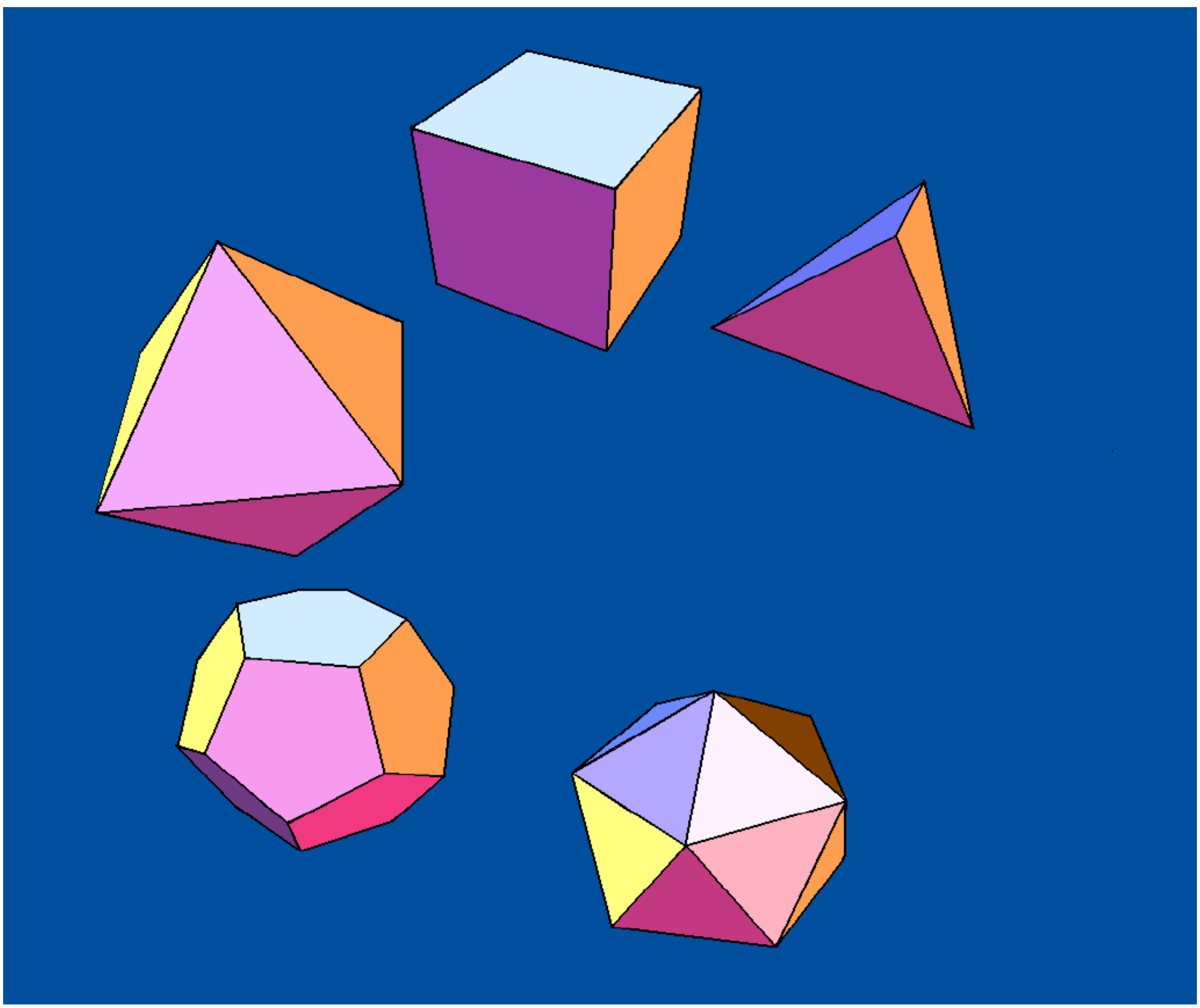
Introduction to Calculus

Volume II

by J.H. Heinbockel



The regular solids or regular polyhedra are solid geometric figures with the same identical regular polygon on each face. There are only five regular solids discovered by the ancient Greek mathematicians. These five solids are the following.

the tetrahedron (4 faces)

the cube or hexadron (6 faces)

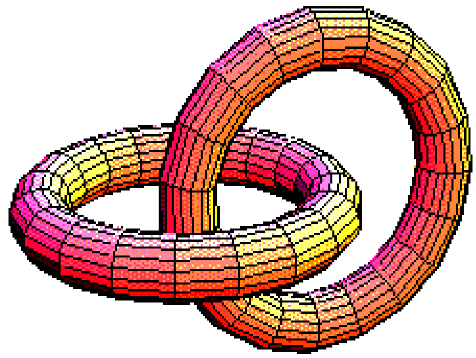
the octahedron (8 faces)

the dodecahedron (12 faces)

the icosahedron (20 faces)

Each figure follows the Euler formula

$$\begin{array}{ccccccc} \text{Number of faces} & + & \text{Number of vertices} & = & \text{Number of edges} & + & 2 \\ F & + & V & = & E & + & 2 \end{array}$$



Introduction to Calculus

Volume II

by J.H. Heinbockel

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Preface

This is the second volume of an introductory calculus presentation intended for future scientists and engineers. Volume II is a continuation of volume I and contains chapters six through twelve. The chapter six presents an introduction to vectors, vector operations, differentiation and integration of vectors with many application. The chapter seven investigates curves and surfaces represented in a vector form and examines vector operations associated with these forms. Also investigated are methods for representing arclength, surface area and volume elements from vector representations. The directional derivative is defined along with other vector operations and their properties as these additional vectors enable one to find maximum and minimum values associated with functions of more than one variable. The chapter 8 investigates scalar and vector fields and operations involving these quantities. The Gauss divergence theorem, the Stokes theorem and Green's theorem in the plane along with applications associated with these theorems are investigated in some detail. The chapter 9 presents applications of vectors from selected areas of science and engineering. The chapter 10 presents an introduction to the matrix calculus and the difference calculus. The chapter 11 presents an introduction to probability and statistics. The chapters 10 and 11 are presented because in todays society technology development is tending toward a digital world and students should be exposed to some of the operational calculus that is going to be needed in order to understand some of this technology. The chapter 12 is added as an after thought to introduce those interested into some more advanced areas of mathematics.

If you are a beginner in calculus, then be sure that you have had the appropriate background material of algebra and trigonometry. If you don't understand something then don't be afraid to ask your instructor a question. Go to the library and check out some other calculus books to get a presentation of the subject from a different perspective. The internet is a place where one can find numerous help aids for calculus. Also on the internet one can find many illustrations of the applications of calculus. These additional study aids will show you that there are multiple approaches to various calculus subjects and should help you with the development of your analytical and reasoning skills.

J.H. Heinbockel
January 2016

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Chapter 6

Introduction to Vectors

Scalars are quantities with **magnitude only** whereas vectors are those quantities having **both a magnitude and a direction.** Vectors are used to model a variety of fundamental processes occurring in engineering, physics and the sciences. The material presented in the pages that follow investigates both scalar and vectors quantities and operations associated with their use in solving applied problems. In particular, differentiation and integration techniques associated with both scalar and vector quantities will be investigated.

Vectors and Scalars

A vector is any quantity which possesses both magnitude and direction.

A scalar is a quantity which possesses a magnitude but does not possess a direction.

Examples of vector quantities are force, velocity, acceleration, momentum, weight, torque, angular velocity, angular acceleration, angular momentum.

Examples of scalar quantities are time, temperature, size of an angle, energy, mass, length, speed, density

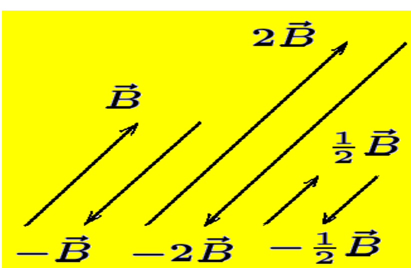


Figure 6-1.
Scalar multiplication.

A vector can be represented by an arrow. The **orientation of the arrow determines the direction of the vector**, and the **length of the arrow is associated with the magnitude of the vector.** The magnitude of a vector $\vec{B}$ is denoted $|\,\vec{B}\,|$ or B and represents the length of the vector. The **tail end of the arrow is called the origin**, and **the arrowhead is called the terminus**. Vectors are usually denoted by letters in bold face type. When a bold face type is inconvenient to use, then a letter with an arrow over it is employed, such as, $\vec{A}$, $\vec{B}$, $\vec{C}$. Throughout this text the arrow notation is used in all discussions of vectors.

Properties of Vectors

Some important properties of vectors are

1. Two vectors $\vec{A}$ and $\vec{B}$ are equal if they have the **same magnitude (length) and direction**. Equality is denoted by $\vec{A} = \vec{B}$.

2. The magnitude of a vector is **a nonnegative scalar quantity**. The magnitude of a vector $\vec{B}$ is denoted by the symbols B or $|\vec{B}|$.
3. A vector $\vec{B}$ is equal to zero only if its magnitude is zero. A vector whose magnitude is zero is called the **zero or null vector** and denoted by the symbol $\vec{0}$.
4. Multiplication of a nonzero vector $\vec{B}$ by a positive scalar m is denoted by $m\vec{B}$ and produces a new vector whose direction is the same as $\vec{B}$ but whose magnitude is m times the magnitude of $\vec{B}$. Symbolically, $|m\vec{B}| = m|\vec{B}|$. If m is a negative scalar the direction of $m\vec{B}$ is opposite to that of the direction of $\vec{B}$. In figure 6-1 several vectors obtained from $\vec{B}$ by scalar multiplication are exhibited.
5. Vectors are considered as **"free vectors".** The term "free vector" is used to mean the following. Any vector may be moved to a new position in space provided that in the new position it is **parallel to and has the same direction as its original position.** In many of the examples that follow, there are times when a given vector is moved to a convenient point in space in order to emphasize a special geometrical or physical concept. See for example figure 6-1.

Vector Addition and Subtraction

Let $\vec{C} = \vec{A} + \vec{B}$ denote **the sum of two vectors** $\vec{A}$ and $\vec{B}$. To find the vector sum $\vec{A}+\vec{B}$, slide the origin of the vector $\vec{B}$ to the terminus point of the vector $\vec{A}$, then draw the line from the origin of $\vec{A}$ to the terminus of $\vec{B}$ to represent $\vec{C}$. Alternatively, start with the vector $\vec{B}$ and place the origin of the vector $\vec{A}$ at the terminus point of $\vec{B}$ to construct the vector $\vec{B}+\vec{A}$. Adding vectors in this way employs the **parallelogram law for vector addition** which is illustrated in the figure 6-2. Note that vector addition is commutative. That is, using the shifted vectors $\vec{A}$ and $\vec{B}$, as illustrated in the figure 6-2, the commutative law for vector addition $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, is illustrated using the parallelogram illustrated. The addition of vectors can be thought of as connecting the origin and terminus of directed line segments.

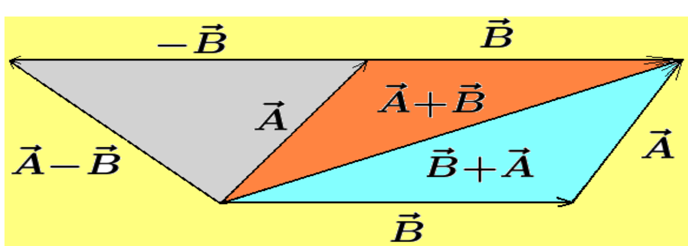


Figure 6-2. Parallelogram law for vector addition

If $\vec{F} = \vec{A} - \vec{B}$ denotes **the difference of two vectors** $\vec{A}$ and $\vec{B}$, then $\vec{F}$ is determined by the above rule for vector addition by writing $\vec{F} = \vec{A} + (-\vec{B})$. Thus, subtraction of the vector $\vec{B}$ from the vector $\vec{A}$ is represented by the addition of the vector $-\vec{B}$ to $\vec{A}$. In figure 6-2 observe that the vectors $\vec{A}$ and $\vec{B}$ are free vectors and have been translated to appropriate positions to illustrate the concepts of addition and subtraction. The sum of two or more force vectors is sometimes referred to as **the resultant force**. In general, the **resultant force** acting on an object is calculated by using **a vector addition** of all the forces acting on the object.

Vectors **constitute a group under the operation of addition**. That is, the following four properties are satisfied.

1. **Closure property** If $\vec{A}$ and $\vec{B}$ belong to a set of vectors, then their sum $\vec{A} + \vec{B}$ must also belong to the same set.
2. **Associative property** The insertion of parentheses or grouping of terms in vector summation is immaterial. That is,
$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \tag{6.50}$$
3. **Identity element** The zero or null vector when added to a vector does not produce a new vector. In symbols, $\vec{A} + \vec{0} = \vec{A}$. The null vector is called the identity element under addition.
4. **Inverse element** If to each vector $\vec{A}$, there is associated a vector $\vec{E}$ such that under addition these two vectors produce the identity element, and $\vec{A} + \vec{E} = \vec{0}$, then the vector $\vec{E}$ is called the inverse of $\vec{A}$ under vector addition and is denoted by $\vec{E} = -\vec{A}$.

