymptotes, but for rational functions f in which the numerator and denominator have no common factors, vertical asymptotes are the only type of discontinuity which can occur. Notice from the graph why we include $x = 0$ but not $x = \pm 1$, when we solve $f(x) \le 0$.

For another perspective justifying the technique for sign charts for rational functions, consider that a ratio of functions can only change signs if the numerator or denominator changes signs. Since both numerator and denominator are polynomials and hence continuous, they can only change signs by passing through zero. Summarizing, we conclude that a ratio of polynomials can only change signs if the numerator passes through zero or the denominator passes through zero. However, for more general functions we have to consider all possible types of discontinuities.

The method above can generalize for solving any rational inequality the same way we generalized for the polynomial case: we rewrite any inequality into an equivalent statement about signs $(+/-)$.

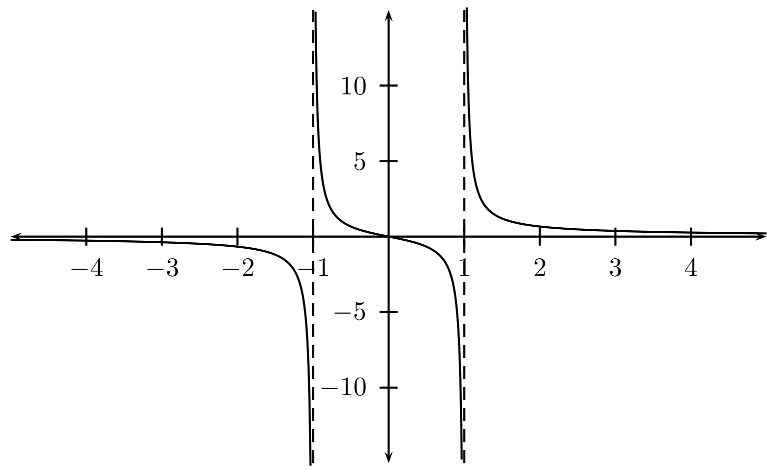


Figure 3.8: Graph of $f(x)=\dfrac{x}{(x+1)(x-1)}$. See Example 3.3.7. Dashed lines are vertical asymptotes at $x=\pm1$, which are those values of x outside the domain of $f(x)$. Vertical asymptotes will be properly developed later in the text.

Example 3.3.8 *Solve the inequality* $\dfrac{x}{x^2-7}\le\dfrac{2x}{x^2-9}$.

Solution: First we do as before—make this into a question about signs—by subtracting the right-hand side from the inequality, and define the difference to be $f(x)$. Hence we have

$$f(x)=\frac{x}{x^2-7}-\frac{2x}{x^2-9}=\frac{x(x^2-9)-2x(x^2-7)}{(x^2-7)(x^2-9)}=\frac{-x^3+5x}{(x^2-7)(x^2-9)}=\frac{-x(x^2-5)}{(x^2-7)(x^2-9)},$$

and we are trying to find where $f(x)\le0$. We see that $f(x)=0$ for $x=0$ and $x=\pm\sqrt5\approx\pm2.23607$, and is discontinuous at $x=\pm3$ and $x=\pm\sqrt7\approx\pm2.64575$. The sign chart follows:

Function: $f(x)=\dfrac{(-x)(x^2-5)}{(x^2-7)(x^2-9)}$

Test $x=$	-10	-2.9	-2.5	-1	1	2.5	2.9	10
Sign Factors:	$\frac{\oplus\oplus}{\oplus\oplus}$	$\frac{\oplus\oplus}{\oplus\ominus}$	$\frac{\oplus\oplus}{\ominus\ominus}$	$\frac{\oplus\ominus}{\ominus\ominus}$	$\frac{\ominus\ominus}{\ominus\ominus}$	$\frac{\ominus\oplus}{\ominus\ominus}$	$\frac{\ominus\oplus}{\oplus\ominus}$	$\frac{\ominus\oplus}{\oplus\oplus}$

-3	$-\sqrt7$	$-\sqrt5$	0	$\sqrt5$	$\sqrt7$	3

Sign $f(x)$:	$\oplus$	$\ominus$	$\oplus$	$\ominus$	$\oplus$	$\ominus$	$\oplus$	$\ominus$

It is important to note that the sign chart gives us the signs on the various **open** *intervals. The endpoints of the interval are either where the function is zero, or undefined (the latter implying discontinuity). Since for this case we want our solution to be those points where $f(x)\le0$, we have to include those endpoints where $f(x)=0$. Putting all this together, we see that*

$$x\in\left(-3,-\sqrt7\right)\cup\left[-\sqrt5,0\right]\cup\left[\sqrt5,\sqrt7\right)\cup(3,\infty).$$

Note where f changes signs continuously (i.e., passing through zero height) at $0,\pm\sqrt5$ and discontinuously (in fact, via vertical asymptotes) at $x=\pm\sqrt7,\pm3$. We do not include a graph of

$f(x)$ here, but much of its behavior is evident by the sign chart and the way f changes signs by passing through height zero at $x = 0, \pm\sqrt{5}$, and by discontinuity (in fact, via vertical asymptotes) at $x = \pm 3, \pm\sqrt{7}$.

Fortunately, most standard calculus problems require only that we know where certain functions (or more precisely, derivatives of functions) are positive (> 0), and where they are negative ($<$). For completeness here we also discussed the inclusive inequalities ($\geq, \leq$). We will round out this section with the following final example.

Example 3.3.9 *Solve the inequality* $\dfrac{x^2+x+1}{x^2-7x+12} > 0$.

Solution: As usual, we first check to see where the numerator is zero (where $f(x) = 0$, if $f(x)$ is the function on the left) and where the denominator is zero (where $f(x)$ does not exist). For the numerator (NUM) *we need the quadratic formula:*

$$\text{NUM} = 0 \iff x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1-\sqrt{-3}}{2} \notin \mathbb{R}.$$

We see that the numerator is never zero for $x \in \mathbb{R}$, and so the numerator does not contribute any points to include in making the sign chart. For the denominator (DEN) *we have*

$$\text{DEN} = 0 \iff x^2 - 7x + 12 = 0 \iff (x-3)(x-4) = 0 \iff (x=3) \vee (x=4).$$

We use these two points to bound the regions of the sign chart:

Function: $f(x) = \dfrac{x^2+x+1}{(x-3)(x-4)}$

Test $x =$	0		3.5		5
Sign Factors:	$\oplus/(\ominus\ominus)$		$\oplus/(\oplus\ominus)$		$\oplus/(\oplus\oplus)$
Sign $f(x)$:	$\oplus$	3	$\ominus$	4	$\oplus$

From the sign chart we see the solution is $x \in (-\infty, 3) \cup (4, \infty)$.

Exercises

1. For each of the following, draw a continuous function $f(x)$ whose domain is $x \in (2,5)$, and whose image of that set (i.e., whose range) is given.

   (a) $(1,4)$
   (b) $\{3\}$
   (c) $\mathbb{R}$
   (d) $[1,4]$
   (e) $(1,4]$
   (f) $(-\infty,1)$
   (g) $(-\infty,1]$

2. Repeat the previous problem except assume the domain for $f(x)$ is $x \in [2,5]$. Some of the cases are impossible (but are interesting to attempt anyhow). See Theorem 3.3.2, page 195.

3. Draw the graph of a function $y = f(x)$, defined for $x \in [2,5]$, where $f(x)$ is continuous on $(2,5)$ but not on $[2,5]$.

4. Show that $x^3 - 6x^2 + 4x + 10 = 0$ has at least three solutions, by checking values

of $f(x) = x^3 - 6x^2 + 4x + 10$ at various x-values.

5. Show that $x^5 - 8x^2 + 15x = 97$ has at least one solution in $\mathbb{R}$.

For each of the following, solve the inequality by means of a sign chart. You may have to first rewrite the inequality.

6. $(x+1)(x-3) \geq 0$
7. $x^2 - 9 < 0$
8. $x^2 + 8 \leq 6x$
9. $x^2 + 3x \geq 2$
10. $x^2 - 15 > 0$
11. $x^2 + 15 > 0$
12. $x^2 + 18 > 11x$
13. $\dfrac{x}{x+5} < 0$
14. $\dfrac{x-7}{x+6} \geq 0$
15. $\dfrac{x^2-16}{x^2+16} > 0$
16. $\dfrac{2x+1}{x^2-16} \leq 0$
17. $\dfrac{27x - x^2}{x^2+11x+30} < 0$
18. $\dfrac{(x+2)^2}{x^2+4} < 1$
19. $\dfrac{x}{x+5} > \dfrac{1}{x-7}$
20. $x^3 + 2x^2 \leq 15x$
21. $\dfrac{x^2-1}{x^4-3x^2-10} \leq 0$
22. $\dfrac{2x}{x+5} < \dfrac{3x}{x+6}$

Use a sign chart to graph the following functions, to the extent that continuity and sign of f are illustrated. You can assume a vertical asymptote will pass through any point in which the denominator is zero (after any cancellation of factors common to both numerator and denominator).

23. $f(x) = x(x-1)^2(x^2-4)$
24. $f(x) = x^3 - x^2 - 2x$
25. $f(x) = \dfrac{x}{x^2-9}$
26. $f(x) = \dfrac{x-1}{x^2-9x+8}$

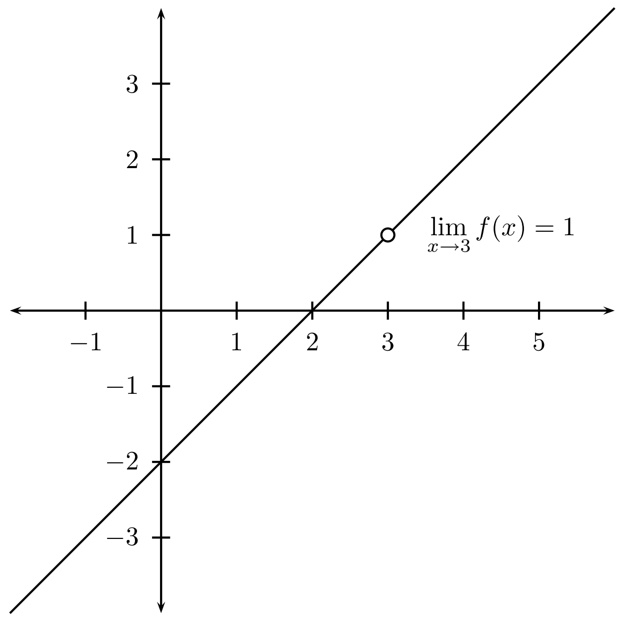


Figure 3.9: Graph of the function $f(x) = (x^2 - 5x + 6)/(x - 3) = (x - 3)(x - 2)/(x - 3)$. Except at $x = 3$, this simplifies to $f(x) = x - 2$. Thus the graph of $y = f(x)$ is the same as the line $y = x - 2$, except for a "hole" at $(3, 1)$. Though $f(x)$ is undefined at $x = 3$, we can at least say $\lim_{x\to 3} f(x) = 1$.

3.4 Finite Limits at Points

The concept of limit is fundamental to calculus. Before its development, mathematicians were able to observe many of the apparent truths of calculus, but were unable to actually *prove* them. The development of a theory of limits, together with the rigorous definition of $\mathbb{R}$, bridged many important theoretical gaps in calculus between what could be observed and what could be proved.[19]

For our simplest cases, limits are just a small step away from continuity; we can compute many limits quickly based upon our knowledge of continuous functions. However, the concept of limit has been greatly expanded to have meaning in many other contexts. This section is devoted to the simplest case—closest to continuity—which is the case of a finite limit at a point.

As with continuity, we will have many theorems which should be remembered and understood, and which are intuitive on their faces though their proofs are somewhat technical. Because of this we leave the proofs until the end of the section.[20]

[19]In fact the ancient Greek mathematician and physicist Archimedes of Syracuse, (287–212 B.C.) used many arguments which are considered calculus today for his mathematical discoveries. Without the foundations of calculus his arguments were very convincing, but fell short of proofs.

[20]To be sure, the proofs are always worth reading and understanding and the techniques involved are accessible and relevant to our reading here, but for now it is more important to be able to understand and apply the principles enunciated by the theorems and to understand the limit definition and examples.

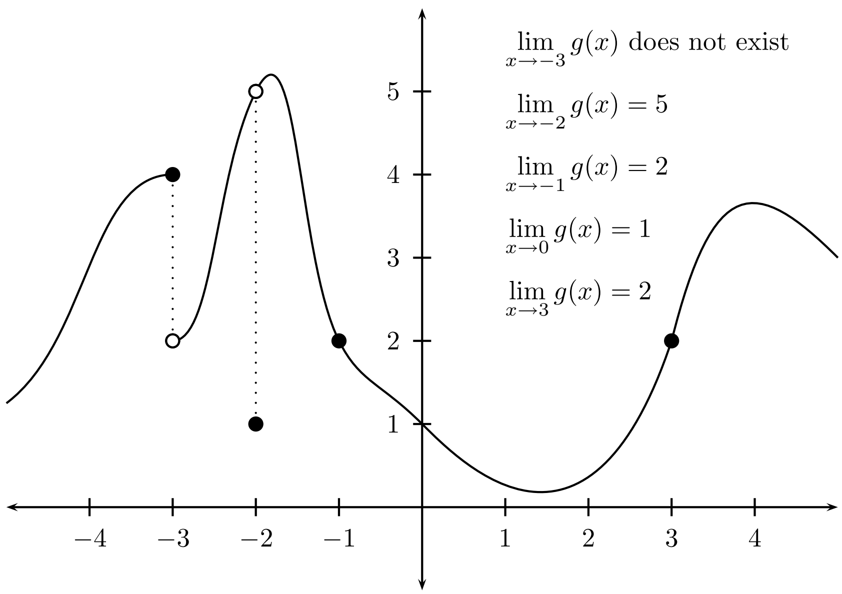


Figure 3.10: A function $g(x)$ to illustrate the concept of limit. (Circles and dotted lines are drawn as visual aids.) See text.

3.4.1 Definition, Theorems and Examples

A typical example of where we might use a limit is the function illustrated in Figure 3.9, on page 208, namely

$$f(x) = \frac{x^2 - 5x + 6}{x - 3}.$$

We see immediately that this function is not defined at $x = 3$. If we look at any point other than $x = 3$—though what is crucial is to look at those points near, but not at, $x = 3$—we can simplify $f(x)$ as follows:

$$f(x) = \frac{x^2 - 5x + 6}{x - 3} = \frac{(x-3)(x-2)}{x-3} = x - 2, \qquad x \neq 3.$$

Thus $f(x) = x - 2$ as long as $x \neq 3$. A legitimate question to ask is, as x gets near the forbidden value at 3, does $f(x)$ get near a particular value? As we see from Figure 3.9, the height $f(x)$ *does* approach the value 1 as x approaches 3. The notation we use to signify this is:

$$\lim_{x\to3} f(x) = 1,$$

read, "the limit, as x approaches 3, of $f(x)$, equals 1."

Of course, one example does not define a concept, but before we give the definition we will sharpen the idea further by considering the function $g(x)$ graphed in Figure 3.10.

- $\lim_{x\to0} g(x) = 1$. That should be clear from the graph; as x nears zero, the height indeed approaches the value 1. (We will note later how continuity, as at $x = 0$, has implications for the limit.) Similarly $\lim_{x\to-1} g(x) = 2$.

- $\lim_{x\to -2} g(x) = 5$. Such a limit is oblivious to what actually occurs *at* $x = -2$, but is instead concerned with what occurs as we *approach* $x = -2$, from both sides. In fact $g(-2) = 1$, but that is irrelevant. We are only interested in the value approached in the output of $g(x)$ as its input x approaches, *but does not equal*, the value -2.
- $\lim_{x\to -3} g(x)$ *does not exist.* As we approach from the left of $x = -3$ the function's value nears 4, but as we approach from the right the function's value nears 2. For such an ambiguous case we simply decline to assign a value to the limit, instead declaring that it *does not exist.* Though not actually relevant, we note that $g(-3) = 4$.
- $\lim_{x\to -1} g(x) = 2$, and (coincidentally) $\lim_{x\to 3} g(x) = 2$. These should be clear from the graph.

Now we will give the technical definition of a finite limit at (or "about") a point.

Definition 3.4.1 *Given a function $f(x)$ defined on the set $0 < |x-a| < d$, for some $d > 0$ ($f(x)$ possibly defined at the point $x = a$ as well), and some $L \in \mathbb{R}$, we define $\lim_{x\to a} f(x) = L$ to mean the following:*

$$\lim_{x\to a} f(x) = L \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(0 < |x-a| < \delta \longrightarrow |f(x) - L| < \varepsilon). \tag{3.22}$$

If there is no such L satisfying (3.22)*, we say that $\lim_{x\to a} f(x)$* **does not exist**.[21]

This is sometimes called the epsilon-delta (ε-δ) definition of limit. It differs from the definition of continuity in the implication:

$$\begin{aligned} &\textbf{continuity:} && |x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon; && (3.23)\\ &\textbf{limit:} && 0 < |x-a| < \delta \longrightarrow |f(x) - L| < \varepsilon. && (3.24)\end{aligned}$$

A simplistic interpretation would see that the part of $f(a)$ is played by L in the limit, so L is, in some sense, where $f(a)$ seems it should be, at least if the function is to be continuous (which it may or may not be) at $x = a$. Equally crucial is that in the limit defintion, the implication in (3.24) is silent on the behavior at $x = a$, i.e., when $0 = |x-a|$. Indeed, neither $x = a$ nor $f(a)$ play a role in the limit definition's implication (3.24), while both are crucial in continuity's implication (3.23). (See again Figure 3.10, page 209.)

This definition (3.22) of limit is nontrivial and justifies periodic revisiting. With all the cases we will study in this section it will become more and more clear that (3.22) is exactly what we need. However it would be unwieldy indeed to use this definition to prove every limit computation, particularly since such computations are ubiquitous in the rest of this text. Fortunately we know something about continuous functions, and indeed will be able to bootstrap much of our knowledge of continuity to complete nearly all of our limit calculations without working with the ε-δ definition directly. The limit-specific theorems of this section will also assist in developing intuition and computational methods.

Our first theorem is on the uniqueness of the limit, if it exists.

Theorem 3.4.1 *If $\lim_{x\to a} f(x)$ exists, it is unique. In other words,*

$$\left(\lim_{x\to a} f(x) = L\right) \wedge \left(\lim_{x\to a} f(x) = M\right) \implies L = M.$$

[21] Note that we will use arrows such as "$\to$" and "$\longrightarrow$" for both implication and "approaches." The meanings should be clear from the contexts. The valid implication "$\implies$" will keep its earlier meaning throughout.

Theorem 3.4.1 really says the obvious (though the proof is not so transparent): that a function cannot simultaneously be within arbitrarily small ε tolerance of two values and still conform to the limit definition. Eventually the function has to "choose" between approaching L or approaching M (or other values, or no values, but never more than one), or the limit definition is violated. (Recall the discussion of Figure 3.10, particularly as $x \to -3$.) The proof is given at the end of the section.

The following theorem is very important for computing limits.[22]

Theorem 3.4.2 *$f(x)$ is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.*

A glance back at Figure 3.10, page 209 (and also Figure 3.9, page 208) makes a good case for the validity of this theorem. The proof is given at the end of the section. A paraphrasing of the theorem could be: if the value which the function approaches is also where the function "ends up," then we have continuity; if these are different then we do not. With this theorem we can compute many limits by just evaluating the functions at the limit points, *if the functions are continuous there*. (To help decide continuity we had several theorems in Section 3.2.)

Example 3.4.1 *Consider the following limits, where we can "plug in" the limit points because the functions are continuous at their respective limit points.*

- $\lim_{x\to 5} \left(x^2 + 6x\right) = 5^2 + 6(5) = 25 + 30 = 55,$
- $\lim_{x\to 9} \dfrac{1}{x-5} = \dfrac{1}{9-5} = \dfrac{1}{4},$
- $\lim_{x\to 4} \sqrt{26 - x^2} = \sqrt{26 - 4^2} = \sqrt{10},$
- $\lim_{x\to -3} \dfrac{x}{x^2+1} = \dfrac{-3}{(-3)^2+1} = \dfrac{-3}{10}.$

These we could quickly calculate by simple evaluation because the functions were continuous at the points in question. However we do need to be careful not to draw the wrong conclusion from the above example; *evaluating a limit by evaluating the function at the limit point is valid if and* **only if** *the function is continuous there.* This is illustrated in the next example.

Example 3.4.2 *$\lim_{x\to 3} \sqrt{9 - x^2}$ does not exist because we cannot approach from the right side of 3; if $x > 3$, then $9 - x^2 < 0$ and so $\sqrt{9 - x^2}$ is undefined. This does not contradict* Theorem 3.4.2, *since the function is not continuous (only left-continuous) at $x = 3$.*

Continuity at $x = a$ is a stronger condition than the limit existing at $x = a$: with continuity the limit must exist *and* equal the function value (which must also exist, i.e., be defined) there. Where limits are truly useful then is with functions which might not be continuous at a given point, but might nonetheless have a limit there. Often the function in question is equivalent to a continuous function *near*, but perhaps not *at*, the limit point. We will make repeated use of this fact through the validity of the following theorem:

[22] Many calculus texts define limits first, using ε-δ and then use the second statement of Theorem 3.4.2, $\lim_{x\to a} f(x) = f(a)$ as the *definition* of continuity of $f(x)$ at $x = a$. This is valid since the definition of limit stands alone without reference to continuity, and (as the proof of the theorem shows) their limit definition of continuity is equivalent to our earlier ε-δ definition.

Recall that our approach was instead to first define continuity with ε-δ, explore continuity theorems, and then define the limit. With our approach we avoid ε-δ in limit calculations since those technicalities are built into the theorems (specifically in the proofs). Ours is the approach of many analysis texts, and seems to the authors less convoluted and (hopefully) more intuitive than the usual calculus textbook approach.

Eventually (Section 3.9) we do state the limit theorems which other authors build upon, but only after we exhaust the problems we can do without those methods, and after we build a strong, foundational understanding of limits in a context which is closest to continuity.

Theorem 3.4.3 *If $f(x) = g(x)$ on a set $0 < |x - a| < d$, where $d > 0$, then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$$

or both limits do not exist.

Rephrased, if $f(x)$ and $g(x)$ agree near, but not necessarily at, $x = a$ then their limits (as x approaches a) will be the same. This follows quickly from the definition of limit, in which we can replace f with g, assuming $\delta \le d$, which is the sort of thing we did in proving several continuity theorems.

Besides having several applications, this theorem also illustrates much of the nature of limits. For instance it reflects the fact that the limit has a built-in blind spot (by definition) at the point $x = a$. Also apparent is the local nature of limits: the fact that f and g could be wildly different farther away from $x = a$, i.e., for $|x - a| \ge d$, is of no importance to the limit.

The most common place where we use Theorem 3.4.3 is when we calculate limits by simplifying the given function to one which is the same near $x = a$ and has a more obvious limit there. We will consider several examples below. For our first example we revisit the limit which began this section. (See Figure 3.9, page 208.)

Example 3.4.3 *Compute the limit* $\displaystyle\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$.

Solution: Assuming $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$, *we see that*

$$f(x) = \frac{(x-3)(x-2)}{x-3} = x - 2, \qquad x \ne 3. \tag{3.25}$$

In other words, for $0 < |x - 3| < \infty$ (thus $0 < |x - 3| < d$ for any positive number d as in the theorem), we have $f(x) = g(x)$, where $g(x) = x - 2$, a function continuous at $x = 3$. Now

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} (x - 2) = 3 - 2 = 1,$$

and since f and g agree except at $x = 3$, we conclude that $\displaystyle\lim_{x \to 3} f(x) = 1$ *as well.*

The explanation in the example above is complete and correct, but rather pedantic. Since it is one of the simpler limit problems we will come across, a less verbose explanation—but one still faithful to the spirit of the theorems used—can suffice. In this text we will write a summary version which will read as follows:

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x \to 3} \frac{(x-3)(x-2)}{x-3} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x \to 3} (x - 2) = 3 - 2 = 1.$$

The "0/0" (usually read "zero over zero") notation over the equality symbols "=" signify that the limit is of *0/0 form*, which we will discuss in the next paragraph. The "ALG" underneath the second equality symbol signifies that an algebraic rewriting was carried out which was legitimate near, but not necessarily at, the limit point (see again (3.25)). That particular step is where we used Theorem 3.4.3, with the original function playing the part of f and the function $(x - 2)$ playing the part of g. (The "ALG" under the first equality symbol just signifies we algebraically rewrote the function, so we will use "ALG" in both contexts.) We will have other comments to write in the convenient spaces above and below the equality symbols for bookkeeping purposes, so we can concisely organize and later check our work.[23]

[23] The comment system we use here was developed by the authors (who doubt that it is unique). It has been very useful to calculus students wishing to follow the instructor's thinking and to clarify their own. One would usually omit such comments in professional publications, where readers are expected to have sufficient knowledge and experience to fully understand each step without explanation. Their knowledge and experience, of course, come from having practice in solving problems themselves as they learned and later applied these principles.

The previous example is a limit of a certain *form*, which is called the "0/0 form," meaning that the function is a fraction in which the numerator and denominator both approach zero as x approaches the limit point. This is one of many *indeterminate forms;* knowing we have 0/0 form tells us nothing about the value of the limit itself, or even if it exists. The reason is that a shrinking numerator by itself tends to shrink a fraction, while a shrinking denominator alone makes a fraction grow in absolute size. Knowing these are happening simultaneously does not tell us the relative rates at which the two influences are operating. In other words, which competing influence (numerator shrinking or denominator shrinking) dominates? Or is there a compromise, as in the previous example?

There will be many other indeterminate forms, and even more *determinate* forms later in the text. Most of the interesting limits we will find here are of the indeterminate forms. Fortunately we can often simplify an indeterminate form to one which is not. We did so in the previous example, finding an equivalent polynomial limit.

For now it is important to note that when we have 0/0 form, we are not ready to "plug in the limit point," but have more work to do, often algebraic, ideally finding a continuous function which is equal to the original function within the limit except possibly at the limit point. With the new, continuous function we can just "plug in" the limit point and be done. This is our strategy in the following examples. In particular note how we work towards cancelling terms in the denominator which cause the denominator to approach zero as x approaches the limit point.

Example 3.4.4 *Compute* $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$.

<u>Solution</u>: Note that $x = 9$ is outside of the domain of the function, but the actual domain is $x \in [0, 9) \cup (9, \infty)$ so we can certainly approach $x = 9$ from both directions. More casually, we can say that we can let x venture small distances to the left or right of $x = 9$ and the function will be defined. The usual technique for a problem such as this is to algebraically rewrite it by multiplying by $(\sqrt{x}+3)/(\sqrt{x}+3)$:

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)}$$
$$\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

As before, we took a 0/0 form and algebraically manipulated it until we found a function equal to the original near, but not at, $x = 9$, and could then evaluate the new function at that point since it was continuous there. A quick alternative method for this limit uses some slightly clever factoring:

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{6}.$$

Example 3.4.5 *Compute* $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4}$.

<u>Solution</u>: This is of form 0/0 as well. We need to simplify the fraction and see how things might cancel. To be rid of "fractions in the numerator" we will multiply by $\frac{4x}{4x}$.

$$\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4}\cdot\frac{4x}{4x} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 4}\frac{\frac{1}{x}\cdot 4x-\frac{1}{4}\cdot 4x}{(x-4)(4x)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 4}\frac{4-x}{(x-4)(4x)}$$
$$\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 4}\frac{(-1)(x-4)}{(x-4)(4x)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 4}\frac{-1}{4x} = \frac{-1}{16}.$$

In the above example, the domain of the original function was $x \neq 0, 4$, so we could approach $x = 4$ locally inside the domain. Such technicalities are important, but one usually does not make mention of them while working a problem unless a quick inspection shows they need to be considered. An alternative algebraic method of simplifying the expression in this limit is to combine the fractions in the numerator: $1/x - 1/4 = (4-x)/(4x)$, so the function simplifies

$$\frac{\frac{1}{x}-\frac{1}{4}}{x-4} = \frac{\frac{4-x}{4x}}{x-4} = \frac{4-x}{4x}\cdot\frac{1}{x-4} = \frac{4-x}{4x(x-4)}.$$

The first method gives the same simplification a step (or two) sooner. Next consider the following rational limit.

Example 3.4.6 *Compute* $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x^6-243x}$.

<u>Solution</u>: This time we will factor both numerator and denominator, using $243 = 3^5$ along the way. Recall $(a^n - b^n) = (a-b)\left(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1}\right)$.

$$\begin{aligned}\lim_{x\to 3}\frac{x^2-9}{x^6-243x} &\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{(x+3)(x-3)}{x(x^5-3^5)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{(x+3)(x-3)}{x(x-3)(x^4+3x^3+9x^2+27x+81)}\\ &\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{x+3}{x(x^4+3x^3+9x^2+27x+81)}\\ &= \frac{6}{3(81+81+81+81+81)} = \frac{3\cdot 2}{3\cdot(5\cdot 81)} = \frac{2}{405}.\end{aligned}$$

In the above example we used the factoring $a^5-b^5 = (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$. Recall earlier we used a version of $a^2-b^2 = (a-b)(a+b)$ in the form of $x-9 = (\sqrt{x}-3)(\sqrt{x}+3)$. Similarly, we can use $a^3-b^3 = (a-b)(a^2+ab+b^2)$, or the related form $a^3+b^3 = (a+b)(a^2-ab+b^2)$. These and related techniques often work with limits containing radicals.

Example 3.4.7 *Compute* $\displaystyle\lim_{x\to -8}\frac{\sqrt[3]{x}+2}{x+8}$.

<u>Solution</u>: This can be accomplished by completing the factorization $a^3+b^3 = (a+b)(a^2-ab+b^2)$ in the numerator, with $a = \sqrt[3]{x}$ and $b = 2$:

$$\begin{aligned}\lim_{x\to -8}\frac{\sqrt[3]{x}+2}{x+8} &\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to -8}\frac{x^{1/3}+2}{x+8}\cdot\frac{x^{2/3}-2x^{1/3}+4}{x^{2/3}-2x^{1/3}+4}\\ &\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to -8}\frac{x+8}{(x+8)(x^{2/3}-2x^{1/3}+4)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to -8}\frac{1}{x^{2/3}-2x^{1/3}+4}\\ &= \frac{1}{(-8)^{2/3}-2(-8)^{1/3}+4} = \frac{1}{4-2(-2)+4} = \frac{1}{12}.\end{aligned}$$

Note that $1/(x^{2/3}-2x^{1/3}+4)$ is continuous at $x=-8$, since this is just $1/\left(\left(\sqrt[3]{x}\right)^2 - 2\sqrt[3]{x}+4\right)$, the denominator's terms being continuous and the denominator not approaching zero. An algebraic alternative is to factor the denominator using $a^3+b^3 = (a+b)(a^2-ab+b^2)$, with $a = \sqrt[3]{x}$ and $b = 2$:

$$\lim_{x\to -8}\frac{\sqrt[3]{x}+2}{x+8} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to -8}\frac{\sqrt[3]{x}+2}{(\sqrt[3]{x}+2)(x^{2/3}-2x^{1/3}+4)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to -8}\frac{1}{x^{2/3}-2x^{1/3}+4} = \frac{1}{12}$$

as before.

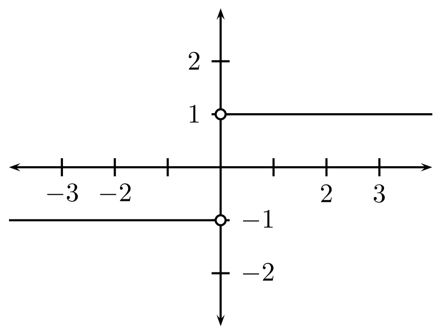


Figure 3.11: Partial graph of $f(x) = x/|x|$. See Example 3.4.9.

We do have to be careful that the limit exists, and there are many instances in which it will not. We will discuss a few below, and add more instances in the next section.

Example 3.4.8 *Consider the limit* $\lim_{x\to 5}\sqrt{x^2-25}$. *This limit does not exist, because the function is not defined for* $x \in (-5,5)$, *and for the limit to exist we need to be able to approach* $x = 5$ *from the immediate left as well as the right. Indeed, the domain is* $x \in (-\infty,-5]\cup[5,\infty)$, *so there is a gap to the left of* $x = 5$ *in the domain.*

The previous example shows again that we can not compute a limit by inputting the limit point just because the function is defined there. Recall we can do so if and only if the function is continuous at that point. The function $f(x) = \sqrt{x^2-25}$ is only right continuous at $x = 5$.

Example 3.4.9 *Consider the limit* $\lim_{x\to 0}\dfrac{x}{\sqrt{x^2}}$. *This limit is of* $0/0$ *form, but ultimately does not exist. For the sake of argument, if it did we would have*

$$\lim_{x\to 0}\frac{x}{\sqrt{x^2}} = \lim_{x\to 0}\frac{x}{|x|}.$$

Although $f(x)$ *is undefined at* $x = 0$, *we can use the piecewise definition of the absolute value function to rewrite the function for* $x \neq 0$*:*

$$f(x) = \frac{x}{|x|} = \begin{cases} x/(x) & \text{for } x > 0 \\ x/(-x) & \text{for } x < 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0. \end{cases}$$

Now that function has height -1 *for* $x < 0$ *and height* 1 *for* $x > 0$, *so we get different heights when we approach from different sides. Thus the limit does not exist. This function is graphed in* Figure 3.11.

Example 3.4.10 *Next consider* $\lim_{x\to -2}\dfrac{\sqrt{x^2}}{x} = \lim_{x\to -2}\dfrac{|x|}{x}$.