   Additional properties satisfied by vectors include
5. **Commutative law** If in addition all vectors of the group satisfy $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, then the set of vectors is said to form a commutative group under vector addition.
6. **Distributive law** The distributive law with respect to scalar multiplication is
$$m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}, \qquad \text{where } m \text{ is a scalar.} \tag{6.51}$$

Definition (Linear combination)

If there exists constants $c_1, c_2, \ldots, c_n$, not all zero, together with a set of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$, such that
$$\vec{A} = c_1\vec{A}_1 + c_2\vec{A}_2 + \cdots + c_n\vec{A}_n,$$
then the vector $\vec{A}$ is said to be a linear combination of the vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$.

Definition (Linear dependence and independence of vectors)
Two nonzero vectors $\vec{A}$ ***and*** $\vec{B}$ ***are said to be*** **linearly dependent** ***if it is possible to find scalars*** k_1, k_2 ***not both zero, such that the equation***

$$k_1\vec{A} + k_2\vec{B} = \vec{0} \tag{6.3}$$

is satisfied. If $k_1 = 0$ ***and*** $k_2 = 0$ ***are the only scalars for which the above equation is satisfied, then the vectors*** $\vec{A}$ ***and*** $\vec{B}$ ***are said to be*** **linearly independent.**

This definition can be interpreted geometrically. If $k_1 \neq 0$, then equation (6.3) implies that $\vec{A} = -\frac{k_2}{k_1}\vec{B} = m\vec{B}$ showing that $\vec{A}$ is a scalar multiple of $\vec{B}$. That is, $\vec{A}$ and $\vec{B}$ have the same direction and therefore, they are called **colinear vectors**. If $\vec{A}$ and $\vec{B}$ are not colinear, then they are linearly independent **(noncolinear)**. If two nonzero vectors $\vec{A}$ and $\vec{B}$ are linearly independent, then any vector $\vec{C}$ lying in the plane of $\vec{A}$ and $\vec{B}$ can be expressed as a linear combination of the these vectors. Construct as in figure 6-3 a parallelogram with diagonal $\vec{C}$ and sides parallel to the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide.

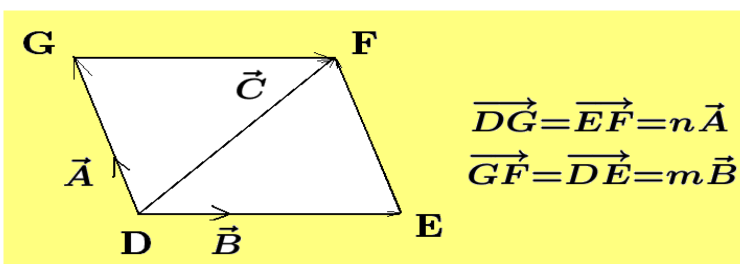


Figure 6-3. Vector $\vec{C}$ is a linear combination of vectors $\vec{A}$ and $\vec{B}$.

Since the vector side $\overrightarrow{DE}$ is parallel to $\vec{B}$ and the vector side $\overrightarrow{EF}$ is parallel to $\vec{A}$, then there exists scalars m and n such that $\overrightarrow{DE} = m\vec{B}$ and $\overrightarrow{EF} = n\vec{A}$. With vector addition,

$$\vec{C} = \overrightarrow{DE} + \overrightarrow{EF} = m\vec{B} + n\vec{A} \tag{6.54}$$

which shows that $\vec{C}$ is a linear combination of the vectors $\vec{A}$ and $\vec{B}$.

Example 6-1. Show that the medians of a triangle meet at a trisection point.

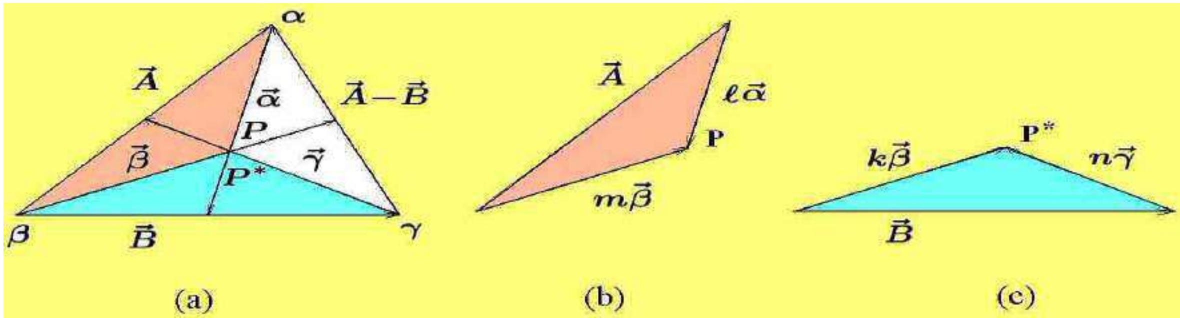


Figure 6-4. Constructing medians of a triangle

Solution: Let the sides of a triangle with vertices α, β, γ be denoted by the vectors $\vec{A}$, $\vec{B}$, and $\vec{A}-\vec{B}$ as illustrated in the figure 6-4. Further, let $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ denote the vectors from the respective vertices of α, β, γ to the midpoints of the opposite sides. By using the above definitions one can construct the following vector equations

$$\vec{A}+\vec{\alpha}=\frac{1}{2}\vec{B} \qquad \vec{B}+\frac{1}{2}(\vec{A}-\vec{B})=\vec{\beta} \qquad \vec{B}+\vec{\gamma}=\frac{1}{2}\vec{A}. \tag{6.5}$$

Let the vectors $\vec{\alpha}$ and $\vec{\beta}$ intersect at a point designated by P, Similarly, let the vectors $\vec{\beta}$ and $\vec{\gamma}$ intersect at the point designated P^*. The problem is to show that the points P and P^* are the same. Figures 6-4(b) and 6-4(c) illustrate that for suitable scalars k, ℓ, m, n, the points P and P^* determine the vectors equations

$$\vec{A}+\ell\vec{\alpha}=m\vec{\beta} \qquad \text{and} \qquad \vec{B}+n\vec{\gamma}=k\vec{\beta}. \tag{6.6}$$

In these equations the scalars k, ℓ, m, n are unknowns to be determined. Use the set of equations (6.5), to solve for the vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ in terms of the vectors $\vec{A}$ and $\vec{B}$ and show

$$\vec{\alpha}=\frac{1}{2}\vec{B}-\vec{A} \qquad \vec{\beta}=\frac{1}{2}(\vec{A}+\vec{B}) \qquad \vec{\gamma}=\frac{1}{2}\vec{A}-\vec{B}. \tag{6.7}$$

These equations can now be substituted into the equations (6.6) to yield, after some simplification, the equations

$$(1-\ell-\frac{m}{2})\vec{A}=(\frac{m}{2}-\frac{\ell}{2})\vec{B} \quad \text{and} \quad (\frac{k}{2}-\frac{n}{2})\vec{A}=(1-n-\frac{k}{2})\vec{B}.$$

Since the vectors $\vec{A}$ and $\vec{B}$ are linearly independent (noncolinear), the scalar coefficients in the above equation must equal zero, because if these scalar coefficients were not zero, then the vectors $\vec{A}$ and $\vec{B}$ would be linearly dependent (colinear) and a triangle would not exist. By equating to zero the scalar coefficients in these equations, there results the simultaneous scalar equations

$$(1-\ell-\frac{m}{2})=0, \qquad (\frac{m}{2}-\frac{\ell}{2})=0, \qquad (\frac{k}{2}-\frac{n}{2})=0, \qquad (1-n-\frac{k}{2})=0$$

The solution of these equations produces the fact that $k=\ell=m=n=\frac{2}{3}$ and hence the conclusion $P=P^*$ is a trisection point.

■

Unit Vectors

A vector having length or magnitude of one is called **a unit vector**. If $\vec{A}$ is a nonzero vector of length $|\vec{A}|$, a unit vector in the direction of $\vec{A}$ is obtained by multiplying the vector $\vec{A}$ by the scalar $m=\frac{1}{|\vec{A}|}$. The unit vector so constructed is denoted

$$\hat{\mathbf{e}}_A=\frac{\vec{A}}{|\vec{A}|} \qquad \text{and satisfies} \qquad |\,\hat{\mathbf{e}}_A|=1.$$

The symbol $\hat{\mathbf{e}}$ is reserved for unit vectors and the notation $\hat{\mathbf{e}}_A$ is to be read **"a unit vector in the direction of $\vec{A}$."** The hat or carat ($\hat{}$) notation is used to represent a **unit vector or normalized vector.**

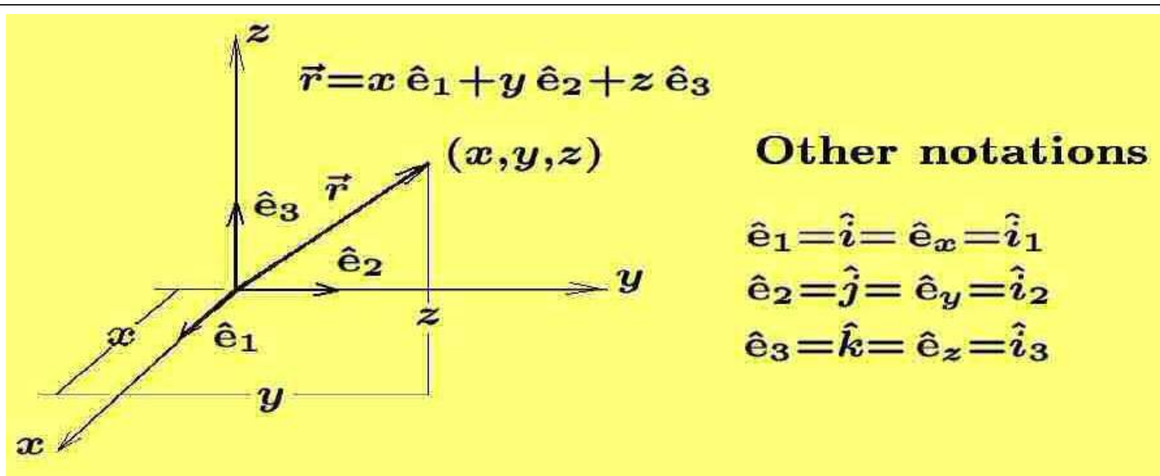


Figure 6-5. Cartesian axes.

The figure 6-5 illustrates unit base vectors $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$ in the directions of the positive x,y,z-coordinate axes in a rectangular three dimensional cartesian coordinate system. These unit base vectors in the direction of the x,y,z axes have historically been represented by a variety of notations. Some of the more common notations employed in various textbooks to denote **rectangular unit base vectors** are

$$\hat{\imath},\ \hat{\jmath},\ \hat{k}, \quad \hat{\mathbf{e}}_x,\ \hat{\mathbf{e}}_y,\ \hat{\mathbf{e}}_z, \quad \hat{\imath}_1,\ \hat{\imath}_2,\ \hat{\imath}_3, \quad \overline{1}_x,\ \overline{1}_y,\ \overline{1}_z, \quad \hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$$

The notation $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$ to represent the unit base vectors in the direction of the x, y, z axes will be used in the discussions that follow as this notation makes it easier to generalize vector concepts to n-dimensional spaces.

Scalar or Dot Product (inner product)

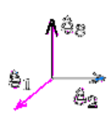
The **scalar or dot product of two vectors** is sometimes referred to as **an inner product of vectors**.

Definition (Dot product) ***The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is denoted***

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\, \cos\theta, \tag{6.8}$$

and represents the magnitude of $\vec{A}$ times the magnitude $\vec{B}$ times the cosine of θ, where θ is the angle between the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide.

The angle between any two of the orthogonal unit base vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in cartesian coordinates is 90° or $\frac{\pi}{2}$ radians. Using the results $\cos\frac{\pi}{2} = 0$ and $\cos 0 = 1$, there results the following dot product relations for these unit vectors



$$\begin{array}{lll} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = 1 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 = 0 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 = 0 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = 1 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2 = 0 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 = 0 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 = 1 \end{array} \tag{6.9}$$

Using **an index notation** the above dot products can be expressed $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ where the subscripts i and j can take on any of the integer values $1, 2, 3$. Here δ_{ij} is the **Kronecker delta symbol**[1] defined by $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

The dot product satisfies the following properties

Commutative law $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

Distributive law $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Magnitude squared $\vec{A} \cdot \vec{A} = A^2 = |\vec{A}|^2$

which are proved using the definition of a dot product.

---

[1] Leopold Kronecker (1823-1891) A German mathematician.

The physical interpretation of **projection** can be assigned to the dot product as is illustrated in figure 6-6. In this figure $\vec{A}$ and $\vec{B}$ are nonzero vectors with $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_B$ **unit vectors in the directions of** $\vec{A}$ **and** $\vec{B}$, respectively. The figure 6-6 illustrates the physical interpretation of the following equations:

$$\hat{\mathbf{e}}_B \cdot \vec{A} = |\vec{A}|\,\cos\theta = \text{Projection of } \vec{A} \text{ onto direction of } \hat{\mathbf{e}}_B$$
$$\hat{\mathbf{e}}_A \cdot \vec{B} = |\vec{B}|\,\cos\theta = \text{Projection of } \vec{B} \text{ onto direction of } \hat{\mathbf{e}}_A.$$

In general, the dot product of a nonzero vector $\vec{A}$ with a unit vector $\hat{\mathbf{e}}$ is given by $\vec{A}\cdot\hat{\mathbf{e}} = \hat{\mathbf{e}}\cdot\vec{A} = |\vec{A}||\,\hat{\mathbf{e}}|\cos\theta$ and represents the projection of the given vector onto the direction of the unit vector. The **dot product of a vector with a unit vector** is a **basic fundamental concept** which arises in a variety of science and engineering applications.

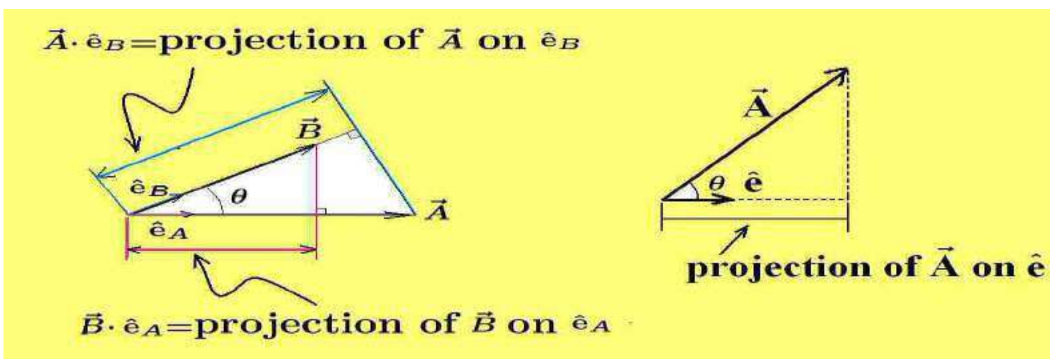


Figure 6-6. Projection of one vector onto another.

Observe that if the dot product of two vectors is zero, $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = 0$, then this implies that either $\vec{A} = \vec{0}$, $\vec{B} = \vec{0}$, or $\theta = \frac{\pi}{2}$. If $\vec{A}$ and $\vec{B}$ are **both nonzero vectors** and their **dot product is zero**, then the angle between these vectors, when their origins coincide, must be $\theta = \frac{\pi}{2}$. One can then say **the vector** $\vec{A}$ **is perpendicular to the vector** $\vec{B}$ or one can state that **the projection of** $\vec{B}$ **on** $\vec{A}$ **is zero.** If $\vec{A}$ and $\vec{B}$ are nonzero vectors and $\vec{A}\cdot\vec{B} = 0$, then the vectors $\vec{A}$ and $\vec{B}$ are said to be **orthogonal vectors**.

Direction Cosines Associated With Vectors

Let $\vec{A}$ be a nonzero vector having its origin at the origin of a rectangular cartesian coordinate system. The dot products

$$\vec{A}\cdot\hat{\mathbf{e}}_1 = A_1 \qquad \vec{A}\cdot\hat{\mathbf{e}}_2 = A_2 \qquad \vec{A}\cdot\hat{\mathbf{e}}_3 = A_3 \tag{6.10}$$

represent respectively, **the components or projections of the vector** $\vec{A}$ onto the x, y and z-axes. The projections $A_1,\ A_2,\ A_3$ of the vector $\vec{A}$ onto the coordinate axes are scalars which are called **the components of the vector** $\vec{A}$. From the definition of the dot product of two vectors, the scalar components of the vector $\vec{A}$ satisfy the equations

$$A_1 = \vec{A}\cdot\hat{\mathbf{e}}_1 = |\vec{A}|\,\cos\alpha, \qquad A_2 = \vec{A}\cdot\hat{\mathbf{e}}_2 = |\vec{A}|\cos\beta, \qquad A_3 = \vec{A}\cdot\hat{\mathbf{e}}_3 = |\vec{A}|\cos\gamma, \tag{6.11}$$

where $\alpha,\ \beta,\ \gamma$ are respectively, the smaller angles between the vector $\vec{A}$ and the $x,\ y,\ z$ coordinate axes. The cosine of these angles are referred to as the direction cosines of the vector $\vec{A}$. These angles are illustrated in figure 6-7.

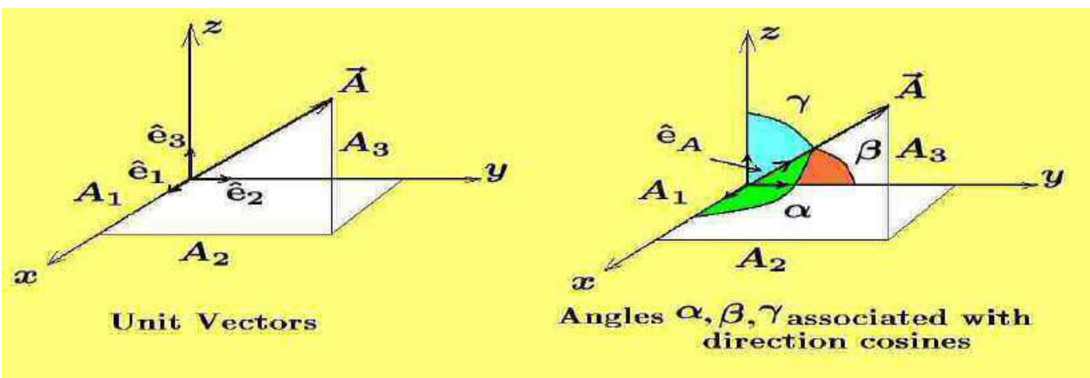


Figure 6-7. Unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_A = \cos\alpha\ \hat{\mathbf{e}}_1 + \cos\beta\ \hat{\mathbf{e}}_2 + \cos\gamma\ \hat{\mathbf{e}}_3$

The vector quantities

$$\vec{A}_1 = A_1\,\hat{\mathbf{e}}_1, \qquad \vec{A}_2 = A_2\,\hat{\mathbf{e}}_2, \qquad \vec{A}_3 = A_3\,\hat{\mathbf{e}}_3 \tag{6.12}$$

are called **the vector components of the vector** $\vec{A}$. From the **addition property of vectors,** the vector components of $\vec{A}$ may be added to obtain

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3 = |\vec{A}|(\cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3) = |\vec{A}|\ \hat{\mathbf{e}}_A \tag{6.13}$$

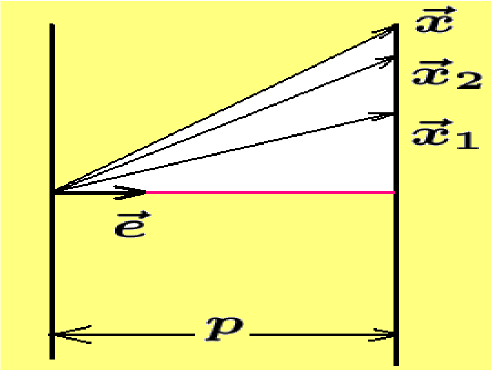
This vector representation $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ is called the **component form of the vector** $\vec{A}$ and the unit vector $\hat{\mathbf{e}}_A = \cos\alpha\ \hat{\mathbf{e}}_1 + \cos\beta\ \hat{\mathbf{e}}_2 + \cos\gamma\ \hat{\mathbf{e}}_3$ is **a unit vector in the direction of** $\vec{A}$**.**

Any numbers proportional to the direction cosines of a line are called **the direction numbers of the line.** Show for $a : b : c$ the direction numbers of a line which are not all zero, then the direction cosines are given by

$$\cos\alpha = \frac{a}{r} \qquad \cos\beta = \frac{b}{r} \qquad \cos\gamma = \frac{c}{r},$$

where $r = \sqrt{a^2 + b^2 + c^2}$.

Example 6-2.



Sketch a large version of the letter H. Consider the sides of the letter H as parallel lines a distance of p units apart. Place a unit vector $\hat{\mathbf{e}}$ perpendicular to the left side of H and pointing toward the right side of H. Construct a vector $\vec{x}_1$ which runs from the origin of $\hat{\mathbf{e}}$ to a point on the right side of the H. Observe that $\hat{\mathbf{e}} \cdot \vec{x}_1 = p$ is a projection of $\vec{x}_1$ on $\hat{\mathbf{e}}$. Now construct another vector $\vec{x}_2$, different from $\vec{x}_1$, again from the origin of $\hat{e}$ to the right side of the H. Note also that $\hat{e} \cdot \vec{x}_2 = p$ is a projection of $\vec{x}_2$ on the vector $\hat{\mathbf{e}}$. Draw still another vector $\vec{x}$, from the origin of $\hat{\mathbf{e}}$ to the right side of H which is different from $\vec{x}_1$ and $\vec{x}_2$. Observe that the dot product $\hat{\mathbf{e}} \cdot \vec{x} = p$ representing the projection of $\vec{x}$ on $\hat{\mathbf{e}}$ still produces the value p.

Assume you are given $\hat{\mathbf{e}}$ and p and are asked to solve the vector equation $\hat{\mathbf{e}} \cdot \vec{x} = p$ for the unknown quantity $\vec{x}$. You might think that there is some operation like vector division, for example $\vec{x} = p/\hat{\mathbf{e}}$, whereby $\vec{x}$ can be determined. However, if you look at the equation $\hat{\mathbf{e}} \cdot \vec{x} = p$ as a projection, one can observe that there would be an infinite number of solutions to this equation and for this reason there is **no division of vector quantities.**

■

Component Form for Dot Product

Let $\vec{A}$, $\vec{B}$ be two nonzero vectors represented in the component form

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3, \qquad \vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_1\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$$

The **dot product** of these two vectors is

$$\vec{A} \cdot \vec{B} = (A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3) \cdot (B_1\,\hat{\mathbf{e}}_1 + B_1\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3) \tag{6.14}$$

and this product can be expanded utilizing the distributive and commutative laws to obtain

$$\begin{aligned}\vec{A}\cdot\vec{B} = & \; A_1B_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 + A_1B_2\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 + A_1B_3\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3\\ & + A_2B_1\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + A_2B_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 + A_2B_3\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_3\\ & + A_3B_1\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1 + A_3B_2\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_2 + A_3B_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_3.\end{aligned} \tag{6.15}$$

From the previous properties of the dot product of unit vectors, given by equations (6.9), the dot product reduces to the form

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3. \tag{6.16}$$

Thus, the dot product of two vectors produces a scalar quantity which is **the sum of the products of like components.**

From the definition of the dot product the following useful relationship results:

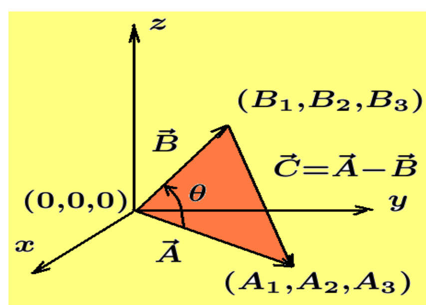
$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = |\vec{A}||\vec{B}|\cos\theta. \tag{6.17}$$

This relation may be used to find the angle between two vectors when their origins are made to coincide and their components are known. If in equation (6.17) one makes the substitution $\vec{A} = \vec{B}$, there results the special formula

$$\vec{A}\cdot\vec{A} = A_1^2 + A_2^2 + A_3^2 = A\cdot A\cos 0 = A^2 = |\vec{A}|^2. \tag{6.18}$$

Consequently, the magnitude of a vector $\vec{A}$ is given by the square root of the sum of the squares of its components or $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_1^2 + A_2^2 + A_3^2}$

The previous dot product definition is motivated by the **law of cosines** as the following arguments demonstrate. Consider three points having the coordinates $(0,0,0)$, (A_1, A_2, A_3), and (B_1, B_2, B_3) and plot these points in a cartesian coordinate system as illustrated. Denote by $\vec{A}$ the directed line segment from $(0,0,0)$ to (A_1, A_2, A_3) and denote by $\vec{B}$ the directed straight-line segment from $(0,0,0)$ to (B_1, B_2, B_3).