Here we look at two methods of computing this. First we note that the function is continuous at $x = -2$, *so we can simply evaluate the function there:*

$$\lim_{x\to -2}\frac{\sqrt{x^2}}{x} = \lim_{x\to -2}\frac{|x|}{x} = \frac{|-2|}{-2} = \frac{2}{-2} = -1.$$

Another method is to replace the function by another which is equal to the original near $x = -2$:

$$\lim_{x\to-2}\frac{\sqrt{x^2}}{x} = \underbrace{\lim_{x\to-2}\frac{|x|}{x} = \lim_{x\to-2}\frac{-x}{x}}_{\text{since } x\to-2<0} = \lim_{x\to-1}(-1) = -1.$$

We include the comment below the brace above for emphasis, and would normally not include it within the actual computation. The point made there is that in the limit computation above, we are not claiming that $|x|/x = -x/x$ for all x, but merely that we can make that substitution here because it **is** *the case "near $x = -2$," that is, we can replace the original function by $g(x) = (-x)/x$ because $|x| = -x$ for $x \in (-\infty, 0)$ and -2 is "safely" inside of $(-\infty, 0)$. Furthermore, that function could be then be replaced by the* **constant** *function $h(x) = -1$—which is trivially continuous—near $x = -2$.*[24]

Example 3.4.11 *Consider the function* $f(x) = \begin{cases} 2x+3 & \text{if} \quad x > 4, \\ x+4 & \text{if} \quad -3 < x \le 4, \\ x^2 - 8 & \text{if} \quad x < 3. \end{cases}$

- $\lim\limits_{x\to5} f(x) = \lim\limits_{x\to5}(2x+3) = 2(5)+3 = 13$. *This is because $5 \in (4, \infty)$, with room to the left and right, so we would claim that $x \to 5 \implies f(x) = 2x + 3 \to 13$. The important point is that 5 is well within the interval on which $f(x)$ is defined to be the continuous function $2x + 3$.*
- $\lim\limits_{x\to-2.9} f(x) = \lim\limits_{x\to-2.9}(x+4) = -2.9 + 4 = 1.1$. *Here $-2.9 \in (-3, 4)$, and -2.9 is safely within the interval on which $f(x) = x + 4$, a continuous function there. Though -2.9 is "only" 0.1 units from where the function's formula changes, that is a postive distance and we can always assume $\delta \le 0.1$ in our limit and continuity proofs to show $f(x)$ is continuous at $x = -2.9$, and thus we can "plug in" $x = -2.9$ to our limit computation.*
- $\lim\limits_{x\to4} f(x)$ *does not exist. From the left-hand side of $x = 4$, we see $f(x) = x + 4 \to 8$, but from the right-hand side of $x = 4$ we have $f(x) = 2x + 3 \to 11$.*
- $\lim\limits_{x\to-3} f(x)$ *requires us to again look at both left-hand side and right-hand side of $x = -3$. From the left we have $f(x) = x^2 - 8 \to 9 - 8 = 1$, and from the right we have $f(x) = x + 4 \to -3 + 4 = 1$. Thus $\lim\limits_{x\to-3} f(x) = 1$ exists. (Here "$\to$" is read "approaches," and is not to be confused with the implication operator from logic.)*

The arguments made regarding the limits as $x \to 4$ and $x \to -3$ will be more systematic in the next section, and we are only showing a glimpse of them here. In the next section we will deal more with piecewise-defined functions, and how better to deal with limit points that are on the boundaries of the "pieces." It is noteworthy that we did not require a graph to analyze any of the limits in the above example, though we may have had aspects of the graph in mind as part of our thinking.

[24]The key fact that *constant* functions are continuous is often overlooked at first by students as they compute limits. Many calculus textbooks emphasize this fact in the limit context by enshrining it in a theorem, the gist of which is

$$\lim_{x\to a} K = K. \tag{3.26}$$

This is obvious when its meaning is understood: that if we define $f(x) = K$, where $K \in \mathbb{R}$ is a constant, then $\lim\limits_{x\to a} f(x) = K$ as well. A quick glance at such a function—whose graph is a horizontal line at height K—shows that such a function is obviously continuous, so we can evaluate the limit by evaluating the (constant) function.

We list one more example here which avoids that issue, but shows the second method of Example 3.4.10 in action again.

Example 3.4.12 *Consider the function* $f(x) = \dfrac{x^2 - 9}{|x+3|(x-3)}$. *Then*

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x+3)(x-3)} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3} 1 = 1;$$
$$\lim_{x\to -5} f(x) = \lim_{x\to -5} \frac{(x+3)(x-3)}{-(x+3)(x-3)} \underset{\text{ALG}}{=\!=} \lim_{x\to -5} (-1) = -1.$$

We used the fact that $|x+3| = x+3$ *for* x *near* 3, *while* $|x+3| = -(x+3)$ *for* x *near* -5. *A quick check shows this limit does not exist for* $x \to -3$ *(in a way similar to* Example 3.4.9, page 215*).*

Example 3.4.13 *Consider the limit* $\lim\limits_{x\to 0} \dfrac{1}{x}$. *This limit does not exist as a finite number. The graph is given at the start of* Section 3.6. *It is easy to see that small inputs into this function quickly return large outputs. For instance,* $f\left(10^{-1}\right) = 10^1$, $f\left(10^{-2}\right) = 10^2$, *and so on, while* $f\left(-10^{-1}\right) = -10^1$, $f\left(-10^{-2}\right) = -10^2$, *and so on. This is sometimes verbally described as* $f(x)$ *"blowing up" as* x *gets nearer to zero:* $1/x$ *is unbounded and negative as* x *approaches zero from the left, and unbounded and positive as* x *approaches zero from the right. The geometric result is a vertical asymptote at* $x = 0$. *When we have vertical asymptotes at our limit points, we do not have finite limits.*[25]

3.4.2 Further Limit Notation

We next take the opportunity to introduce some convenient notation, already alluded to previously. Unfortunately it resembles some notation from logic, but it is usually obvious which meaning is intended from the context. The notation is defined as follows:

$$\lim_{x\to a} f(x) = L \iff \big[(x \longrightarrow a) \longrightarrow (f(x) \longrightarrow L)\big]. \tag{3.27}$$

For reasons of style, it is often less awkward to write, "$x^2 \longrightarrow 9$ as $x \longrightarrow 3$," than to write $\lim\limits_{x\to 3} x^2 = 9$. We might also write, "as $x \longrightarrow 3$, $x^2 \longrightarrow 9$," or "$x \longrightarrow 3 \implies x^2 \longrightarrow 9$." (Arrow lengths are often chosen for convenience or aesthetics, both here and in logic notation.) It is important, but usually obvious, where the expressions would be placed in our original limit notation. The usefulness of this notation will become more apparent in the next section. One nice use we can put it to here is with the following, perhaps more elegant restatement of Theorem 3.4.2:

$$f(x) \text{ continuous at } x = a \iff (x \longrightarrow a) \longrightarrow (f(x) \longrightarrow f(a)). \tag{3.28}$$

We will have much more use for this kind of notation as we look at other limit forms, and other contexts where we use limits. It is only mentioned here so that the reader will be aware of it well before it becomes ubiquitous.

[25] In subsequent sections we will define infinite limits, and left- and right-side limits. For this section we only concern ourselves with finite limits at (or "about," i.e., from both the left and the right) a point. Nothing we do here contradicts subsequent sections.

3.4.3 Proofs of Limit Theorems

Now we prove Theorem 3.4.2, which states that $f(x)$ is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

Proof: We prove this in two parts. First we show the "only if" part ($\Longrightarrow$):

Suppose that $f(x)$ is continuous at $x = a$. Then by definition,

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\left(|x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon\right).$$

Now for a pair ε, δ from continuity, we get

$$0 < |x-a| < \delta \longrightarrow |x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon,$$

and so the definition of limit works here with the ε, δ from (assumed) continuity, and $f(a)$ for L.

For the "if" part ($\Longleftarrow$), suppose $\lim_{x\to a} f(x) = f(a)$. Thus

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(0 < |x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon).$$

We only need to show that

$$0 = |x-a| \implies |f(x) - f(a)| < \varepsilon.$$

But that is obvious from the definition of a function, since

$$|x-a| = 0 \iff (x = a) \implies f(x) = f(a) \implies |f(x) - f(a)| = 0 < \varepsilon.$$

Thus the case $0 < |x-a| < \delta$ is taken care of by the limit assumption, and the $0 = |x-a|$ by the definition of function, giving us the full case $|x-a| < \delta$, as required by the continuity definition, q.e.d.

Now we prove Theorem 3.4.1, that if $\left(\lim_{x\to a} f(x) = L\right) \wedge \left(\lim_{x\to a} f(x) = M\right) \implies L = M$.

Proof: We will prove this by contradiction. Suppose that the theorem is false, i.e., that there are two numbers, $L, M \in \mathbb{R}$, different, which satisfy the definition of the limit at $x = a$ (if necessary see (1.21)). In other words,

$$(\forall \varepsilon > 0)(\exists \delta_1 > 0)(\forall x)\left(0 < |x-a| < \delta_1 \longrightarrow |f(x) - L| < \varepsilon\right), \tag{3.29}$$

$$(\forall \varepsilon > 0)(\exists \delta_2 > 0)(\forall x)\left(0 < |x-a| < \delta_2 \longrightarrow |f(x) - M| < \varepsilon\right). \tag{3.30}$$

Since we are assuming $L \neq M$, we must have $|L - M| > 0$. Choose $\varepsilon = \frac{1}{3}|L - M|$.

Now pick $\delta_1 > 0$ which satisfies the implication in (3.29) for this particular ε, and $\delta_2 > 0$ which satisfies the implication in (3.30) for this particular ε. Now define

$$\delta = \min\{\delta_1, \delta_2\}. \tag{3.31}$$

Now choose any x satisfying $0 < |x-a| < \delta$. This also gives $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ by (3.31). This ultimately implies

$$\begin{aligned}|L - M| &= |L - f(x) + f(x) - M| \leq |L - f(x)| + |f(x) - M| \\ &= |f(x) - L| + |f(x) - M| < \varepsilon + \varepsilon = 2\varepsilon = \frac{2}{3}|L - M| < |L - M| \implies \mathcal{F}.\end{aligned}$$

> The reason that this is a contradiction is the we ultimately showed that if the thereom is false, then $L \neq M \implies |L-M| < |L-M|$, which is impossible. Hence the assumption that the theorem is false is itself false, proving that the theorem must be true.[26] This completes the proof of Theorem 3.4.1, q.e.d.

From the symbolic logic point of view, it is interesting to analyze the above proof, which an advanced student would recognize immediately as a classic "proof by contradiction," which is a minor variation of *modus tollens*. We could also outline it as

$$(\sim P) \to \mathcal{F} \implies \sim (\sim P) \iff P,$$

where

$$P : \text{Theorem } 3.4.1.$$

Of course we have to refer to our discussion of quantifiers in order to analyze what exactly would be the statement of $\sim P$, and that sets up our proof that $\sim P \implies \mathcal{F}$.

Note also that $\sim P \to \mathcal{F} \iff [\sim (\sim P)] \vee \mathcal{F} \iff P \vee \mathcal{F} \iff P$, so we actually have logical equivalence instead of $\implies$ in the above symbolic logic computation. We showed $\sim P \to \mathcal{F}$ is a tautology (by proving the truth of it), hence proving P.

[26] We need to be careful about the last step. It is because $|L-M| > 0$ that we can say $\frac{2}{3}|L-M| < |L-M|$. Clearly this is false if $L = M$. If we do not know that $L \neq M$ (and thus $|L-M| > 0$), then we can only say $\frac{2}{3}|L-M| \le |L-M|$, as it is always true that $0 \le \frac{2}{3}|L-M| \le |L-M|$ regardless of whether $L = M$ or $L \neq M$.

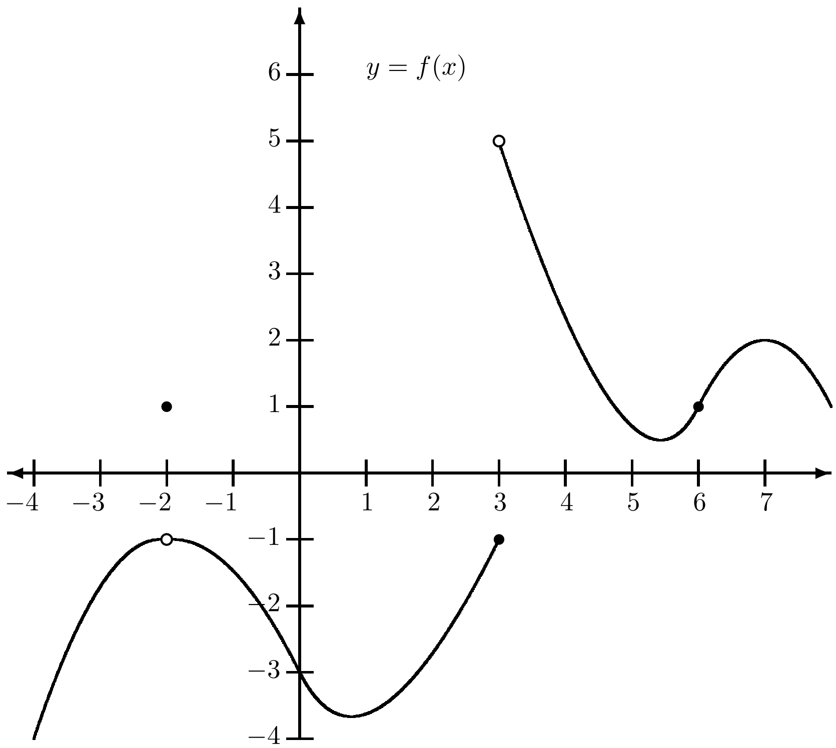


Figure 3.12: Figure for Exercises 1–3. Dotted lines like those found in Figure 3.10, page 209 are omitted here.

Exercises

Consider the graph given in Figure 3.12 above.

1. For each limit, see if it exists. If not, state so. If so, evaluate it.

   (a) $\lim_{x\to -2} f(x)$.

   (b) $\lim_{x\to 0} f(x)$.

   (c) $\lim_{x\to 3} f(x)$.

   (d) $\lim_{x\to 6} f(x)$.

2. Find $f(-2)$, $f(0)$, $f(3)$ and $f(6)$.

3. Decide if $f(x)$ is continuous at the given point. Explain why or why not, in light of Theorem 3.4.2.

   (a) $x = -2$.

   (b) $x = 0$.

   (c) $x = 3$.

   (d) $x = 6$.

Compute the following limits, if they exist, using Theorem 3.4.2 (page 211). If the limit does not exist, explain why.

4. $\displaystyle\lim_{x\to 0} \frac{x+2}{x-5}$.

5. $\displaystyle\lim_{x\to 5} \sqrt{x^2-16}$.

6. $\displaystyle\lim_{x\to 4} \sqrt{x^2-16}$.

7. $\displaystyle\lim_{x\to 3} \sqrt{x^2-16}$.

8. $\displaystyle\lim_{x\to 81} \sqrt[4]{x}$.

9. $\displaystyle\lim_{x\to -1} \left(x^{100} - x^2 + 3\right)$.

10. $\displaystyle\lim_{x\to 0} \sqrt{x^2+1}$.

11. $\displaystyle\lim_{x\to 0} \sqrt[3]{x}$.

12. $\displaystyle\lim_{x\to 7} \big((x^2-9)(x+4)\big)$.

13. Here we consider why the 0/0-form is indeterminate, i.e., why knowing the numerator and denominator both approach zero does not tell us immediately what the limit is. Compute each of the following limits.

    (a) $\displaystyle\lim_{x\to 0} \frac{x}{2x}$.

    (b) $\displaystyle\lim_{x\to 0} \frac{x}{3x}$.

    (c) $\displaystyle\lim_{x\to 0} \frac{x^2}{x}$.

    (d) $\displaystyle\lim_{x\to 0} \frac{x}{x^2}$.

    (e) $\displaystyle\lim_{x\to 0} \frac{x}{|x|}$.

    (f) $\displaystyle\lim_{x\to 0} \frac{x^2}{|x|}$.

    (g) Explain why the above computations show that the 0/0-form is indeed indeterminate.

Compute the given limits where possible. Be sure to indicate any 0/0 forms and any algebraic steps (preferably using the spaces above and below the "=" as shown in previous examples). If a particular limit does not exist, state so and explain why.

14. $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{2x^2-5x-3}$.

15. $\displaystyle\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}$.

16. $\displaystyle\lim_{x\to\sqrt{3}} \frac{x^4-9}{x^2-3}$.

17. $\displaystyle\lim_{x\to 0} \frac{\frac{1}{x+3}-\frac{1}{3}}{x}$.

18. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

19. $\displaystyle\lim_{x\to 1} \frac{x^8-1}{x^2-1}$.

20. $\displaystyle\lim_{x\to 4} \frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}$.

21. $\displaystyle\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16}$.

22. $\displaystyle\lim_{x\to -3} \frac{x^2-9}{x^3+27}$.

23. Let $f(x) = \dfrac{(x-2)|2x-5|}{(x-1)(2x-5)}$. Compute the following where possible.

    (a) $\displaystyle\lim_{x\to 2} f(x)$.

    (b) $\displaystyle\lim_{x\to 0} f(x)$.

    (c) $\displaystyle\lim_{x\to 3} f(x)$.

    (d) $\displaystyle\lim_{x\to 5/2} f(x)$.

24. For each of the following, compute the given limit if it exists. Otherwise state and explain why it does not.

    (a) $\displaystyle\lim_{x\to 3} \sqrt{x^2-4}$.

    (b) $\displaystyle\lim_{x\to -2} \sqrt{x^2-4}$.

25. Compute the limit

$$\lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}.$$

It can help to rewrite the limit

$$\lim_{x\to 0} \frac{1}{x}\left[\frac{\sqrt{1+x}-1}{x} + \frac{\sqrt{1-x}-1}{x}\right].$$

26. Retrace the proof of Theorem 3.4.1 and show that the proof still works if we instead take $\varepsilon = \dfrac{1}{2}|L-M|$ to arrive at a contradiction. (You should be able to then observe why the factor 1/2 is the largest factor which works in the proof of the theorem.)

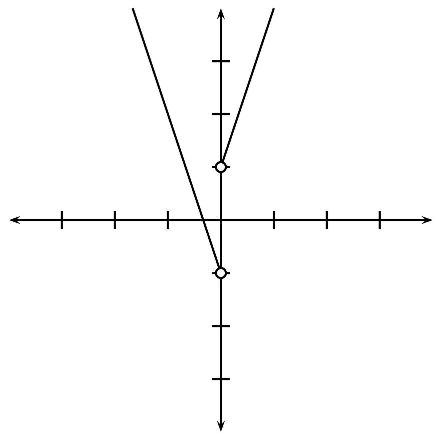


Figure 3.13: Partial graph of $f(x) = (3x^2 + x)/|x|$.

3.5 One-Sided Finite Limits

Just as we found a use for one-sided continuity, so we also have use for so-called left-side (or left-hand side) and right-side (or right-hand side) limits. For instance, consider the function $f(x) = (3x^2 + x)/|x|$. Now $|x|$ can be defined piecewise to be $-x$ for $x \le 0$, and x for $x > 0$, for instance.[27] Although $f(x)$ is undefined at $x = 0$, for $x \neq 0$ we can write

$$f(x) = \frac{3x^2 + x}{|x|} = \begin{cases} (3x^2+x)/(x) & \text{for } x > 0 \\ (3x^2+x)/(-x) & \text{for } x < 0 \end{cases} = \begin{cases} 3x+1 & \text{for } x > 0 \\ -3x-1 & \text{for } x < 0. \end{cases}$$

The function is graphed in Figure 3.13. Of course this function is undefined at $x = 0$, but we might be interested in what occurs when we approach zero from one side or the other. We see that approaching zero from the left side (thus moving right towards zero) we travel along a line whose height is approaching -1, while from the right of zero (moving left) we travel a different line whose height approaches 1. The notation we use to reflect this is the following:

$$\lim_{x\to 0^-} f(x) = -1, \qquad \lim_{x\to 0^+} f(x) = 1.$$

We read the notation $x \to 0^-$ as "x approaches zero from the left," and $x \to 0^+$ as "x approaches zero from the right." To actually work such a problem, in particular without resorting to a graph, we could write the following, as the two cases reflect where $|x| = -x$ and where $|x| = -x$:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{3x^2+x}{|x|} = \lim_{x\to 0^-} \frac{3x^2+x}{-x} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 0^-} (-3x-1) = -1, \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{3x^2+x}{|x|} = \lim_{x\to 0^+} \frac{3x^2+x}{x} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 0^+} (3x+1) \quad = 1. \end{aligned}$$

Notice that $\lim_{x\to 0} f(x)$ does not exist, since approaching the limit point (zero) from one side gives us one value (-1), while approaching from the other side gives another (1). We next make several clarifications regarding one-sided limits.

[27] Of course we can let $|x|$ be x or $-x$ for the case $x = 0$, but that will not come to play in our discussion here.

1. When we analyze a left-side limit, we disregard behavior at and to the right of the limit point; when we analyze a right-side limit, we disregard behavior at and to the left of the limit point.

2. We have analogs (see the theorem below) of Theorem 3.2.11 (page 188), and Theorems 3.4.2 and 3.4.3 (page 212) except that here it is enough to have one sided continuity and equality of the replacement function. We also have the very important theorem in (a) below.

Theorem 3.5.1 *The following hold (where we assume that $d > 0$ where it appears):*

(a) $\lim_{x\to a} f(x) = L \iff \left(\lim_{x\to a^-} f(x) = L\right) \wedge \left(\lim_{x\to a^+} f(x) = L\right)$.

(b) $\lim_{x\to a^-} f(x) = f(a) \iff f(x)$ *is left-continuous at $x = a$.*

(c) $\lim_{x\to a^+} f(x) = f(a) \iff f(x)$ *is right-continuous at $x = a$.*

(d) $f(x) = g(x)$ *on* $x \in (a-d, a) \implies \lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x)$ *(or both do not exist).*

(e) $f(x) = g(x)$ *on* $x \in (a, a+d) \implies \lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x)$ *(or both do not exist).*

These should become clear as we proceed, but are rather technical to prove.[28] For now we will pursue these ideas in several examples.

Example 3.5.1 *If possible find* $\lim_{x\to 5^-} \sqrt{x^2-25}$ *and* $\lim_{x\to 5^+} \sqrt{x^2-25}$.

Solution: Note that the domain of $\sqrt{x^2-25}$ *is* $(-\infty,-5]\cup[5,\infty)$. *Thus we can not approach* $x = 5$ *from the (immediate) left, so* $\lim_{x\to 5^-} \sqrt{x^2-25}$ *does not exist.*

On the other hand, the function is right-continuous at $x = 5$, so we can follow the height at x-values on the right of $x = 5$, and move left in x, and the height will change continuously to the height at $x = 5$ as we move down to that x-value.

$$\lim_{x\to 5^+} \sqrt{x^2-25} = \sqrt{5^2-25} = 0.$$

What gave away the fact that we could just "plug in" $x = 5$ for the right-side limit was that the values inside the square root were nonnegative for x in that range. That was not the case as $x \to 5^-$; we have "wiggle room" to the right, but not to the left, of $x = 5$. For a more concise presentation, we could write

$$\begin{aligned} &\lim_{x\to 5^-} \sqrt{\underbrace{x^2-25}_{<0}} && \text{does not exist,} \\ &\lim_{x\to 5^+} \sqrt{\underbrace{x^2-25}_{>0}} && = \sqrt{5^2-25} = 0. \end{aligned}$$

Note that the example $f(x) = (3x^2+x)/|x|$ at the beginning of this section did not allow us to "plug in" $x = 0$ for either limit, because the function is neither left-continuous nor right-continuous at $x = 0$. But in both cases we replaced the given function with functions which were: $g(x) = -3x - 1$ for the left-side limit, and $g(x) = 3x + 1$ for the right-side limit.

[28]Actually most of these can be proven quickly by modifications of the proofs of earlier theorems, but those proofs each took enough space that to modify them here would require a rather distracting effort. We leave them as exercises (but not listed in the regular exercises) for the interested reader.

Example 3.5.2 *Compute if possible* $\displaystyle\lim_{x\to 3}\frac{(x+2)\sqrt{x^2-6x+9}}{x^2-7x+12}$.

Solution: First we will do some algebraic simplification:

$$\lim_{x\to 3}\frac{(x+2)\sqrt{x^2-6x+9}}{x^2-7x+12}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 3}\frac{(x+2)\sqrt{(x-3)^2}}{(x-4)(x-3)}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 3}\frac{(x+2)|x-3|}{(x-4)(x-3)}.$$

Because we have an absolute value which is zero at our limit point, the (continuous) function inside the absolute value may change sign there and we should check left and right limits.

$$\lim_{x\to 3^-}\frac{(x+2)|\overbrace{x-3}^{<0}|}{(x-4)(x-3)}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 3^-}\frac{(x+2)[-(x-3)]}{(x-4)(x-3)}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 3^-}\frac{-(x+2)}{(x-4)}=\frac{-5}{-1}=5,$$

$$\lim_{x\to 3^+}\frac{(x+2)|\overbrace{x-3}^{>0}|}{(x-4)(x-3)}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 3^+}\frac{(x+2)[(x-3)]}{(x-4)(x-3)}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 3^-}\frac{(x+2)}{(x-4)}=\frac{5}{-1}=-5.$$

Since the left and right limits do not agree, the original (two-sided) limit does not exist.

With absolute value problems like the above example, it is often possible to see quickly if it should be replaced by the quantity inside, or its opposite (additive inverse). One only needs to check if it is positive or negative just left or just right of the limit point (depending upon the side from which we are approaching). Since $x-3$ is negative just left of $x=3$, we can use $|x-3|=-(x-3)$ as $x\to 3^-$. (Recall the piecewise definition of $|x|$.)

It is not the case that the two-sided limit will not exist whenever an absolute value is involved. Consider the next two examples.

Example 3.5.3 *Compute, if possible,* $\displaystyle\lim_{x\to 0}\frac{x^2}{|x|}$.

Solution: As before, we will see what limits we get from both sides.

$$\lim_{x\to 0^-}\frac{x^2}{|x|}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 0^-}\frac{x^2}{-x}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 0^-}(-x)=0,$$

$$\lim_{x\to 0^+}\frac{x^2}{|x|}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 0^+}\frac{x^2}{x}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 0^+}(x)\quad=0.$$

Since both are the same, we conclude[29] $\displaystyle\lim_{x\to 0}\frac{x^2}{|x|}=0$.

Example 3.5.4 *Find* $\displaystyle\lim_{x\to 9}\frac{(2x-18)|2x-23|}{x^2-11x+18}$.

Solution: For this function, the expression inside the absolute value does not change signs at $x=9$ *and so we can use older techniques:*

$$\lim_{x\to 9}\frac{(2x-18)|2x-23|}{x^2-11x+18}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 9}\frac{2(x-9)[-(2x-23)]}{(x-9)(x-2)}\overset{0/0}{\underset{\text{ALG}}{=\!=}}\lim_{x\to 9}\frac{2(23-2x)}{x-2}=\frac{2(23-18)}{9-2}=\frac{10}{7}.$$

[29]Actually this function simplifies to $|x|$ for $x\neq 0$, since $\dfrac{x^2}{|x|}=\dfrac{|x|^2}{|x|}=|x|$, assuming $|x|\neq 0$, i.e., $x\neq 0$.

Example 3.5.5 *Suppose* $f(x) = \begin{cases} x+1 & \text{for } x > 3 \\ 5x-11 & \text{for } x \in [-2,3] \\ x+3 & \text{for } x < -2. \end{cases}$
Find all left, right, and two-sided limits, where possible, at $x = 3$*,* $x = 1$ *and* $x = -2$*.*

Solution: First we look at $x = 3$*. In each case the key is to see which "piece"* x *is on in its approach to the limit point. (See* Theorem 3.5.1(d),(e), page 223.*)*

$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-}(5x-11) = 5(3) - 11 = 4,\\
\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+}(x+1) = 4.
\end{aligned}$$

Since these limits' values are the same, $\lim_{x\to 3} f(x) = 4$ *also. (See* Theorem 3.5.1(a), page 223.*)*

Next are the limits at $x = 1$*. In all cases the computation is the same, but we will go ahead and write them out here. (With practice, for such a case only the third would likely be computed, the other two following.)*

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-}(5x-11) = 5(1) - 11 = -6,\\
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+}(5x-11) = 5(1) - 11 = -6,\\
\lim_{x\to 1} f(x) &= \lim_{x\to 1}(5x-11) = 5(1) - 11 = -6.
\end{aligned}$$

Those were easier because $x = 1$ *is safely inside the interval* $[-2,3]$*, on which* f *is defined by the (continuous) function* $5x - 12$*. Finally we turn our attention to* $x = -2$*.*

$$\begin{aligned}
\lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-}(x+3) = -2 + 3 = 1,\\
\lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+}(5x-11) = 5(-2) - 11 = -21.
\end{aligned}$$

Since the left and right limits do not agree, we conclude $\lim_{x\to -2} f(x)$ *does not exist.*

If we collect the information in the previous example, along with the values of $f(x)$ at the limit points, we can use Theorem 3.5.1(b),(c), and the two-sided analogs, to make some conclusions regarding various types of continuity at these points:

- At $x = 3$, we have $\lim_{x\to 3} f(x) = 4 = f(3)$, so $f(x)$ is continuous at $x = 3$.
- At $x = 1$, we have $\lim_{x\to 1} f(x) = -6 = f(1)$, so $f(x)$ is continuous at $x = 1$.
- $\lim_{x\to -2^-} f(x) = 1 \neq f(-2) = -21$, so $f(x)$ is not left-continuous at $x = -2$.
- $\lim_{x\to -2^+} f(x) = -21 = f(-2)$, so $f(x)$ **is** right-continuous at $x = -2$.
- $f(x)$ is not continuous at $x = -2$, since it is **not both** left- and right-continuous at $x = -2$. Furthermore, since $\lim_{x\to -2} f(x)$ does not exist, it cannot equal $f(-2)$, so Theorem 3.4.2, page 211 also gives us a discontinuity at $x = -2$.
- Since the function is "piecewise linear," it is continuous at all other x-values. Thus it is continuous for $x \neq -2$, i.e., for all $x \in \mathbb{R} - \{-2\} = (-\infty, -2) \cup (-2, \infty)$.

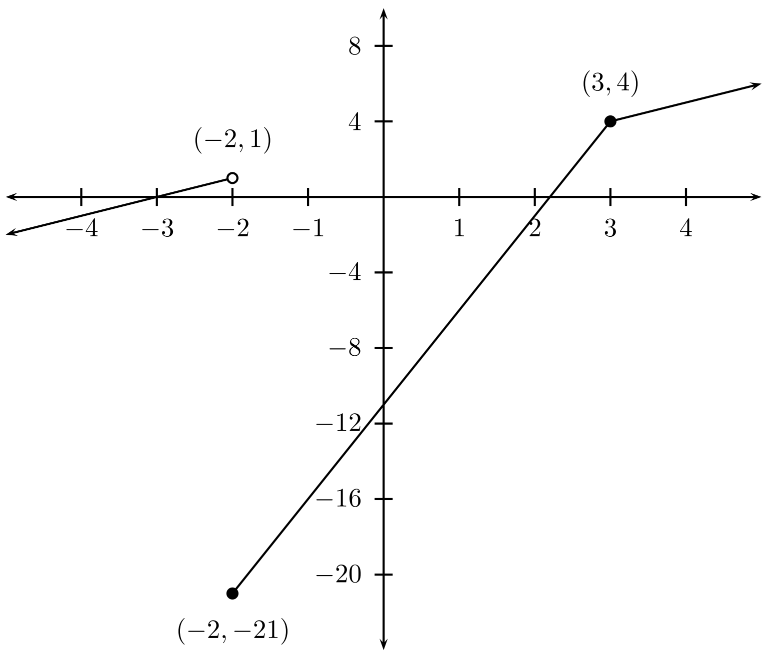


Figure 3.14: Partial graph of function from Example 3.5.5, page 225.

Just as with continuity and limits, the graph of a function can often indicate one-sided continuity and the values of one-sided limits. If we can "ride along" the graph towards the limit point from the prescribed direction, we can visually observe if some height is approached. While the function in Example 3.5.5 is graphed in Figure 3.14, we should be able to read many of the limit and continuity properties from the graph itself.

- $\displaystyle\lim_{x\to-2^-} f(x) = 1$
- $\displaystyle\lim_{x\to-2^+} f(x) = -21$
- $\displaystyle\lim_{x\to-2} f(x)$ DNE
- $f(-2) = -21$
- $f(x)$ is right-continuous at $x = -2$
- $\displaystyle\lim_{x\to-3} f(x) = 0$
- $f(-3) = 0$.
- $f(x)$ is continuous at $x = -3$
- $\displaystyle\lim_{x\to3} f(x) = 4$
- $f(3) = 4$
- $f(x)$ is continuous at $x = 3$
- $f(x)$ is continuous on $(-\infty, -2)$
- $f(x)$ is continuous on $[-2, \infty)$.

Note that the continuity on $[-2, \infty)$, by definition, means $f(x)$ is continuous at each $x \in (-2, \infty)$ (the open interval) and right-continuous at $x = -2$. (This is Definition 3.3.2, page 195.) Continuity on $(-\infty, -2)$ simply means (two-sided) continuity at each $x \in (-\infty, -2)$ (Definition 3.3.1, page 194).

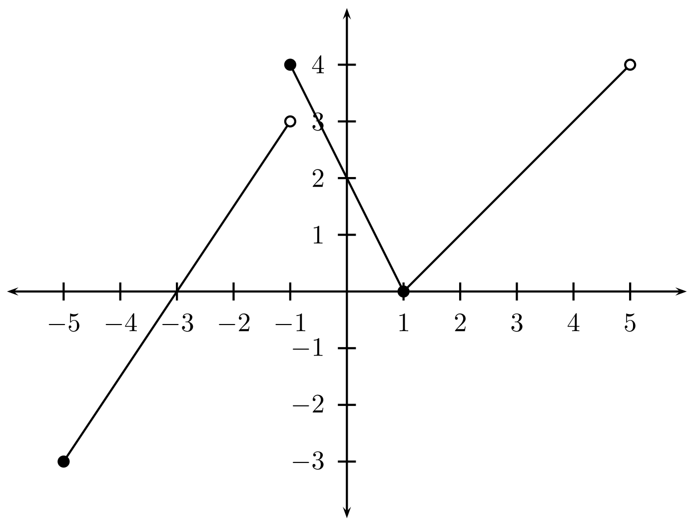


Figure 3.15: Graph of the function $f(x)$ given in Exercises 1 and 2.

Exercises

1. Consider the function graphed in Figure 3.15. *From looking at the graph,* answer the following questions. (It is possible that a requested limit does not exist.)
   (a) $\lim_{x\to -5^+} f(x) =$
   (b) $\lim_{x\to -5^-} f(x) =$
   (c) $\lim_{x\to -5} f(x) =$
   (d) $\lim_{x\to 5^+} f(x) =$
   (e) $\lim_{x\to 5^-} f(x) =$
   (f) $\lim_{x\to 5} f(x) =$
   (g) $\lim_{x\to -1^-} f(x) =$
   (h) $\lim_{x\to -1^+} f(x) =$
   (i) $\lim_{x\to -1} f(x) =$
   (j) $\lim_{x\to 0^-} f(x) =$
   (k) $\lim_{x\to 0^+} f(x) =$
   (l) $\lim_{x\to 0} f(x) =$
   (m) $\lim_{x\to 1} f(x) =$
   (n) $\lim_{x\to 1^-} f(x) =$
   (o) $\lim_{x\to 1^+} f(x) =$
   (p) $\lim_{x\to -3} f(x) =$
   (q) Is $f(x)$ continuous at $x = -5$? Is $f(x)$ left-continuous at $x = -5$? Is $f(x)$ right-continuous at $x = -5$?
   (r) Repeat for $x = -1$.
   (s) Repeat for $x = 1$.
   (t) Repeat for $x = 5$.
   (u) Repeat for $x = 0$.

2. The function in Figure 3.15 can be defined piecewise as follows:

$$f(x) = \begin{cases} x - 1 & \text{for } x \in (1, 5) \\ -2(x-1) & \text{for } x \in [-1, 1] \\ \frac{3}{2}(x+3) & \text{for } x \in [-5, -1). \end{cases}$$

   Using this fact (and not referring directly to the graph), answer (a)–(t) as in the previous exercise. (For example,

$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x-1) = 2-1 = 1$.
For continuity and other questions here, see also Theorem 3.5.1 and the previous examples.)

3. Consider the function $f(x) = \sqrt{x^2-16}$.
   (a) If possible compute $\lim_{x\to 4^-} f(x) =$
   (b) If possible compute $\lim_{x\to 4^+} f(x) =$
   (c) If possible compute $\lim_{x\to 4} f(x) =$
   (d) Do we have continuity, left-continuity, right-continuity or neither at $x = 4$?
   (e) If possible compute $\lim_{x\to -4^-} f(x) =$
   (f) If possible compute $\lim_{x\to -4^+} f(x) =$
   (g) If possible compute $\lim_{x\to -4} f(x) =$
   (h) Do we have continuity, left-continuity, right-continuity or neither at $x = -4$?
   (i) If possible compute $\lim_{x\to 5^-} f(x) =$
   (j) If possible compute $\lim_{x\to 5^+} f(x) =$
   (k) If possible compute $\lim_{x\to 5} f(x) =$
   (l) Do we have continuity, left-continuity, right-continuity or neither at $x = 5$?

For each of the following limits 4–8, compute its value or (if appropriate) state why it does not exist. Show details. (For some, you may need to check both left-side and right-side limits.)

4. $\lim_{x\to 3} \dfrac{|x-5|}{x+4}$

5. $\lim_{x\to 1^-} \dfrac{|x-1|}{x^2+3x-4}$

6. $\lim_{x\to -4} \dfrac{|x+4|}{x^2-16}$

7. $\lim_{x\to 4} \dfrac{|x^2-16|}{x-4}$

8. $\lim_{x\to 4} \dfrac{x^2-8x+16}{|x-4|}$

9. Consider the following function (where m and b will be determined later).

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x > 4 \\ mx+b & \text{for } -4 \le x \le 4 \\ -3-(x+4)^2 & \text{for } x < -4. \end{cases}$$

   (a) Find $\lim_{x\to 4^+} f(x)$.
   (b) Find $\lim_{x\to 4^-} f(x)$.
   (c) Find $\lim_{x\to -4^+} f(x)$.
   (d) Find $\lim_{x\to -4^-} f(x)$.
   (e) Use these to find m and b so that $f(x)$ is continuous on $\mathbb{R}$.

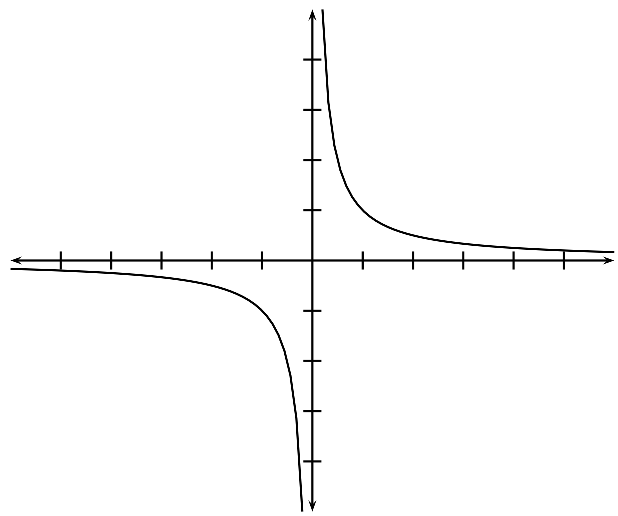


Figure 3.16: Partial graph of $f(x) = 1/x$. We see $f(x) \longrightarrow \infty$ as $x \longrightarrow 0^+$, while $f(x) \longrightarrow -\infty$ as $x \longrightarrow 0^-$.

3.6 Infinite Limits at Points

In this section we extend the notion of limit so it can describe quantities which are growing without bound in many circumstances. The prototype function to help define what we mean here will be the familiar $f(x) = 1/x$, which we graph in Figure 3.16 above. As x approaches zero from the right, we have $1/x$ returning larger and larger positive numbers, with unbounded growth in $1/x$. Similarly as x approaches zero from the left we have $1/x$ returning larger and larger negative numbers, growing without bound. For this function we would write

$$\lim_{x\to 0^+} \frac{1}{x} \overset{1/0^+}{=\!=} +\infty \text{ (or just } \infty), \tag{3.32}$$

$$\lim_{x\to 0^-} \frac{1}{x} \overset{1/0^-}{=\!=} -\infty, \tag{3.33}$$

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist } (1/0^\pm). \tag{3.34}$$

The above limits are the most important such examples. The first two limits are particular *determinate* forms, $1/0^+$ and $1/0^-$ respectively, which we will discuss later in this section. The third limit does not exist because the left-side and right-side limits do not agree, as often happened with finite limits. We will still label its form $1/0^\pm$. For completeness and future reference we now give definitions of what it means for a limit at a point, from the left or right, to be ∞:

Definition 3.6.1 *For $a \in \mathbb{R}$, we say*

$$\lim_{x\to a} f(x) = \infty \iff (\forall M \in \mathbb{R})(\exists\delta > 0)(\forall x)(0 < |x-a| < \delta \longrightarrow f(x) > M), \tag{3.35}$$

$$\lim_{x\to a^+} f(x) = \infty \iff (\forall M \in \mathbb{R})(\exists\delta > 0)(\forall x)(x \in (a, a+\delta) \longrightarrow f(x) > M), \tag{3.36}$$

$$\lim_{x\to a^-} f(x) = \infty \iff (\forall M \in \mathbb{R})(\exists\delta > 0)(\forall x)(x \in (a-\delta, a) \longrightarrow f(x) > M). \tag{3.37}$$

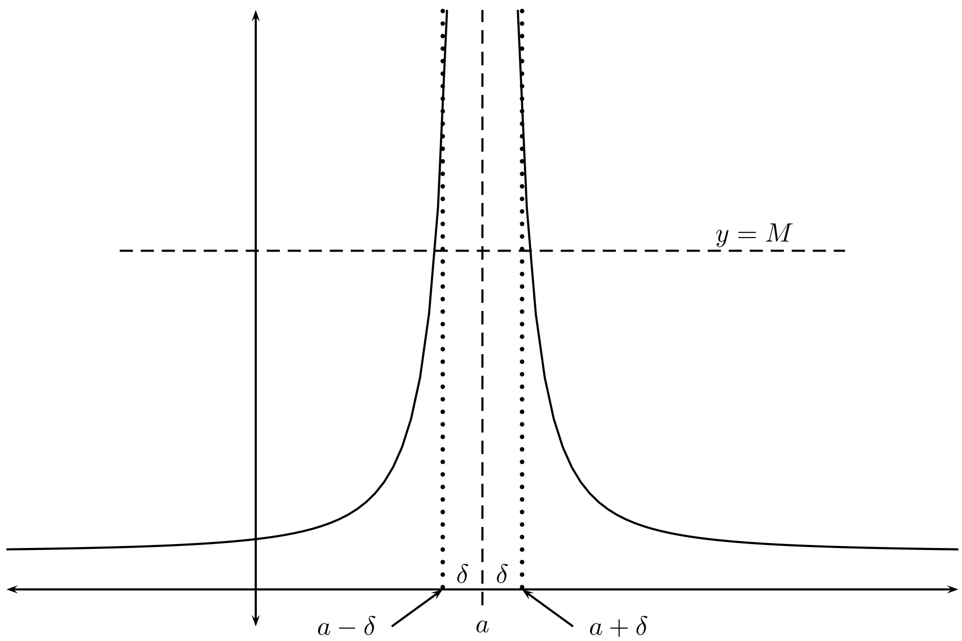


Figure 3.17: A function $f(x)$ with $\lim_{x \to a} f(x) = \infty$. By (3.35), for any height M we can find $\delta > 0$ so that $0 < |x - a| < \delta \implies f(x) > M$. In other words, we can choose any height, and then force $f(x)$ to be still higher than that height by forcing x to be within some δ of a, but (as always with limits) never actually equal to a. Notice that since $f(x)$ is continuous near $x = a$ (but not at $x = a$), the limit as $x \to a$ being infinite gives a vertical asymptote there (given by the dashed vertical line at $x = a$).

In other words, to say that a limit is ∞ is to say that we can force $f(x)$ to be greater than any previously chosen number M by forcing x to be within δ of a (but not equal to a), from one or both directions depending upon if it is a right, left, or two-sided limit. (Of course the choice of δ depends upon the choice of M.) A graphical example of (3.35) is given in Figure 3.17.

Proofs using these definitions are interesting, and we will include one here, but we will have numerous shortcuts based upon general observations.

Example 3.6.1 *Prove that* $\displaystyle\lim_{x \to 0^+} \frac{1}{x} = \infty$.

Solution: Here we start with $M \in \mathbb{R}$ *and try to work back towards some* $\delta > 0$, *so that* $x \in (0, \delta) \implies f(x) = \frac{1}{x} > M$, *as in (3.36). Upon reflection, solving this inequality for* x *seems to be the correct strategy to find* δ, *though we cannot immediately because we do not know if* $M > 0$.

In fact this does not have to be an issue, in the sense that if we can prove that the implication is true for some pair $\delta, M > 0$, *the same* δ *will work for lesser (in particular nonpositive)* M *values as well; if* $M_1 < M_2$, *then any* δ *which makes the implication* $x \in (0, \delta) \implies f(x) > M$ *true for* $M = M_2$ *will also make it true for* $M = M_1$, *as* $f(x) > M_2 \implies f(x) > M_1$.[30] *When*

[30] This is similar to some freedoms we have with our choices of $\delta, \varepsilon > 0$ in the continuity definition and proofs. We already took advantage of the fact that we could *a priori* restrict the values of δ to be bounded by a fixed positive number (we used 1 usually), because if $0 < \delta_1 < \delta_2$ and δ_2 worked for δ in the implication $|x - a| < \delta \to |f(x) - f(a)| < \varepsilon$, then so did δ_1 because $|x - a| < \delta_1 \implies |x - a| < \delta_2$. Furthermore, we could

thus streamlining the proof argument, the usual phrasing used would in this situation read, "without loss of generality, we will assume $M > 0$."

We will eventually use the following fact, which we have used before (as in Example 3.1.7, page 175*) regarding the effect of the reciprocal function on inequalities of positive numbers:* $(\forall a, b > 0)\,[a < b \longleftrightarrow 1/a > 1/b]$*, which itself is easily proved if one simply divides both sides of* $a < b$ *by* $ab > 0$*, giving* $1/b < 1/a$*, essentially* $1/a > 1/b$ *as claimed, written backwards.*

So, assuming $M > 0$ *and* $x \in (0, \delta)$ *for some* $\delta > 0$*, we now compute a value of* δ *which guarantees* $f(x) > M$*. We thus begin with* $f(x) > M$ *and work backwards towards* δ *(note* $x, M > 0$*):*

$$f(x) > M \iff \frac{1}{x} > M \iff x < \frac{1}{M}.$$

We will take $\delta = 1/M$ *in our proof.*

<u>Proof</u>: *Let* $M \in \mathbb{R}$*. Without loss of generality assume* $M > 0$*. Take* $\delta = 1/M$*. Then*

$$x \in (0, \delta) \iff 0 < x < \delta \implies f(x) = \frac{1}{x} > \frac{1}{\delta} = \frac{1}{1/M} = M, \text{ q.e.d.}$$

In this section we will not emphasize proofs, in which we would need to show that a particular limit conforms to a particular limit definition. While interesting in their own rights, from the exmaple above we can see that such proofs will often contain interesting but perhaps distracting technicalities.

Instead we will emphasize the limit forms $1/0^+$, $1/0^-$, $1/0^\pm$ and their variations, and argue intuitively what these limits' values will be.

However there is value in being aware of the various limit definitions as they are given precisely, and so for completeness we next list the similar definitions for the limits at a point being $-\infty$ and invite the reader to make some sense of these as well as the previous definitions.

Definition 3.6.2 *For* $a \in \mathbb{R}$*, we say*

$$\lim_{x\to a} f(x) = -\infty \iff (\forall N \in \mathbb{R})(\exists \delta > 0)(\forall x)(0 < |x - a| < \delta \longrightarrow f(x) < N), \tag{3.38}$$

$$\lim_{x\to a^+} f(x) = -\infty \iff (\forall N \in \mathbb{R})(\exists \delta > 0)(\forall x)(x \in (a, a + \delta) \longrightarrow f(x) < N), \tag{3.39}$$

$$\lim_{x\to a^-} f(x) = -\infty \iff (\forall N \in \mathbb{R})(\exists \delta > 0)(\forall x)(x \in (a - \delta, a) \longrightarrow f(x) < N). \tag{3.40}$$

Thus, to say a limit is $-\infty$ is to say that the function's height can be forced to be below any given (earlier) fixed level by forcing x to be within some distance of a, ignoring the case $x = a$ as we always do with limits.

Earlier we had the *indeterminate* limit form $0/0$ (meaning that knowing the numerator and denominator both approach zero does not tell us the value of the limit). In many of these next cases we will have variations of forms $1/0^+$, $1/0^-$ and $1/0^\pm$, which are *not indeterminate*, i.e., are *determinate* and as such should become intuitive upon reflection. In the definitions below, we will write the functions NUM for the numerator, and DEN for the denominator. (Note that here "$\longrightarrow$" means "approaches.")

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have "without loss of generality" assumed ε was bounded from above. Indeed, if $0 < \varepsilon_1 < \varepsilon_2$, and δ, ε_1 work in the implication $|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon_1$, so do δ, ε_2: $|f(x) - f(a)| < \varepsilon_1 \implies |f(x) - f(a)| < \varepsilon_2$. So we actually had some freedom to restrict our ε range as well as our δ range, in the event it were advantageous.

Definition 3.6.3 *For a function* $f(x) = \dfrac{\text{NUM}(x)}{\text{DEN}(x)}$*, define the following forms, as* x *approaches the limit point in the prescribed way:*

1. $1/0^+$*: any limit in which* $\text{NUM}(x) \longrightarrow 1$ *and* $\text{DEN}(x) \longrightarrow 0^+$*, the latter meaning the denominator is positive but approaching zero.*

2. $1/0^-$*: any limit in which* $\text{NUM}(x) \longrightarrow 1$ *and* $\text{DEN}(x) \longrightarrow 0^-$*, the latter meaning the denominator is negative but approaching zero.*

3. $1/0^\pm$*: any limit in which* $\text{NUM}(x) \longrightarrow 1$ *and* $\text{DEN}(x) \longrightarrow 0$*, with* $\text{DEN}(x) \neq 0$ *but sometimes* $\text{DEN}(x)$ *is positive, other times negative.*

Theorem 3.6.1 *Any limit of the form* $1/0^+$ *will return* $+\infty$*; any limit of the form* $1/0^-$ *will return* $-\infty$*; and any limit of the form* $1/0^\pm$ *will not exist.*

This is well illustrated in the behavior of $f(x) = 1/x$ as zero is approached, as in Figure 3.16 at the opening of this section (page 229), and the corresponding limits (3.32), (3.33) and (3.34).

Example 3.6.2 *Consider the following limits and their methods of computation:*

1. $\displaystyle\lim_{x\to 5^+} \frac{1}{\sqrt{x^2-25}} \stackrel{1/0^+}{=\!=} \infty.$

2. $\displaystyle\lim_{x\to 9^+} \frac{10-x}{9-x} \stackrel{1/0^-}{=\!=} -\infty.$

3. $\displaystyle\lim_{x\to 3} \frac{1}{x-3}$ *does not exist* $(1/0^\pm)$.

4. $\displaystyle\lim_{x\to 4} \frac{x}{(x-4)^2} \stackrel{4/0^+}{=\!=} \infty.$

In the last limit, the denominator is positive for $x \longrightarrow 4$ from both sides, because it is a perfect (polynomial) square. Notice also that, strictly speaking, it is of the form $4/0^+$, but that is also *determinate* (i.e., not indeterminate). In fact, it is just 4 times a limit which is of the form $1/0^+$ if we factor out the 4 (and accept that $4 \cdot \infty = \infty$):

$$\lim_{x\to 4} \frac{x}{(x-4)^2} = 4 \lim_{x\to 4} \frac{x/4}{(x-4)^2} \stackrel{4(1/0^+)}{=\!=\!=} 4 \cdot \infty = \infty. \tag{3.41}$$

Notice that limits allow us to—somewhat—extend our arithmetic when it is understood that it is really a statement about limit forms. For instance, we could say that

$$(\forall a > 0)[a \cdot \infty = \infty], \qquad \text{and} \qquad (\forall a > 0)[a \cdot (-\infty) = -\infty]. \tag{3.42}$$

These mean that if part of the function approaches $a > 0$, and the other "approaches" ∞, then so does the limit of the product approach ∞. Similarly $a \cdot (-\infty)$ gives us $-\infty$. For (3.41), we used the fact that if a function grows positive without bound as we approach the limit point, then so will something which is roughly 4 times that function (it will just "grow" roughly four times as fast). On the other hand,

$$(\forall a < 0)[a \cdot \infty = -\infty], \qquad \text{and} \qquad (\forall a < 0)[a \cdot (-\infty) = \infty]. \tag{3.43}$$

Multiplying a function, which "blows up" as we approach the limit point, by another function which approaches a negative number, does not change the fact that the product will still blow up, but it will occur in the other direction, i.e., with the other sign.

In terms of division, we can write[31]

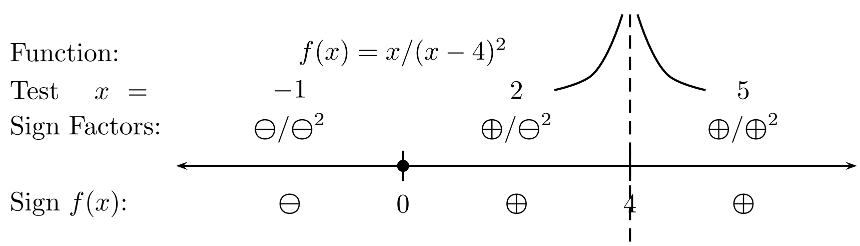
$$(\forall a > 0)\left[\left(\frac{a}{0^+} = a\cdot\frac{1}{0^+} = a\cdot\infty = \infty\right) \wedge \left(\frac{a}{0^-} = a\cdot\frac{1}{0^-} = a\cdot(-\infty) = -\infty\right)\right], \tag{3.44}$$

$$(\forall a < 0)\left[\left(\frac{a}{0^+} = a\cdot\frac{1}{0^+} = a\cdot\infty = -\infty\right) \wedge \left(\frac{a}{0^-} = a\cdot\frac{1}{0^-} = a\cdot(-\infty) = \infty\right)\right]. \tag{3.45}$$

We take this opportunity to point out that knowing where these infinite limits occur is useful in sketching a graph of the function. Consider for instance the limit from the previous example:

$$\lim_{x\to 4}\frac{x}{(x-4)^2} = \infty.$$

If we were to construct a sign chart for this function, and note the vertical asymptote at $x = 4$, we get some idea of what its graph looks like near that vertical asymptote.



In fact this function also has horizontal asymptotes, a topic which will occur in a later section. A computer-generated graph is given in Figure 3.18, page 234, and so our sign chart at least accurately displays the sign and the behavior as $x \to 4$. (Note also the x-intercept at $x = 0$.)

Knowing the function "blows up" at $x = 4$, along with the sign chart, gives us the limits from *both* sides. For another example, we revisit the function $f(x) = x/((x+1)(x-1))$ which we encountered in Example 3.3.7, page 204.

Example 3.6.3 *The following limits can be found by considering the sign chart for the function* $f(x) = x/((x+1)(x-1))$*:*

$$\lim_{x\to -1^-}\frac{x}{(x+1)(x-1)} = -\infty, \qquad \lim_{x\to 1^-}\frac{x}{(x+1)(x-1)} = -\infty,$$
$$\lim_{x\to -1^+}\frac{x}{(x+1)(x-1)} = \infty, \qquad \lim_{x\to 1^+}\frac{x}{(x+1)(x-1)} = \infty,$$

All of these are of forms $1/0^+$, $1/0^-$, $-1/0^+$ *or* $-1/0^-$, *since the numerators are approaching* ± 1 *and the denominators are approaching zero from left or right. Thus we have the function "blowing up" at* ± 1, *so we just need to know what are its signs as we approach* ± 1 *from either side. We repeat the sign chart from* Example 3.3.7, page 204 *for convenience, though we add features to the chart to illustrate locations of the* x*-intercept and vertical asymptotes:*

[31] Another way to look at these limits is to first make note that they are blowing up—for instance NUM $\to a \neq 0$ but DEN $\to 0$—and then just take note of the sign of the fraction as a whole. If it is positive and blowing up, the limit must be ∞; if negative and blowing up, the limit must be $-\infty$; if both signs occur consistently as x approaches the limit point, *and* the fraction is blowing up in absolute size, then the limit must not exist (part of the function blows up towards ∞, and another part towards $-\infty$, as $x \to a$).

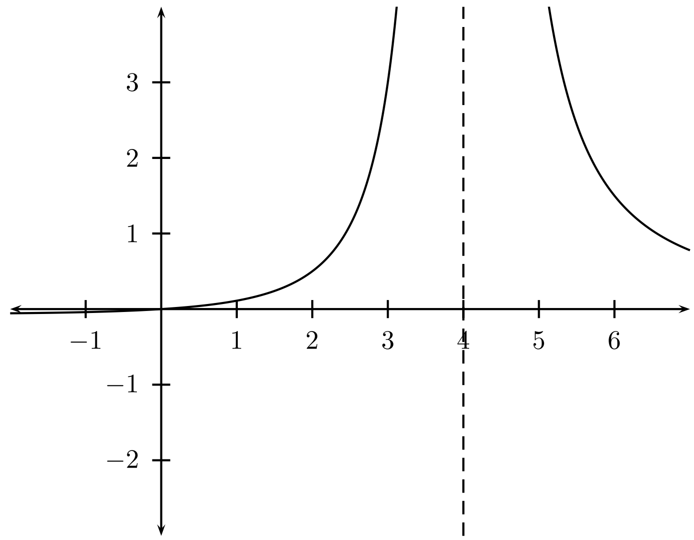


Figure 3.18: Partial graph of $f(x) = x/(x-4)^2$. Besides the vertical asymptote at $x = 4$, note the sign change at $x = 0$, the latter being not easily discerned due to the scale being true. (For instance, $f(-1) = -1/25 = -0.04$.)

Function: $$f(x) = \frac{x}{(x+1)(x-1)}$$

Test $x =$	-2		-0.5		0.5		2
Sign Factors:	$\frac{\ominus}{\ominus\ominus}$		$\frac{\ominus}{\oplus\ominus}$		$\frac{\oplus}{\oplus\ominus}$		$\frac{\oplus}{\oplus\oplus}$
Sign $f(x)$:	$\ominus$	-1	$\oplus$	0	$\ominus$	1	$\oplus$

The graph of $f(x) = x/((x+1)(x-1))$ *is given in* Figure 3.19, page 235.

While the sign chart can help in computing limits, its construction is often more work than required. If we are already building it for some other reason, we can go ahead and use it to compute our limits at the vertical asymptotes, but first we need to know that there is in fact "blow up" as we approach those input values, and so we need an eye to what a computation without a sign chart would look like. Thus, more examples of such computations are in order.

Example 3.6.4 *Consider the following limits.*

1. $\displaystyle \lim_{x\to 9^+} \frac{x}{3-\sqrt{x}} \overset{9/0^-}{=\!=} -\infty,$

2. $\displaystyle \lim_{x\to 6^-} \frac{3-2x}{x^2+2x-48} = \lim_{x\to 6^-} \frac{3-2x}{(x+8)(x-6)} \overset{\frac{-9}{(14)(0^-)}}{=\!=} \infty.$

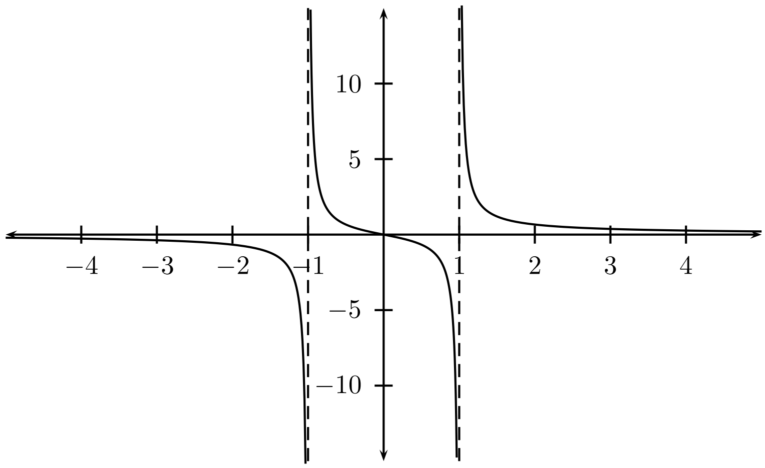


Figure 3.19: Graph of $f(x) = \dfrac{x}{(x+1)(x-1)}$. See Example 3.6.3.

These are fairly routine, but did use some subtlety to get the correct signs. In the first one, it was important to notice that the denominator $3-\sqrt{x}$ is negative for $x > 9$. The second one can be thought of as a form $\frac{-9}{14}\cdot(-\infty) = \infty$. (It also helped to have the denominator factored.)

It is possible that a $0/0$ form can simplify to one of these determinate forms, as in the following example. (We will start to use the shorthand "=D.N.E." when a limit does not exist.)

Example 3.6.5 *Consider the following limits. (The algebra is the same for each so we only show details for the first limit.)*

1. $$\lim_{x\to 3}\frac{81-x^4}{(x^2-6x+9)^2} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{(9-x^2)(9+x^2)}{\left[(x-3)^2\right]^2} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{(3-x)(3+x)(9+x^2)}{(x-3)^4}$$
$$\overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{-(x-3)(3+x)(9+x^2)}{(x-3)^4} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3}\frac{-(3+x)(9+x^2)}{(x-3)^3} \overset{\frac{-6\cdot 18}{0^{\pm}}}{=\!=} \textit{D.N.E.}$$

2. $$\lim_{x\to 3^+}\frac{81-x^4}{(x^2-6x+9)^2} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3^+}\frac{-(3+x)(9+x^2)}{(x-3)^3} \overset{\frac{-6\cdot 18}{0^+}}{=\!=} -\infty.$$

3. $$\lim_{x\to 3^-}\frac{81-x^4}{(x^2-6x+9)^2} \overset{0/0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to 3^-}\frac{-(3+x)(9+x^2)}{(x-3)^3} \overset{\frac{-6\cdot 18}{0^-}}{=\!=} \infty.$$

Considerations also need to be taken for limits involving rational exponents. Recall that if a/b is a simplified fraction (so we do not allow $4/6$ but do allow $2/3$, for example), then

$$x^{a/b} = (x^a)^{1/b} = \left(x^{1/b}\right)^a. \tag{3.46}$$

Also recall that $x^{1/b} = \sqrt[b]{x}$. Finally, $(x^2)^{1/2} = \sqrt{x^2} = |x|$, for instance, while $(x^3)^{1/3} = \sqrt[3]{x^3} = x$. As usual, the odd roots are simpler to deal with then the even roots, at least for abstract computations.

Example 3.6.6 *Consider the following limits:*

1. $\displaystyle\lim_{x\to 0^-}\frac{1}{x^{2/3}}=\lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}}\overset{1/0^+}{=\!=\!=}\infty,$

2. $\displaystyle\lim_{x\to 0^-}x^{-5/3}=\lim_{x\to 0^-}\frac{1}{\sqrt[3]{x^5}}\overset{1/0^-}{=\!=\!=}-\infty,$

3. $\displaystyle\lim_{x\to -4}\frac{x}{(x+4)^{4/3}}=\lim_{x\to -4}\frac{x}{\left[(x+4)^4\right]^{1/3}}\overset{-4/0^+}{=\!=\!=}-\infty,$

4. $\displaystyle\lim_{x\to -4^-}\frac{x}{(x+4)^{1/3}}\overset{-4/0^-}{=\!=\!=}\infty,$

5. $\displaystyle\lim_{x\to -4^-}\frac{x}{\left[(x+4)^2\right]^{1/2}}=\lim_{x\to -4^-}\frac{x}{|x+4|}\overset{-4/0^+}{=\!=\!=}-\infty.$

There are applications where it is interesting to know what occurs in the "extreme," or limit, case. We now consider two of these.

Example 3.6.7 *According to electrostatic theory, if two protons are brought to within the distance $d>0$ of each other, the magnitude of the repelling force they would exert on each other will be given by $F=k/d^2$, where k is a positive constant. Then since $k>0$ we can write*

$$\lim_{d\to 0^+}F=\lim_{d\to 0^+}\frac{k}{d^2}\overset{k/0^+}{=\!=\!=}\infty.$$

Thus, according to electrostatic theory it would require infinite force to bring two protons together such that the distance between them is zero.[32]

Example 3.6.8 *Suppose an object lies along a line running through the center of a thin double convex lens, and further suppose that the line is perpendicular to the plane containing the outer edges of the lens. Also suppose the lens has a focal length of f, and d_o is the distance from the object to the center of the lens, where $d_o>f>0$. If d_i is the distance from the lens center at which the resulting image of the object would be located on the opposite side of the lens, then the* **thin lens equation** *states that*

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}. \tag{3.47}$$

Find the trend in the distance d_i as the object is placed closer and closer to the focal length. Assume $d_o>f>0$ throughout.

Solution: *We wish to compute $\displaystyle\lim_{d_o\to f^+}d_i$, so first we solve for d_i:*

$$\begin{aligned}
&& \frac{1}{d_i}&=\frac{1}{f}-\frac{1}{d_o}\\
\implies && d_i&=\frac{1}{\frac{1}{f}-\frac{1}{d_o}}\cdot\frac{d_of}{d_of}=\frac{d_of}{d_o-f}\\
\implies && \lim_{d_o\to f^+}d_i&=\lim_{d_o\to f^+}\frac{d_of}{d_o-f}\overset{d_of/0^+}{\underset{\text{ALG}}{=\!=\!=}}\infty.
\end{aligned}$$

[32]The electrostatic theory here is far from complete, because it does not take into account other forces and quantum mechanical effects, nor is it clear what it means for the distance to be zero. One should ask if d is only the distances between their "centers," for instance, or is it truly the distances to their edges, and is the "edge" of a proton well-defined? Still it is interesting to see what does the theory say about what occurs to the force "in the limit."

Thus, as the object approaches the focal length position, the image appears to move farther and farther away (and more rapidly away) on the opposite side of the lens. One might conclude that the image of an object placed at the focal length will never be seen on the opposite side of the lens. This effect can be observed in a laboratory to the extent we can approach the ideal of having an object lie precisely and only at the focal length, and numerically using some sample values. Suppose $f = 20\text{cm}$. Testing several relevent values of d_o, using $d_i = d_o f/(d_o - f)$

$$\begin{aligned}
d_o &= 25\text{cm} &&\Longrightarrow d_i = \frac{(25\text{cm})(20\text{cm})}{25\text{cm} - 20\text{cm}} &&= \frac{500\text{cm}^2}{5\text{cm}} = 100\text{cm},\\
d_o &= 22\text{cm} &&\Longrightarrow d_i = \frac{(22\text{cm})(20\text{cm})}{22\text{cm} - 20\text{cm}} &&= \frac{440\text{cm}^2}{2\text{cm}} = 220\text{cm},\\
d_o &= 21\text{cm} &&\Longrightarrow d_i = \frac{(21\text{cm})(20\text{cm})}{21\text{cm} - 20\text{cm}} &&= \frac{420\text{cm}^2}{1\text{cm}} = 420\text{cm},\\
d_o &= 20.5\text{cm} &&\Longrightarrow d_i = \frac{(20.5\text{cm})(20\text{cm})}{20.5\text{cm} - 20\text{cm}} &&= \frac{410\text{cm}^2}{0.5\text{cm}} = 820\text{cm}.
\end{aligned}$$

Finally, we would again see (for the case $f = 20\text{cm}$) that (suppressing units)

$$d_o \to 20^+ \implies d_i = \frac{d_o \cdot 20}{d_o - 20} = \frac{20}{1 - \frac{20}{d_o}} \xrightarrow[20/0^+]{20/(1-1^-)} \infty.$$

Exercises

Compute each limit (stating which ones do not exist) without reference to a sign chart, unless otherwise instructed. In computing these limits, write the form which allows you to make the conclusion where appropriate. (See examples throughout this section.)