One can now apply the distance formula from analytic geometry to represent the lengths of these line segments. We find these lengths can be represented by

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \quad \text{and} \quad |\vec{B}| = \sqrt{B_1^2 + B_2^2 + B_3^2}.$$

Let $\vec{C} = \vec{A} - \vec{B}$ denote the directed line segment from (B_1, B_2, B_3) to (A_1, A_2, A_3). The length of this vector is found to be

$$|\vec{C}| = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2}.$$

If θ is the angle between the vectors $\vec{A}$ and $\vec{B}$, the law of cosines is employed to write

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta.$$

Substitute into this relation the distances of the directed line segments for the magnitudes of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Expanding the resulting equation shows that the law of cosines takes on the form

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - 2|\vec{A}||\vec{B}|\cos\theta.$$

With elementary algebra, this relation simplifies to the form

$$A_1B_1 + A_2B_2 + A_3B_3 = |\vec{A}||\vec{B}|\cos\theta$$

which suggests the definition of a dot product as $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$.

Example 6-3. If $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ is a given vector in component form, then

$$\vec{A} \cdot \vec{A} = A_1^2 + A_2^2 + A_3^3 \quad \text{and} \quad |\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

The vector

$$\widehat{e}_A = \frac{1}{|\vec{A}|}\vec{A} = \frac{A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}} = \cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3$$

is **a unit vector in the direction of $\vec{A}$**, where

$$\cos\alpha = \frac{A_1}{|\vec{A}|}, \qquad \cos\beta = \frac{A_2}{|\vec{A}|}, \qquad \cos\gamma = \frac{A_3}{|\vec{A}|}$$

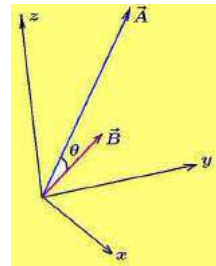
are the direction cosines of the vector $\vec{A}$. The dot product

$$\widehat{e}_A \cdot \widehat{e}_A = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

shows that **the sum of squares of the direction cosines is unity.**

■

Example 6-4. Given the vectors



$$\vec{A} = 2\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3 \quad \text{and} \quad \vec{B} = \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$$

Find:

(a) $|\vec{A}|,\ |\vec{B}|,\ \vec{A}\cdot\vec{B},\ |\vec{A}+\vec{B}|$

(b) The angle between the vectors $\vec{A}$ and $\vec{B}$

(c) The direction cosines of $\vec{A}$ and $\vec{B}$

(d) A unit vector in the direction $\vec{C} = \vec{A} - \vec{B}$.

Solution

$$(a)\quad |\vec{A}| = \sqrt{(2)^2+(3)^2+(6)^2} = \sqrt{49} = 7$$
$$|\vec{B}| = \sqrt{(1)^2+(2)^2+(2)^2} = \sqrt{9} = 3$$
$$\vec{A}\cdot\vec{B} = (2)(1)+(3)(2)+(6)(2) = 20$$
$$\vec{A}+\vec{B} = 3\,\hat{\mathbf{e}}_1 + 5\,\hat{\mathbf{e}}_2 + 8\,\hat{\mathbf{e}}_3$$
$$|\vec{A}+\vec{B}| = \sqrt{(3)^2+(5)^2+(8)^2} = \sqrt{98}$$

$$(b)\qquad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta \quad\Longrightarrow\quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{20}{7\cdot 3} = \frac{20}{21}$$
$$\theta = \arccos\left(\frac{20}{21}\right) = 0.3098446 \text{ radians } = 17.753 \text{ degrees}$$

or one can determine that

$$\tan\theta = \frac{\sqrt{(21)^2-(20)^2}}{20} = \frac{\sqrt{41}}{20} \quad\Longrightarrow \theta = 0.3098446 \quad \text{radians}$$

(c) A unit vector in the direction of the vector $\vec{A}$ is obtained by multiplying $\vec{A}$ by the scalar $\frac{1}{|\vec{A}|}$ to obtain

$$\hat{\mathbf{e}}_A = \frac{\vec{A}}{|\vec{A}|} = \cos\alpha_1\,\hat{\mathbf{e}}_1 + \cos\beta_1\,\hat{\mathbf{e}}_2 + \cos\gamma_1\,\hat{\mathbf{e}}_3 = \frac{2}{7}\,\hat{\mathbf{e}}_1 + \frac{3}{7}\,\hat{\mathbf{e}}_2 + \frac{6}{7}\,\hat{\mathbf{e}}_3$$

which implies the direction cosines are $\cos\alpha_1 = \frac{2}{7},\ \cos\beta_1 = \frac{3}{7},\ \cos\gamma_1 = \frac{6}{7}$ In a similar fashion one can show $\hat{\mathbf{e}}_B = \frac{\vec{B}}{|\vec{B}|} = \cos\alpha_2\,\hat{\mathbf{e}}_1 + \cos\beta_2\,\hat{\mathbf{e}}_2 + \cos\gamma_2\,\hat{\mathbf{e}}_3 = \frac{1}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{2}{3}\,\hat{\mathbf{e}}_3$ which implies the direction cosines are $\cos\alpha_2 = \frac{1}{3},\ \cos\beta_2 = \frac{2}{3},\ \cos\gamma_2 = \frac{2}{3}$

(d) $\vec{C} = \vec{A} - \vec{B} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3$ and $|\vec{C}| = |\vec{A}-\vec{B}| = \sqrt{(1)^2+(1)^2+(4)^2} = \sqrt{18} = 3\sqrt{2}$ Unit vector in direction of $\vec{C}$ is $\hat{\mathbf{e}}_C = \frac{\vec{C}}{|\vec{C}|} = \frac{\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3}{3\sqrt{2}}$. Make note of the fact that the sum of the squares of the direction cosines equals unity.

■

Example 6-5. (The Schwarz inequality)

Show that for any two vectors $\vec{A}$ and $\vec{B}$ one can write the Schwarz inequality $|\vec{A}\cdot\vec{B}| \le |\vec{A}||\vec{B}|$ the equality holding if $\vec{A}$ and $\vec{B}$ are colinear.

Solution If $\vec{A}$ and $\vec{B}$ are nonzero quantities, then $|\vec{A}\cdot\vec{B}|$ must be a positive quantity. Consider a graph of the function

$$\begin{aligned} y = y(x) =& |\vec{A}+x\vec{B}|^2 = (\vec{A}+x\vec{B})\cdot(\vec{A}+x\vec{B}) \\ y(x) =& \vec{A}\cdot\vec{A} + x\vec{A}\cdot\vec{B} + x\vec{B}\cdot\vec{A} + x^2\vec{B}\cdot\vec{B} \\ y(x) =& |\vec{B}|^2\ x^2 + 2(\vec{A}\cdot\vec{B})x + |\vec{A}|^2 = ax^2+bx+c \end{aligned}$$

Note that if $y(x) > 0$ for all values of x, then this would imply the graph of $y(x)$ must not cross the $x-$axis. If $y(x)$ did cross the $x-$axis, then the equation $y(x)=0$ would have the two roots

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

in which case the discriminant b^2-4ac would be positive. If $y(x)$ does not cross the $x-$axis, then the discriminant would satisfy $b^2-4ac \le 0$. Here $b = 2(\vec{A}\cdot\vec{B})$, $a = |\vec{B}|^2$ and $c = |\vec{A}|^2$ and the condition that the discriminant be less than or equal zero can be expressed

$$b^2 - 4ac = 4(\vec{A}\cdot\vec{B})^2 - 4|\vec{B}|^2|\vec{A}|^2 \le 0$$

or
$$|\vec{A}\cdot\vec{B}| \le |\vec{A}||\vec{B}|$$

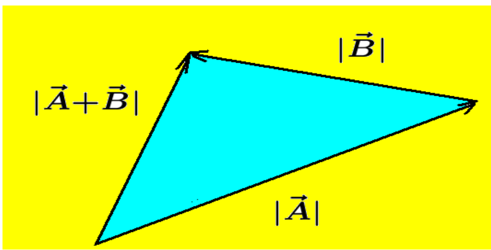
an inequality known as the **Schwarz inequality**.

■

Example 6-6. The triangle inequality

Show that for two vectors $\vec{A}$ and $\vec{B}$ the inequality $|\vec{A}+\vec{B}| \le |\vec{A}| + |\vec{B}|$ must hold.

This inequality is known as **the triangle inequality** and indicates that the length of one side of a triangle is always less than the sum of the lengths of the other two sides.



Solution To prove the triangle inequality one can use the Schwarz inequality from the previous example. Observe that

$$|\vec{A}+\vec{B}|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$

or

$$|\vec{A}+\vec{B}|^2 = |\vec{A}|^2 + 2(\vec{A}\cdot\vec{B}) + |\vec{B}|^2 \le |\vec{A}|^2 + 2|\vec{A}\cdot\vec{B}| + |\vec{B}|^2 \tag{6.19}$$

Using the Schwarz inequality $|\vec{A}\cdot\vec{B}| \le |\vec{A}||\vec{B}|$ the equation (6.19) can be expressed

$$|\vec{A}+\vec{B}|^2 \le |\vec{A}|^2 + 2|\vec{A}||\vec{B}| + |\vec{B}|^2 = (|\vec{A}| + |\vec{B}|)^2 \tag{6.20}$$

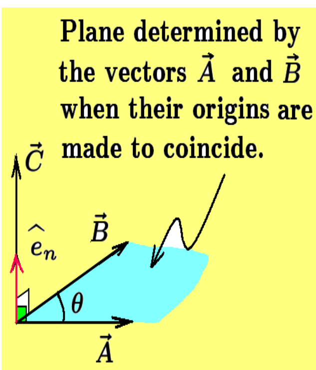
Taking the square root of both sides of the equation (6.20) gives **the triangle inequality** $|\vec{A}+\vec{B}| \le |\vec{A}| + |\vec{B}|$.

■

The Cross Product or Outer Product

The **cross or outer product of two nonzero vectors** $\vec{A}$ and $\vec{B}$ is denoted using the notation $\vec{A}\times\vec{B}$ and represents the construction of a new vector $\vec{C}$ defined as

$$\vec{C} = \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{\mathbf{e}}_n, \tag{6.21}$$



where θ is the smaller angle between the two nonzero vectors $\vec{A}$ and $\vec{B}$ when their origins coincide, and $\hat{\mathbf{e}}_n$ is a unit vector perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ when their origins are made to coincide. The direction of $\hat{\mathbf{e}}_n$ is determined by the **right-hand rule**. Place the fingers of your right-hand in the direction of $\vec{A}$ and rotate the fingers toward the vector $\vec{B}$, then the thumb of the right-hand points in the direction $\vec{C}$.

The vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ then form a **right-handed system.**[2] Note that the cross product $\vec{A}\times\vec{B}$ is a vector which will always be perpendicular to the vectors $\vec{A}$ and $\vec{B}$, whenever $\vec{A}$ and $\vec{B}$ are linearly independent.

A special case of the above definition occurs when $\vec{A}\times\vec{B} = \vec{0}$ and in this case one can state that either $\theta = 0$, which implies the vectors $\vec{A}$ and $\vec{B}$ **are parallel** or $\vec{A} = \vec{0}$ or $\vec{B} = \vec{0}$.

Use the above definition of a cross product and show that the orthogonal unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ satisfy the relations

$$\begin{array}{lll} \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1 = \vec{0} & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3 & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_2 = \vec{0} & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3 = -\hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_3 = \vec{0} \end{array} \tag{6.22}$$

[2] Note many European technical books use left-handed coordinate systems which produces results different from using a right-handed coordinate system.

Properties of the Cross Product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \text{(noncommutative)}$$
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad \text{(distributive law)}$$
$$m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B}) \quad m \text{ a scalar}$$
$$\vec{A} \times \vec{A} = \vec{0} \quad \text{since } \vec{A} \text{ is parallel to itself.}$$

Let $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ and $\vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$ be two nonzero vectors in component form and form the cross product $\vec{A} \times \vec{B}$ to obtain

$$\vec{A} \times \vec{B} = (A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3) \times (B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3). \tag{6.23}$$

The cross product can be expanded by using the distributive law to obtain

$$\begin{aligned}\vec{A} \times \vec{B} = &\ A_1B_1\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1 + A_1B_2\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 + A_1B_3\,\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 \\ &+ A_2B_1\,\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_1 + A_2B_2\,\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2 + A_2B_3\,\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 \\ &+ A_3B_1\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 + A_3B_2\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2 + A_3B_3\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_3.\end{aligned} \tag{6.24}$$

Simplification by using the previous results from equation (6.22) produces the important cross product formula

$$\vec{A} \times \vec{B} = (A_2B_3 - A_3B_2)\,\hat{\mathbf{e}}_1 + (A_3B_1 - A_1B_3)\,\hat{\mathbf{e}}_2 + (A_1B_2 - A_2B_1)\,\hat{\mathbf{e}}_3, \tag{6.25}$$

This result that can be expressed in **the determinant form**[3]

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \hat{\mathbf{e}}_1 - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \hat{\mathbf{e}}_2 + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \hat{\mathbf{e}}_3. \tag{6.26}$$

In summary, the cross product of two vectors $\vec{A}$ and $\vec{B}$ is a new vector $\vec{C}$, where

$$\vec{C} = \vec{A} \times \vec{B} = C_1\,\hat{\mathbf{e}}_1 + C_2\,\hat{\mathbf{e}}_2 + C_3\,\hat{\mathbf{e}}_3 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

with components

$$C_1 = A_2B_3 - A_3B_2, \qquad C_2 = A_3B_1 - A_1B_3, \qquad C_3 = A_1B_2 - A_2B_1 \tag{6.27}$$

[3] For more information on determinants see chapter 10.

Geometric Interpretation

A geometric interpretation that can be assigned to the magnitude of the cross product of two vectors is illustrated in figure 6-8.

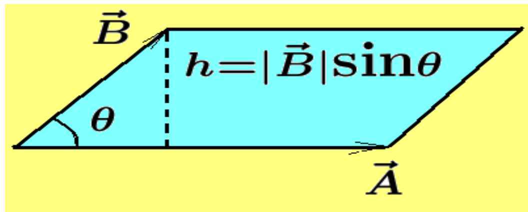


Figure 6-8. Parallelogram with sides $\vec{A}$ and $\vec{B}$.

The area of the parallelogram having the vectors $\vec{A}$ and $\vec{B}$ for its sides is given by

$$\text{Area} = |\vec{A}| \cdot h = |\vec{A}||\vec{B}| \sin\theta = |\vec{A} \times \vec{B}|. \tag{6.28}$$

Therefore, the magnitude of the cross product of two vectors represents the **area of the parallelogram formed from these vectors when their origins are made to coincide.**

Vector Identities

The following vector identities are often needed to simplify various equations in science and engineering.

1. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (6.29)
2. $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$ (6.30)

   An identity known as the **triple scalar product**.
3. $$\begin{aligned}(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) &= \vec{C}\left[\vec{D} \cdot (\vec{A} \times \vec{B})\right] - \vec{D}\left[\vec{C} \cdot (\vec{A} \times \vec{B})\right] \\ &= \vec{B}\left[\vec{A} \cdot (\vec{C} \times \vec{D})\right] - \vec{A}\left[\vec{B} \cdot (\vec{C} \times \vec{D})\right]\end{aligned} \tag{6.31}$$
4. $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ (6.32)

   The quantity $\vec{A} \times (\vec{B} \times \vec{C})$ is called a **triple vector product**.
5. $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$ (6.33)
6. The triple vector product satisfies

   $$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0} \tag{6.34}$$

Note that in the triple scalar product $\vec{A} \cdot (\vec{B} \times \vec{C})$ the parenthesis is sometimes omitted because $(\vec{A} \cdot \vec{B}) \times \vec{C}$ is meaningless and so $\vec{A} \cdot \vec{B} \times \vec{C}$ can have only one meaning. The parenthesis just emphasizes this one meaning.

A physical interpretation can be assigned to the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C})$ is that **its absolute value represents the volume of the parallelepiped formed by the three noncoplaner vectors $\vec{A},\vec{B},\vec{C}$ when their origins are made to coincide.** The absolute value is needed because sometimes the triple scalar product is negative. This physical interpretation can be obtained from the following analysis.

In figure 6-9 note the following.

(a) The magnitude $|\vec{B}\times\vec{C}|$ represents the area of the parallelogram $PQRS$.

(b) The unit vector $\hat{\mathbf{e}}_n = \dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$ is normal to the plane containing the vectors $\vec{B}$ and $\vec{C}$.

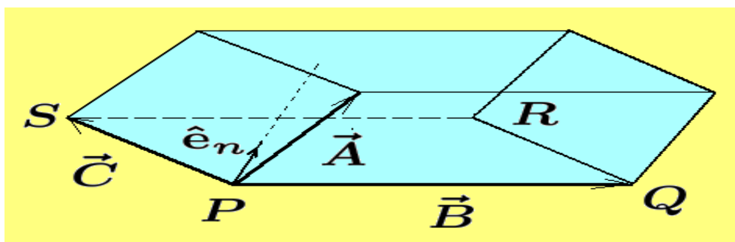


Figure 6-9. Triple scalar product and volume.

(c) The dot product $\vec{A}\cdot\hat{\mathbf{e}}_n = \vec{A}\cdot\dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|} = h$ represents the projection of $\vec{A}$ on $\hat{\mathbf{e}}_n$ and produces the height of the parallelepiped. These results demonstrate that

$$\left|\vec{A}\cdot(\vec{B}\times\vec{C})\right| = |\vec{B}\times\vec{C}|\,h = (\text{Area of base})(\text{Height}) = \text{Volume}.$$

so that the magnitude of the triple scalar product is the volume of the parallelepiped formed when the origins of the three vectors are made to coincide.

Example 6-7. Show that the triple scalar product satisfies the relations

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Note **the cyclic rotation of the symbols** in the above relations where the first symbol is moved to the last position and the second and third symbols are each moved to the left. This is called **a cyclic permutation** of the symbols.

Solution Use the determinant form for the cross product and express the triple scalar product as a determinant as follows.

$$\begin{aligned}
\vec{A}\cdot(\vec{B}\times\vec{C}) =&(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}\\
\vec{A}\cdot(\vec{B}\times\vec{C}) =&(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3)\cdot[(B_2C_3 - B_3C_2)\,\hat{\mathbf{e}}_1 - (B_1C_3 - B_3C_1)\,\hat{\mathbf{e}}_2 + (B_1C_2 - B_2C_1)\,\hat{\mathbf{e}}_3]\\
\vec{A}\cdot(\vec{B}\times\vec{C}) =&A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)\\
\vec{A}\cdot(\vec{B}\times\vec{C}) =&A_1\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - A_2\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + A_3\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}
\end{aligned}$$

Determinants have the property[4] that the interchange of two rows of a determinant changes its sign. One can then show

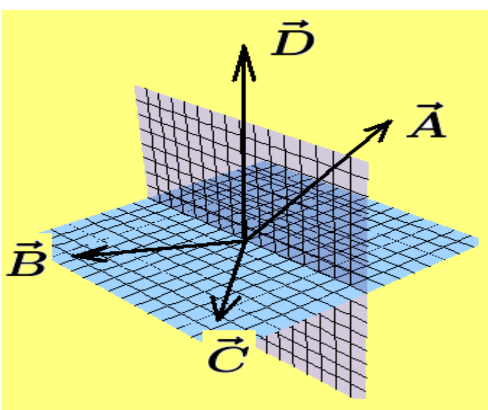
$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

or

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

■

Example 6-8. For nonzero vectors $\vec{A}, \vec{B}, \vec{C}$ show that the triple vector product satisfies $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$
That is, the triple vector product is not associative and the order of execution of the cross product is important.



Solution Let $\vec{B}\times\vec{C} = \vec{D}$ denote the vector perpendicular to the plane determined by the vectors $\vec{B}$ and $\vec{C}$. The vector $\vec{A}\times\vec{D} = \vec{E}$ is a vector perpendicular to the plane determined by the vectors $\vec{A}$ and $\vec{D}$ and therefore must lie in the plane of the vectors $\vec{B}$ and $\vec{C}$. One can then say the vectors $\vec{B}, \vec{C}$ and $\vec{A}\times(\vec{B}\times\vec{C})$ are coplanar and consequently there must exist scalars α and β such that

$$\vec{A}\times(\vec{B}\times\vec{C}) = \alpha\vec{B} + \beta\vec{C} \tag{6.35}$$

[4] See chapter 10 for properties of determinants.

In a similar fashion one can show that the vectors $(\vec{A}\times\vec{B})\times\vec{C}$, $\vec{A}$ and $\vec{B}$ are coplanar so that there exists constants γ and δ such that

$$(\vec{A}\times\vec{B})\times\vec{C} = \gamma\vec{A} + \delta\vec{B} \tag{6.36}$$

The equations (6.35) and (6.36) show that in general

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

■

Example 6-9. Show that the triple vector product satisfies

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{A}\cdot\vec{B})\,\vec{C}$$

Solution Use the results from the previous example showing there exists scalars α and β such that

$$\vec{A}\times(\vec{B}\times\vec{C}) = \alpha\vec{B} + \beta\vec{C} \tag{6.37}$$

Let $\vec{B}\times\vec{C} = \vec{D}$ and write

$$\vec{A}\times\vec{D} = \alpha\vec{B} + \beta\vec{C} \tag{6.38}$$

Take the dot product of both sides of equation (6.38) with the vector $\vec{A}$ to obtain the triple scalar product

$$\vec{A}\cdot(\vec{A}\times\vec{D}) = \alpha(\vec{A}\cdot\vec{B}) + \beta(\vec{A}\cdot\vec{C})$$

By the permutation properties of the triple scalar product one can write

$$\vec{A}\cdot(\vec{A}\times\vec{D}) = \vec{A}\cdot(\vec{D}\times\vec{A}) = \vec{D}\cdot(\vec{A}\times\vec{A}) = 0 = \alpha(\vec{A}\cdot\vec{B}) + \beta(\vec{A}\cdot\vec{C}) \tag{6.39}$$

The above result holds because $\vec{A}\times\vec{A} = \vec{0}$ and implies

$$\alpha(\vec{A}\cdot\vec{B}) = -\beta(\vec{A}\cdot\vec{C}) \quad \text{or} \quad \frac{\alpha}{\vec{A}\cdot\vec{C}} = \frac{-\beta}{\vec{A}\cdot\vec{B}} = \lambda$$

where λ is a scalar. This shows that the equation (6.37) can be expressed in the form

$$\vec{A}\times(\vec{B}\times\vec{C}) = \lambda(\vec{A}\cdot\vec{C})\vec{B} - \lambda(\vec{A}\cdot\vec{B})\vec{C} \tag{6.40}$$

which shows that the vectors $\vec{A}\times(\vec{B}\times\vec{C})$ and $(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ are colinear. The equation (6.40) must hold for all vectors $\vec{A}, \vec{B}, \vec{C}$ and so it must be true in the special case $\vec{A} = \hat{\mathbf{e}}_2$, $\vec{B} = \hat{\mathbf{e}}_1$, $\vec{C} = \hat{\mathbf{e}}_2$ where equation (6.40) reduces to

$$\hat{\mathbf{e}}_2\times(\,\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2) = \hat{\mathbf{e}}_1 = \lambda\,\hat{\mathbf{e}}_1 \quad \text{which implies} \quad \lambda = 1$$

■

Example 6-10.

Derive the **law of sines** for the triangle illustrated in the figure 6-10.

Solution The sides of the given triangle are formed from the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and since these vectors are free vectors they can be moved to the positions illustrated in figure 6-10. Also sketch the vector $-\vec{C}$ as illustrated. The new positions for the vectors $\vec{A}, \vec{B}, \vec{C}$ and $-\vec{C}$ are constructed to better visualize certain vector cross products associated with the law of sines.

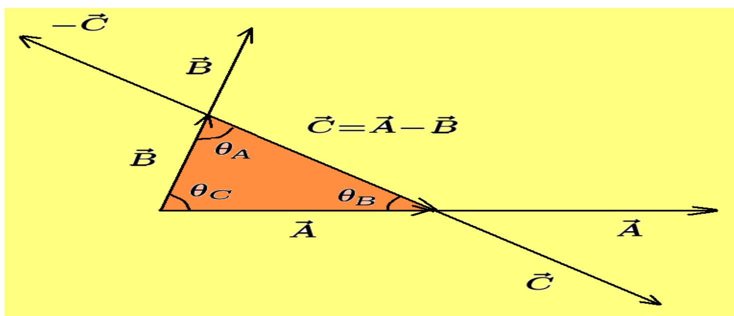


Figure 6-10. Triangle for law of sines.

Examine figure 6-10 and note the following cross products

$$\vec{C} \times \vec{A} = (\vec{A} - \vec{B}) \times \vec{A} = \vec{A} \times \vec{A} - \vec{B} \times \vec{A} = -\vec{B} \times \vec{A} = \vec{A} \times \vec{B}$$

$$\text{and} \quad \vec{B} \times (-\vec{C}) = \vec{B} \times (-\vec{A} + \vec{B}) = \vec{B} \times (-\vec{A}) + \vec{B} \times \vec{B} = \vec{A} \times \vec{B}.$$

Taking the magnitude of the above cross products gives

$$|\vec{C} \times \vec{A}| = |\vec{A} \times \vec{B}| = |\vec{B} \times (-\vec{C})|$$

or

$$AC \sin\theta_B = AB \sin\theta_C = BC \sin\theta_A.$$

Dividing by the product of the vector magnitudes ABC produces the law of sines

$$\frac{\sin\theta_A}{A} = \frac{\sin\theta_B}{B} = \frac{\sin\theta_C}{C}.$$

■

Example 6-11. Derive the **law of cosines** for the triangle illustrated.