1. $\displaystyle\lim_{x\to-5} \frac{1}{(x+5)^2}$
2. $\displaystyle\lim_{x\to1^+} \frac{x}{x^2-1}$
3. $\displaystyle\lim_{x\to1^-} \frac{x}{x^2-1}$
4. $\displaystyle\lim_{x\to1} \frac{x}{x^2-1}$
5. $\displaystyle\lim_{x\to-1^+} \frac{x}{x^2-1}$
6. $\displaystyle\lim_{x\to-1^-} \frac{x}{x^2-1}$
7. $\displaystyle\lim_{x\to-1} \frac{x}{x^2-1}$
8. $\displaystyle\lim_{x\to2^+} \frac{x^2-4x+4}{x^2-4}$
9. $\displaystyle\lim_{x\to2^-} \frac{x^2-4x+4}{x^2-4}$
10. $\displaystyle\lim_{x\to2} \frac{x^2-4x+4}{x^2-4}$
11. $\displaystyle\lim_{x\to-2^+} \frac{x^2-4x+4}{x^2-4}$
12. $\displaystyle\lim_{x\to-2^-} \frac{x^2-4x+4}{x^2-4}$
13. $\displaystyle\lim_{x\to-2} \frac{x^2-4x+4}{x^2-4}$
14. $\displaystyle\lim_{x\to-3} \frac{x}{|x-3|}$
15. $\displaystyle\lim_{x\to3} \frac{x}{|x-3|}$
16. $\displaystyle\lim_{x\to3^+} \frac{|x-3|}{x^2-9}$
17. $\displaystyle\lim_{x\to3^-} \frac{|x-3|}{x^2-9}$
18. $\displaystyle\lim_{x\to3} \frac{|x-3|}{x^2-9}$
19. $\displaystyle\lim_{x\to-3^+} \frac{|x-3|}{x^2-9}$

20. $\lim_{x \to -3^-} \frac{|x-3|}{x^2-9}$

21. $\lim_{x \to -3} \frac{|x-3|}{x^2-9}$

22. Suppose $f(x) = \frac{x}{\sqrt{1-x^2}}$.

    (a) Discuss all possible points $a \in \mathbb{R}$ at which $f(x)$ may have infinite limits as x approaches a from one side or both sides. List all such limits and their values.

    (b) Draw a sign chart for this function. (First, note its domain; where is it defined?)

    (c) Use all the information above to sketch a rough graph of the function.

23. Suppose $f(x) = \frac{1}{x^4-9}$.

    (a) Make a sign chart for $f(x)$.

    (b) Discuss all possible points $a \in \mathbb{R}$ at which $f(x)$ may have infinite limits as x approaches a from one side or both sides. List all such limits and their values, using the sign chart above.

    (c) Use all the information above to sketch a rough graph of the function.

24. According to Einstein's Special Relativity theory, the mass of an object with resting mass m and velocity v is given by

$$M(v) = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

    where c is the speed of light. Assume $m > 0$ and $0 \leq v < c$. Compute the following (for (b) and (c) write your final answer as some decimal number times m, using three significant digits):

    (a) $M(0)$

    (b) $M(c/2)$

    (c) $M(0.9c)$

    (d) $\lim_{v \to c^-} M(v)$

    (e) What physical insight(s) should this limit provide us?

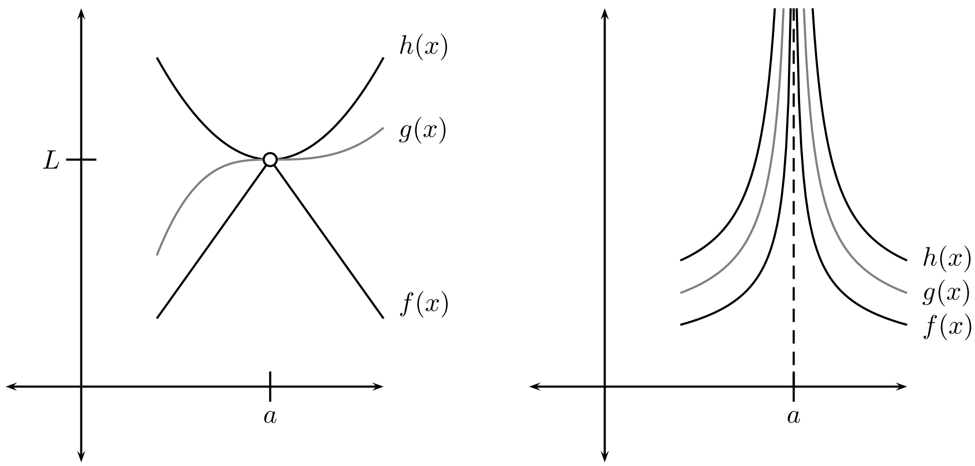


Figure 3.20: Examples illustrating the Sandwich Theorem. In both cases, $f(x) \le g(x) \le h(x)$ near $x = a$, i.e., for some set $0 < |x - a| < \delta$, some $\delta > 0$. Since the limits, as $x \to a$, of $f(x)$ and $h(x)$ are the same (L and $+\infty$ for the respective graphs), the function $g(x)$ must also have the same limit as $x \to a$.

3.7 Sandwich, Composition and Trigonometric Continuity Theorems

In this section we will state the Sandwich Theorem[33] and use it for computing several limits, including those which prove that the trigonometric functions are continuous where defined.

3.7.1 Sandwich Theorem

Theorem 3.7.1 *(Sandwich Theorem) Suppose that there exists some $d > 0$ such that for every $x \in (a - d, a) \cup (a, a + d)$, i.e., for $0 < |x - a| < d$ we have*

$$f(x) \le g(x) \le h(x). \tag{3.48}$$

Then

$$\left(\lim_{x\to a} f(x) = L\right) \wedge \left(\lim_{x\to a} h(x) = L\right) \implies \lim_{x\to a} g(x) = L.$$

The idea is that f and h "sandwich" g between them, and so if f and h both approach L, then g has nowhere to go but L. This is graphed for two cases in Figure 3.20, where L first is a finite real number, and then where $L = \infty$. The functions $f(x)$ and $h(x)$ can be thought of as variable lower and upper bounds for the function $g(x)$ in between by (3.48). Thus the behavior of $f(x)$ and $h(x)$ can, in some circumstances (as in the theorem) force behavior from $g(x)$. The logic of the argument for the theorem is often graphed in various ways. We will employ the style of Figure 3.21, page 240 to illustrate our arguments, except that we will not include the labels "(Hypothesis)" and "(Conclusion)," as they will become apparent in context.

There are several variations of the Sandwich Theorem, in which behavior of one or more *bounding functions* $f(x)$ and $h(x)$ can force behavior upon a (variably) *bounded* function $g(x)$.

[33]The Sandwich Theorem is also called the Squeeze Theorem and the Pinching Theorem in other texts.

$$\begin{array}{rccccc}
\text{(Hypothesis) As } x \to a: & \underbrace{f(x)} & \leq & g(x) & \leq & \underbrace{g(x)} \\
 & \big\downarrow & & & & \big\downarrow \\
 & L & & & & L \\
 & & & & & \\
\text{(Conclusion):} & & & \therefore g(x) \longrightarrow L & &
\end{array}$$

Figure 3.21: Figure illustrating the argument for the Sandwich Theorem.

These variations are perhaps most clearly seen by graphing their respective situations. For instance, it is easily seen that we can replace $f(x) \leq g(x) \leq h(x)$ with $f(x) < g(x) < h(x)$ (see again Figure 3.20 at the beginning of this section).[34] One-sided versions of the theorem also hold, as in for instance the left-sided limit version:

$$\left((\exists d > 0)(\forall x) \left[x \in (a-d, a) \longrightarrow f(x) \leq g(x) \leq h(x) \right] \right) \wedge \left(\lim_{x \to a^-} f(x) = L = \lim_{x \to a^-} h(x) \right)$$
$$\Longrightarrow \lim_{x \to a^-} g(x) = L.$$

The following limit is a very traditional example for the original statement of the Sandwich Theorem. Note that it relies on the fact that $\sin\theta$ is defined for every $\theta \in \mathbb{R}$, and that $-1 \leq \sin\theta \leq 1$.

Example 3.7.1 *Compute the limit* $\displaystyle\lim_{x \to 0} x \sin \frac{1}{x}$.

Solution: Note that $x \sin \frac{1}{x}$ is always between $x \cdot 1$ and $x \cdot (-1)$, but these switch roles as top and bottom bounding functions depending upon the sign of x. However, we can always write

$$-|x| \leq x \sin \frac{1}{x} \leq |x|.^{35} \tag{3.49}$$

By continuity of $|x|$ and $-|x|$, we have

$$\lim_{x \to 0} (-|x|) = -|0| = 0, \qquad \textit{and} \qquad \lim_{x \to 0} |x| = |0| = 0,$$

so by the Sandwich Theorem we must conclude as well that

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

[34] Note that $f(x) < g(x) < h(x) \implies f(x) \leq g(x) \leq h(x)$, so the fact that we can replace the latter with the former follows quickly from logic; if we have the strict inequalities, then we also have the non-strict inequalities and that hypotheses of the Sandwich Theorem will still hold.

[35] We can make the argument leading to (3.49) more precise. For $x \neq 0$, we can say

$$\begin{aligned}
&\left| x \sin \frac{1}{x} \right| = |x| \cdot \left| \sin \frac{1}{x} \right| \leq |x| \cdot 1 = |x|, \\
\Longrightarrow\ &\left| x \sin \frac{1}{x} \right| \leq |x| \\
\Longleftrightarrow\ &-|x| \leq x \sin \frac{1}{x} \leq |x|,
\end{aligned}$$

where the " $\Longleftrightarrow$ " comes from the general rule that $|z| \leq K \iff -K \leq z \leq K$.

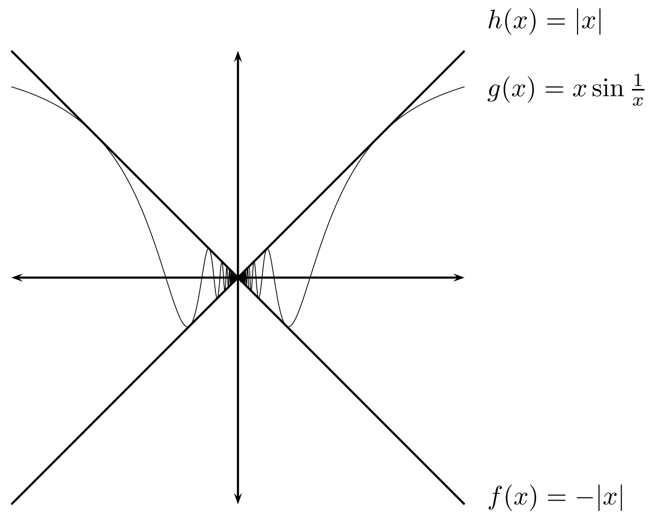


Figure 3.22: Partial graph of $g(x) = x \sin \dfrac{1}{x}$, which is bounded from above by $h(x) = |x|$ and from below by $f(x) = -|x|$. It oscillates wildly, running through infinitely many periods in the argument $1/x$ of the sine function as $x \to 0$, but is bounded in amplitude by functions which shrink to zero as $x \to 0$. ($g(x)$ is undefined at $x = 0$ but that fact is not apparent in the graph above.)

The Sandwich Theorem argument for the limit of Example 3.7.1 above can be summarized graphically as follows:

$$\text{As } x \to 0: \qquad \underbrace{-|x|}_{\downarrow \atop 0} \quad \le \quad x \sin \tfrac{1}{x} \quad \le \quad \underbrace{|x|}_{\downarrow \atop 0}$$

$$\therefore x \sin \tfrac{1}{x} \longrightarrow 0.$$

The function $g(x) = x \sin \frac{1}{x}$ is graphed in Figure 3.22 above, together with the bounding functions $-|x|$ and $|x|$. It has some interesting features which make it very valuable for later examples which clarify some limit principles. We note how the argument $1/x$ of the sine function here runs through infinitely many periods of sine as $x \to 0$, so the function oscillates with infinitely increasing rapidity as $x \to 0$. However the "amplitude" $|x|$ is variable and shrinking to zero.

One use of the above function is in illustrating a rather general theorem, based upon the Sandwich Theorem, regarding limits of products where one factor approaches zero while the other factor, however else it is ill-behaved, is at least bounded and defined as we approach the limit point.

Theorem 3.7.2 *Suppose that $f(x)$ is defined for $0 < |x-a| < d$ for some $d > 0$, and that for such x, $f(x)$ is of the form $f(x) = g(x)h(x)$ where $|g(x)| \le M$ and $h(x) \longrightarrow 0$ as $x \to a$. Then $\lim\limits_{x\to a} f(x) = 0$.*

The proof consists of noting that $-M|h(x)| \le f(x) \le M|h(x)|$, so $\pm M|h(x)| \longrightarrow 0$ as $x \to a$ implies $f(x) \longrightarrow 0$ as $x \to a$ as well:

$$\text{As } x \to a: \qquad \underbrace{-M|h(x)|}_{\substack{\big\downarrow \\ 0}} \le \ g(x)h(x) \ \le \ \underbrace{M|h(x)|}_{\substack{\big\downarrow \\ 0}}$$

$$\therefore\ f(x) = g(x)h(x) \longrightarrow 0.$$

This theorem could have been used in the previous example to compute that limit immediately. We will have more use for this theorem in the next section. For instance, there we will use "B" to refer to a function which is defined and *bounded* as we approach the limit point. Then we point out that "$B \cdot 0$" is in fact a *determinate* form which yields zero in the limit. For the previous example, we would note the fact that for $x \neq 0$ we have $\left|\sin\frac{1}{x}\right| \le 1$ and so, aside from being defined, $\sin\frac{1}{x}$ is bounded as $x \to 0$, and we can write

$$\lim_{x\to 0} x \sin\frac{1}{x} \overset{0\cdot B}{=\!=} 0.$$

While based upon the Sandwich Theorem, the above argument is somewhat intuitive and certainly more concise.

3.7.2 "Approaches" for Independent Versus Dependent Variables

This was not such an important issue before (though a reader might have wondered about this point), so it was deferred until now, and given its own subsection here to be sure it is clarified.

The point is that when we consider the *independent* variable x "approaching" some point, say $x \longrightarrow a$, we should visualize it gradually getting closer to that point a—as close as we like and then even closer—but never actually achieving the value $x = a$. That is built into, for instance, the definition of limit, in the antecedant $0 < |x-a| < \delta$ of the defining implication. On the other hand, we have more flexibility in the consequent $|f(x)-L| < \varepsilon$, though we still write $f(x) \longrightarrow L$. For instance, in our latest example, $x\sin\frac{1}{x} \longrightarrow 0$, that function not only go closer to zero consistently, but also achieved the value zero repeatedly (infinitely many times!) as $x \to 0$. So the independent variable x is forced to approach *and avoid* its limiting value, but the dependent variable need not actually avoid its limiting value when we write $x \to a \implies f(x) \to L$.

For another, rather trivial example, consider that $f(x) = 0 \cdot \sin x \longrightarrow 0$ as $x \to \frac{\pi}{2}$. In fact, $f(x) = 0$ for all $x \in \mathbb{R}$, so the function not only approaches zero, it is never anything but zero. Still we use the notation $f(x) \longrightarrow 0$.

3.7.3 "Sandwiching" From One Side

One only needs one bounding function in an infinite limit case, to have a valid Sandwich Theorem-type argument In (3.50) below, f is the bounding function pushing h, while in (3.51) h is the bounding function pushing f.

Theorem 3.7.3 *Suppose $f(x) \le h(x)$ on $0 < |x-a| < d$ for some $d > 0$. Then (separately) we have*

$$\lim_{x\to a} f(x) = \infty \implies \lim_{x\to a} h(x) = \infty, \tag{3.50}$$

$$\lim_{x\to a} h(x) = -\infty \implies \lim_{x\to a} f(x) = -\infty. \tag{3.51}$$

In other words, if the lesser (lower) function blows up towards ∞, then so must the greater (upper) function, while if the greater function has blowup towards $-\infty$, then so must the lesser function. Such arguments can be verified easily by graphing the situations and seeing how the "blowup" of one function can force a similar behavior of another. It is also useful to see the following, somewhat visual style of the arguments of (3.50) and (3.51). Note that both diagrams below are, for now, hypothetical; instead of $\therefore$ we could instead write $\implies$.

$$\text{As } x \to a: \quad \underbrace{g(x)}_{\substack{\Big\downarrow \\ \infty}} \le h(x) \qquad\qquad \text{As } x \to a: \quad f(x) \le \underbrace{h(x)}_{\substack{\Big\downarrow \\ -\infty}}$$

$$\therefore h(x) \longrightarrow \infty \qquad\qquad\qquad \therefore f(x) \longrightarrow -\infty$$

Of course we always have to be careful. For instance, suppose $f(x) \le h(x)$ and $h(x) \longrightarrow \infty$. It is not necessarily the case that $f(x) \longrightarrow \infty$ as well, since $h(x)$ is above $f(x)$, and thus unable to "push" $f(x)$ up with it.

Example 3.7.2 *Compute* $\displaystyle\lim_{x\to 2^+} \frac{x^2+\sin x}{x-2}$.

<u>Solution</u>: *Since $-1 \le \sin x \le 1$, and for $x > 2$ we have $x^2 > 0$, $x-2 > 0$ we can write*

$$\underbrace{\frac{x^2-1}{x-2}}_{\text{“}f(x)\text{”}} \le \underbrace{\frac{x^2+\sin x}{x-2}}_{\text{“}g(x)\text{”}} \le \underbrace{\frac{x^2+1}{x-2}}_{\text{“}h(x)\text{”}}. \tag{3.52}$$

In other words, the least that $\frac{x^2+\sin x}{x-2}$ can be as $x \to 2^+$ is $\frac{x^2-1}{x-2}$, i.e., where $\sin x = -1$, and the greatest it can be is $\frac{x^2+1}{x-2}$, the case where $\sin x = 1$. Next we notice that

$$\lim_{x\to 2^+} \frac{x^2-1}{x-2} \overset{3/0^+}{=\!=} \infty, \qquad\qquad \lim_{x\to 2^+} \frac{x^2+1}{x-2} \overset{5/0^+}{=\!=} \infty,$$

and so $\displaystyle\lim_{x\to 2} \frac{x^2+\sin x}{x-2} = \infty$ *as well.*[36]

As noted before, the first inequality in (3.52) is in fact enough to "push" the desired limit to be ∞:

[36]After the next subsection, where we prove trigonometric functions continuous where defined, the limit in Example 3.7.2 can be done directly:

$$\lim_{x\to 2^+} \frac{x^2+\sin x}{x-2} \overset{\frac{4+\sin 2}{0^+}}{=\!=\!=} \infty,$$

since $x^2 + \sin x \longrightarrow 4 + \sin 2 > 0$. Though not necessary here, it is perhaps easier to see we have the correct sign if we note that $\sin 2 \approx 0.909297426$, and thus $x^2 + \sin x \longrightarrow 4 + \sin 2 \approx 4.909297426$. (The sign of $x^2 + \sin 2$ was going to be positive as $x \to 2$ anyways because $\sin 2 \in [-1, 1]$, while $x^2 \longrightarrow 4$.)

$$\text{As } x \to 2^+: \qquad \underbrace{\frac{x^2-1}{x-2}}_{\substack{\big\downarrow \\ \infty}} \le \frac{x^2+\sin x}{x-2}$$

$$\therefore \frac{x^2+\sin x}{x-2} \longrightarrow \infty$$

In fact the example above will not need this Sandwich Theorem-type argument if we note that $\sin 2 \approx 0.909297426 > 0.9$. That is because we could have written

$$\lim_{x\to 2^+} \frac{x^2+\sin x}{x-2} = \lim_{x\to 2^+} \left[\frac{x^2}{x-2} + \frac{\sin x}{x-2}\right] \overset{\frac{4}{0^+}+\frac{\sin 2}{0^+}}{\underset{\infty+\infty}{=\!=}} \infty.$$

This used the fact that, as a form, $\infty + \infty$ yields limits which are ∞, as we will note in later sections and should be intuitive now. That does not mean we can avoid using such arguments altogether (recall our first example, $x \sin \frac{1}{x} \longrightarrow 0$ as $x \to 0$). A small change makes the case for such a method: consider a similar limit but where the numerator of the function is now $x^2 - \sin x$. Then we still have

$$\lim_{x\to 2^+} \frac{x^2+\sin x}{x-2} = \lim_{x\to 2^+} \left[\frac{x^2}{x-2} + \frac{\sin x}{x-2}\right] \overset{\frac{4}{0^+}-\frac{\sin 2}{0^+}}{\underset{\infty-\infty}{=\!=}} ?,$$

that is, $\infty - \infty$ form which is indeterminate (first discussed properly in Section 3.8). However the earlier Sandwich Theorem-type argument still applies, since $-\sin x \in [-1, 1]$:

$$\text{As } x \to 2^+: \qquad \underbrace{\frac{x^2-1}{x-2}}_{\substack{\big\downarrow \\ \infty}} \le \frac{x^2-\sin x}{x-2}$$

$$\therefore \frac{x^2-\sin x}{x-2} \longrightarrow \infty$$

In fact the above limit still does not require the Sandwich Theorem-type argument if it is noticed that $x^2 - \sin x \longrightarrow 4 - 0.909297426 > 0$, but with practice it is a reasonably quick argument leading to the conclusion that the limit is in fact ∞.

3.7.4 Limits with Compositions of Functions

Theorem 3.7.4 *Suppose that, for some limiting behavior of x, we have $g(x) \longrightarrow L$, and $f(x)$ continuous at $x = L$. Then for the same limiting behavior of x we have $f(g(x)) \longrightarrow f(L)$.*

So for instance, if $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f(L),$$

i.e.,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L). \tag{3.53}$$

In other words, if the limit of the "inside" function is a point of continuity of the "outside" function, then we can "move the limit notation (lim) inside." (The reader is invited to consider this idea in light of a function diagram for $f(g(x))$.)

In fact in the interest of avoiding errors students are often discouraged from using (3.53) as it is written. One reason is that it is crucial that f be continuous at the limit of g, or the theorem is false, as we will show in the exercises, while reading (3.53) without context may give the impression that we can always move the limit inside. Instead we will concentrate on the theorem itself, and the limit forms which arise from its proper application. Still, the theorem is very general in terms of the type of limit (basic, left, right or the "at infinity" variety we will have in Section 3.8). We will give a proof for the most basic case at the end of this section. That proof will be a simple modification of the proof of Theorem 3.2.5, page 183. The theorem can be illustrated graphically as follows:

$$\begin{tikzcd}
\text{As } x\to a: & f(\underbrace{g(x)}) \arrow[d] \arrow[rr, "\therefore\ \text{(by continuity)}"] & & f(L) \\
& L & &
\end{tikzcd}$$

Again, the diagram above is not necessarily true if $f(x)$ is not continuous at $x = L$. When f is, we might not always give the justification "(by continuity)" within the diagram.

Example 3.7.3 *Suppose that* $\lim_{x\to a} g(x) = 12$, *while* $\lim_{x\to b} g(x) = 0$. *Then*

- $\displaystyle\lim_{x\to a} \sqrt[3]{g(x)} \overset{\sqrt[3]{12}}{=\!=\!=} \sqrt[3]{12}$,
- $\displaystyle\lim_{x\to b} \sqrt[3]{g(x)} \overset{\sqrt[3]{0}}{=\!=} \sqrt[3]{0} = 0$,
- $\displaystyle\lim_{x\to a} \sqrt{g(x)} \overset{\sqrt{12}}{=\!=\!=} \sqrt{12} = 2\sqrt{3}$,
- $\displaystyle\lim_{x\to b} \sqrt{g(x)}$ *cannot be determined by the given information. It depends upon how* $g(x)$ *approaches* 0 *as* $x \to b$*:*
  - *If* $g(x) \to 0^+$ *then* $\lim_{x\to b} \sqrt{g(x)} = \sqrt{0} = 0$*:* $\displaystyle\lim_{x\to b} \sqrt{g(x)} \overset{\sqrt{0^+}}{=\!=\!=} \sqrt{0} = 0$.
  - *In fact, if we just have* $g(x) \geq 0$ *as* $x \to b$*, we have the limit being* $\sqrt{0} = 0$.
  - *However, if* $g(x) \to 0^{\pm}$*, then* $\displaystyle\lim_{x\to b} \sqrt{g(x)} \overset{\sqrt{0^{\pm}}}{=\!=\!=}$ *does not exist.*

In the first two limits above, we have $g(x) \longrightarrow 12$ or 0, which are well-within the set on which $\sqrt[3]{x}$ is continuous, namely $\mathbb{R}$. For the third limit above, we recall that $\sqrt{x}$ is continuous for $x > 0$, and so $g(x) \longrightarrow 12 \implies \sqrt{g(x)} \longrightarrow \sqrt{12}$. Since $\sqrt{x}$ is defined for $x \geq 0$, and is in fact right-continuous at $x = 0$, it is possible that if $g(x) \to 0$, then $\lim_{x\to b} \sqrt{g(x)}$ is zero or does not exist. The right-continuity of $\sqrt{x}$ at $x = 0$ guarantees that $\sqrt{g(x)} \longrightarrow 0$ if $g(x) \to 0^+$, but does not exist if $g(x)$ is sometimes negative as $x \to b$.

This theorem will prove to be more useful—and in fact will be crucial—in later sections, but we will make some use of it in this section as an alternative method for some limit computations for which earlier methods are not quite as efficient.

3.7.5 Continuity Considerations for Trigonometric Functions

Our theorem is as follows:

Theorem 3.7.5 *The six basic trigonometric functions,* $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ *and* $\csc x$ *are continuous everywhere they are defined. Thus*

1. $\sin x$ *and* $\cos x$ *are continuous for* $x \in \mathbb{R}$.
2. $\tan x$ *and* $\sec x$ *are continuous* **except** *where* $\cos x = 0$, *and are thus continuous for all*
$$x \neq \frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \frac{\pm 5\pi}{2}, \cdots.$$
3. $\cot x$ *and* $\csc x$ *are continuous* **except** *where* $\sin x = 0$, *and are thus continuous for all*
$$x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots.$$

Considering the unit circle definitions of $\sin\theta$ and $\cos\theta$, it is reasonable that these are continuous (as functions of θ). The continuity of the other trigonometric functions, which are quotients of 1, $\sin\theta$ and $\cos\theta$ where they are defined then follows immediately. Because the results listed in Theorem 3.7.5 are intuitive, we will defer the proof until the end of the section. The proofs that $\sin\theta$ and $\cos\theta$ are continuous use the sandwich theorem and a geometric argument, and are interesting in their own rights, but for now we will concentrate our efforts in *applications* of the theorem.

Example 3.7.4 *The following limits follow directly from continuity of the trigonometric functions (where defined):*

- $\displaystyle\lim_{x\to\pi} \sin x = \sin\pi = 0.$
- $\displaystyle\lim_{x\to\pi/4} \tan x = \tan\frac{\pi}{4} = 1.$
- $\displaystyle\lim_{x\to\sqrt{\pi}} \cos x^2 = \cos\left(\sqrt{\pi}\right)^2 = \cos\pi = -1.$

The last limit above was computable as shown since x^2 *is continuous on all of* $\mathbb{R}$, *and so is* $\cos x$, *so the composition* $\cos x^2$ *is continuous on all of* $\mathbb{R}$, *including* $x = \sqrt{\pi}$ *(see* Theorem 3.2.5, page 183*).*

In the last example above, the cosine function was the "outer" function, but the trigonometric functions can be combined with other functions in a variety of ways.

Example 3.7.5 *Consider the following limit computations.*

- $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{1-\sin^2 x}{\cos x} \stackrel{0/0}{=\!=} \lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{\cos x} \stackrel{0/0}{=\!=} \lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0.$

- $\lim_{x\to 0}\sqrt{1-\underbrace{\cos^2 x}_{<1}} \overset{\sqrt{0^+}}{=\!=} \sqrt{1-\cos^2 0} = \sqrt{1-1^2} = \sqrt{0} = 0.$

The first was a standard $0/0$-form simplification using the trigonometric identity $\cos^2 x + \sin^2 x = 1$. The second relied upon the fact that $\cos^2\theta < 1$ as $x \to 0$. In fact it was enough that $\cos x \in [-1,1]$, so $\cos^2 x \in [0,1]$ and so, though we are only interested in behavior as we approach zero, in fact the expression inside the square root, $1-\cos^2 x$ is never negative (and is everywhere continuous): $\mathbb{R} \xrightarrow{\cos x} [-1,1] \xrightarrow{x^2} [0,1] \xrightarrow{1-x} [0,1]$.

Example 3.7.6 *Consider the following limit computation. Note that since $0/0$ is indeterminate, so is $\sin\frac{0}{0}$. (Note that all angles are assumed to be measured in radians.)*

$$\lim_{x\to 3}\sin\left(\frac{x^2-9}{x^2-5x+6}\right) \overset{\sin\frac{0}{0}}{=\!=\!=} \lim_{x\to 3}\sin\left(\frac{(x+3)(x-3)}{(x-2)(x-3)}\right) \overset{\sin\frac{0}{0}}{=\!=\!=} \lim_{x\to 3}\sin\left(\frac{x+3}{x-2}\right)$$
$$= \sin\frac{3+3}{3-2} = \sin 6 \approx -0.279415498.$$

Since $0/0$-form is indeterminate, any "function" of it is also, so we have to deal with the argument ("inside") of the sine function. Of course the **exact** *answer is $\sin 6$, but may naturally be curious about what is the approximate value of this limit so a nine-digit approximation is also included.*

In Theorem 3.7.4, page 244 we had an alternative method for analyzing the previous example's limit: we could instead exploit the everywhere-continuity of the sine function to allow manipulations such as

$$\lim_{x\to 3}\sin\left(\frac{x^2-9}{x^2-5x+6}\right) = \sin\left(\lim_{x\to 3}\frac{x^2-9}{x^2-5x+6}\right) = \cdots = \sin 6.$$

We will usually opt for the first method—as in Example 3.7.6 above—where possible for reasons stated previously.[37]

Trigonometric functions can also give rise to infinite-valued limits. In such cases it is crucial to determine from within which quadrant the argument of the function is approaching the limit point, and thus the sign of the trigonometric functions. If the argument is x, then $x \to 0^+$ means the "angle" x approaches zero from within the first quadrant (where x is the measure of an angle in standard position). For $x \to \pi^+$, we have x approaching π from angles with measure slightly greater than π, i.e., from the third quadrant. For $x \to \frac{\pi}{2}^+$, we are in the second quadrant, and so on. See Figure 3.23, page 248 where the part of "x" is played by the angle θ.

Example 3.7.7 *Consider the following trigonometric limits:*

1. $\lim_{x\to 0^+}\csc x = \lim_{x\to 0^+}\frac{1}{\sin x} \overset{1/0^+}{=\!=\!=} \infty,$

2. $\lim_{x\to \pi^+}\csc x = \lim_{x\to \pi^+}\frac{1}{\sin x} \overset{1/0^-}{=\!=\!=} -\infty,$

[37] Many textbooks prescribe exactly this second method (not preferred here) for such a problem. Instead we solved it by replacing the original function with one which was continuous at $x = 3$, and were thus able to call upon Theorem 3.4.3, page 212. We will require this alternative kind of manipulation later, particularly with limits "at infinity," but we will otherwise usually avoid it when possible because it requires very specific hypotheses. It is acceptable here since we know that if the limit inside exists and is finite, then the sine function is continuous there (since it is continuous everywhere!). We need more care if the outer function is not continuous on all of $\mathbb{R}$.

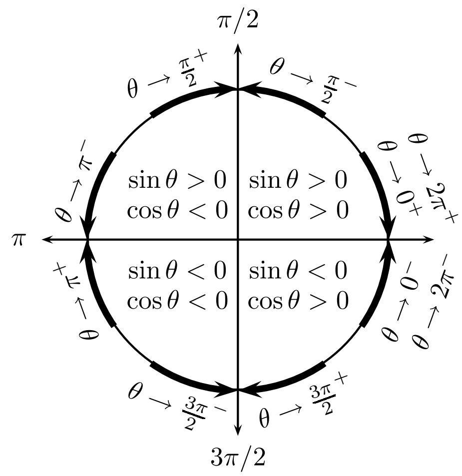


Figure 3.23: Unit circle graph showing the signs of $\sin\theta$ and $\cos\theta$ as θ approaches various axial from the left and right. For instance, as $x \to \frac{\pi}{2}^+$, we have $\cos x \longrightarrow 0^-$, since in such a case x is approaching the angle $\pi/2$ from angles within the second quadrant, in which the cosine is negative. As a result, $\sec x \longrightarrow -\infty$ as $x \to \frac{\pi}{2}^+$.

3. $\displaystyle\lim_{x\to\frac{\pi}{2}^-} \tan x = \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} \overset{1/0^+}{=\!=} +\infty,$

4. $\displaystyle\lim_{x\to\frac{\pi}{2}^+} \tan x = \lim_{x\to\frac{\pi}{2}^+} \frac{\sin x}{\cos x} \overset{1/0^-}{=\!=} -\infty.$

5. $\displaystyle\lim_{x\to\frac{\pi}{2}} \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} \overset{1/0^\pm}{=\!=}$ *does not exist.*

3.7.6 Proofs

First we prove Theorem 3.7.4, page 244, which states that if $g(x) \longrightarrow L$, and $f(x)$ is continuous at $x = L$, then $f(g(x)) \longrightarrow f(L)$. Here we will prove the basic case where the limiting behavior is as $x \to a \in \mathbb{R}$ (a not infinite).

<u>Proof</u>: Here we will prove the basic case where $a \in \mathbb{R}$ (not infinite):

$$\left(\lim_{x\to a} g(x) = L\right) \wedge (f(x) \text{ continuous at } x = L) \implies \left(\lim_{x\to a} f(g(x)) = f(L)\right).$$

To show this, we have to show that for any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(g(x)) - f(L)| < \varepsilon.$$

By our continuity and limit assumptions, we know that

$$(\forall\varepsilon_1 > 0)(\exists\delta_1 > 0)(\forall x)(|x - L| < \delta_1 \longrightarrow |f(x) - f(L)| < \varepsilon_1), \tag{3.54}$$
$$(\forall\varepsilon_2 > 0)(\exists\delta_2 > 0)(\forall x)(0 < |x - a| < \delta_2 \longrightarrow |g(x) - L| < \varepsilon_2). \tag{3.55}$$

So for this ε, choose $\varepsilon_1 = \varepsilon$, which gives a $\delta_1 > 0$ so that

$$|x - L| < \delta_1 \implies |f(x) - f(L)| < \varepsilon.$$

Next set $\varepsilon_2 = \delta_1 > 0$. This gives a $\delta_2 > 0$ so that

$$0 < |x - a| < \delta_2 \implies |g(x) - g(a)| < \varepsilon_2 = \delta_1.$$

Finally, let $\delta = \delta_2$, corresponding to ε_2 in the limit requirement for $g(x) \longrightarrow L$. This gives (with the part of "x" in (3.54) played by $g(x)$ in the third and fourth lines below):

$$\begin{aligned} 0 < |x - a| < \delta &\iff 0 < |x - a| < \delta_2 \\ &\implies |g(x) - L| < \varepsilon_2 \\ &\iff |g(x) - L| < \delta_1 \\ &\implies |f(g(x)) - f(L)| < \varepsilon_1 = \varepsilon, \qquad \text{q.e.d.} \end{aligned}$$

Next we prove that the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are continuous wherever they are defined.