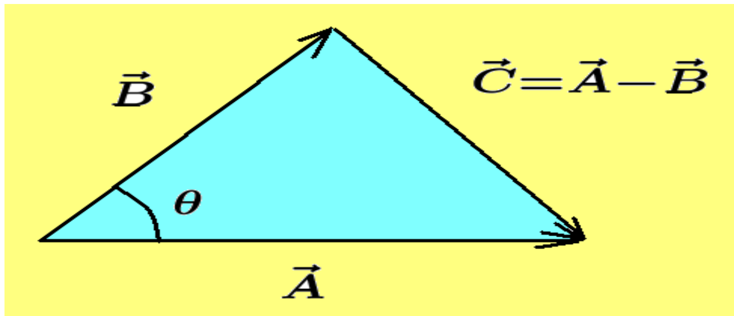


Figure 6-11. Triangle for law of cosines.

Solution Let $\vec{C} = \vec{A} - \vec{B}$ so that the dot product of $\vec{C}$ with itself gives

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\vec{A} \cdot \vec{B}$$

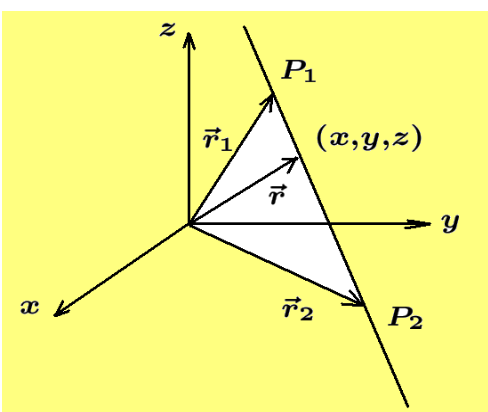
or

$$C^2 = A^2 + B^2 - 2AB\cos\theta,$$

where $A = |\vec{A}|, \quad B = |\vec{B}|, \quad C = |\vec{C}|$ represent the magnitudes of the vector sides.

■

Example 6-12. Find the **vector equation of the line** which passes through the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.



Solution Let

$$\begin{aligned} \vec{r}_1 &= x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3 \\ \text{and} \quad \vec{r}_2 &= x_2\,\hat{\mathbf{e}}_1 + y_2\,\hat{\mathbf{e}}_2 + z_2\,\hat{\mathbf{e}}_3 \end{aligned}$$

denote position vectors to the points P_1 and P_2 respectively and let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector of any other variable point on the line. Observe that the vector $\vec{r}_2 - \vec{r}_1$ is parallel to the line through the points P_1 and P_2. By vector addition the (x, y, z) position on the line is given by

$$\vec{r} = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1) \qquad -\infty < \lambda < \infty \tag{6.41}$$

where λ is a scalar parameter. Note that as λ varies from 0 to 1 the position vector $\vec{r}$ moves from $\vec{r}_1$ to $\vec{r}_2$. An alternative form for the equation of the line is given by

$$\vec{r} = \vec{r}_2 + \lambda^*(\vec{r}_1 - \vec{r}_2) \qquad -\infty < \lambda^* < \infty$$

where λ^* is some other scalar parameter. This second form for the line has the position vector $\vec{r}$ moving from $\vec{r}_2$ to $\vec{r}_1$ as λ^* varies from 0 to 1. The vector $\pm(\vec{r}_2 - \vec{r}_1)$ is called **the direction vector of the line**. Equating the coefficients of the unit vectors in the equation (6.41) there results the scalar parametric equations representing the line. These parametric equations have the form

$$x = x_1 + \lambda(x_2 - x_1), \qquad y = y_1 + \lambda(y_2 - y_1), \qquad z = z_1 + \lambda(z_1 - z_2) \qquad -\infty < \lambda < \infty$$

If the quantities $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are different from zero, then the equation for the line can be represented in **the symmetric form**

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda \tag{6.42}$$

Note that the equation of a line can also be represented as **the intersection of two planes**

$$\begin{aligned} N_1 x + N_2 y + N_3 z + D_1 &= 0 \\ M_1 x + M_2 y + M_3 z + D_2 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} \vec{N} \cdot (\vec{r} - \vec{r}_0) &= 0 \\ \vec{M} \cdot (\vec{r} - \vec{r}_1) &= 0 \end{aligned}$$

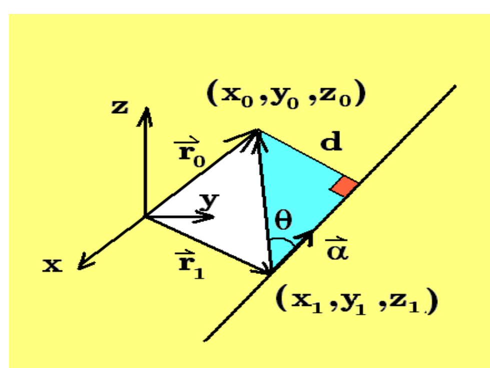
provided the planes are not parallel or $\vec{N} \neq k\vec{M}$, for k a nonzero constant.

■

Example 6-13. Show the **perpendicular distance from a point** (x_0, y_0, z_0) **to a given line** defined by $x = x_1 + \alpha_1 t, \quad y = y_1 + \alpha_2 t, \quad z = z_1 + \alpha_3 t$ is given by

$$d = \left| (\vec{r}_0 - \vec{r}_1) \times \frac{\vec{\alpha}}{|\vec{\alpha}|} \right| \quad \text{where} \quad \vec{\alpha} = \alpha_1\,\hat{\mathbf{e}}_1 + \alpha_2\,\hat{\mathbf{e}}_2 + \alpha_3\,\hat{\mathbf{e}}_3$$

Solution The **vector equation of the line** is $\vec{r} = \vec{r}_1 + \vec{\alpha}\,t$, where (x_1, y_1, z_1) is a point on the line described by the position vector $\vec{r}_1$ and $\vec{\alpha}$ is **the direction vector of the line**. The vector $\vec{r}_0 - \vec{r}_1$ is a vector pointing from (x_1, y_1, z_1) to the point (x_0, y_0, z_0). These vectors are illustrated in the accompanying figure.

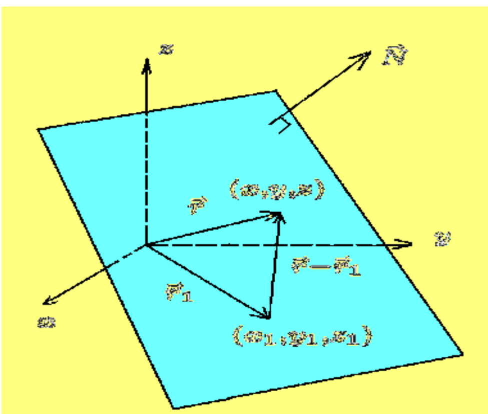


Define the unit vector $\hat{\mathbf{e}}_\alpha = \frac{1}{|\vec{\alpha}|}\vec{\alpha}$ and construct the line from (x_0, y_0, z_0) which is perpendicular to the given line and label this distance d. Our problem is to find the distance d. From the geometry of the right triangle with sides $\vec{r}_0 - \vec{r}_1$ and d one can write $\sin\theta = \dfrac{d}{|\vec{r}_0 - \vec{r}_1|}$. Use the fact that by definition of a cross product one can write

$$|(\vec{r}_0 - \vec{r}_1) \times \hat{\mathbf{e}}_\alpha| = |\vec{r}_0 - \vec{r}_1||\hat{\mathbf{e}}_\alpha|\sin\theta = d = \left|(\vec{r}_0 - \vec{r}_1) \times \frac{\vec{\alpha}}{|\vec{\alpha}|}\right|$$

■

Example 6-14. Find the **equation of the plane which passes through the point** $P_1(x_1, y_1, z_1)$ **and is perpendicular to the given vector** $\vec{N} = N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3$.



Solution Let $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ denote the position vector to the point P_1 and let the vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to any variable point (x, y, z) in the plane. If the vector $\vec{r} - \vec{r}_1$ lies in the plane, then it must be perpendicular to the given vector $\vec{N}$ and consequently the dot product of $(\vec{r} - \vec{r}_1)$ with $\vec{N}$ must be zero and so one can write

$$(\vec{r} - \vec{r}_1) \cdot \vec{N} = 0 \tag{6.43}$$

as the equation representing the plane. In scalar form, the equation of the plane is given as

$$(x - x_1)N_1 + (y - y_1)N_2 + (z - z_1)N_3 = 0 \tag{6.44}$$

■

Example 6-15. Find **the perpendicular distance** d **from a given plane**

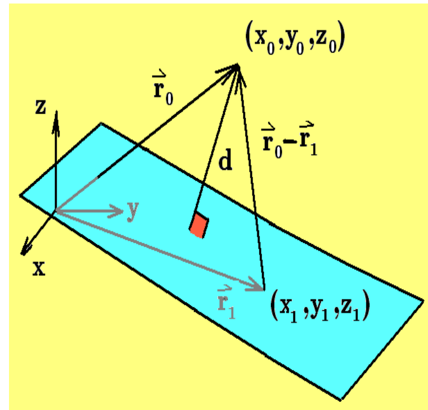
$$(x - x_1)N_1 + (y - y_1)N_2 + (z - z_1)N_3 = 0$$

to a given point (x_0, y_0, z_0).

Solution Let the vector $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ point to the given point (x_0, y_0, z_0) and the vector $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ point the point (x_1, y_1, z_1) lying in the plane.

Construct the vector $\vec{r}_0 - \vec{r}_1$ which points from the terminus of $\vec{r}_1$ to the terminus of $\vec{r}_0$ and construct the unit normal to the plane which is given by

$$\hat{\mathbf{e}}_N = \frac{N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}}$$



Observe that the dot product $\hat{\mathbf{e}}_N \cdot (\vec{r}_0 - \vec{r}_1)$ equals the projection of $\vec{r}_0 - \vec{r}_1$ onto $\hat{\mathbf{e}}_N$. This gives the distance

$$d = |\,\hat{\mathbf{e}}_N \cdot (\vec{r}_0 - \vec{r}_1)| = \left| \frac{(x_0 - x_1)N_1 + (y_0 - y_1)N_2 + (z_0 - z_)N_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}} \right|$$

where the absolute value signs guarantee that the sign of d is always positive and does not depend upon the direction selected for the unit vector $\hat{\mathbf{e}}_N$.

■

Moment Produced by a Force

The **moment of a force with respect to a line is a measure of the forces tendency to produce a rotation about the line**. Let a force $\vec{F}$ acting at the point (x_1, y_1, z_1) be resolved into components parallel to the coordinate axes by expressing $\vec{F}$ in the component form

$$\vec{F} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$$

That component of the force which is parallel to an axis has no tendency to produce a rotation about that axis. For example, the F_1 component is parallel to the $x-$axis and does not produce a rotation about this axis. For a chosen axis, the moment about that axis is the **product of the force component times the perpendicular distance of the force from the axis. By using the right-hand screw rule, one can assign a negative sign to the moment if it acts clockwise and a positive sign to the moment if it acts counterclockwise.** The moment of a force is a vector quantity which produces a definite sense of rotation about an axis.

With the use of figure 6-12 let us calculate the moment of a force $\vec{F}$, acting at the point (x_1, y_1, z_1), about the x-, y- and z-axes.

(a) For the moment about the x-axis produces

$$\begin{aligned}&F_1 \text{ component parallel to } x\text{-axis does not produce moment}\\ &(\text{Force})(\perp \text{ distance}) = +F_3 y_1 \ (\text{Counterclockwise rotation})\\ &(\text{Force})(\perp \text{ distance}) = -F_2 z_1 \ \ (\text{Clockwise rotation})\end{aligned}$$

The total moment about the x-axis is therefore the sum of these moments and given by

$$M_1 = F_3 y_1 - F_2 z_1.$$

(b) For the moment about the y-axis, one finds

(Force)($\perp$ distance) $= +F_1 z_1$ (Counterclockwise rotation)

F_2 component parallel to the y-axis does not produce a moment

(Force)($\perp$ distance) $= -F_3 x_1$ (Clockwise rotation)

The total moment about the y-axis is therefore

$$M_2 = F_1 z_1 - F_3 x_1.$$

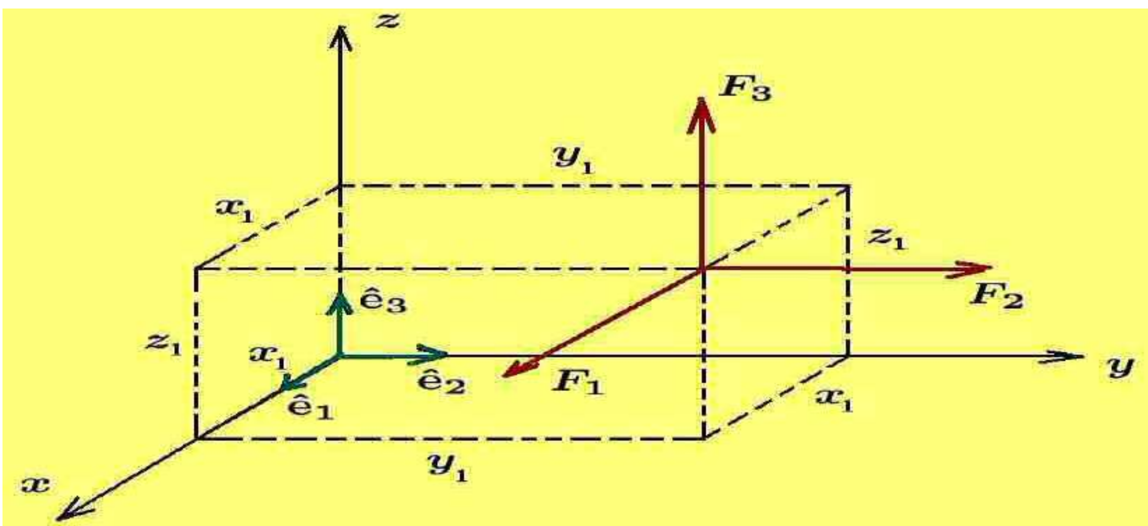


Figure 6-12. Moments produced by a force $\vec{F} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$

(c) For the moment about the z-axis show that

(Force)($\perp$ distance) $= -F_1 y_1$ (Clockwise rotation)

(Force)($\perp$ distance) $= +F_2 x_1$ (Counterclockwise rotation)

F_3 component parallel to the z-axis does not produce a moment

The total moment about the z-axis is given by

$$M_3 = F_2 x_1 - F_1 y_1.$$

The total moment about the origin is a vector quantity represented as the vector sum of the above moments in the form

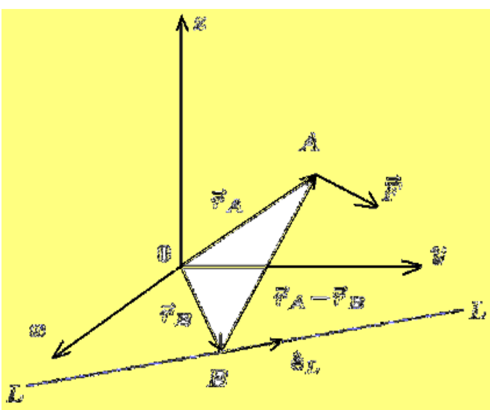
$$\begin{aligned}\vec{M}_0 &= M_1\,\hat{\mathbf{e}}_1 + M_2\,\hat{\mathbf{e}}_2 + M_3\,\hat{\mathbf{e}}_3\\ &= (F_3y_1 - F_2z_1)\,\hat{\mathbf{e}}_1 + (F_1z_1 - F_3x_1)\,\hat{\mathbf{e}}_2 + (F_2x_1 - F_1y_1)\,\hat{\mathbf{e}}_3.\end{aligned} \tag{6.45}$$

If $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ is the position vector from the origin to the point (x_1, y_1, z_1), then the moment about the origin produced by the force $\vec{F}$ can be expressed as a cross product of the vectors $\vec{r}_1$ and $\vec{F}$ and written as

$$\vec{M}_0 = \vec{r}_1 \times \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ x_1 & y_1 & z_1\\ F_1 & F_2 & F_3 \end{vmatrix}. \tag{6.46}$$

This is readily verified by expanding the equation (6.46) and showing the result is given by equation (6.45).

Moment About Arbitrary Line



Assume one has a force $\vec{F}$ acting through a given point A and $\vec{r}_A$ is the position from the origin to the point A. The moment about the origin is given by $\vec{M}_0 = \vec{r}_A \times \vec{F}$. The moment of the given force $\vec{F}$ about the lines representing the x, y and z axes are given by the projections of $\vec{M}_0$ on each of these axes. One finds these moments

$$\vec{M}_0 \cdot \hat{\mathbf{e}}_1 = M_1, \quad \vec{M}_0 \cdot \hat{\mathbf{e}}_2 = M_2, \quad \vec{M}_0 \cdot \hat{\mathbf{e}}_3 = M_3$$

To find the moment about a given line L, choose any point B on the line L and construct the position vector $\vec{r}_B$ from the origin to the point B. The vector $\vec{r}_A - \vec{r}_B$ then points from point B to the force $\vec{F}$ acting at point A as illustrated in the previous figure.

The moment of the force $\vec{F}$ about the point B is given by

$$\vec{M}_B = (\vec{r}_A - \vec{r}_B) \times \vec{F}$$

Observe that this equation for $\vec{M}_B$ represents a position vector from point B to the force $\vec{F}$ crossed with $\vec{F}$ and has the exact same form as equation (6.46). The only difference being where the position vector to the force $\vec{F}$ is constructed. The moment about the line L is then the projection of the vector moment $\vec{M}_B$ on this line. If $\hat{\mathbf{e}}_L$ is a unit vector along the line, then $\vec{M}_B \cdot \hat{\mathbf{e}}_L$ represents the projection of $\vec{M}_B$ on L. The direction of the unit vector $\hat{\mathbf{e}}_L$ on the line L can point in one of two directions (i.e. $\hat{\mathbf{e}}_L$ or $-\hat{\mathbf{e}}_L$). However, once the direction of $\hat{\mathbf{e}}_L$ has been chosen one must be careful to analyze the dot product $\vec{M}_B \cdot \hat{\mathbf{e}}_L$ as its algebraic sign determines the rotation sense produced by the moment (i.e., clockwise or counterclockwise).

A **resultant force** is the algebraic sum of the forces associated with a system. The moment of a resultant force with respect to some axis is equal to the algebraic sum of the moments of the system forces with respect to the same axis.

Example 6-16. If $\vec{F} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$ is a force acting at the end of the position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ then the moment of the force about the origin is $\vec{M} = \vec{r} \times \vec{F} = (yF_3 - zF_2)\,\hat{\mathbf{e}}_1 + (zF_1 - xF_3)\,\hat{\mathbf{e}}_2 + (xF_2 - yF_1)\,\hat{\mathbf{e}}_3$. Make note that the moments of the force components are

$$\begin{aligned}
\vec{M}_1 =& \vec{r} \times (F_1\,\hat{\mathbf{e}}_1) = (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3) \times (F_1\,\hat{\mathbf{e}}_1) = -yF_1\,\hat{\mathbf{e}}_3 + zF_1\,\hat{\mathbf{e}}_2\\
\vec{M}_2 =& \vec{r} \times (F_2\,\hat{\mathbf{e}}_2) = (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3) \times (F_2\,\hat{\mathbf{e}}_2) = xF_2\,\hat{\mathbf{e}}_3 - zF_2\,\hat{\mathbf{e}}_1\\
\vec{M}_3 =& \vec{r} \times (F_3\,\hat{\mathbf{e}}_3) = (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3) \times (F_3\,\hat{\mathbf{e}}_3) = -xF_3\,\hat{\mathbf{e}}_2 + yF_3\,\hat{\mathbf{e}}_1
\end{aligned}$$

so that $\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3$

■

Differentiation of Vectors

Let us define what is meant by a derivative associated with a vector and consider some applications of these derivatives. Again notation plays an important part in the representation of the derivatives and therefore many examples are given to help clarify concepts as they arise.

The equation of a space curve can be described in terms of a position vector from the origin of a chosen coordinate system. For example, in cartesian coordinates the position vector of a space curve can have the form

$$\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3, \tag{6.47}$$

where the space curve is defined by the parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t). \tag{6.48}$$

where t represents some convenient parameter, say time. The derivative of the position vector $\vec{r}$ with respect to the parameter t is defined as

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}\end{aligned} \tag{6.49}$$

In component form the derivative is represented in a form where one can recognize the previous definition of a derivative of a scalar function. One finds

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \lim_{\Delta t\to 0}\frac{[x(t+\Delta t)\,\hat{\mathbf{e}}_1 + y(t+\Delta t)\,\hat{\mathbf{e}}_2 + z(t+\Delta t)\,\hat{\mathbf{e}}_3] - [x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3]}{\Delta t}\\ \frac{d\vec{r}}{dt} &= \lim_{\Delta t\to 0}\left[\frac{x(t+\Delta t)-x(t)}{\Delta t}\,\hat{\mathbf{e}}_1 + \frac{y(t+\Delta t)-y(t)}{\Delta t}\,\hat{\mathbf{e}}_2 + \frac{z(t+\Delta t)-z(t)}{\Delta t}\,\hat{\mathbf{e}}_3\right]\\ \frac{d\vec{r}}{dt} &= \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 = x'(t)\,\hat{\mathbf{e}}_1 + y'(t)\,\hat{\mathbf{e}}_2 + z'(t)\,\hat{\mathbf{e}}_3\end{aligned}$$

This shows that the derivative of the position vector (6.47) is obtained by differentiating each component of the vector. It will be shown that this derivative represents a vector tangent to the space curve at the point $(x(t), y(t), z(t))$ for any fixed value of the parameter t. Second-order and higher order derivatives are defined as derivatives of derivatives.

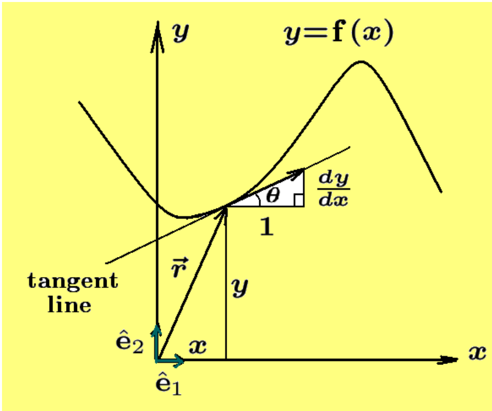
Example 6-17.

The two dimensional curve $y = f(x)$ can be represented by the position vector

$$\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$$

with the derivative

$$\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + \frac{df}{dx}\,\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 + \frac{dy}{dx}\,\hat{\mathbf{e}}_2$$

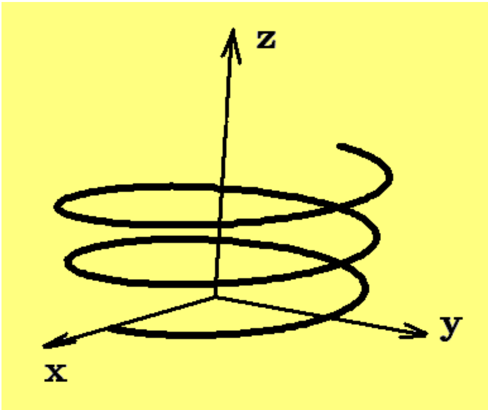


Note that at the point $(x, f(x))$ on the curve one can draw the derivative vector and show that it lies along the tangent line to the curve at the point $(x, f(x))$. This shows that the derivative $\frac{d\vec{r}}{dx}$ is a tangent vector to the curve $y = f(x)$.

In general, if $\vec{r} = \vec{r}(t)$ is the position vector of a three dimensional curve, then the vector $\frac{d\vec{r}}{dt}$ will be a tangent vector to the curve. This can be illustrated by drawing the secant line through the points $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ and showing the secant line then approaches the tangent line as Δt approaches zero.

■

Example 6-18.



Consider the space curve defined by the position vector

$$\vec{r} = \vec{r}(t) = \cos t\,\hat{\mathbf{e}}_1 + \sin t\,\hat{\mathbf{e}}_2 + t\,\hat{\mathbf{e}}_3.$$

This curve sweeps out a spiral called a helix[5]. The projection of the position vector $\vec{r}$ on the plane $z = 0$ generates a circle with unit radius about the origin.

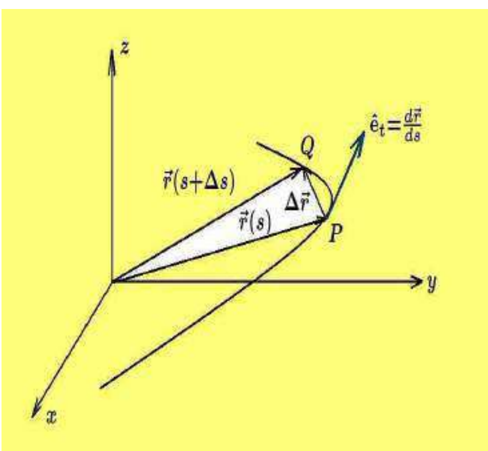
The first and second derivatives of the position vector with respect to the parameter t are

$$\frac{d\vec{r}}{dt} = -\sin t\,\hat{\mathbf{e}}_1 + \cos t\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$
$$\frac{d^2\vec{r}}{dt^2} = -\cos t\,\hat{\mathbf{e}}_1 - \sin t\,\hat{\mathbf{e}}_2.$$

The vector $\frac{d\vec{r}}{dt}$ is tangent to the curve at the point $(\cos t, \sin t, t)$ for any fixed value of the parameter t.

■

Tangent Vector to Curve



Let s denote the distance along a curve measured from some fixed point on the curve and let the position vector of a point P on the curve be represented as a function of this distance. If the position vector is given by

$$\vec{r} = \vec{r}(s) = x(s)\,\hat{\mathbf{e}}_1 + y(s)\,\hat{\mathbf{e}}_2 + z(s)\,\hat{\mathbf{e}}_3$$

then the derivative with respect to arc length s is defined

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \vec{r}}{\Delta s} = \lim_{\Delta s \to 0} \frac{\vec{r}(s+\Delta s) - \vec{r}(s)}{\Delta s}$$

This limiting statement can be interpreted by the illustration above with the vector $\vec{r}(s)$ pointing to some point P and the vector $\vec{r}(s+\Delta s)$ pointing to some near point

[5] The given equation sweeps out a right-handed helix. Can you determine the equation for a left-handed helix?