Proof: Our "proof" will be in four parts, and will be cut somewhat shorter than a proof from "first principles" would be by using an observation about the geometry of the unit circle. In this abbreviated proof we will see the sandwich theorem in action, in particular as applied to a useful inequality, (3.56), which will be our "observation."

The order in which we will prove our results is as follows: 1. continuity of $\sin x$ at $x = 0$, implying 2. continuity of $\cos x$ at $x = 0$, together implying 3. continuity of $\sin x$ and $\cos x$ at every $x \in \mathbb{R}$, which implies 4. continuity of the other trigonometric functions wherever they are defined.

1. $\sin x$ is continuous at $x = 0$.

   Consider the unit circle graphed in Figure 3.24. Now $|\sin x|$ is the distance from the horizontal axis to a point P on the terminal side of the angle. The arc is another, but non-straight path of length $|x|$ from the horizontal axis to P. Thus

   $$|\sin x| \le |x|, \tag{3.56}$$

   which is the same as $-|x| \le \sin x \le |x|$. Letting $x \to 0$, we get the following:

   $$\underbrace{-|x|}_{\displaystyle\downarrow \atop \displaystyle 0} \quad \le \quad \sin x \quad \le \quad \underbrace{|x|}_{\displaystyle\downarrow \atop \displaystyle 0}$$

   The Sandwich Theorem then gives us $\lim\limits_{x\to 0} \sin x = 0$. Since $\sin 0 = 0$ as well, we have $\sin x$ is continuous at $x = 0$, q.e.d.[38]

[38] Recall that $f(x)$ is continuous at $x = a$ if and only if $\lim\limits_{x\to a} f(x) = f(a)$. See Theorem 3.4.2, page 211.

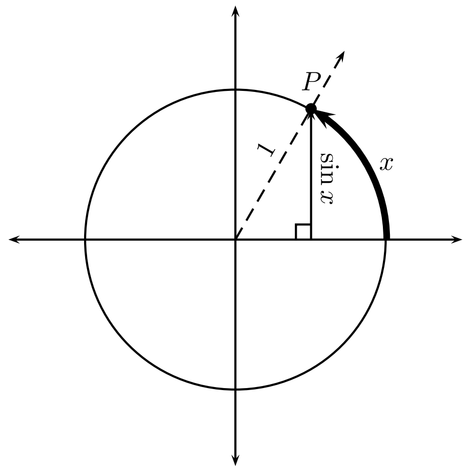


Figure 3.24: Unit circle graph showing the relative sizes of $\sin x$ and x, where x is the angle measure in radians, i.e., the directed length of the arc. More generally, the distance from the horizontal axis to P on the terminal side is $|\sin x|$, and the arc length distance from $(1, 0)$ to P is given by $|x|$.

2. $\cos x$ is continuous at $x = 0$. This follows immediately, since near $x = 0$ (so the "angle" x terminates in the first or fourth quadrants) we have $\cos x > 0$ and thus (again, near $x = 0$) $\cos x = \sqrt{1 - \sin^2 x}$, and so we can replace $\cos x$ with that expression (according to Theorem 3.4.3):

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{1-\sin^2 x} = \sqrt{1-\sin^2 0} = \sqrt{1} = 1 = \cos 0, \text{ q.e.d.}$$

We will take a moment here to explain why we could compute the above limit as we did. Because $\sin x$ is continuous at $x = 0$, so is $1 - \sin^2 x$, and since that function approaches $1 > 0$ as $x \to 0$, its square root is also continuous at $x = 0$.

3. $\sin x$ and $\cos x$ are continuous for all $x \in \mathbb{R}$. These follow from the two results above and the trigonometric identities (**??**) and (**??**) as below:

$$\begin{aligned}\lim_{x\to a} \sin x &= \lim_{x\to a} \sin(a + (x - a)) \\ &= \lim_{x\to a} (\sin a \cos(\underbrace{x-a}_{\substack{\downarrow \\ 0}}) + \cos a \sin(\underbrace{x-a}_{\substack{\downarrow \\ 0}})) = \sin a \cos 0 + \cos a \sin 0 \\ &= (\sin a)(1) + (\cos a)(0) = \sin a,\end{aligned}$$

$$\begin{aligned}\lim_{x\to a} \cos x &= \lim_{x\to a} \cos(a + (x - a)) \\ &= \lim_{x\to a} (\cos a \cos(\underbrace{x-a}_{\substack{\downarrow \\ 0}}) - \sin a \sin(\underbrace{x-a}_{\substack{\downarrow \\ 0}})) = \cos a \cos 0 - \sin a \sin 0 \\ &= (\cos a)(1) - (\sin a)(0) = \cos a, \text{ q.e.d.}\end{aligned}$$

Here we used what we will later call a *substitution argument*, which will be introduced properly in Section 3.9 (though we could also invoke Theorem 3.7.4, page 244). The idea is, roughly, that $x \to a \iff x - a \to 0$ in the sense of limit (where x is never actually equal to a, and $x - a$ is never equal to zero).

4. All six trigonometric functions are continuous where they are defined.

   Of course $\sin x$ and $\cos x$ were already shown continuous for all $x \in \mathbb{R}$, i.e., where defined, earlier. The other functions are defined by quotients where the numerators are either $\sin x$, $\cos x$ or 1, which are continuous everywhere, while the denominators are either $\sin x$ or $\cos x$, again continuous everywhere. Since a ratio of two functions is continuous if both numerator and denominator are continuous and the denominator is nonzero, the functions $\tan x$ and $\sec x$ are continuous except where $\cos x = 0$, and $\cot x$ and $\csc x$ are continuous except where $\sin x = 0$. Summarizing, all trigonometric functions are continuous where defined, q.e.d.

Exercises

1. Compute $\displaystyle\lim_{x\to 0^+}\left[\sqrt{x}\sin\left(\frac{1}{x}\right)\right]$.

2. Compute $\displaystyle\lim_{x\to 1^+}\frac{\sin x}{x-1}$. (Hint: $\sin 1 \approx 0.841470985$.)

3. Compute using a Sandwich Theorem-type argument $\displaystyle\lim_{x\to 5^+}\frac{x+\cos x}{x^2-25}$.

4. Compute $\displaystyle\lim_{x\to 2}\cos\left(\frac{x^2-4x+4}{x^2-4}\right)$.

5. Compute the following limits.
   (a) $\displaystyle\lim_{x\to\frac{\pi}{2}^+}\sec x$.
   (b) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}\sec x$.
   (c) $\displaystyle\lim_{x\to\frac{3\pi}{2}^+}\sec x$.
   (d) $\displaystyle\lim_{x\to\frac{3\pi}{2}^-}\sec x$.

6. Compute the following limits.
   (a) $\displaystyle\lim_{x\to 0^+}\cot x$.
   (b) $\displaystyle\lim_{x\to 0^-}\cot x$.
   (c) $\displaystyle\lim_{x\to \pi^+}\cot x$.
   (d) $\displaystyle\lim_{x\to \pi^-}\cot x$.

7. Compute $\displaystyle\lim_{x\to 0^+}\left[\sqrt{x}\sin(\csc x)\right]$.

8. Suppose that $f(x)\le h(x)$ for $0<|x-a|<d$, for some $d>0$, and that $\displaystyle\lim_{x\to a}h(x)=\infty$. By drawing several graphs, show that $\displaystyle\lim_{x\to a}f(x)$ can be anything: finite, ∞, $-\infty$, or nonexistent.

9. Suppose that $-x^3+2x^2-x+2<f(x)<x^2-2x+3$ for all $x\in[0,2]$, except for $x\neq 1$. Find $\displaystyle\lim_{x\to 1}f(x)$ if possible.

10. Suppose for all $x\neq 2$ we have $4\le f(x)\le (x-2)^2+4$. Find $\displaystyle\lim_{x\to 2}f(x)$ if possible.

For the following, compute each limit which exists, state which do not, and if you use the Sandwich Theorem to prove one exists, show all details.

11. $\displaystyle\lim_{x\to 0}\sqrt[3]{x\sin\frac{1}{x}}$

12. $\displaystyle\lim_{x\to 0}\sqrt{x\sin\frac{1}{x}}$

13. $\displaystyle\lim_{x\to 0}\sqrt{\left|x\sin\frac{1}{x}\right|}$

14. $\displaystyle\lim_{x\to 0}\sqrt{x^2\sin^2\frac{1}{x}}$

15. $\displaystyle\lim_{x\to 0^+}\left[\sqrt{x}\sin(\csc x)\right]$

16. $\displaystyle\lim_{x\to 0^+}\left[\sqrt{x}\sin\left(\csc\left(\frac{1}{x}\right)\right)\right]$

17. $\displaystyle\lim_{x\to 0}x^2\cos\frac{1}{\sqrt[3]{x}}$

18. $\displaystyle\lim_{x\to 0}\sqrt[3]{x}\sin\left(\frac{1}{x}\right)$

19. $\displaystyle\lim_{x\to 0}\frac{\cos\frac{1}{x}}{x^2}$

20. $\displaystyle\lim_{x\to 0}\sin x\csc x$

21. $\displaystyle\lim_{x\to 0}\cot x\csc x$

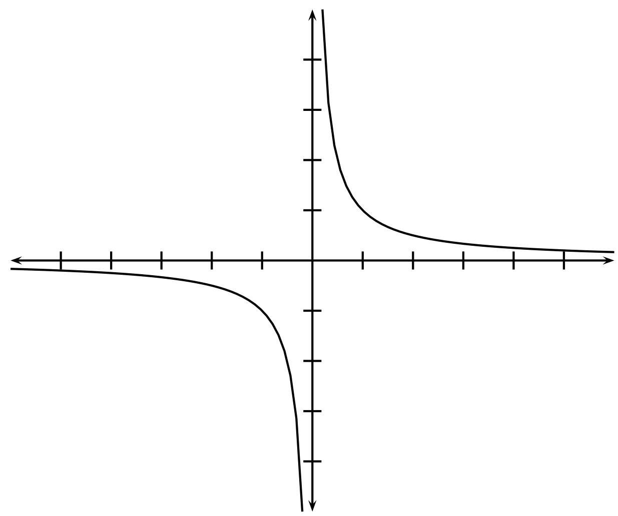


Figure 3.25: Partial graph of $f(x) = 1/x$. We see here that $f(x) \longrightarrow 0$ as $x \to \infty$ and as $x \to -\infty$.

3.8 Limits "At Infinity"

The limits we introduce here differ from previous limits in that here we are interested in the behavior of functions $f(x)$ as x grows without bound, rather than as x approaches a finite point. There are new "forms" we will come across here, such as $1/\infty$, $\infty \cdot \infty$, ∞/∞, $0 \cdot \infty$ and $\infty - \infty$. (Only the first two *determinate.*)

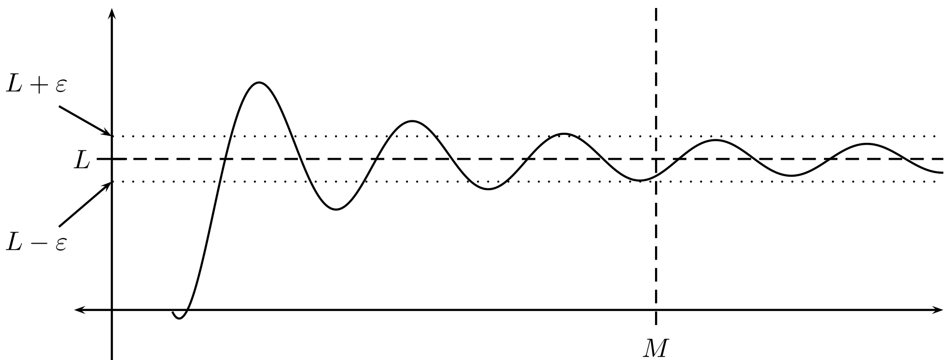
The first forms we will look at are $1/\infty$ and $1/(-\infty)$. For these we look again to the function $f(x) = 1/x$. Due to the importance of this function we produce it here for the third time, in Figure 3.25. We see that as x moves to the right through values like $x = 1$, 2, 3, 10, 100, 1000, 10^6 and so on, the function takes on respective values $f(x) = 1$, 1/2, 1/3, 1/10, 1/100, 1/1000, 10^{-6} and so on. So as x grows without bound, the function's output shrinks towards (though is never equal to, for this case) zero. A similar phenomenon occurs when we take x-values $x = -1, -2, -3, -10, -100, -1000, -10^6$, etc., except the values of $f(x)$ are then $f(x) = -1, -1/2, -1/3, -1/10, -1/100, -1/1000, -10^{-6}$ etc. So as x moves left without bound, the function values are negative numbers shrinking in absolute size. The fact that in both cases we can get as close to zero in the values of $f(x)$ as we could like (without necessarily *achieving* the value zero) by choosing x large enough is reflected in the statements

$$\lim_{x\to\infty} \frac{1}{x} \overset{\frac{1}{\infty}}{=\!=} 0, \tag{3.57}$$

$$\lim_{x\to-\infty} \frac{1}{x} \overset{\frac{1}{(-\infty)}}{=\!=\!=} 0. \tag{3.58}$$

The forms $1/\infty$ and $1/(-\infty)$ are determinate, both yielding zero limits. Recall that a growing denominator will produce a shrinking fraction.[39] Furthermore reciprocals of large numbers give small numbers. From earlier discussions of the graph of $y = 1/x$ we can see how, as x gets arbitrarily large, $1/x$ gets arbitrarily small (though never quite zero) in absolute size.

[39] Unless there is another effect to counteract the growing denominator, such as a growing numerator. We will soon see that ∞/∞ is indeterminate.



$(\forall\varepsilon>0)(\exists M\in\mathbb{R})(\forall x\in\mathbb{R})(x>M\longrightarrow|f(x)-L|<\varepsilon)$

Figure 3.26: Illustration of the definition of a finite limit L of a function as $x\to\infty$.

It is common to read the left-hand side of (3.57) as, "the limit, as x approaches infinity, of $1/x$." Of course x does not "get close" to ∞, but the notation means that we are computing what the behavior of $1/x$ will be as x grows positive without bound. Similarly for $x\to-\infty$. To make these precise, we give the following definitions.

Definition 3.8.1 *For a finite number $L\in\mathbb{R}$, we say*

$$\lim_{x\to\infty} f(x)=L \qquad\Longleftrightarrow\qquad (\forall\varepsilon>0)(\exists M\in\mathbb{R})(\forall x\in\mathbb{R})(x>M\longrightarrow|f(x)-L|<\varepsilon), \tag{3.59}$$

$$\lim_{x\to-\infty} f(x)=L \qquad\Longleftrightarrow\qquad (\forall\varepsilon>0)(\exists N\in\mathbb{R})(\forall x\in\mathbb{R})(x<N\longrightarrow|f(x)-L|<\varepsilon). \tag{3.60}$$

In (3.59), we could also write $f((M,\infty))\subseteq(L-\varepsilon,L+\varepsilon)$, while in (3.60), we could write $f((-\infty,N))\subseteq(L-\varepsilon,L+\varepsilon)$. A case of (3.59) for a particular ε is illustrated in Figure 3.26. We will leave the illustrations of (3.60) to the reader.

Next we point out that it is natural to have a notion of an *infinite* limit as $x\to\infty$ or $x\to-\infty$. For instance,

$$\lim_{x\to\infty} x=\infty \tag{3.61}$$

seems quite reasonable, as does

$$\lim_{x\to-\infty} x^2=\infty. \tag{3.62}$$

There are many common functions which grow without bound as x grows without bound. Note that (3.62) can be thought of as a form $(-\infty)\cdot(-\infty)$ or $(-\infty)^2$, which reasonably yields the limit ∞.[40] On the other hand,

$$\lim_{x\to-\infty} x^3=-\infty, \tag{3.63}$$

since $x^3<0$ and x^3 grows without bound as x grows larger, without bound but negative. We could think of the above limit as a form $(-\infty)^3$, giving the limit as $-\infty$ as we should expect. In general, all positive powers of x will grow to $+\infty$ as $x\to\infty$, while even powers will grow to $+\infty$ as $x\to-\infty$ and odd powers will grow to $-\infty$ as $x\to-\infty$.[41] Constant factors behave as

[40] Of course $(-\infty)\cdot(-\infty)$ is a particular *form* representing a product of two functions which are both negative and growing without bound. The product is naturally positive and also growing without bound, the resulting limit then being ∞.

[41] Noninteger powers of x are more complicated for $x\to-\infty$. Some approach $+\infty$, some $-\infty$ and some are undefined as $x\to-\infty$. Such things will be discussed as they come up in the text.

before (see (3.44) and (3.45), page 233), as in

$$\lim_{x\to\infty} 5x \stackrel{5\cdot\infty}{=\!=} \infty, \qquad\qquad \lim_{x\to-\infty}(-3x) \stackrel{-3\cdot(-\infty)}{=\!=\!=\!=} \infty.$$

The definition of $\lim\limits_{x\to\infty} f(x) = \infty$ is given below:

Definition 3.8.2 *We make the following definition:*

$$\lim_{x\to\infty} f(x) = \infty \iff (\forall M)(\exists N)(\forall x)(x > N \longrightarrow f(x) > M). \tag{3.64}$$

In other words, for any fixed M, we can force $f(x)$ to be greater than M by taking $x > N$, so that $f((N,\infty)) \subseteq (M,\infty)$. The definition of $f(x) \longrightarrow -\infty$ as $x \to \infty$, and similar definitions, are left as exercises. To see (3.64) in action, consider proving $\lim\limits_{x\to\infty} x^2 = \infty$. For any M, we can take $N = \sqrt{|M|} \geq 0$ to get

$$x > N = \sqrt{|M|} \implies f(x) = x^2 > N^2 = \left(\sqrt{|M|}\right)^2 = |M| \geq M.$$

We needed $N \geq 0$ so that $x > N \implies x^2 > N^2$.

Some relevant limit forms which occur in this and other contexts, and which are not indeterminate include the following:

1. $\infty + a = \infty$ for any fixed $a \in \mathbb{R}$,
2. $\infty + \infty = \infty$,
3. $a \cdot \infty = \infty$ if $a > 0$, but $a \cdot \infty = -\infty$ if $a < 0$.

As before, we can perform *some* "arithmetic" of limit forms, though we always have to be careful (see Example 3.8.1 below).

The cases mentioned in 3. above was also mentioned in the previous sections, first on page 232. Note again that $a\cdot\infty$ as a limit form means that we have a limit where one function is approaching a, and the other ∞ (postive and growing without bound in the limit), and so their product approaches ∞ if $a > 0$, and $-\infty$ if $a < 0$. If $a = 0$ the form is indeterminate, and we have to attempt to rewrite it algebraically to see if it can be written in a determinate form.

The above forms are relatively intuitive. The following are more subtle, and in fact indeterminate:

$$\infty - \infty, \qquad 0\cdot\infty, \qquad \infty/\infty, \qquad 0/0.$$

To see the first is indeterminate, consider for instance the following limits of the form $\infty - \infty$:[42]

Example 3.8.1

$$\lim_{x\to\infty}\left[(x^2+1)-(x^2)\right] \underset{\text{ALG}}{\stackrel{\infty-\infty}{=\!=\!=}} \lim_{x\to\infty}(1) = 1,$$
$$\lim_{x\to\infty}\left(x^3-x^2\right) \underset{\text{ALG}}{\stackrel{\infty-\infty}{=\!=\!=}} \lim_{x\to\infty} x^2(x-1) \stackrel{\infty\cdot\infty}{=\!=\!=} \infty,$$
$$\lim_{x\to\infty}\left(x^4-x^6\right) \underset{\text{ALG}}{\stackrel{\infty-\infty}{=\!=\!=}} \lim_{x\to\infty} x^4(1-x^2) \stackrel{\infty\cdot(-\infty)}{=\!=\!=} -\infty.$$

[42]The first limit shows that it is possible to come up with a limit of the form $\infty - \infty$ which when evaluated gives any predetermined real value we would like (just replace the number 1 with the desired value). The second and third show we can, furthermore, find limits of form $\infty - \infty$ which return infinite limits as well.

The question for $\infty - \infty$ form becomes, which "infinity" is larger, i.e., which function grows faster when we have a difference $f(x) - g(x)$ of functions f and g which both grow without bound? Or is there ultimately a compromise? Similar examples can be found for forms $0 \cdot \infty$ (or $\infty \cdot 0$) and ∞/∞. The former we look at next, with a few examples to show that it is in fact indeterminate.

Example 3.8.2 *Consider the following limits:*

$$\begin{aligned}
\lim_{x\to\infty}\left[x\cdot\frac{1}{x^2}\right] &\overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty}\frac{1}{x} = 0,\\
\lim_{x\to\infty}\left[x^2\cdot\frac{1}{x}\right] &\overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty} x = \infty,\\
\lim_{x\to\infty}\left[x\cdot\frac{5}{x}\right] &\overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=\!=}} \lim_{x\to\infty} 5 = 5.
\end{aligned}$$

Now let us turn to polynomial and rational functions. Our first theorem is the following:

Theorem 3.8.1 *For a polynomial function $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_n \neq 0$ (so the polynomial really is of degree n), we have*

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_nx^n, \tag{3.65}$$

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_nx^n. \tag{3.66}$$

In other words, for $x \to \infty$ and $x \to -\infty$, a polynomial function's growth is ultimately dictated by its leading (highest degree) term. Rather than prove this in general, we can see the essence of a proof in the following examples and leave the actual proof as an exercise.

Example 3.8.3 *Consider the following limits. (Forms are first given above the "=," and then simplified below.)*

$$\begin{aligned}
\lim_{x\to\infty}\left(3x^2 - 5x + 11\right) &= \lim_{x\to\infty} x^2\left(3 - \frac{5}{x} + \frac{11}{x^2}\right) \overset{\infty\cdot(3-0+0)}{\underset{\infty\cdot 3}{=\!=\!=\!=}} \infty,\\
\lim_{x\to-\infty}\left(x^3 + 95x^2 - 15x + 1000\right) &= \lim_{x\to-\infty} x^3\left(1 + \frac{95}{x} - \frac{15}{x^2} + \frac{1000}{x^3}\right) \overset{-\infty(1+0-0)}{\underset{-\infty\cdot 1}{=\!=\!=\!=}} -\infty.
\end{aligned}$$

When we factor out the highest power, the lower-order terms we are left with have negative powers of x which then shrink to zero, leaving only the coefficient of the highest-order term as a factor in the limit. This phenomenon is very useful when we look at rational limits as $x \to \pm\infty$, which are often of the form ∞/∞, $(-\infty)/(-\infty)$ and so on.

Example 3.8.4 *Consider the following limits.*

$$\lim_{x\to\infty}\frac{3x^2+5x-9}{6x+11} \overset{\infty/\infty}{=\!=} \lim_{x\to\infty}\frac{x^2\left(3+\frac{5}{x}-\frac{9}{x^2}\right)}{x\left(6+\frac{11}{x}\right)} \underset{\text{ALG}}{\overset{\infty/\infty}{=\!=}} \lim_{x\to\infty} x\cdot\frac{3+\frac{5}{x}-\frac{9}{x^2}}{6+\frac{11}{x}} \overset{\infty\cdot\frac{3}{6}}{=\!=} \infty,$$

$$\lim_{x\to\infty}\frac{9x^2+2x+1}{16x^2+3x-100} \overset{\infty/\infty}{=\!=} \lim_{x\to\infty}\frac{x^2\left(9+\frac{2}{x}+\frac{1}{x^2}\right)}{x^2\left(16+\frac{3}{x}-\frac{100}{x^2}\right)} \underset{\text{ALG}}{\overset{\infty/\infty}{=\!=}} \lim_{x\to\infty}\frac{9+\frac{2}{x}+\frac{1}{x^2}}{16+\frac{3}{x}-\frac{100}{x^2}}$$
$$=\frac{9+0+0}{16+0-0}=\frac{9}{16},$$

$$\lim_{x\to-\infty}\frac{5-3x}{2x^2+x+1} \overset{\infty/\infty}{=\!=} \lim_{x\to-\infty}\frac{x\cdot\left(\frac{5}{x}-3\right)}{x^2\left(2+\frac{1}{x}+\frac{1}{x^2}\right)} \underset{\text{ALG}}{\overset{\infty/\infty}{=\!=}} \lim_{x\to-\infty}\left[\frac{1}{x}\cdot\frac{\frac{5}{x}-3}{2+\frac{1}{x}+\frac{1}{x^2}}\right]$$
$$\overset{\frac{1}{-\infty}\cdot\frac{-3}{2}}{=\!=} 0\cdot\frac{-3}{2}=0.$$

A quick corollary—which we must be careful not to abuse—to our theorem is the following:

Theorem 3.8.2 *For any rational function $f(x)=\frac{p(x)}{q(x)}$, where $p(x)=a_nx^n+\cdots+a_1x+a_0$ and $q(x)=b_mx^m+\cdots+b_1x+b_0$, with $a_n,b_m\neq 0$, we have*

$$\lim_{x\to\infty}\frac{p(x)}{q(x)}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}, \tag{3.67}$$
$$\lim_{x\to-\infty}\frac{p(x)}{q(x)}=\lim_{x\to-\infty}\frac{a_nx^n}{b_mx^m}. \tag{3.68}$$

So as we take $x\to\infty$ or $x\to-\infty$, the limiting behavior of a rational function is governed by the leading terms of the numerator and denominator.[43] We will use this theorem for anticipating results, but will work the actual limits as in Example 3.8.4.[44]

It is common for trigonometric limits, and variations of the Sandwich Theorem (originally Theorem 3.7.1, page 239) to appear with limits "at infinity."

Example 3.8.5 *Consider the limit $\lim_{x\to\infty}\frac{\cos x}{x}$. This yields to the Sandwich Theorem quickly:*

[43] Note that the "leading term" means the nonzero term of the highest degree, not necessarily the first term appearing. For instance, in the polynomial $6-5x^2$, the leading term is $-5x^2$.

[44] We will continue to compute the limits longhand for three reasons. First it is good reinforcement of the underlying principles. Second, it is not entirely standard to write, for instance,

$$\lim_{x\to\infty}\frac{5x^2+3x-11}{7x^2-9x+1,000}=\lim_{x\to\infty}\frac{5x^2}{7x^2}=\lim_{x\to\infty}\frac{5}{7}=5/7.$$

A reader might be confused about the whereabouts of the terms that were dropped, and generally lose confidence that the writer's understanding is correct. Finally, the theorem requires that $x\to\infty$ or $x\to-\infty$, so if we reflexively drop terms we may be tempted to do so for a limit at a finite point, where the theorem does not hold. That said, it is not uncommon for a trained mathematician to simply drop all steps above and write

$$\lim_{x\to\infty}\frac{5x^2+3x-11}{7x^2-9x+1,000}=5/7.$$

$$\text{As } x \to \infty: \quad \underbrace{\frac{-1}{x}}_{\downarrow\, 0} \leq \frac{\cos x}{x} \leq \underbrace{\frac{1}{x}}_{\downarrow\, 0} \qquad \therefore \frac{\cos x}{x} \longrightarrow 0.$$

One would usually then summarize: $\lim_{x\to\infty} \frac{\cos x}{x} = 0.$

One could also look at the previous limit as one of a product of two functions, one which is bounded ($\cos x$), and the other which approaches zero ($1/x$), yielding $B \cdot 0$ form, which is a determinate form giving zero in the limit. Furthermore we could define a form, "B/∞" which will always yield zero since the denominator grows without bound (shrinking the fraction) while the numerator is unable to compensate (by growing the fraction) since it is bounded. We could also write $B/\infty = B \cdot \frac{1}{\infty} = B \cdot 0$. The "algebra" of forms is interesting and intuitive, but one needs to be careful to understand the underlying mechanisms to perform such calculations on forms.

Example 3.8.6 *Consider the limit* $\lim_{x\to\infty}(x + \sin x)$. *Here we have a sum of functions, the first growing without bound and the second being bounded. Intuitively this sum should grow without bound since the function* $\sin x$ *is unable to check the growth of* x. *We can again use the Sandwich Theorem:*

$$\text{As } x \to \infty: \quad \underbrace{x-1}_{\downarrow\, \infty} \leq x + \sin x \leq \underbrace{x+1}_{\downarrow\, \infty} \qquad \therefore (x + \sin x) \longrightarrow \infty.$$

In fact, recall that in such a case we only need the first inequality above to form our conclusion.

We could look at the limit above as an example of a form we could define as "$\infty + B$," which will always give us the actual limit being ∞. To see this, note that for such a case we are looking at sums $f(x)+g(x)$ where $g(x)$ is defined and bounded, i.e., $|g(x)| \leq M$ for some finite fixed M, and $f(x) \longrightarrow \infty$. By the boundedness of $g(x)$, we get

$$f(x) - M \leq f(x) + g(x) \leq f(x) + M.$$

Since $f(x) - M, f(x) + M \longrightarrow \infty$, we would conclude $f(x) + g(x) \longrightarrow \infty$ as well.

It should be pointed out that the limits $\lim_{x\to\infty} \sin x$ and $\lim_{x\to\infty} \cos x$ both do not exist. This is because these functions oscillate between -1 and 1, and do not approach any particular value to the exclusion of others (recall that a limit must be unique). However the limits above show that such functions can still be involved in limits "at infinity," especially when their (bounded) oscillations can be checked by, or absorbed into, the influences of other functions in the limits.

The methods of of above two examples are important and should be mastered, but we can use observations about forms (proved the same ways) and have more abbreviated computations:

$$\lim_{x\to\infty} \frac{\cos x}{x} \stackrel{B/\infty}{=\!=\!=} 0,$$
$$\lim_{x\to\infty} (x + \sin x) \stackrel{\infty+B}{=\!=\!=} \infty.$$

Here B stands for any bounded function, including constants. In both cases, the "B" can not check the growth of the other ("∞") function, and so the other function's influence ultimately prevails in the limit. Note that B/∞ and $\infty+B$ are *determinate* forms. We list these and some others below. Note that the left sides are *forms,* and the right sides are final limit values.

$$B/\infty = 0, \tag{3.69}$$
$$B/(-\infty) = 0, \tag{3.70}$$
$$B+\infty = \infty, \tag{3.71}$$
$$B-\infty = -\infty. \tag{3.72}$$

All these are intuitive and provable using the Sandwich Theorem and its variations. As always it is important that we are aware of technicalities. For instance,

$$\lim_{x\to\infty}\frac{\tan x}{x\sec x}\qquad \text{DNE.}$$

Thought $(\tan x)/(x\sec x) = (\sin x/\cos x)/(x/\cos x) = (\sin x)/x$, this simplification is only valid if $\tan x$, $\sec x$ are defined, i.e., when $\cos x\neq 0$. But there are infinitely many times $\cos x = 0$ as $x\to\infty$, and for that matter within any interval (M,∞), so none of our definitions for limits as $x\to\infty$ can hold true. This is despite the fact that if we (naively) simplify the function within the limit we would get $\lim_{x\to\infty}\frac{\sin x}{x} = 0$. We cannot say the same about the original limit because the function is undefined infinitely many times as $x\to\infty$, so its limit can not exist.

Also, knowing we have a bounded function combined with one which blows up does not always tell us the limit (unless it is one of the forms above). For instance we can not really say anything about $B\cdot\infty$ without more information. If the first function is $f(x)$ defined by $(\forall x)(f(x)=0)$, then we have a zero limit. If $f(x)\to 0$, we definitely need more information.[45] If we instead know $1\leq f(x)\leq 2$, and $g(x)\longrightarrow\infty$ we have $g(x)\leq f(x)g(x)\leq 2g(x)$, and $g(x)\to\infty, 2g(x)\longrightarrow\infty$ so we can say in this case that $f(x)g(x)\longrightarrow\infty$.[46] The upshot of all this is the fact that we sometimes *do* need to refer back to the Sandwich Theorem-type computations for these, unless the form gives us an obvious answer.

The continuity of the trigonometric functions (where they are defined) can also come into play with these limits, for instance in light of Theorem 3.7.4, page 244 on the compositions of functions, namely $(f(x)$ continuous at $x=L)\wedge(g(x)\to L)\implies f(g(x))\to f(L)$:

Example 3.8.7 *Consider* $\lim_{x\to\infty}\sec\left(\dfrac{x}{x^2+1}\right)$. *From what we know of rational functions, the input of the secant function here is approaching zero. Since the secant (=1/cosine) is continuous at zero, our answer should be* $\sec 0 = 1/\cos 0 = 1/1 = 1$. *For a more computational argument we might write*

$$\lim_{x\to\infty}\sec\left(\frac{x}{x^2+1}\right)\overset{\sec(\infty/\infty)}{\underset{\text{ALG}}{=\!=\!=}}\lim_{x\to\infty}\sec\left(\frac{x(1)}{x\left(x+\frac{1}{x}\right)}\right)=\lim_{x\to\infty}\sec\left(\frac{1}{x+\frac{1}{x}}\right)\overset{\sec\left(\frac{1}{\infty+0}\right)}{=\!=\!=}\sec 0 = 1.$$

(We used the fact that $1/x\longrightarrow 0$ *as* $x\to\infty$ *in the denominator of the input of the secant function.) Again, the last step utilized the fact that* $\sec x$ *is continuous at* $x=0$.

---

[45]For example, consider $f(x)=K/x$, $g(x)=x$ and $x\to\infty$. This gives a limit of K, i.e., $f(x)g(x)=(K/x)\cdot x = K\longrightarrow K$, and we can choose K to be anything real number we like.

[46]For example, this is exactly what occurs with the limit $\lim_{x\to\infty}\left[\left(\dfrac{3+\sin x}{2}\right)\cdot x\right]=\infty$. This is because for $x>0$,

$$\left(\frac{3-1}{2}\right)\cdot x\leq\left(\frac{3+\sin x}{2}\right)\cdot x\leq\left(\frac{3+1}{2}\right)\cdot x,$$

and so the function is sandwiched between $x\to\infty$ and $2x\to\infty$.

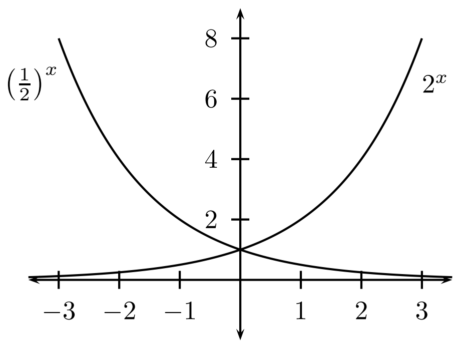


Figure 3.27a.

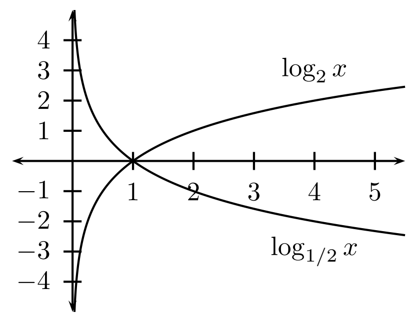


Figure 3.27b.

Figure 3.27: Partial graphs of exponential functions 2^x and $(1/2)^x$, along with logarithmic fuctions $\log_2 x$ and $\log_{1/2} x$, showing continuity and limiting behaviors.

For our next example we will return to a form $\infty - \infty$. When problematic terms do not cancel from subtraction, a rewriting of the expression as a quotient will often achieve some useful cancellation or a determinate form. (This theme will return several times in the text.) In the case below, we use a conjugate multiplication step.

Example 3.8.8 *Consider the limit* $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)$. *Clearly* $x^2+x+1 \longrightarrow \infty$, *and so we are taking square roots of numbers as large as we like. In fact, since* $x>0$ *we can write* $\sqrt{x^2+x+1} > \sqrt{x^2} = |x| = x \longrightarrow \infty$ *as* $x\to\infty$.[47]

We solve this using the following method:

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right) &\overset{\infty-\infty}{\underset{ALG}{=\!=}} \lim_{x\to\infty}\left[\left(\sqrt{x^2+x+1}-x\right)\cdot\frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x}\right]\\
&= \lim_{x\to\infty}\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \lim_{x\to\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}\\
&\overset{\infty/\infty}{\underset{ALG}{=\!=}} \lim_{x\to\infty}\frac{x\left(1+\frac{1}{x}\right)}{x\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right]}\\
&= \lim_{x\to\infty}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{1+0}{\sqrt{1+0+0}+1} = 1/2.
\end{aligned}$$

The limit above is correct, but probably not at all obvious from the original form. Only after finding a useful fractional form could we use our earlier techniques to compute its value. Limits which are writable as ratios are often easier to solve than other forms. Here it allowed us to compare the powers of x in the numerator and denominator. In our first limit section we had many $0/0$ forms which we could easily simplify to get determinate forms.

There are applications, both conceptual and practical, for limits as the input variable "blows up." Many interesting applications involve exponential functions $f(x)=a^x$, or their variants

[47] It is also valid to define a limit form $\sqrt{\infty}$, which will return limits of ∞. (See Exercise 22, which gives some idea of a proof for this fact.)

such as $f(x) = C \cdot a^{kx}$. These are continuous for $x \in \mathbb{R}$, and their limits as $x \to \infty$ or $x \to -\infty$ are as follow:

$$a > 1: \qquad x \to \infty \implies a^x \to \infty, \tag{3.73}$$
$$x \to -\infty \implies a^x \to 0^+, \tag{3.74}$$
$$a \in (0,1) \qquad x \to \infty \implies a^x \to 0^+, \tag{3.75}$$
$$x \to -\infty \implies a^x \to \infty. \tag{3.76}$$

See Figure 3.27a.

This gives rise to limit forms, so for examples (recalling that $e \approx 2.71828 > 1$), we can write

$$\lim_{x\to\infty} 2^x \overset{2^\infty}{=\!=} \infty, \qquad \lim_{x\to\frac{\pi}{2}^-} e^{\tan x} \overset{e^\infty}{=\!=} \infty,$$

$$\lim_{x\to\infty} 1.5^{-x} \overset{1.5^{-\infty}}{=\!=} -\infty, \qquad \lim_{x\to\frac{\pi}{2}^+} e^{\tan x} \overset{e^{-\infty}}{=\!=} 0,$$

$$\lim_{x\to\infty} \frac{e^x}{e^{-x}+1} \overset{\frac{e^\infty}{e^{-\infty}+1}}{\underset{\infty/(0+1)}{=\!=}} \infty, \qquad \lim_{x\to\infty} \frac{2^{x+1}}{3^x} \overset{\infty/\infty}{\underset{\text{ALG}}{=\!=}} \lim_{x\to\infty} \frac{2^x \cdot 2}{3^x}$$

$$= \lim_{x\to\infty} 2 \cdot \left(\frac{2}{3}\right)^x \overset{2\cdot\left(\frac{2}{3}\right)^\infty}{=\!=} 0.$$

Related to the behaviors of the exponential functions are those of the logarithmic functions. Recall

$$\log_a x = y \iff a^y = x,$$

so when looking at $y = \log_a x$ is the same as looking at $x = a^y$, or $y = a^x$ but with x and y trading roles. We can see from the graphs in Figure 3.27b that

$$a > 1: \qquad x \to \infty \implies \log_a x \to \infty, \tag{3.77}$$
$$x \to 0^+ \implies \log_a x \to -\infty, \tag{3.78}$$
$$a \in (0,1) \qquad x \to \infty \implies \log_a x \to -\infty, \tag{3.79}$$
$$x \to 0^+ \implies \log_a x \to \infty. \tag{3.80}$$

These are the logarithmic analogs of (3.73)–(3.76). In fact it is not immediately clear from the figure that $\log_2 x \to \infty$ as $x \to \infty$, but we can go back to our definition in (3.64), page 255, and so for $M > 0$ we can take $N = 2^M$, and get $x > N = 2^M \implies \log_2 x > \log_2 2^M = M$. So the logarithmic graphs do "blow up" for $(a > 0) \wedge (a \neq 1)$, though they do so very slowly (for instance for $\log_2 x > 10$ we need $x > 2^{10} = 1024$). We will thus get limit forms such as $\log_2(\infty)$ yielding a limit of ∞, $\log_2(0^+)$ yielding $-\infty$, and others. Recall that $\ln x = \log_e x$, with $e \approx 2.71828 > 1$ and so $\ln x$ has a similar shape and asymptotics as $\log_2 x$, which is shown in Figure 3.27b, page 260.

We can now quickly compute some limits involving logarithms:

$$\lim_{x\to 0^+} \ln(\sin x) \overset{\ln 0^+}{=\!=} -\infty, \qquad \lim_{x\to\infty} \ln\left(\frac{x^2+5x-9}{3x^2-8x+27}\right) \overset{\ln\frac{1}{3}}{=\!=} \ln\frac{1}{3}$$

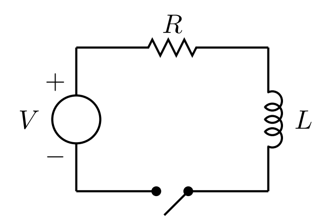
$$\lim_{x\to\infty} \ln(x^2+5x-9) \overset{\ln\infty}{=\!=} \infty, \qquad \lim_{x\to\frac{\pi}{2}^-} \ln(\tan x) \overset{\ln\infty}{=\!=} \infty$$

$$\lim_{x\to\infty} \frac{1}{\ln x} \overset{1/\ln\infty}{\underset{1/\infty}{=\!=}} = 0 \qquad \lim_{x\to 0^+} \sqrt{\ln\frac{1}{x}} \overset{\sqrt{\ln\infty}}{\underset{\sqrt{\infty}}{=\!=}} \infty.$$

Note that $\lim\limits_{x\to\frac{\pi}{2}^+} \ln(\tan x)$ does not exist, because $\tan x < 0$ as $x \to \pi/2^+$, i.e., when x is in the second quadrant; recall that logarithms can only process positive numbers.

Next we consider an application of such limits. Limits as the input variable grows towards $+\infty$ are particularly valuable in the analysis of expected long-term behaviors of different systems. It is interesting because it can describe the state of a system as it seems to mostly "settle down." For many systems it does not take unreasonably long for the state of the system to be near its limit. Put another way, if $x \to \infty \implies f(x) \to L$, then for large enough x we should have $f(x) \approx L$. Thus the limit point L is interesting even though we cannot, in fact, "travel to infinity" in x (or whatever we call the input variable) to experience the limit, but may be able to experience the state of the system where x is large enough that $f(x) \approx L$ satisfactorally.

Example 3.8.9 *In* Section 4.3 *we will consider electrical circuits which contain a resistor and an inductor in series, as seen below right.*



With a circuit having voltage V, a resistor with resistance R, and an inductor with inductance L, and a switch which is first "closed" at $t = 0$, the current flowing through the circuit will be given by the following, for $t \geq 0$:

$$I(t) = \frac{V}{R}\left(1 - e^{-tR/L}\right).$$

(a) What is the current at time $t = 0$?

(b) What are the current values at times $t = \frac{L}{R}, \frac{2L}{R}, \frac{3L}{R}$?

(c) As $t \to \infty$, what value does I approach?

Solution:

(a) The current at $t = 0$ is $I(0) = \frac{V}{R}\left(1 - e^0\right) = \frac{V}{R}(1 - 1) = 0$.

(b) For the other times we get

$$\begin{aligned}
I\left(\frac{L}{R}\right) &= \frac{V}{R}\left(1 - e^{-\left(\frac{L}{R}\right)\cdot R/L}\right) = \frac{V}{R}\left(1 - e^{-1}\right) \approx 0.63\left(\frac{V}{R}\right), \\
I\left(\frac{2L}{R}\right) &= \frac{V}{R}\left(1 - e^{-\left(\frac{2L}{R}\right)\cdot R/L}\right) = \frac{V}{R}\left(1 - e^{-2}\right) \approx 0.86\left(\frac{V}{R}\right), \\
I\left(\frac{3L}{R}\right) &= \frac{V}{R}\left(1 - e^{-\left(\frac{3L}{R}\right)\cdot R/L}\right) = \frac{V}{R}\left(1 - e^{-3}\right) \approx 0.95\left(\frac{V}{R}\right),
\end{aligned}$$

(c) Here we compute $\lim\limits_{t\to\infty} I(t)$:

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{V}{R}\left(1 - e^{-tR/L}\right) = \lim_{t\to\infty} \frac{V}{R}\left(1 - e^{-tR/L}\right) \overset{\frac{V}{R}\left(1-e^{-\infty}\right)}{=\!=\!=\!=} \frac{V}{R}(1 - 0) = \frac{V}{R}.$$

In Chapter 4 we will introduce Ohm's Law, which can be written $V = IR$. Note that as $t \to \infty$ above we have $I \to \frac{V}{R}$, i.e., $I = V/R$ "in the limit," which is equivalent to Ohm's Law. An inductor will resist any sudden voltage change, in fact countering that change with a back voltage of its own, but in the presence of a steady voltage an inductor will behave like a conductor. When $t = 0$ and the switch is thrown, the rate of voltage change felt by the inductor is most sudden, and for that instant no current flows as the inductor completely counters the voltage source. However its capacity to resist (L) is not unlimited, and the voltage change it experiences (and its reactance as well) fades until the inductor behaves more and more like an conductor, so that nearly all (and in fact all, in the limit for the ideal case) of the resistance in the circuit comes from the resistor.

Sometimes computing the limit as the input approaches infinity is also useful just to indicate what could theoretically occur if the variable were allowed to get large enough. For instance, if some process's output is logarithmic, with a base greater than 1, even though growth may be very slow it is theoretically possible for it to be as large as we like. Just that piece of information is at times valuable.

In the meantime, the limit computations in the exercises help us to gain further "number sense" and "function sense," as we explore more aspects of the behaviors of functions so we can better analyze these things both in theory, and in the practice of analyzing real-world problems.

Exercises

For problems 1–15, compute the limits where they exist (and if not, state so), showing all steps. You may wish to use Theorem 3.8.1 (page 256) or Theorem 3.8.2 (page 257) to **anticipate** an answer, but perform all the computations as in Examples 3.8.3 and 3.8.4, starting on page 257.

1. $\displaystyle\lim_{x\to\infty} x^5$
2. $\displaystyle\lim_{x\to-\infty} x^5$
3. $\displaystyle\lim_{x\to-\infty} x^4$
4. $\displaystyle\lim_{x\to\infty} \left(x^4 - 5x^5\right)$.
5. $\displaystyle\lim_{x\to-\infty} \left(x^4 - 5x^5\right)$.
6. $\displaystyle\lim_{x\to\infty} \left(x^4 - 5x^6\right)$.
7. $\displaystyle\lim_{x\to-\infty} \left(x^4 - 5x^6\right)$.
8. $\displaystyle\lim_{x\to\infty} \frac{1}{x^3}$.
9. $\displaystyle\lim_{x\to-\infty} \frac{1}{x^3}$.
10. $\displaystyle\lim_{x\to\infty} \frac{x^2}{x^4+1}$.
11. $\displaystyle\lim_{x\to-\infty} \frac{1-2x^2}{x^3+x^2+x+9}$.
12. $\displaystyle\lim_{x\to\infty} \frac{3x^2+5x-11}{2x^2-27x+100}$.
13. $\displaystyle\lim_{x\to-\infty} \frac{3x^2+5x-11}{2x^2-27x+100}$.
14. $\displaystyle\lim_{x\to\infty} \frac{x^2+3x-7}{x+5}$.
15. $\displaystyle\lim_{x\to-\infty} \frac{x^2+3x-7}{x+5}$.
16. Compute the limits, showing logical steps to justify answers.
    (a) $\displaystyle\lim_{x\to\infty} (x + \cos x)$.
    (b) $\displaystyle\lim_{x\to\infty} (x - \cos x)$.
    (c) $\displaystyle\lim_{x\to-\infty} (x + \cos x)$.
    (d) $\displaystyle\lim_{x\to-\infty} \left(x^2 + \cos x\right)$
17. Compute the limits, showing logical steps to justify answers.

(a) $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x^2}$.

(b) $\displaystyle\lim_{x\to\infty} \frac{x+\sin x}{x^2+1}$.

(c) $\displaystyle\lim_{x\to\infty} \frac{x^2}{x-\sin x}$.

(d) $\displaystyle\lim_{x\to\infty} \frac{x^2+2x+1-\sin x}{3x^2+2x-1}$.

(e) $\displaystyle\lim_{x\to\infty} x\sin x$.

18. Compute the limit

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x+9}-x\right).$$

19. Compute the limit

$$\lim_{x\to-\infty}\left(\sqrt{x^2-5x+9}-x\right).$$

It is actually simpler than the previous limit (one line!).

20. Compute the following.

(a) $\displaystyle\lim_{x\to\infty} \sin\frac{1}{x}$.

(b) $\displaystyle\lim_{x\to\infty} \cos\frac{1}{x}$.

(c) $\displaystyle\lim_{x\to\infty} \sin\left[\frac{x^2+2x+9}{6x^2-11x+45}\right]$.

21. Write definitions for the following (see (3.64), page 255). It may help to graph situations where these are true.

(a) $\displaystyle\lim_{x\to\infty} f(x)=\infty$.

(b) $\displaystyle\lim_{x\to-\infty} f(x)=\infty$.

(c) $\displaystyle\lim_{x\to\infty} f(x)=-\infty$.

(d) $\displaystyle\lim_{x\to-\infty} f(x)=-\infty$.

22. Prove that $\displaystyle\lim_{x\to\infty}\sqrt{x}=\infty$ using the the definition found in (3.64), page 255. (For a hint, see the example in the paragraph immediately following that definition.)

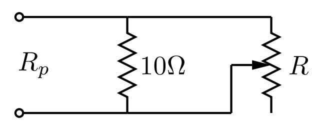
23. Prove Theorem 3.8.1, page 256.

24. Prove Theorem 3.8.2, page 257.

25. A 10Ω resistor and a variable resistor R are placed in parallel in a circuit. The equivalent resistance R_p is related by the equation

$$\frac{1}{R_p}=\frac{1}{10}+\frac{1}{R},$$

where all resistances are in ohms (Ω):



(a) Solve for R_p.

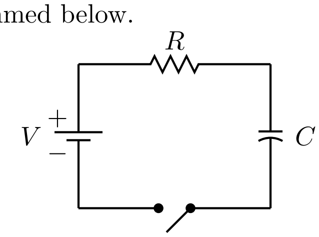
(b) Compute $\displaystyle\lim_{R\to\infty} R_p$, the limiting value of R_p as $R\to\infty$.

26. An employee can produce approximately

$$N(x)=\frac{50x+7}{2x+5}$$

items per day on the production line after x days on the job. In Chapter 4 we will be able to show that this is an increasing function. Find the maximum number of items that can be produced per day by computing $\displaystyle\lim_{x\to\infty} N(x)$.

27. A series circuit consisting of a voltage source, a resistor, a capacitor (initially discharged) and a switch is diagrammed below.



If the switch is first closed ("on") at $t=0$ the charge on the capacitor is given by

$$q(t)=CV\left(1-e^{-t/RC}\right).$$

(a) What is the charge at $t=0$?

(b) Expressed as a percentage of CV, what is the charge at $t = RC$, $t = 2RC$, $t = 3RC$?

(c) What is the trend in the charge as $t \to \infty$?

28. Compute the following:

(a) $\lim\limits_{x\to\infty} 2^{1-\sqrt{x}}$

(b) $\lim\limits_{x\to 0^+} e^{1/x}$

(c) $\lim\limits_{x\to 0^-} e^{1/x}$

(d) $\lim\limits_{x\to\infty} e^{1/x}$

(e) $\lim\limits_{x\to\infty} e^{\frac{2x^2+6x-5}{-2x^2+9x+56}}$

29. Compute the following:

(a) $\lim\limits_{x\to 0^+} \ln \csc x$

(b) $\lim\limits_{x\to 0^+} \dfrac{1}{\ln x}$

(c) $\lim\limits_{x\to 5^-} \ln\left(\dfrac{1}{5-x}\right)$

(d) $\lim\limits_{x\to\infty} \ln(\ln x)$

(e) $\lim\limits_{x\to\infty} \ln(\ln(\ln x))$

(f) $\lim\limits_{x\to\infty} x \ln\left(\dfrac{1}{x}\right)$

3.9 Further Limit Theorems and Trigonometric Limits

In this section we wrap up our discussion of fundamental methods for computing limits of functions. We also look at some very general theorems on limits, ranging from very intuitive to rather sophisticated. Into the mix we introduce and prove an interesting trigonometric limit, (3.81) which will be the basis for another fundamental trigonometric limit, (3.86) and many consequent trigonometric limits we could not have proved with previous methods. Finally we will develop our most sophisticated (so far) limit method, which is substitution.[48] The aforementioned trigonometric limits will give rise to many interesting limits which require the new methods here, as well as the methods of previous sections.

3.9.1 Simple Limit Theorems

Many of the limit computations that we have already performed in this textbook were based upon what we knew from continuity arguments. We took that approach because it is more intuitive than the usual treatment found in most calculus textbooks. (See Footnote 22, page 211.) In fact, the more common treatment is to instead rely upon theorems about limits independent of possible underlying continuity (or continuity of replacement functions). These basic facts we combine below into one theorem, the different parts of which can be proved in ways very similar to the proofs of corresponding continuity theorems (see especially Section 3.2).

Theorem 3.9.1 *Suppose that, as x approaches some value, we have $f(x) \longrightarrow L \in \mathbb{R}$ and $g(x) \longrightarrow M \in \mathbb{R}$. Then*

(i) $f(x) + g(x) \longrightarrow L + M$,

(ii) $f(x) - g(x) \longrightarrow L - M$,

(iii) $Cf(x) \longrightarrow CL$,

(iv) $f(x)g(x) \longrightarrow LM$,

(v) *and if $M \neq 0$, then $f(x)/g(x) \longrightarrow L/M$.*

(vi) $\lim\limits_{x\to a} C = C$, *where C is any fixed constant.*

The theorem above assumes $L, M \in \mathbb{R}$, so L and M exist and are finite. The theorem can be extended to include several (but not all!) cases where L or M do not exist or are infinite. Furthermore the theorem extends in the obvious ways to numerous *forms*, so for instance if $f(x) \xrightarrow{\infty} \infty$ and $g(x) \xrightarrow{\infty} \infty$, then $f(x) + g(x) \xrightarrow{\infty+\infty} \infty$. We still have to be careful: for such f and g we have $f(x) - g(x)$ is of form $\infty - \infty$, which is indeterminate, as demonstrated in Example 3.8.1, page 255.

Textbooks include part (vi) for theoretical reasons we will consider momentarily, but also give it special emphasis because it is so simple it sometimes confuses. For an example illustrating how to interpret (vi) properly, consider the statement that $\lim_{x\to 5} 10 = 10$. What this means is that a function $h(x)$, where $h(x) = 10$ for all $x \in \mathbb{R}$, will give $\lim_{x\to a} h(x) = 10$ regardless of a (finite or infinite). This $h(x)$ is a "constant" function, which always returns the value 10 regardless of the input; its graph is the horizontal line $y = 10$. When graphed it is clear that

[48] Actually, we will add other methods after we develop derivatives. Still, what we finish in this section are the *foundational* methods for limits of functions. Even with the methods we will introduce after derivatives, we can not avoid the requirements of the methods of this section or this chapter. Indeed, the later methods are quite powerful where applicable, but have very limited scopes and therefore cannot replace what we develop here.

$h(x) \longrightarrow 10$ as $x \to a$ (regardless of a, chosen in advance). Rather than explicitly defining such an h, it is customary to simply write, for example, $\lim_{x\to a} 10 = 10$.

The usual method—found in most of today's calculus textbooks—for obtaining a preliminary theory of finite limits is in fact based upon Theorem 3.9.1 above using the following scheme:

1. First prove (with ε-δ or by graphical demonstration) that $\lim_{x\to a} x = a$.

2. Then use (i), (iii), (iv) and (vi) from the theorem above repeatedly to show that polynomials $p(x)$ have the property that $\lim\limits_{x\to a} p(x) = p(a)$, using the limit version of the argument given in the proof of Theorem 3.2.4, page 183 but without reference to continuity (defined later in that approach).

3. From there (v) gives that rational functions $p(x)/q(x)$, i.e., where p and q are polynomials are continuous where defined, that being where $q(x) \neq 0$.

4. Then one looks at rational cases with $0/0$ form, mentioning Theorem 3.4.3, page 212 on replacing functions with other functions that agree near the limit point and are continuous there.

5. One then progresses through the more sophisticated cases (radicals, infinite limits, limits at infinity, Sandwich Theorem, etc.).

6. *Define* continuity at $x = a$ by the criterion that $\lim\limits_{x\to a} f(x) = f(a)$. (Our definition of continuity is logically equivalent, according to Theorem 3.4.2, page 211.)

This is mentioned here so the reader will be aware of this common alternative treatment. Though we did not follow that logical scheme (instead opting for the advanced calculus and real analysis style of continuity before limits), we will occasionally have use for Theorem 3.9.1 above in the rest of the text. Now we will look at an (admittedly) abstract example.

Example 3.9.1 *Suppose that*

$$\lim_{x\to 3} f(x) = 5, \qquad \textit{and} \qquad \lim_{x\to 3} g(x) = 7.$$

Then

$$\begin{aligned}
\lim_{x\to 3}[f(x)+g(x)] &\overset{5+7}{=\!=} 5+7 = 12, &\qquad \lim_{x\to 3}[f(x)g(x)] &\overset{5\cdot 7}{=\!=} 5\cdot 7 = 35,\\
\lim_{x\to 3}[f(x)-g(x)] &\overset{5-7}{=\!=} 5-7 = -2, & \lim_{x\to 3}\frac{f(x)}{g(x)} &\overset{5/7}{=\!=} \frac{5}{7},\\
\lim_{x\to 3}[4f(x)] &\overset{4\cdot 5}{=\!=} 4\cdot 5 = 20, & \lim_{x\to 3} 191 &= 191.
\end{aligned}$$

The last equation above, of course, has nothing to do with the functions f or g. For a more concrete example, consider the following:

Example 3.9.2 *Compute* $\lim\limits_{x\to\infty}\left[17 + \dfrac{x}{x^2+1} + \dfrac{3x^2+5x+9}{2-x^2}\cdot\dfrac{2x-3}{6x-5}\right]$.

Solution: There are several subexpressions here whose limits exist. In this particular example we are lucky that we can partition the whole expression into such well-behaved subexpressions:

$$\lim_{x\to\infty}\left[\underbrace{17}_{\downarrow\ 17}+\underbrace{\frac{x}{x^2+1}}_{\downarrow\ 0}+\underbrace{\frac{3x^2+5x+9}{2-x^2}}_{\downarrow\ (-3)}\cdot\underbrace{\frac{2x-3}{6x-5}}_{\downarrow\ 1/3}\right]=17+0-3\cdot\frac{1}{3}=17-1=16.$$

Note that we relied upon our previous experience with limits at infinity, as outlined in Section 3.8, *especially for rational expressions cases, where only the highest powers were ultimately relevant, as $x\to\infty$ (or $x\to-\infty$).*

The above limit could be computed as it was because all the limits of subexpressions, as we organized them, existed and so our general Theorem 3.9.1, page 266 allows us to so combine them. Clearly the argument above is easier than combining the subexpressions into a single, rational expression. Sometimes recombining can not be avoided, but often the complete computation can be avoided, as in what follows.

Example 3.9.3 *Compute* $\displaystyle\lim_{x\to\infty}\left[\frac{x-2}{x^3-9x+5}\cdot\frac{x^4+10x^3-9x^2+11x+5}{2x^2+x-9}\right]$.

Solution: As it stands, the form of this limit is $0\cdot\infty$ (by Theorem 3.8.2, page 257 *and the thinking surrounding that result) which is indeterminate. One brute-force method of computing this limit is to combine the two fractions into one, but this requires some lengthy multiplication calculations. Instead we offer the two methods below, which work well because it is a limit at infinity.*

1. *We can factor the largest power of x which appears in each term and cancel:*

$$\begin{aligned}
&\lim_{x\to\infty}\left[\frac{x-2}{x^3-9x+5}\cdot\frac{x^4+10x^3-9x^2+11x+5}{2x^2+x-9}\right]\\
&\quad=\lim_{x\to\infty}\left[\frac{x\left(1-\frac{2}{x}\right)}{x^3\left(1-\frac{9}{x^2}+\frac{5}{x^3}\right)}\cdot\frac{x^4\left(1+\frac{10}{x}-\frac{9}{x^2}+\frac{11}{x^3}+\frac{5}{x^4}\right)}{x^2\left(1+\frac{1}{x}-\frac{9}{x^2}\right)}\right]\\
&\quad=\lim_{x\to\infty}\left[\frac{x^5\left(1-\frac{2}{x}\right)\left(1+\frac{10}{x}-\frac{9}{x^2}+\frac{11}{x^3}+\frac{5}{x^4}\right)}{x^5\left(1-\frac{9}{x^2}+\frac{5}{x^3}\right)\left(2+\frac{1}{x}-\frac{9}{x^2}\right)}\right]\\
&\quad=\lim_{x\to\infty}\left[\frac{\left(1-\frac{2}{x}\right)\left(1+\frac{10}{x}-\frac{9}{x^2}+\frac{11}{x^3}+\frac{5}{x^4}\right)}{\left(1-\frac{9}{x^2}+\frac{5}{x^3}\right)\left(2+\frac{1}{x}-\frac{9}{x^2}\right)}\right]=\frac{1\cdot 1}{1\cdot 2}=\frac{1}{2}.
\end{aligned}$$

2. *Another method is to observe what the leading terms of the numerator and denominator polynomials would be if we were to multiply and simplify them. For a limit at infinity, as we know we need only look at the highest-order terms in the numerator and denominator:*

$$\lim_{x\to\infty}\left[\frac{x-2}{x^3-9x+5}\cdot\frac{x^4+10x^3-9x^2+11x+5}{2x^2+x-9}\right]\quad=\quad\lim_{x\to\infty}\frac{x^5+\cdots-10}{2x^5+\cdots-45}\quad=\quad\frac{1}{2}.$$

   The highest- and lowest-order terms are the easiest to compute for a polynomial product (whereas—recall—the intermediate-order terms may be complicated sums). In fact it is only the highest-order terms which are relevant, again, **because the limit is at infinity.**

Next we consider what to do if our partition of the limit's function yields subexpressions whose limits do not necessarily exist. It is true that knowing that one of the component limits does not exist can sometimes allow us to conclude that the entire limit does not. However this is not always the case. In the next example we give an argument where the nonexistence of a component limit can, in that context, imply nonexistence of the full limit. We follow that example with one in which nonexistence of a component limit does not wreck the full limit. Both types should become intuitive, but the perennial lesson that we must be careful not to be too cavalier with our limit arguments should be apparent below.

Example 3.9.4 *Suppose that* $\lim_{x\to a} f(x) = 5$ *and* $\lim_{x\to a} g(x)$ *D.N.E. Then*

$$\lim_{x\to a}[f(x)+g(x)] \overset{5+DNE}{=\!=\!=} D.N.E.$$

To see this, we can argue that since $g(x)$ is not approaching a well-defined limit value for whatever reason (perhaps being undefined near $x = a$, or oscillating, or having different left and right limits), then adding $f(x)$, which **is** *approaching a number, will not compensate for the behavior of $g(x)$, and the final limit can not exist. A more rigorous argument is given next.*

> **<u>Proof</u>:** *Suppose again $\lim_{x\to a} f(x) = 5$ and $\lim_{x\to a} g(x)$ D.N.E. We will prove by contradiction that $\lim_{x\to a}[f(x)+g(x)]$ can not exist, for suppose that it does. Then according to our general* Theorem 3.9.1, page 266 *since $\lim_{x\to a}[f(x)+g(x)]$ exists (by our assumption to be contradicted), we have*
>
> $$\begin{aligned} f(x)+g(x)\longrightarrow L &\implies -f(x)+(f(x)+g(x))\longrightarrow -5+L && \text{(by Theorem 3.9.1(i))}\\ &\implies (-f(x)+f(x))+g(x)\longrightarrow -5+L && \textit{(algebra)}\\ &\implies (0+g(x))\longrightarrow -5+L && \textit{(algebra near limit point)}^{49}\\ &\iff g(x)\longrightarrow -5+L \textit{ exists} && \textit{(algebra).}\end{aligned}$$
>
> *But that contradicts our original information that $\lim_{x\to a} g(x)$ must not exist. Thus we have to conclude that our assumption $\lim_{x\to a}[f(x)+g(x)]$ exists—which leads to a contradiction—must be false, and so $\lim_{x\to a}[f(x)+g(x)]$ does not exist, q.e.d.*[50]

In the above example, the fact that the limit of $f(x)$ was finite meant that it could not—through simple addition—compensate for "bad behavior" of $g(x)$ in the limit. It is possible, however, for f to still compensate, in the sense that $f(x)$ can also simply dominate $g(x)$ in the expression $f(x)+g(x)$ if $g(x)$ is defined but bounded and $f(x)$ blows up. (The reader may wish to revisit (3.69)–(3.72), page 259 as we work the next example.)

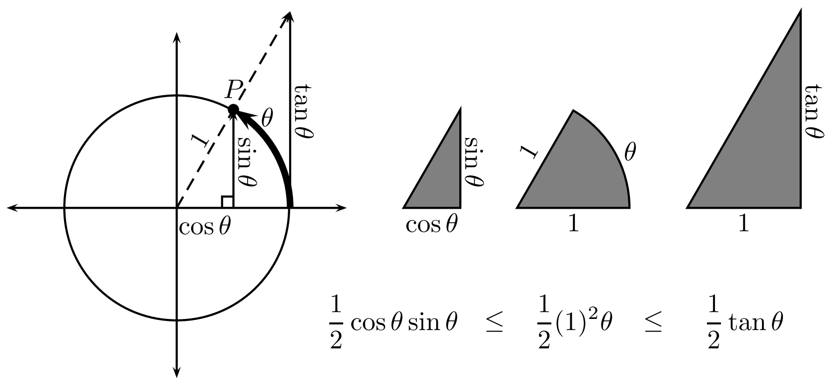
Example 3.9.5 *Suppose that $f(x)\longrightarrow\infty$ and $|g(x)|\le M\in\mathbb{R}$ for some $M>0$ (but real and therefore finite) as $x\to a$. Then we employ a Sandwich Theorem argument, with $-M+f(x)\le f(x)+g(x)\le M+f(x)$, with the first and third terms approaching ∞ (though the first of the three is enough) carrying $f(x)+g(x)$ to an infinite limit as well. We could write*

$$\lim_{x\to a}[f(x)+g(x)] \overset{\infty+B}{=\!=\!=} \infty.$$

For a specific example, consider: $\lim_{x\to\infty}(x+\sin x) \overset{\infty+B}{=\!=\!=} \infty.$

[49] At $x = a$ we may have that $f(x)$ is undefined, so there it may be incorrect to say $-f(x)+f(x) = 0$. However, near the limit point we know $f(x)$ is defined, from the assumption $f(x)\to 5$ as $x\to a$. Implied in our definition of limits ((3.22), page 210) is that $f(x)$ exists for $0<|x-a|<\delta$, for some $\delta>0$.

[50] Note how we used $P\longrightarrow(\sim P)\implies\sim P$.



$$\frac{1}{2}\cos\theta\sin\theta \quad \le \quad \frac{1}{2}(1)^2\theta \quad \le \quad \frac{1}{2}\tan\theta$$

Figure 3.28: Illustration of relative sizes of $\frac{1}{2}\cos\theta\sin\theta$, $\frac{1}{2}\theta$ and $\frac{1}{2}\tan\theta$ for an angle $\theta \in [0, \pi/2)$. Note that each area is a superset of any preceding area illustrated above, as most clearly illustrated in the far-left figure. Recall also that $\theta = s/r$, where s is the directed arc length and r the radius, so on the unit circle $\theta = s/1 = s$. Also recall that the area of a circular wedge is given by $A = \frac{1}{2}r^2\theta$. (To help remember this formula, think of what this means if $\theta = 2\pi$: Area $= \frac{1}{2}r^2 \cdot 2\pi = \pi r^2$ if θ sweeps the whole circle.)

Actually we worked exactly this example during the discussion of the form $B + \infty$, introduced on page 259.

The lesson to be gleaned from the above examples is that if one part of the function "inside the limit" has nonexistent limit (for the prescribed approach in the independent variable), sometimes we can conclude the same about the whole limit and sometimes we can not. We usually need to dig deeper into the behavior of the other parts of the function, and take into account how their influences combine. Sometimes the form is enough to determine the actual limit (or its nonexistence). With experience, the various cases become intuitive. (Recall also Example 3.7.1, page 241 and its associated Figure 3.22.)

3.9.2 A Trigonometric Limit

An interesting limit, which is surprisingly useful for future results, is the following:

Theorem 3.9.2 *With θ given in radians, we have the following limit:*

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1. \tag{3.81}$$

Proof:

The proof relies upon the Sandwich Theorem and a geometric observation which is given in Figure 3.28. In that figure, θ is the radian measure of the angle which terminates in the first quadrant. The observation involves three areas defined by this angle θ: a right triangle, contained within a circular wedge, which is in turn contained in another right triangle. The smaller triangle has "base" $\cos\theta$ and "height" $\sin\theta$, and thus has area $\frac{1}{2}\cos\theta\sin\theta$. For the circular wedge, recall that a wedge with radius r and radian-measure angle θ has area $\frac{1}{2}r^2\theta$. The other triangle has base 1 and height $\tan\theta$. To see this, note that it is similar to the smaller triangle, and so we have the proportion of sides: $\sin\theta/\cos\theta = h/1$, where h is the height of the larger triangle. It follows that $h = \tan\theta$. Thus the larger triangle has area $\frac{1}{2}\cdot 1\cdot\tan\theta$. Now for $\theta\in[0,\pi/2)$, we have

$$\frac{1}{2}\cos\theta\sin\theta \le \frac{1}{2}\theta \le \frac{1}{2}\tan\theta.$$

As $\theta\to 0^+$, we have $\sin\theta > 0$ so we can not only multiply by 2, but also then divide by $\sin\theta$, giving us

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}. \tag{3.82}$$

This gives us a Sandwich Theorem type of argument as $\theta\to 0^+$:

$$\text{As } \theta\to 0^+: \qquad \underbrace{\cos\theta}_{\substack{\big\downarrow \\ 1}} \quad\le\quad \frac{\theta}{\sin\theta} \quad\le\quad \underbrace{\frac{1}{\cos\theta}}_{\substack{\big\downarrow \\ 1}}$$

giving us $\frac{\theta}{\sin\theta} \longrightarrow 1$ as $\theta\to 0^+$, i.e.,

$$\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} = 1. \tag{3.83}$$

Next we use dispatch with the left-side limit. In fact the same inequality (3.82) holds as $\theta\to 0^-$, because all three expressions are the same if we replace θ with $-\theta$. This follows because all three functions are "even," i.e., $\cos(-\theta)=\cos\theta$, $(-\theta)/\sin(-\theta) = \theta/\sin\theta$, and $1/\cos(-\theta) = 1/\cos\theta$, and $\theta\to 0^+ \iff (-\theta)\to 0^-$. With this one can perform the above computations with $(-\theta)\to 0^+$, and thus $\theta\to 0^-$.

Perhaps a less convoluted approach is to more explicitly borrow the substitution method from upcoming Subsection 3.9.3. Here we let $\phi = -\theta$ so that $\theta\to 0^- \iff \phi\to 0^+$. Thus

$$\lim_{\theta\to 0^-}\frac{\theta}{\sin\theta} = \lim_{\phi\to 0^+}\frac{-\phi}{\sin(-\phi)} = \lim_{\phi\to 0^+}\frac{-\phi}{-\sin\phi} = \lim_{\phi\to 0^+}\frac{\phi}{\sin\phi} = 1, \tag{3.84}$$

the last limit being, of course, (3.83) with the variable renamed. Putting (3.83) together with (3.84), of course, gives us

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1. \tag{3.85}$$

Finally, we can then get our result based upon the above limit, looking at the reciprocal function:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{\left(\frac{\theta}{\sin\theta}\right)} \overset{1/1}{=\!=} 1, \qquad \text{q.e.d.}$$

Notice that the limit we proved, (3.81), says something about how fast $\sin\theta$ approaches zero as θ approaches zero: that $\sin\theta$ and θ approach zero at approximately the same rate. In fact, many physics problems use the approximation $\sin\theta\approx\theta$ (following from $(\sin\theta)/\theta\approx 1$) for $|\theta|$ small and in radians. This approximation is graphed in Figure 3.29, page 272 but using x instead

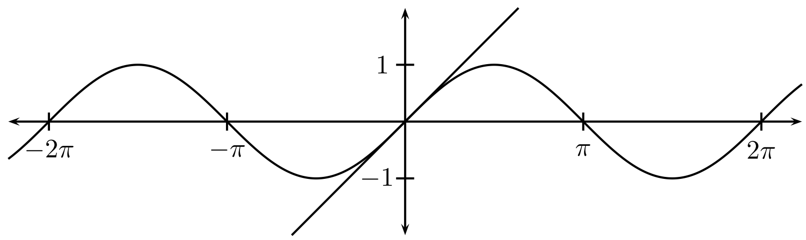


Figure 3.29: Partial graphs of $y = x$ and $y = \sin x$. Since $(\sin x)/x \longrightarrow 1$ as $x \to 0$, we get $(\sin x)/x \approx 1$ for x near zero, and thus $\sin x \approx 1 \cdot x$ for small x. In the graph above we see how quickly $\sin x$ and x become close enough that the limited resolution of the graphic rendering device makes it difficult to distinguish the two functions for small x. (However, it should be noticed that the two functions x and $\sin x$ only agree at $x = 0$. Indeed, a closer examination of Figure 3.28 on page 270 shows that for $\theta > 0$ we have $\theta > \sin\theta$, and for $\theta < 0$ we have $\theta < \sin\theta$.)

of θ as the independent (domain) variable. We will see how this very important approximation, and the underlying limit, arise from other calculus techniques in later chapters.[51]

Many other interesting limits follow from this limit (3.81), as we can see in the following example. Note that the second limit computation below utilizes the trigonometric identities $(1 - \cos x)(1 + \cos x) = 1 - \cos^2 x = \sin^2 x$.

Example 3.9.6 *Consider the following limits, which require both our basic trigonometric limit* (3.81) *and our general limit theorem,* Theorem 3.9.1 *with which we began the section, on* page 266.

- $\displaystyle\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right] \overset{1\cdot\frac{1}{1}}{=\!=} 1.$
- $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} \overset{0/0}{=\!=} \lim_{x\to 0}\left[\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$

$$= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right] \overset{1\cdot\frac{0}{2}}{=\!=} 1\cdot\frac{0}{2} = 0.$$

The first limit is also used in physics in the form $\tan\theta \approx \theta$ for $|\theta|$ small. The second limit occurs enough to warrant being set aside as its own theorem, though as we see above it is easily derived from the more basic $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Theorem 3.9.3 *The following limit (proved above) holds:*

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0. \tag{3.86}$$

[51]While (3.85) is the immediate result of our analysis, and is interesting in its own right, most texts (as here) go ahead and present the reciprocal limit (3.81). The reason is that $\lim_{\theta\to 0}(\sin\theta/\theta)$ gives a (perhaps) more intuitive comparison of the behavior of $\sin\theta$ *versus* that of the independent variable θ. We have already had many limits of ratios of functions $f(x)/g(x)$ which compare the two functions' behaviors in the sense of limits. A very useful way to analyze a function is to compare it to its input variable by ratios $f(x)/x$, $f(x)/(x-a)$, or $(f(x)-f(a))/(x-a)$, for examples. Though this can often be accomplished instead by looking the reciprocals of these, it is less intuitive to do so.

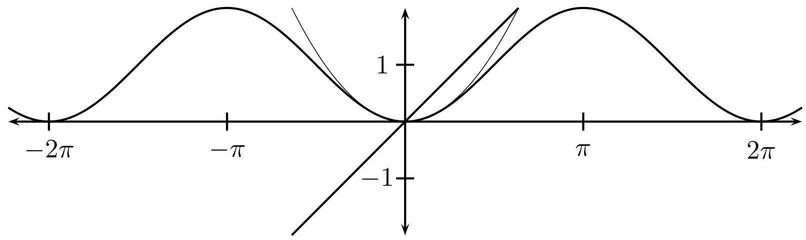


Figure 3.30: Partial graphs of $y = x$ and $y = 1 - \cos x$, showing how $1 - \cos x$ approaches zero faster than x as $x \to 0$—so much so that $(1 - \cos x)/x \longrightarrow 0$ (see (3.86)). The graph of $y = \frac{1}{2}x^2$ is also given (thinner curve), illustrating how $(1 - \cos x)/x^2 \approx \frac{1}{2}$, i.e., $\cos x \approx 1 - \frac{1}{2}x^2$ for x small, as derived in Example 3.9.7.

In other words, $1 - \cos\theta$ shrinks to zero faster than θ does, as $\theta \to 0$. This is reasonable when we see the graphs of $y = 1 - \cos x$ and $y = x$, given (as darker curves) in Figure 3.30.[52]

For the rest of this section, we will assume (3.81) and (3.86) and compute other trigonometric limits based upon these, the work from previous sections, and Theorem 3.9.1. As with all limits, it is important to be careful and not jump to incorrect conclusions; our basic trigonometric limits (3.81) and (3.86)—which the reader should commit to memory—given here again as

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1, \qquad\qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0,$$

are very specific in their scopes. Moreover, with trigonometric limits it is sometimes still necessary to exploit the algebraic identities among those functions. Consider the following (perhaps surprising) trigonometric limit calculation:

Example 3.9.7 *Compute* $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

<u>Solution</u>: *A first, perhaps more obvious attempt is quickly seen to be a dead-end:*

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\left[\frac{1-\cos x}{x}\cdot\frac{1}{x}\right] \overset{0\cdot(\pm\infty)}{=\!=\!=} ?,$$

of the indeterminate form $0\cdot(\pm\infty)$. *As happens so frequently with limits, we look for some other way of rewriting the function. The usual method of computing this limit is to again exploit the fact that*

$$(1-\cos x)(1+\cos x) = 1-\cos^2 x = \sin^2 x.$$

By multiplying the function inside the limit by $(1+\cos x)/(1+\cos x)$, *we can compute the limit as follows:*

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\left[\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}\\ &= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}\right] = 1\cdot 1\cdot\frac{1}{1+1} = \frac{1}{2}.\end{aligned}$$

[52] One could also say the $\cos\theta \to 1$ very rapidly as $\theta \to 0$, but of course neither description is as precise as (3.86), i.e., that $(1-\cos\theta)/\theta \to 0$ as $\theta \to 0$.

In the last step of the above example, we used Theorem 3.9.1 (the limit of a product being the product of the limits, when they exist), and the fact that the factor $1/(1 + \cos x)$ is continuous at $x = 0$.

One could argue using this limit that, for $|x|$ small, $1 - \cos x \approx \frac{1}{2}x^2$, which is also illustrated in Figure 3.30, page 273. This fact we will derive later in the form $\cos x \approx 1 - \frac{1}{2}x^2$, which is sometimes used in applications. It is much more accurate than the earlier approximation that $\cos x \approx 1$ for x small (though they coincide at $x = 0$).

We will compute many more trigonometric limits in this section, but first we need a new limit technique which we introduce next.

3.9.3 Limits by Substitution

To bring our analytical methods of computing limits to the next level, we now develop some substitution techniques. These techniques require delicacy, but with care they are quite powerful and sophisticated. To motivate the discussion, we first give two examples below:

Example 3.9.8 *Compute* $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{x}$.

Solution: The usual method is to multiply by $\frac{5}{5}$ as below.

$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} \frac{5\sin 5x}{5x} = \lim_{x\to 0}\left[5\cdot\frac{\sin 5x}{5x}\right] = 5\cdot 1 = 5.$$

Notice that the original limit was of the form "$(\sin 0)/0$" (i.e., $0/0$) which is indeterminate. It is important in our basic trigonometric limit (3.81), i.e., $x \to 0 \implies (\sin x)/x \to 1$, that the two "zeros" are terms approaching zero at the same rate (though even that is not always quite enough, as we will eventually see). By multiplying the fraction by $5/5$, we were able to get a form "$5 \cdot [(\sin 0)/0]$," but where the rates of the "zeros" were exactly the same. Most presentations of the computation of the above limit are exactly as given above, but it is to be understood that we are transforming this by way of a substitution. One might instead write

$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0}\left[5\cdot\frac{\sin 5x}{5x}\right] = \lim_{(5x)\to 0}\left[5\cdot\frac{\sin 5x}{5x}\right] = 5\cdot 1 = 5,$$

or

$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0}\left[5\cdot\frac{\sin 5x}{5x}\right] = \lim_{\theta\to 0}\left[5\cdot\frac{\sin \theta}{\theta}\right] = 5\cdot 1 = 5,$$

where $\theta = 5x$, and $x \to 0 \iff \theta \to 0$. The nature of the mechanism whereby $\theta \to 0 \iff x \to 0$ is crucial, but we will explain this in due course. The next example displays a slightly more sophisticated argument.

Example 3.9.9 *Compute* $\displaystyle\lim_{x\to 0} \frac{\cos x^2 - 1}{x^4}$. *(Compare to* Example 3.9.7, page 273.*)*

Solution: Here we will make a substitution $\theta = x^2$, so that $x \to 0 \implies \theta = x^2 \to 0^+$. Then we can write

$$\lim_{x\to 0} \frac{\cos x^2 - 1}{x^4} \overset{0/0}{=\!=} \lim_{\theta\to 0^+} \frac{\cos\theta - 1}{\theta^2} = \lim_{\theta\to 0^+}\left[\frac{\cos\theta - 1}{\theta^2}\cdot\frac{\cos\theta + 1}{\cos\theta + 1}\right] = \lim_{\theta\to 0^+} \frac{\cos^2\theta - 1}{\theta^2(\cos\theta + 1)}$$

$$= \lim_{\theta\to 0^+} \frac{-\sin^2\theta}{\theta^2(\cos\theta + 1)} = \lim_{\theta\to 0^+}\left[-\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta + 1}\right] = -1\cdot 1\cdot\frac{1}{1+1} = -\frac{1}{2}.$$

Notice that in the above example we used the fact that full limits imply one-sided limits. Specifically, above we used that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \implies \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1.$$

Now we consider circumstances where we can make a substitution to compute a limit. First we take another look at what it means for $x \to a$. (Subsection 3.7.2, page 242 began this discussion.) When we write $x \to a$ under "lim,", we understand this to mean that x gets arbitrarily close to **but not equal to** the value a. But we also assume that x does so over a continuum of values, so that x does not "skip over" any values as we approach a either, as we see from the definition of $\lim_{x\to a} f(x)$, as for example in the case this limit is a finite number L:

$$(\forall\varepsilon>0)(\exists\delta>0)(\forall x)[\underbrace{0<|x-a|<\delta}_{x\in(a-\delta,a)\cup(a,a+\delta)} \longrightarrow |f(x)-L|<\varepsilon].$$

In order to substitute algebraically to produce another limit, say $\lim_{u\to\beta} F(u)$ and claim it is the same as $\lim_{x\to a} f(x)$, we would like to be sure that $x \to a \implies u \to \beta$, and that the latter "approach" is still somehow a **proper** approach. Usually what is obvious is only that $x \to a \implies u \to \beta$ in the sense that $\lim_{x\to a} u = \beta$, which allows for the approach of u to β to be quite sloppy. It may be that the approach u takes to β is one-sided, or has gaps as $x \to a$, or actually *achieves* $u = \beta$ while we *require* $x \neq a$. Indeed when we developed the convention that $\lim_{x\to a} u = \beta$ would also be written $x \to a \implies u \to \beta$, we interpreted the approach of u to β quite loosely, while that of x to a was more strict. To force $u \to \beta$ to occur in the same way that $x \to a$ requires much more structure in the relationship between x and u than we usually wish to have to accommodate. One notable example where this does happen is when $u = mx + b$, where $m \neq 0$, giving that $x \to a \iff u \to (ma + b)$, and both approaches are proper in every way. Another example is $x \to 0 \implies x^2 \to 0^+$ (note the absence of the converse $\Longleftarrow$). Other cases can be quite complicated.

What we can do to produce a useful theorem is to not make a strong statement about the relationship between x and u, but rather qualify the result in a different way (which also makes a proof easier to formulate, were we to include it), still yielding a useful result. What we settle on here is the following, the proof of which is similar to that of Theorem 3.7.4, page 244, and is left as an exercise for the interested reader.

Theorem 3.9.4 (Limit Substitution Theorem) *Suppose that the variables x and u are related is such a way that*

(a) $\lim\limits_{x\to a} u = \beta$,

(b) $(\exists d>0)(\forall x)[0<|x-a|<d \longrightarrow (u\neq\beta)]$,

(c) $(\exists d>0)(\forall x)[0<|x-a|<d \longrightarrow (f(x)=F(u))]$.

Then

$$\lim_{x\to a} f(x) = \lim_{u\to\beta} F(u) \qquad \textit{if this second limit exists.} \tag{3.87}$$

There are many theorems we will come across where we have an equality like (3.87) which is qualified by the criterion that the second quantity exists, or otherwise makes sense in the context. To remove that criterion would require a much more complicated set of conditions than (a)–(c) above. That the second limit exists is key, but also that $u \to \beta$ to be an "approach" of a similar kind in the sense that not only should $u \to \beta$, but $u \neq \beta$ as $x \to a$ according to (b).[53] In fact, the validity of (a) and (c) usually are clear from the actual algebraic substitution; it is (b) that requires a bit more scrutiny but is also usually not difficult to see.

Example 3.9.10 *Compute* $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\sin\left(x-\frac{\pi}{2}\right)}{x-\frac{\pi}{2}}$.

Here we make the substitution $u = x - \frac{\pi}{2}$, *so that* $x \to \frac{\pi}{2} \implies u \to 0$ *in the sense of the hypotheses (a)–(c) of the theorem. Thus we can write (as long as the final limit exists!)*

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin\left(x-\frac{\pi}{2}\right)}{x-\frac{\pi}{2}} = \lim_{u\to 0}\frac{\sin u}{u} = 1.$$

In the earlier Examples 3.9.8 and 3.9.9 above we used θ to play the role of u in the theorem, but that is mostly a matter of taste. We will usually use u because it is the traditional variable of substitution later in the calculus.[54]

The theorem can be extended in obvious ways to cases such as

$$\begin{aligned} x \to a &\implies u \to \infty \\ x \to a &\implies u \to \beta^+ \\ x \to a^+ &\implies u \to \beta^+ \end{aligned}$$

and so on.

It should be pointed out that if $x \to a$ implies a "proper" one-sided approach, for instance $u \to \beta^+$, then we can perform a substitution based upon that, except then we would compute the one-sided limit $\lim_{u\to\beta^+} F(u)$. This was the case in Example 3.9.9, page 274. Furthermore, a one-sided approach in x to a may yield a proper approach in a variable u of some kind. Infinite "approaches" can also arise from substitutions, or give rise to substitutions, as we see below.

Example 3.9.11 *Consider* $\displaystyle\lim_{x\to\frac{\pi}{2}^+} \frac{2\tan^2 x + 3\tan x + 7}{\tan^2 x - 6\tan x + 30}$.

Here we let $u = \tan x$. *Then* $x \to \frac{\pi}{2}^+ \implies u \to -\infty$. *(Because we will never have* $u = -\infty$, *we do not need an analog to the criterion* (b) *in the* Limit Substitution Theorem, Theorem 3.9.4, page 275.*) Thus*

$$\lim_{x\to\frac{\pi}{2}^+} \frac{2\tan^2 x + 3\tan x + 7}{\tan^2 x - 6\tan x + 30} = \lim_{u\to-\infty}\frac{2u^2+3u+7}{u^2-6u+30} = \lim_{u\to-\infty}\frac{\cancel{u^2}\left(2+\frac{3}{u}+\frac{7}{u^2}\right)}{\cancel{u^2}\left(1-\frac{6}{u}+\frac{30}{u^2}\right)} = \frac{2+0+0}{1-6+0} = 2.$$

In fact the above example can also be computed by multiplying numerator and denominator by $\cos^2 x$. A similar computation is left to the exercises.

The next example illustrates that if one type of indeterminate form is not easily dealt with, an algebraic manipulation can likely give one which is more easily computed.

Example 3.9.12 *Compute* $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x}$.

<u>Solution</u>: Note that $\frac{1}{x} \longrightarrow 0^+$ *as* $x \to \infty$, *so the form here is essentially* $\infty \cdot \sin 0 = \infty \cdot 0$ *(more precisely,* $\infty \cdot \sin 0^+ = \infty \cdot 0^+$*), which is indeterminate (see* Example 3.8.2, page 256*).*

[53] Later in the subsection we will have an example to show why (b) is necessary. In that example, $x \to a \implies u \to \beta$, but u oscillates, passing through the value β infinitely many times as $x \to a$. In that example the naive substitution is invalid: the new limit exists but the original does not, so obviously they are not equal.

[54] Actually θ is becoming increasingly common as a variable of substitution, and we will have occasion to use it as we did in our first substitution examples.

However we can rewrite the limit with a power of x in the denominator, instead of having x as a multiplicative factor. Then we will perform a substitution and use (3.81) *for the final computation.*

$$\lim_{x\to\infty} x\sin\frac{1}{x} \overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \qquad \textit{(form } \sin 0^+/0^+\textit{)}.$$

Now we let $u = 1/x$, so that $x \to \infty \implies u \to 0^+$ properly, so that

$$\lim_{x\to\infty} x\sin\frac{1}{x} \overset{\infty\cdot 0}{\underset{\text{ALG}}{=\!=}} \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{u\to 0^+} \frac{\sin u}{u} = 1.$$

Substitution is a very powerful method, but sometimes the mechanics of it become needlessly complicated and a certain amount of "hand-waving" becomes appropriate. For instance, depending upon the author and the audience the u-limit above might be omitted, but the substitution principle should be understood. When we think of the limits

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \qquad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0,$$

what is important is that the "θ" inside the sine and cosine functions approaches (but never achieves!) zero at the same rate as the "θ" in the denominator. Thus one may just write

$$\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} = 1$$

since the substitution $\theta = \sin x$ gives us $x \to 0 \implies \theta \to 0$ properly, and $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ is known. To be cautious one can explicitly write this substitution step:

$$\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1,$$

where $\theta = \sin x \to 0$ is a manner consistent with our theorem (as $x \to 0$), as an examination of the graph (or just the nature) of the sine curve indicates.

The next example shows how we can save some effort by just noting the approaches to zero are at the same rate. Still we should have the limit substitution theorem, and its criteria (a)–(c) in mind. We will show the careful substitution method, and then the more terse argument.

Example 3.9.13 *Consider the limit $\lim_{x\to 0} \dfrac{1-\cos x^2}{x}$. Again we know what this limit would be if we had x^2 in the denominator, so we will rewrite the function in a form where that is the case, and compensate in the numerator:*

$$\lim_{x\to 0} \frac{1-\cos x^2}{x} = \lim_{x\to 0} \frac{x(1-\cos x^2)}{x^2} = \lim_{x\to 0} \left[x \cdot \frac{1-\cos x^2}{x^2} \right].$$

At this point we could use a substitution, say $u = x^2$, and so $u \to 0 \implies u \to 0^+$, but we have different expressions for our function as $x \to 0^+$ and $x \to 0^-$:

$$\lim_{x\to 0^+} \left[x \cdot \frac{1-\cos x^2}{x^2} \right] = \lim_{u\to 0^+} \left[\sqrt{u} \cdot \frac{1-\cos u}{u} \right] \overset{0^+\cdot 0}{=\!=} 0,$$

$$\lim_{x\to 0^-} \left[x \cdot \frac{1-\cos x^2}{x^2} \right] = \lim_{u\to 0^+} \left[-\sqrt{u} \cdot \frac{1-\cos u}{u} \right] \overset{-0^+\cdot 0}{=\!=} 0.$$

This is all correct, but instead we will simply rewrite the function so that we get the same rate of approach to zero inside the cosine and in the denominator:

$$\lim_{x\to 0}\frac{1-\cos x^2}{x}=\lim_{x\to 0}\left[x\cdot\frac{1-\cos x^2}{x^2}\right]\stackrel{0\cdot 0}{=\!=}0.$$

Rather than making the substitution, we rewrote the function and noticed we have matching rates of approach to zero in the $(1-\cos x^2)/x^2$ *term: we see that* $x^2\to 0^+$ *as in our limit substitution theorem so we can just cite the result* $(1-\cos\theta)/\theta\to 0$ *as* $\theta\to 0$ *(which includes left and right limits).*

As we saw in the above example, sometimes substitution requires us to consider cases, where instead we could do some hand-waving (based upon anticipating what would happen if we did perform the substitution). The next example perhaps makes the case for selective, "informed handwaving" more strongly:

Example 3.9.14 *Consider* $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}$.

<u>Solution</u>: *We need a* $2x$ *to oppose the* $\sin 2x$*, and* $5x$ *to oppose the* $\sin 5x$*, all the while not actually changing the value of the function. We do this by multiplying the numerator by* $2x/2x$*, and the denominator by* $5x/5x$*, with factors arranged strategically:*

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}=\lim_{x\to 0}\frac{2x\cdot\frac{\sin 2x}{2x}}{5x\cdot\frac{\sin 5x}{5x}}=\lim_{x\to 0}\frac{2\cdot\frac{\sin 2x}{2x}}{5\cdot\frac{\sin 5x}{5x}}\stackrel{\frac{2\cdot 1}{5\cdot 1}}{=\!=}2/5.$$

To actually use a verbose substitution method, we would have to factor the function first into three factors, $\frac{2}{5}\cdot\frac{\sin 2x}{2x}\cdot\frac{5x}{\sin 5x}$ and invoke two separate substitutions, one each for the second and third factors, and use our general limit Theorem 3.9.1, page 266. Such an approach would be correct, but quite cumbersome.

When we have $x\to a\implies u\to\beta$, it is the exceptional cases when the u-variable approach to β is problematic. We always need to be aware that we require $u\neq\beta$ as $x\to a$, but again it is rare that there is a problem when the substitutions are routine.[55] The next two examples give an idea of what kind of substitutions to avoid.

Example 3.9.15 *We claim that the limit* $\displaystyle\lim_{x\to 0}\frac{\sin\left[x\sin\frac{1}{x}\right]}{\left[x\sin\frac{1}{x}\right]}$ *does not exist.*

Now the function $x\sin\frac{1}{x}$ *is plotted in* Figure 3.22, page 241 *and has two relevant features for our discussion here:*

1. $x\sin\dfrac{1}{x}\longrightarrow 0$ *as* $x\to 0$*, which was proved by the Sandwich Theorem, however*
2. $x\sin\dfrac{1}{x}$ *has infinitely many zeroes (that is, points* x *where the function is zero) as* $x\to 0$.

From the first point, we see that if we let $\theta=x\sin\frac{1}{x}$*, then* $x\to 0\implies\theta\to 0$*. Unfortunately,* $\theta=0$ *infinitely many times as* $x\to 0$*. Thus, even though* $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$*, we cannot say that the original limit is the same value. Indeed, the original limit does not exist, because as we have* $x\to 0$ *there are infinitely many points where the original function is not even defined, forcing us to conclude*

$$\lim_{x\to 0}\frac{\sin\left[x\sin\frac{1}{x}\right]}{\left[x\sin\frac{1}{x}\right]}\qquad\textit{does not exist.}$$

[55] In fact it is not necessary for $u\neq\beta$ if f is continuous at $x=a$, but then the value of the limit is just $f(a)$. These more advanced limit techniques such as substitution are for dealing with the cases that continuity is "broken," or the quantities are not all finite.

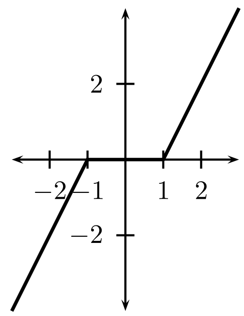


Figure 3.31: Graph of $y = g(x)$ from Example 3.9.16.

The reason the above limit does not exist compares to the reason $\lim_{x\to5}\sqrt{x^2-25}$ does not exist, though the latter is more obvious: that $\sqrt{x^2-25}$ is *undefined* for part of the path of approach, in particular for $x \in (-5,5)$ so that $\sqrt{x^2-25}$ is undefined as $x \to 5^-$. With our example's limit function above, the function does not exist everywhere that the denominator, $x\sin\frac{1}{x}$ is zero, i.e., wherever $1/x = n\pi$, or $x = \pi/n$ where $n \in \mathbb{Z} - \{0\}$, and this happens infinitely many times inside of any interval $(-\delta, 0)$ or $(0, \delta)$ if $\delta > 0$. As with $\lim_{x\to5}\sqrt{x^2-25}$, we cannot *approach* the limit point on a continuum (from both sides in the example $\sqrt{x^2-25}$) and be sure to stay in the domain of the function, and thus we are forced to conclude the limit does not exist.

For a less subtle example, consider the following.

Example 3.9.16 *Suppose* $g(x) = \begin{cases} x-1 & \text{if} \quad x \in (1,\infty) \\ 0 & \text{if} \quad x \in [-1,1] \\ x+1 & \text{if} \quad x \in (-\infty,-1). \end{cases}$

This is graphed in Figure 3.31. *Next consider the limit*

$$\lim_{x\to0}\frac{\sin(g(x))}{g(x)}.$$

If we let $u = g(x)$, then one could say $x \to 0 \implies u \to 0$, but the latter will not be a proper approach, because $u = 0$ for all $x \in [-1,1]$ (not just at the limit point $x = 0$). In other words, u is not simply "approaching" *zero for x near zero, but rather u is the* **constant** *zero for such x close enough to zero. Here $(\sin g(x))/g(x)$ is undefined for $x \in [-1,1]$, which contains the final path of x as $x \to 0$. Thus we* **cannot** *say $\lim_{x\to0}(\sin g(x))/g(x) = \lim_{u\to0}(\sin u)/u$, as the former limit DNE while the latter is just 1.*

3.9.4 Epilogue

The point of this section is that there are limits which we can rightfully summarize how variables and functions approach values, though we have to be very clear to do so in the spirit of limits: we see what values are approached without necessarily having those values actually occur. We also have to consider how the values are approached (including for our approaches "to infinity"): along a continuum of values, or in fits and spurts. At times one kind of "approach" is necessary, and at other times in a computation either way suffices. This depends mainly upon what is used as an input variable and what is used as an output variable, and if there are variables "in between" (as in the case of substitution), how are they behaving. With training and depth of understanding many complicated limits can be dispatched quickly, but attempts do so without the requisite understanding can (and likely will) cause some serious errors in limit computations.

Our development of limits of functions ends here temporarily, and resumes in Chapter 9. There these techniques will be revisited and expanded, and new techniques based upon the calculus developed in between will also be introduced.

We will use the limit techniques developed here in the meantime, and the concept of limit will be embedded in much of what we do throughout the text.

Concerning the expansion of the techniques developed here, the interested student might enjoy perusing Section 9.1 at this point, as it does not depend on any calculus developed in between.

We also offer two sections regarding limits of sequences next, though they will not be necessary until much later in the text.

Exercises

1. Compute $\displaystyle\lim_{x\to 0}\frac{\sin 9x}{x}$.
2. Compute $\displaystyle\lim_{x\to 0}\frac{x}{\sin 9x}$.
3. Compute $\displaystyle\lim_{x\to 0}\frac{\sin 9x}{\sin 7x}$.
4. Compute $\displaystyle\lim_{x\to 0}\frac{\sin x^2}{x}$.
5. Compute $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x^2}$.
6. Compute $\displaystyle\lim_{x\to 0}\frac{\sin^3 5x}{x^3}$.
7. Compute $\displaystyle\lim_{x\to 0}\sqrt[3]{\frac{\sin x}{x}}$.
8. Compute $\displaystyle\lim_{x\to 0}\sqrt{\frac{\sin x}{x}}$.
9. Compute $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$.
10. Compute $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{x}$.
11. Compute $\displaystyle\lim_{x\to 0}\frac{1-\cos 2x}{x}$.
12. Compute $\displaystyle\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$.
13. Compute $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{1-\cos 2x}{x^2}$.
14. Compute $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^4}$.
15. Compute $\displaystyle\lim_{x\to 0}\frac{\sin(\tan x)}{\tan x}$.
16. Compute $\displaystyle\lim_{x\to \frac{\pi}{2}^+}\frac{\sin(\tan x)}{\tan x}$.
17. Compute $\displaystyle\lim_{x\to 0}\frac{\sin|x|}{|x|}$.
18. Show that $\displaystyle\lim_{x\to 0}\frac{\sin|x|}{x}$ does not exist.
19. Compute $\displaystyle\lim_{x\to 0}\frac{x}{\sin x}$.
20. Compute $\displaystyle\lim_{x\to 0}\frac{x^2}{\sin x}$.
21. Compute $\displaystyle\lim_{x\to -\infty}\frac{\sin e^x}{e^x}$.
22. Compute $\displaystyle\lim_{x\to \infty}\frac{\sin e^x}{e^x}$.
23. Compute $\displaystyle\lim_{x\to -\infty}\frac{1-\cos e^x}{e^x}$.
24. Compute $\displaystyle\lim_{x\to 0}\frac{1-\cos\left(x\sin\frac{1}{x}\right)}{x\sin\frac{1}{x}}$.
25. Recompute the limit of Example 3.9.11, page 276, this time by multiplying the numerator and denominator of the function by $\cos^2 x$.
26. Compute $\displaystyle\lim_{x\to 0^+}\frac{2\csc^2 x+3\csc x+11}{5\csc^2 x+4\csc x+7}$ using a substitution argument.
27. Compute $\displaystyle\lim_{x\to 0}\frac{2\csc^2 x+3\csc x+11}{5\csc^2 x+4\csc x+7}$ without a substitution argument.

28. Compute $\lim_{x\to\infty}\left[\frac{2x-9}{3x+5}\cdot\frac{6x^2-9x+10}{x^2-7x+8}\right.$
$\left.+\frac{7x^2+6}{3x^3+4x-3}\cdot\cos\left(\frac{1}{x}\right)\right].$

29. Compute $\lim_{x\to-\infty}\left[\frac{x^2+8x-9}{5x^2-6x+42}+\sin e^x\right].$

30. Compute $\lim_{x\to\infty}\left[\frac{x}{x^2+1}+\cos x\right].$

31. Compute $\lim_{x\to\infty}\left[\frac{x^2+1}{x-1}+\cos x\right].$

3.10 Limits of Sequences: A First Look

Sequences offer a different set of challenges than functions on intervals, and so this section and the next are devoted to the nuances particular to sequences. In fact, a whole course could be devoted to sequences, but the amount required for this text is more modest. What we need for Riemann Sums in Chapter 6 is very similar to some earlier concepts (in particular, limits as $x \to \infty$) and can be quickly dispatched here. Some of our study of sequences here will be required to prepare for the development of series, introduced in Chapter 10 and continued in Chapter 11. Recall for what follow that $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

3.10.1 Definitions and First Examples

Here we give a formal definition of an infinite sequence.

Definition 3.10.1 *An* **infinite sequence** *is a function whose domain is $\mathbb{N}$, or a similar (bounded from below) infinite subset of $\mathbb{Z}$, and whose range is a subset of $\mathbb{R}$. For instance, any function*

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$

will define a sequence.

In such a case, for $n \in \mathbb{N}$ we usually write, for instance, $f(n) = a_n$; the *subscript* n is used in place of the argument of the function, and the letter a is used to "name" the sequence. Another notation used to denote such a sequence is as follows:

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \cdots . \tag{3.88}$$

The notation on the left is read, "the sequence (of numbers) a_n as n ranges from 1 to ∞." The notation on the right is useful only to show whatever pattern may be contained in the sequence, and though common, is not ideal (since it is open to misinterpretation). A simple example where a sequence is "named," a formula is given for each term, and the pattern is established as in the right hand side of (3.88) is

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots . \tag{3.89}$$

So here $a_n = 1/n$ defines each term of the sequence.

For a sequence as in (3.89), we can see immediately that the terms are shrinking in size, and we would naturally enough want to describe this by stating something to the effect that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \overset{1/\infty}{=\!=\!=} 0.$$

First we need definitions for such a statement to make sense. Consider the following definition for a finite limit of a sequence:

Definition 3.10.2 *For a sequence $\{a_n\}_{n=1}^{\infty}$ and a finite number $L \in \mathbb{R}$, we define $\lim\limits_{n\to\infty} a_n = L$ by the following:*

$$\lim_{n\to\infty} a_n = L \iff (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n) \left[n > N \longrightarrow |a_n - L| < \varepsilon \right]. \tag{3.90}$$

Furthermore, in such a case we say the sequence $\{a_n\}_{n=1}^{\infty}$ **converges** *to L.*

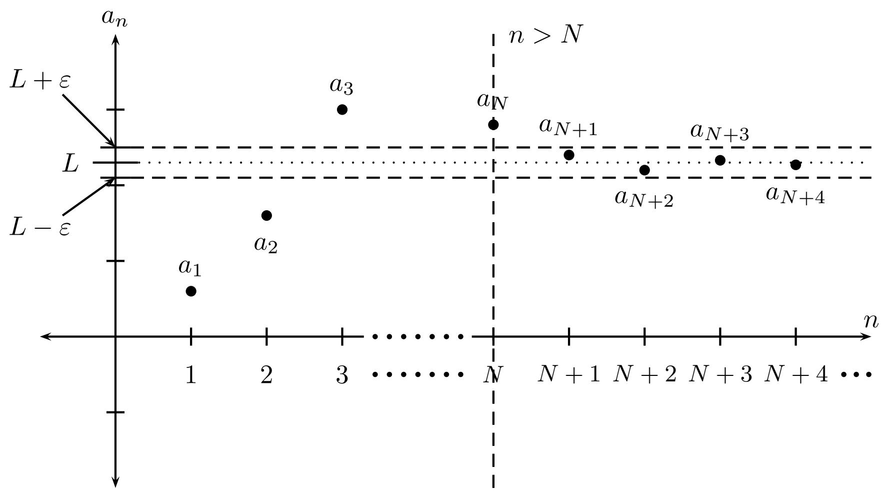


Figure 3.32: Illustration of the definition of a finite limit of a sequence, (3.90). In this type of graph, a_n versus n is plotted. If the sequence converges to L, then once a tolerance $\varepsilon > 0$ is chosen, we can find N (depending upon ε) so that $n > N \longrightarrow |a_n - L| < \varepsilon$, i.e., $a_n \in (L-\varepsilon, L+\varepsilon)$. Here the points are labeled by their heights, i.e., the values of the respective sequence elements a_n.

Compare this definition to (3.59). In fact, this is nearly identical to the definition for $\lim_{x\to\infty} f(x) = L$, except that the part of the *continuous* variable x is now played by the *discrete* variable n. An illustration of what (3.90) prescribes is given below:

$$a_1, a_2, a_3, \cdots, a_N, \underbrace{a_{N+1}, a_{N+2}, a_{N+3}, \cdots}_{\in(L-\varepsilon,L+\varepsilon)}.$$

Thus we can find a "tail end" of the sequence to be within ε of L if we travel far enough down the sequence. Another common illustration of this is given in Figure 3.32.

Other definitions from Section 3.8 have analogs for infinite sequences. For instance the case where $f(x) \longrightarrow \infty$ as $x \to \infty$, (3.64), becomes the following definition:

Definition 3.10.3 *For an infinite sequence* $\{a_n\}_{n=1}^{\infty}$, *we say the sequence* **diverges to infinity**, *and write* $\lim\limits_{n\to\infty} a_n = \infty$, *according to the following:*

$$\lim_{n\to\infty} a_n = \infty \iff (\forall M)(\exists N)(\forall n)(n > N \longrightarrow a_n > M). \tag{3.91}$$

The vocabulary of limits of sequences is slightly different from that of limits of functions on intervals. If $a_n \longrightarrow L \in \mathbb{R}$, we say the sequence *converges to* L, as we mentioned before. If no such real (and therefore *finite*) L exists, we say the sequence *diverges*. If, however, we can say $a_n \longrightarrow \infty$, we say the series *diverges to* ∞.[56] (Of course we similarly say a_n diverges to $-\infty$ when $a_n \longrightarrow -\infty$.)

We can employ all the relevant theorems for limits in x on intervals through their analogs in n. The theorem which shows earlier function limit theorems imply their sequence counterparts is the following:

[56]It would seem strange to say a sequence *converges* to ∞, since the verb *to converge* indicates getting close (or *approaching*). Of course we can not really "get close" to infinity.

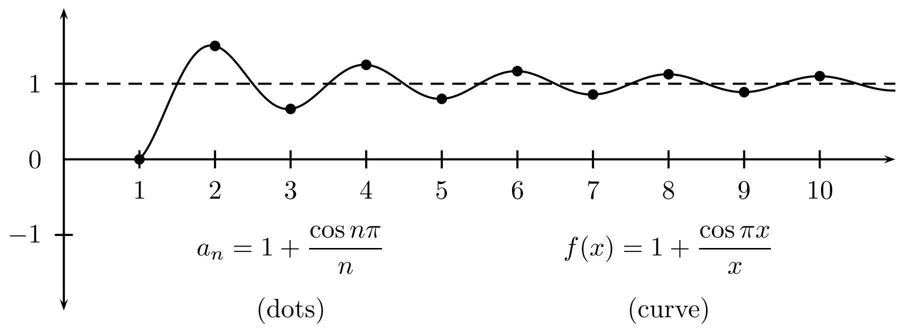


Figure 3.33: Illustration of a sequence $\{a_n\}_{n=1}^{\infty}$ (the dots) where $a_n = 1 + \dfrac{\cos n\pi}{n}$, along with the function (the curve) $f(x) = 1 + \dfrac{\cos \pi x}{x}$. Since $f(x) \longrightarrow 1$ as $x \to \infty$, that behavior carries the sequence, i.e., implying $a_n \longrightarrow 1$ as $n \to \infty$.

Theorem 3.10.1 *Suppose that* $\lim\limits_{x\to\infty} f(x) = L$*, where* L *is either a real number,* ∞ *or* $-\infty$*, and* $a_n = f(n)$ *for* $n \in \mathbb{N}$*. Then* $\lim\limits_{n\to\infty} a_n = L$ *as well. In symbolic logic, we can write*

$$\left(\lim_{x\to\infty} f(x) = L\right) \wedge \Big((\forall n \in \mathbb{N})\ a_n = f(n)\Big) \implies \left(\lim_{n\to\infty} a_n = L\right). \tag{3.92}$$

We will not prove this since first, it is intuitive on its face upon reflection, and second, because it is not difficult to see that it should be the case based upon the similarities of the limit definitions with functions $f : [1, \infty) \longrightarrow \mathbb{R}$ and sequences $\{a_n\}_{n=1}^{\infty}$. A nice way to summarize Theorem 3.92 is the phrase "the function carries the sequence." Figure 3.33 shows this mechanism in action for a particular example (Example 3.10.1 below). In fact there is much more information in the statement $f(x) \longrightarrow L$ than in $a_n \longrightarrow L$, since the function outputs values for a continuum of x-values, including between the positive integer values. However, as we will eventually see, the sequence can not "carry" the function. But for now let us look at a simple example of the theorem.

Example 3.10.1 *Suppose* $a_n = 1 + \dfrac{\cos n\pi}{n}$*. Then, just as*

$$\lim_{x\to\infty}\left[1 + \frac{\cos \pi x}{x}\right] \overset{1+\frac{B}{\infty}}{=\!=\!=} 1 + 0 = 1,$$

we have the sequence $\{a_n\}_{n=1}^{\infty}$ *converges to 1 also, i.e.,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[1 + \frac{\cos \pi n}{n}\right] \overset{1+\frac{B}{\infty}}{=\!=\!=} 1 + 0 = 1.$$

This is illustrated in Figure 3.33. (Recall that the "B" refers to a term which is "bounded," in this case referring to the fact that $|\cos \pi x| \le 1$*, and so* B/∞ *is a determinate form which yields zero in the limit.)*

In the above example, we assumed that B/∞ is a determinate form for sequences, which is the case with functions of a continuous variable (such as our usual x). Theorem 3.10.1 guarantees that this does, indeed, work with sequences as with functions defined on $[1, \infty)$. We will not bother to chase through the details here, but with what we have developed earlier in the chapter, it should ring true.[57] The limits of other sequences can also be found using earlier methods (though later we will see the limitations of Theorem 3.10.1).

Example 3.10.2 *Consider the sequence* $\{a_n\}_{n=1}^{\infty} = \left\{\dfrac{n^3-n}{n^2+1}\right\}_{n=1}^{\infty}$. *Now*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3-n}{n^2+1} = \lim_{n\to\infty} \frac{n^2\left(n-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n-\frac{1}{n}}{1+\frac{1}{n^2}} \overset{\frac{\infty-0}{1+0}}{=\!=\!=} \infty.$$

Thus the sequence $\{a_n\}$ *diverges to* ∞.

Anytime we can meaningfully replace $n \in \mathbb{N}$ with $x \in [1,\infty)$ (or a similar interval, unbounded from above), and the limit as $x \to \infty$ exists, we can use the methods of earlier sections. However, there are examples where the sequence is better behaved than the function.

Example 3.10.3 *Define* $\{a_n\}_{n=1}^{\infty}$ *so that* $a_n = n \sin n\pi$. *Now*

$$\lim_{x\to\infty} x\sin(x\pi) \text{ does not exist,}$$

since the function $f(x) = x\sin(x\pi)$ *oscillates in an unbounded way as* $x \to \infty$. *This behavior is illustrated in Figure 3.34. However, a closer look at* $\{a_n\}_{n=1}^{\infty}$ *shows that*

$$\{a_n\}_{n=1}^{\infty} = 0, 0, 0, 0, 0, \cdots \implies \lim_{n\to\infty} a_n = 0.$$

The example above does not contradict Theorem 3.10.1; the theorem is of the form $P \longrightarrow Q$, while here we have $(\sim P)$, a case the theorem does not address.

Because we can often replace $a_n = f(n)$ with a function $f : [1,\infty) \longrightarrow \mathbb{R}$ which has an existing limit as $x \to \infty$, many of the theorems which were available to us for functions on the continuum have relevance here as well. For example, the Sandwich Theorem (page 239) and its variants apply to sequences as well (just replacing x with n). Consider for example the following:

Example 3.10.4 *Consider* $\left\{\dfrac{n^2\sin(n^2+1)}{n^3+1}\right\}$. *Using algebraic methods we can write*

$$\lim_{n\to\infty} \frac{n^2\sin(n^2+1)}{n^3+1} = \lim_{n\to\infty} \frac{n^2\sin(n^2+1)}{n^3\left(1+\frac{1}{n^3}\right)} = \lim_{n\to\infty} \left[\sin(n^2+1)\cdot\frac{1}{n\left(1+\frac{1}{n^3}\right)}\right] \overset{B\cdot 0}{=\!=\!=} 0.$$

The form $B \cdot 0 = 0$, recall, represents a bounded function multiplying one which approaches zero, implying the product approaches zero, but the argument was essentially a Sandwich Theorem argument; here one could write $-\frac{n^2}{n^3+1} \le a_n \le \frac{n^2}{n^3+1}$, and since $\pm\frac{n^2}{n^3+1} \longrightarrow 0$ as $n \to \infty$, we conclude $a_n \to 0$. Other more explicitly sandwich theorem applications will occur as we proceed.

Example 3.10.5 *Consider the sequence* $\{a_n\}_{n=1}^{\infty} = \left\{\sqrt{\dfrac{6n-1}{n}}\right\}_{n=1}^{\infty}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n\left(6-\frac{1}{n}\right)}{n}} = \lim_{n\to\infty} \sqrt{6-\frac{1}{n}} = \sqrt{6-0} = \sqrt{6}.$$

[57]It is interesting to modify earlier proofs for functions of x to the language of sequences and thus actually *prove* the same theorems for sequences, but we will not do so here. In fact, earlier in the chapter we skipped many of the proofs for the cases $x \to \infty$ because they produced forms we analyzed for limits with $x \to a$ where a was a finite number.

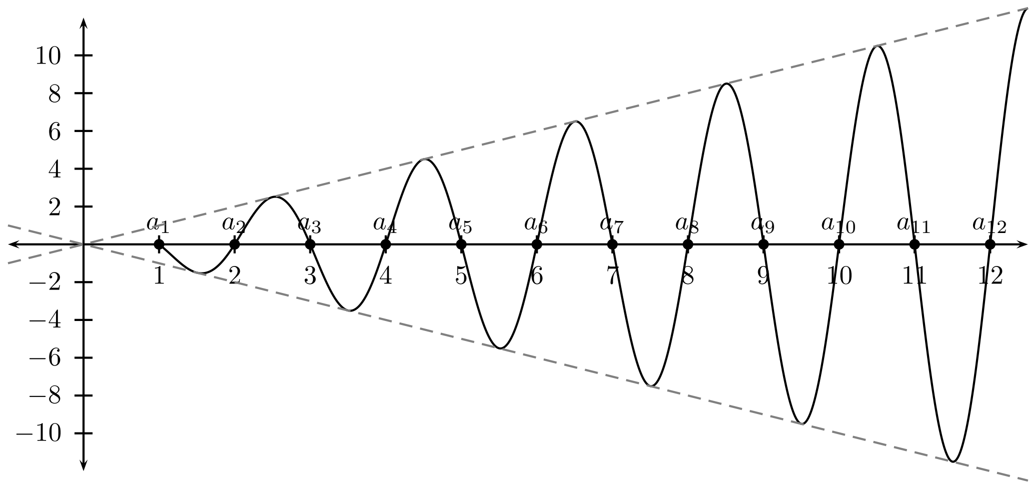


Figure 3.34: The sequence $\{a_n\}_{n=1}^{\infty} = \{n \sin n\pi\}_{n=1}^{\infty}$ is just a sequence of zeros (given by the dots above), while the function $f(x) = x\sin(x\pi)$ oscillates with growing distance between peaks and adjacent troughs. As $x \to \infty$, there is no limit for $f(x)$, though $a_n \longrightarrow 0$ as $n \to \infty$. This does not contradict Theorem 3.10.1, since that states that the function can carry the sequence, not that the sequence can carry the function. (Note $-x \le x\sin(\pi x) \le x$, so $f(x)$ lies between the dashed lines $y = \pm x$.)

Example 3.10.6 *Consider the sequence* $\{a_n\}_{n=1}^{\infty} = \left\{n^2 \cos \dfrac{1}{n}\right\}_{n=1}^{\infty}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 \cos \frac{1}{n} \overset{\infty\cdot 1}{=\!=} \infty.$$

In such a case we would say that a_n diverges to ∞ (or just diverges, to be less descriptive).

Theorem 3.10.2 *If $a_n \longrightarrow L$ and $f(x)$ is continuous at L, then $f(a_n) \longrightarrow f(L)$.*

This theorem can often be avoided because of other methods. For instance, it was not necessary in Example 3.10.5, but it could have been used with $f(x) = \sqrt{x}$, and the sequence $a_n = \frac{6n-1}{n} \longrightarrow 6$ implying the sequence $\sqrt{a_n} = f(a_n) \longrightarrow \sqrt{6}$, since $f(x) = \sqrt{x}$ is continuous at $x = 6$. In the next example the theorem is more useful.

Example 3.10.7 *Consider* $a_n = \cos\left(n \sin \frac{1}{n}\right)$.

Since $\cos x$ is continuous for all $x \in \mathbb{R}$, we can first work "inside" that function to compute (as we have done before):

$$\lim_{n\to\infty} n \sin \frac{1}{n} \overset{\infty\cdot 0}{=\!=} \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1,$$

where we made the substitution $\theta = \frac{1}{n} \to 0^+$ as $n \to \infty$. Thus $\cos\left(n \sin \frac{1}{n}\right) \to \cos 1 \approx 0.540302306$ as $n \to \infty$.

We usually do not mention such a theorem as we make these computations, but its statement provides a useful fact to call upon in abstract arguments.[58] A professional mathematician might

[58] In fact its continuum analog had much to do with our "limits by substitution" methods.

phrase the theorem something like, "continuous functions preserve convergence of a sequence," or "continuous functions take (or map) convergent sequences to convergent sequences." Still, an example like the above may be summarized without direct reference to the theorem:

$$\lim_{n\to\infty} \cos\left(n \sin\frac{1}{n}\right) \xlongequal{\cos(\infty\cdot 0)} \lim_{n\to\infty} \cos\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) \xlongequal{\cos 1} \cos 1.$$

In doing so recall that we have to be sure that the argument of the sine function approached zero at the same rate as the denominator, as occurs here.

3.10.2 Subsequences

As Example 3.10.3 (page 285) showed, there is more to studying sequences than just a rephrasing of our earlier study of limits "at infinity" that we began in Section 3.8. In fact the ability to easily assign a function on $[1,\infty)$ with the same limiting behavior as $\{a_n\}_{n=1}^{\infty}$ is a rather special (though very common) case. Consider the following example.

Example 3.10.8 *Define $a_n = (-1)^n + 1$, for $n = 1, 2, 3, \cdots$.*

First note that it is impossible to have n replaced by $x \in [1,\infty)$ in the formula, since $(-1)^x$ is undefined for x of the form p/q, where $p, q \in \mathbb{N}$, p/q is simplified and q is even. It ($(-1)^x$) is also undefined for $x \in \mathbb{R} - \mathbb{Q}$, i.e., irrational values of x. For this sequence we really do have to analyze it on its face. One method is to list the terms and see the pattern:

$$\begin{aligned} \{a_n\}_{n=1}^{\infty} &= \quad -1+1, \quad 1+1, \quad -1+1, \quad 1+1, \quad -1+1, \quad 1+1, \quad \cdots \\ &= \qquad 0, \qquad 2, \qquad 0, \qquad 2, \qquad 0, \qquad 2, \cdots. \end{aligned}$$

We see that the sequence never converges to a unique number, and therefore the limit does not exist, and so the sequence is divergent.

The example above shows more than just the fact that there are sequences which are best analyzed on their faces. It also shows that sequences diverge in ways other than towards ∞ or $-\infty$. Finally we see that the sequence is really a union of two sequences, one which is all zeros and the other is all twos. These can be listed as $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n-1}\}_{n=1}^{\infty}$:

$$\begin{aligned} \{a_{2n}\}_{n=1}^{\infty} &= a_2, a_4, a_6, a_8, \cdots = 2, 2, 2, 2, \cdots, \\ \{a_{2n-1}\}_{n=1}^{\infty} &= a_1, a_3, a_5, a_7, \cdots = 0, 0, 0, 0, \cdots. \end{aligned}$$

What we have above are two examples of *subsequences* of a given sequence $\{a_n\}_{n=1}^{\infty}$. Now we give the formal definition.

Definition 3.10.4 *Given an infinite sequence $\{a_n\}_{n=1}^{\infty}$ and a set of natural numbers $n_1 < n_2 < n_3 < n_4 < \cdots$, another sequence $\{b_k\}_{k=1}^{\infty} = \{a_{n_k}\}_{k=1}^{\infty}$ is called a* **subsequence** *of the original sequence $\{a_n\}_{n=1}^{\infty}$.*

Thus the sequence $b_k = a_{n_k}$ is gotten by moving along the sequence a_n and picking out the terms a_{n_1}, a_{n_2}, a_{n_3}, etc.

The following theorem contains three results which partially explain the convergence relationships among a sequence and its subsequences. Though all three results are very closely related, their emphases are somewhat different and so we list them separately. The final result, Theorem 3.10.3c, will be the most useful.

Theorem 3.10.3 *Given a sequence* $\{a_n\}_{n=1}^{\infty}$.

(a) *If* $a_n \longrightarrow L$*, then for any subsequence* $\{a_{n_k}\}_{k=1}^{\infty}$*, we also have* $a_{n_k} \longrightarrow L$ *as well (as* $k \to \infty$*).*

(b) *Moreover,* $a_n \longrightarrow L$ *if and only if for every subsequence* $\{a_{n_k}\}_{k=1}^{\infty}$*, we have* $a_{n_k} \longrightarrow L$.

(c) *Finally, suppose that* $n_{k_1} < n_{k_2} < n_{k_3} < \cdots$ *and* $m_{j_1} < m_{j_2} < m_{j_3} < \cdots$*, where* $\{n_k\} \cup \{m_j\} = \{1, 2, 3, 4, \cdots\}$*, and that* $b_k = a_{n_k}$*,* $c_j = a_{m_j}$*. In other words,* $\{b_k\}$ *and* $\{c_j\}$ *are subsequences of* $\{a_n\}$ *which exhaust that whole sequence. Then (as* $k, j, n \to \infty$*)*

$$(b_k \longrightarrow L) \wedge (c_j \longrightarrow L) \iff (a_n \longrightarrow L).$$

Rephrased, if we have two subsequences who collectively contain the whole original sequence, then the convergence of the original sequence to L *is equivalent to the convergence of the subsequences also to* L.

The first (a) can be rephrased to state that "the sequence carries all subsequences." The second statement (b) needs no rephrasing, and is mostly useful for junior or senior real analysis courses. The last one (c) is the most useful here, stating that if we want to check for a limit of the full sequence, it is enough to check two subsequences whose members exhaust the full sequence. For completeness we include a proof below, but the truths of these should be apparent on their faces upon reflection.[59]

Proof: We will prove all of these for the case that the limit in question is some finite $L \in \mathbb{R}$. Simple modifications of the proof given here will cover the cases $L = \pm\infty$.

(a) First assume $a_n \longrightarrow L$ and $b_k = a_{n_k}$ is any subsequence. Now we make the observation that $n_k \geq k$, since the kth choice as we move down the original sequence cannot happen before we come to the kth term of that sequence. Put another way, if $n_1, n_2, n_3, \cdots \in \mathbb{N}$, and $n_1 < n_2 < n_3 < \cdots$, then $n_k \geq k$. By the convergence of the original sequence to L we have

$$(\forall \varepsilon > 0)(\exists N)(\forall n)\,[n > N \longrightarrow |a_n - L| < \varepsilon]. \tag{3.93}$$

But then once ε and N are chosen so that (3.93) holds true, we have

$$k > N \implies n_k \geq k > N \implies |a_{n_k} - L| < \varepsilon \iff |b_k - L| < \varepsilon.$$

In other words, for any $\varepsilon > 0$, the N from (3.93) gives us $k > N \longrightarrow |b_k - L| < \varepsilon$. Thus the definition for $a_{n_k} = b_k \longrightarrow L$ holds. This shows that every subsequence of $\{a_n\}_{n=1}^{\infty}$ also converges to L, q.e.d.

(b) By (a), we know that $a_n \longrightarrow L \implies a_{n_k} \longrightarrow L$. We need to show the converse ($\Longleftarrow$), i.e., that if every every subsequence converges to L, this forces the sequence to also converge to L. But this is trivial, since, if *every* subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to L, then the subsequence a_{n_k} defined by $n_1 = 1, n_2 = 2, n_3 = 3, \cdots$ must converge to L. But that is just the statement $\{a_n\}_{n=1}^{\infty}$ converges to L, q.e.d. (In other words, the sequence is itself a subsequence, so if every subsequence converges to L, then so must the sequence itself, q.e.d.)

[59] Indeed the reader should not let the proof distract, but can in good conscience bypass the proof for the moment, and return later when more familiar with sequences in general.

(c) By (a) (and, for that matter, (b)), we have ($\Longleftarrow$). To show ($\Longrightarrow$), suppose $b_k, c_j \longrightarrow L$ and that the union of these subsequences is the full sequence $\{a_n\}_{n=1}^{\infty}$. For $\varepsilon > 0$, choose n_b and n_c so that

$$\begin{aligned} n > n_b &\longrightarrow |b_n - L| < \varepsilon, \\ n > n_c &\longrightarrow |c_n - L| < \varepsilon. \end{aligned}$$

Now choose N_b, N_c so that a_{N_b} is the term chosen from the original sequence to represent b_{n_b} in the first subsequence, and a_{N_c} from the original is the term c_{n_c} in the second subsequence. In other words, the n_bth term in $\{b_k\}_{k=1}^{\infty}$ is the N_bth term in $\{a_n\}_{n=1}^{\infty}$, and the n_cth term in $\{c_j\}_{j=1}^{\infty}$ is the chosen to be the N_cth term in $\{a_n\}_{n=1}^{\infty}$. Now let $N = \max\{N_b, N_c\}$. Then

$$\begin{aligned} n > N &\Longrightarrow (a_n = b_k,\ \text{some } k > n_b) \vee (a_n = c_k,\ \text{some } k > n_c) \\ &\Longrightarrow (|a_n - L| = |b_k - L| < \varepsilon) \vee (|a_n - L| = |c_k - L| < \varepsilon) \\ &\Longrightarrow |a_n - L| < \varepsilon,\ \text{q.e.d.} \end{aligned}$$

Note where the proof of (c) required that the subsequences $\{b_k\}_{k=1}^{\infty}, \{c_j\}_{j=1}^{\infty}$ exhaust the original sequence $\{a_n\}_{n=1}^{\infty}$ (two lines up from "q.e.d."). A quick look at the proof indicates the following corollary can be proved the same way:

Corollary 3.10.1 *Given a sequence and any finite set of subsequences whose entries exhaust the original sequence (in the sense of Theorem 3.10.3c), we then have the limit of the original sequence is L if and only if each subsequence also has limit L.*

It sometimes occurs that we analyze a sequence $\{a_n\}_{n=1}^{\infty}$ by looking at the odd- and even-indexed subsequences separately, since together they exhaust the original sequence. This is especially useful when the original sequence can be simplified differently for odd and even terms. Note how

$$\{a_n\}_{n=1}^{\infty} = \{a_{2n-1}\}_{n=1}^{\infty} \cup \{a_{2n}\}_{n=1}^{\infty}.$$

If instead we wish to look at three subsequences, each of every third term, we could write

$$\{a_n\}_{n=1}^{\infty} = \{a_{3n-2}\}_{n=1}^{\infty} \cup \{a_{3n-1}\}_{n=1}^{\infty} \cup \{a_{3n}\}_{n=1}^{\infty},$$

and so on.

Example 3.10.9 *Discuss the limiting behavior of the sequence* $\left\{\dfrac{(-1)^n n}{n+1}\right\}$.

Solution: Because this sequence alternates signs, we will look at the even and odd terms separately.

$$\begin{aligned} a_{2n-1} &= \frac{(-1)^{2n-1}(2n-1)}{(2n-1)+1} = \frac{(-1)(2n-1)}{2n} = \frac{(-1)n\left(2-\frac{1}{n}\right)}{n(2)} \\ &= \frac{(-1)\left(2-\frac{1}{n}\right)}{2} \longrightarrow -1 \cdot \frac{2}{2} = -1; \\ a_{2n} &= \frac{(-1)^{2n}(2n)}{(2n)+1} = \frac{2n}{2n+1} = \frac{n(2)}{n\left(2+\frac{1}{n}\right)} \\ &= \frac{2}{2+\frac{1}{n}} \longrightarrow \frac{2}{2} = 1. \end{aligned}$$

Thus the subsequence $\{a_{2n-1}\}_{n=1}^{\infty}$ of odd terms approaches -1 while the subsequence $\{a_{2n}\}_{n=1}^{\infty}$ of even terms approaches 1. Since two subsequences have different limits, we conclude the original sequence diverges.[60]

In the example above we used the fact that odd powers of (-1) (e.g., $(-1)^{2n-1}$) yield (-1), and even powers (e.g., $(-1)^{2n}$) yield 1. Some other ways to achieve alternation of signs use trigonometric functions, which have the conceptual advantage that they are continuous on all of $\mathbb{R}$:

$$\cos n\pi = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even}, \end{cases}$$

$$\sin \frac{(2n-1)\pi}{2} = \begin{cases} 1, & n \text{ odd} \\ -1, & n \text{ even}. \end{cases}$$

Exercises

1. Show that an alternating sequence $\{a_n\}$ converges if and only if $|a_n| \longrightarrow 0$, and thus $\{a_n\}$ converges if and only if $a_n \longrightarrow 0$.

[60]This is a case one can say the sequence diverges "by oscillation," meaning one subsequence goes to L, another to $M \neq L$, and these subsequences exhaust the original sequence. See also Example 3.10.8, page 287.

3.11 Sequences II

Here we examine some of the more sophisticated arguments regarding sequences. In particular we will revisit the least upper bound property of $\mathbb{R}$ (page 83), and its implications for bounded and so-called **monotonic** sequences. These will be of crucial importance theoretically for many of the convergence theorems for series in Chapter 10 (and thus Chapter 11). First we need some definitions.

Definition 3.11.1 *We call a sequence* $\{a_n\}_{n=1}^{\infty}$

nondecreasing *if and only if* $a_1 \leq a_2 \leq a_3 \leq \cdots$, *i.e,*

$$(\forall n \in \mathbb{N})\,[a_n \leq a_{n+1}]\,;$$

nonincreasing *if and only if* $a_1 \geq a_2 \geq a_3 \geq \cdots$, *i.e.,*

$$(\forall n \in \mathbb{N})\,[a_n \geq a_{n+1}]\,;$$

increasing *if and only if* $a_1 < a_2 < a_3 < \cdots$, *i.e.,*

$$(\forall n \in \mathbb{N})\,[a_n < a_{n+1}]\,;$$

decreasing *if and only if* $a_1 > a_2 > a_3 > \cdots$, *i.e.,*

$$(\forall n \in \mathbb{N})\,[a_n > a_{n+1}]\,.$$

Note that

$$\begin{aligned} \{a_n\} \text{ increasing} &\implies \{a_n\} \text{ nondecreasing}, \\ \{a_n\} \text{ decreasing} &\implies \{a_n\} \text{ nonincreasing}. \end{aligned}$$

We will often have increasing or decreasing sequences, but it turns out that our important theorems only require that the sequences are either nondecreasing or nonincreasing, and so these two weaker categories (and thus all four categories) above are collected into one concept:

Definition 3.11.2 *Any sequence* $\{a_n\}_{n=1}^{\infty}$ *which is either nondecreasing or nonincreasing is called monotonic.*

Again we point out that any of the four categories of sequences from the first definition are therefore monotonic.[61]

We also need to recall some definitions regarding boundedness of a set of numbers, except this time the set will be a sequence.

[61] It should be pointed out that there are two camps of writers when it comes to classifying monotonic sequences. Some writers—from the other camp—use "increasing" more loosely to mean what we call "nondecreasing" (meaning "never decreasing") here, and similarly use "decreasing" for our "nonincreasing." On its face this seems inaccurate, but one can argue from negations that these uses make perfect sense. For example, if one thinks of an increasing sequence as one which "never decreases," we need the definition

$$\sim [(\exists n \in \mathbb{N})\,(a_n > a_{n+1})] \equiv [(\forall n \in \mathbb{N})\,(a_n \leq a_{n+1})]\,,$$

the right-hand side of which is exactly this other camp's definition of "increasing."

Using these different definitions of "increasing" and "decreasing," this other camp (which will not coincide with this text's terminology) can say a monotonic sequence is thus one which is increasing (throughout the entire sequence), or decreasing (throughout the entire sequence).

In order to distinguish the cases where $a_n < a_{n+1}$ and $a_n \leq a_{n+1}$, this other camp calls the former case "strictly increasing," and the latter (again) simply "increasing."

Definition 3.11.3 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *is called*

- **bounded from above** *if and only if* $(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})\,[a_n \leq M]$*;*
- **bounded from below** *if and only if* $(\exists m \in \mathbb{R})(\forall n \in \mathbb{N})\,[m \leq a_n]$*.*
- **bounded** *if and only if it is bounded from above and below, i.e., if and only if* $(\exists m, M \in \mathbb{R})(\forall n \in \mathbb{N})\,[m \leq a_n \leq M]$*.*

Our crucial theorem—and main result of this section—is the following:

Theorem 3.11.1 *A bounded, monotonic sequence converges.*

The proof we give here contains many smaller results which are interesting in their own rights. For instance, we have

1. A nondecreasing function which is bounded from above necessarily converges.
2. A nonincreasing function which is bounded from below necessarily converges.

For this particular theorem we will defer the proof until the end of the section, to avoid distraction from the usual intuition, which is quite visual.