Q and the vector $\Delta\vec{r}$ representing the direction of the secant line through the points P and Q.

Letting the point Q approach the point P one finds the direction of the secant line vector $\Delta\vec{r}$ approaches the direction of the tangent to the curve at the point P. In this limiting process one can write

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s\to 0}\frac{\Delta\vec{r}}{\Delta s} = \frac{dx}{ds}\,\hat{\mathbf{e}}_1 + \frac{dy}{ds}\,\hat{\mathbf{e}}_2 + \frac{dz}{ds}\,\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_t$$

where $\hat{\mathbf{e}}_t$ represents a unit tangent vector to the curve. Note that this tangent vector is a **unit vector** since the magnitude of this derivative is

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds}} = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\frac{(dx)^2+(dy)^2+(dz)^2}{(ds)^2}} = 1$$

since an element of arc length is given by $ds^2 = dx^2 + dy^2 + dz^2$. This shows the vector $\frac{d\vec{r}}{ds}$ is a **unit vector** which is tangent to the space curve $\vec{r} = \vec{r}(s)$.

By using chain rule differentiation one can assign a geometric interpretation to the derivative of a space curve $\vec{r} = \vec{r}(t)$ which is expressed in terms of a time parameter t. Using the chain rule one finds

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\frac{d\vec{r}}{ds} = v\,\hat{\mathbf{e}}_t = \vec{v}$$

Here $v = \frac{ds}{dt}$ is a scalar called speed and represents the change in distance with respect to time. The above equation shows the velocity vector is also tangent to the curve at any instant of time.

Differentiation Formulas

The derivative of any vector $\vec{v} = \vec{v}(t)$ is defined $\lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}$. Note the derivative of a constant vector is zero. Using the property that the limit of a sum is the sum of the limits, the above differentiation formula indicates that **each of the components of a vector must be differentiated**. Here it is assumed the unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_3$ are fixed constants and so their derivatives are zero.

For vector functions of the parameter t

$$\vec{u} = \vec{u}(t) = u_1(t)\,\hat{\mathbf{e}}_1 + u_2(t)\,\hat{\mathbf{e}}_2 + u_3(t)\,\hat{\mathbf{e}}_3$$
$$\vec{v} = \vec{v}(t) = v_1(t)\,\hat{\mathbf{e}}_1 + v_2(t)\,\hat{\mathbf{e}}_2 + v_3(t)\,\hat{\mathbf{e}}_3,$$
$$\vec{w} = \vec{w}(t) = w_1(t)\,\hat{\mathbf{e}}_1 + w_2(t)\,\hat{\mathbf{e}}_2 + w_3(t)\,\hat{\mathbf{e}}_3$$

where the components $u_i(t)$,$v_i(t)$ and $w_i(t)$, $i = 1, 2, 3$ are continuous and differentiable, the following differentiation rules can be verified using the definition of a derivative as given by equation (6.49).

The derivative of a sum is the sum of the derivatives and $\frac{d}{dt}(\vec{u}+\vec{v}) = \frac{d\vec{u}}{dt} + \frac{d\vec{u}}{dt}$

The derivative of a dot product of two vectors is the first vector dotted with the derivative of the second vector plus the derivative of the first vector dotted with the second vector and one can write

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\cdot\vec{v}$$

The derivative of a cross product of two vectors gives a similar result

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$$

The derivative of a scalar function times a vector is similar to the product rule and one finds

$$\frac{d}{dt}(f(t)\vec{u}) = f(t)\frac{d\vec{u}}{dt} + \frac{df}{dt}\vec{u}$$

where $f = f(t)$ is a scalar function. In the special case $f = c$ is a constant one finds $\frac{d}{dt}(c\vec{u}) = c\frac{d\vec{u}}{dt}$.

If $\vec{u} = \vec{u}(s)$ and $s = s(t)$, then the **chain rule for differentiating vector functions** is given by

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{ds}\frac{ds}{dt}$$

The derivative of a triple scalar product is found to be

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}\times\vec{w}) = \vec{u}\cdot\vec{v}\times\frac{d\vec{w}}{dt} + \vec{u}\cdot\frac{d\vec{v}}{dt}\times\vec{w} + \frac{d\vec{u}}{dt}\cdot\vec{v}\times\vec{w}$$

Each of the above derivative relations can be derived using the definition of a derivative.

Kinematics of Linear Motion

In the study of dynamics or physics one encounters Newton's three laws of motion. These three laws are sometimes expressed in the following form.

1. *A body at rest remains at rest and a body in motion remains in motion, unless acted upon by an external force.*
2. *The time rate of change of the linear momentum of a body is proportional to the force acting on the body, with the body moving in the direction of the applied force.*
3. *For every action there is an equal and opposite reaction.*

If $\vec{r}$ represents the length and direction of a line drawn to the center of mass of a body, then $\dfrac{d\vec{r}}{dt} = \vec{v}$ represents the instantaneous velocity of the body and $|\vec{v}| = v = |\frac{d\vec{r}}{dt}|$ represents the speed of the body. Let m denote the scalar mass of the body and let $\vec{w}$ denote the vector weight of the body. Here weight is a force given by $\vec{w} = m\vec{g}$, where $\vec{g}$ is the acceleration of gravity[6] Denote by $m\vec{v}$ the linear momentum of the body and let $\vec{F}$ denote the force acting on a body. Using these symbols Newton's second law can be expressed in the form

$$\frac{d}{dt}(m\vec{v}) = k\vec{F}$$

and if the mass m is a constant, then $m\dfrac{d\vec{v}}{dt} = k\vec{F}$ or $m\vec{a} = k\vec{F}$ where k is a proportionality constant and $\vec{a} = \dfrac{d\vec{v}}{dt}$ denotes the acceleration of the body. The value of the constant k depends upon the units used to measure distance, time and force.

The following is a set of units for force, mass, distance and time which allow for the proportionality constant to have the value $k = 1$. The notation of brackets around a quantity is used to denote "the dimensions of" the quantity. For example, the notation, $[y] =$ meters, is read, "The dimension of y is meters.[7]

(fps) System

In the foot (ft), pound (lb), second (sec) system of measurements, one uses

$$[distance] = \text{ft}, \qquad [mass] = \text{lb}, \qquad [time] = \text{sec}, \qquad [Force] = \text{slugs}$$

where $1\ slug \cdot ft/sec^2 = 1\ lb\ force$

(cgs) System

In the centimeter (cm), gram (g), second (sec) system of measurements, one uses

$$[distance] = \text{cm}, \qquad [mass] = \text{g}, \qquad [time] = \text{sec}, \qquad [Force] = \text{dynes}$$

where $1\ dyne = 1\ g \cdot m/sec^2$

(mks) System

In the meter (m), kilogram (kg), second (sec) system of measurements, one uses

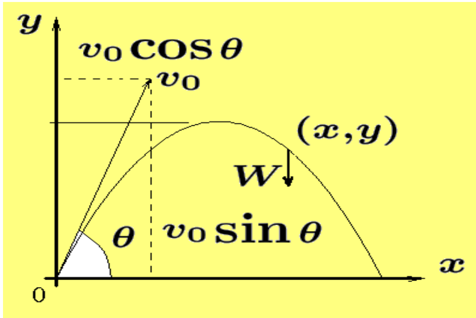
$$[distance] = \text{m}, \qquad [mass] = \text{kg}, \qquad [time] = \text{sec}, \qquad [Force] = \text{N}$$

where $1\ N = 1\ Newton = 1\ kg \cdot m/sec^2$

[6] The magnitude of the acceleration of gravity g varies between 9.78 $\frac{m}{sec^2}$ and 9.82 $\frac{m}{sec^2}$ and depends upon the position of latitude of the body. In this introduction, all particles and bodies are assumed to accelerate in a gravitational field at the same rate with a value of g=32 $\frac{ft}{sec^2}$ or g=980 $\frac{cm}{sec^2}$ or g=9.8 $\frac{m}{sec^2}$.

[7] Bracket notation for dimensions of a quantity was introduced by J.B.J. Fourier, theorie analytique de la chaleur, Paris, 1822.

Example 6-19.
A cannon ball of mass m is fired from a cannon with an initial velocity v_0 inclined at an angle θ with the horizontal as illustrated. Neglect air resistance and find the equations of motion, maximum height, and range of the cannon ball.



Solution: Let $y = y(t)$ denote the vertical height at any time t and let $x = x(t)$ denote the horizontal distance at any time t. Consider the cannon ball at a position (x, y) and examine the forces acting on it. In the y-direction the force due to the weight of the cannon ball is $W = mg$, ($g = 32\,\text{ft/sec}^2$). The equation of motion in the y-direction is represented as

$$m\frac{d^2y}{dt^2} = -W = -mg. \tag{6.50}$$

Forces in the x-direction like air resistance are neglected. Newton's second law can then be expressed

$$m\frac{d^2x}{dt^2} = 0. \tag{6.51}$$

Make note of the fact that whenever time t is the independent variable, the dot notation

$$\dot{x} = \frac{dx}{dt}, \qquad \ddot{x} = \frac{d^2x}{dt^2}, \qquad \dot{y} = \frac{dy}{dt}, \qquad \ddot{y} = \frac{d^2y}{dt^2} \tag{6.52}$$

is often employed to denote derivatives. Using the dot notation the equations (6.50) and (6.51) would be represented

$$\ddot{y} = -g \qquad \text{and} \qquad \ddot{x} = 0 \tag{6.53}$$

Calculating the x and y-components of the initial velocity, the equations (6.50) and (6.51) are solved subject to the initial conditions:

$$x(0) = 0, \qquad y(0) = 0$$
$$\dot{x}(0) = v_0 \cos\theta, \qquad \dot{y}(0) = v_0 \sin\theta,$$

where v_0 is the initial speed and θ is the angle of inclination of the cannon. Solving the differential equations (6.50) and (6.51) by successive integrations gives

$$\begin{aligned} \dot{y} =& -gt + c_1, & \dot{x} =& c_3 \\ y =& -g\frac{t^2}{2} + c_1 t + c_2, & x =& c_3 t + c_4 \end{aligned}$$

where c_1, c_2, c_3, c_4 are constants of integration. The solution satisfying the initial conditions can be expressed as

$$\begin{aligned} y &= y(t) = -\frac{g}{2}t^2 + (v_0 \sin\theta)\, t \\ x &= x(t) = (v_0 \cos\theta)\, t. \end{aligned} \tag{6.54}$$

These are parametric equations describing the position of the cannon ball. The position vector describing the path of the cannon ball is given by

$$\vec{r} = \vec{r}(t) = (v_0 \cos\theta)\, t\, \hat{\mathbf{e}}_1 + (-\frac{g}{2}t^2 + (v_0 \sin\theta)\, t)\, \hat{\mathbf{e}}_2$$

The maximum height occurs where the derivative $\frac{dy}{dt}$ is zero, and the maximum range occurs when the height y returns to zero at some time $t > 0$. The derivative $\frac{dy}{dt}$ is zero when t has the value $t_1 = v_0 \sin\theta / g$, and at this time,

$$y_{max} = y(t_1) = \frac{v_0^2 \sin^2\theta}{2g}, \qquad\qquad x = x(t_1) = \frac{v_0^2 \sin 2\theta}{2g} \tag{6.55}$$

The maximum range occurs when $y = 0$ at time $t_2 = 2\dfrac{v_0 \sin\theta}{g}$, and at this time,

$$x_{max} = x(t_2) = \frac{v_0^2 \sin 2\theta}{g}.$$

Eliminating t from the parametric equations (6.54), demonstrates that the trajectory of the cannon ball is a parabola.

■

Example 6-20. (Circular motion)

Consider a particle moving on a circle of radius r with **a constant angular velocity** $\omega = \frac{d\theta}{dt}$. Construct a cartesian set of axes with origin at the center of the circle. Assume the position of the particle at any given time t is given by the position vector

$$\vec{r} = \vec{r}(t) = r\cos\omega t\ \hat{\mathbf{e}}_1 + r\sin\omega t\ \hat{\mathbf{e}}_2 \qquad r \text{ and } \omega \text{ are constants.}$$

The **displacement of the particle** as it moves around the circle is given by $s = r\theta$ and the speed of the particle is $\dfrac{ds}{dt} = v = r\dfrac{d\theta}{dt} = r\omega$. The velocity of the particle is a vector quantity given by

$$\vec{v} = \frac{d\vec{r}}{dt} = -r\omega\sin\omega t\ \hat{\mathbf{e}}_1 + r\omega\cos\omega t\ \hat{\mathbf{e}}_2 \tag{6.56}$$

The velocity vector is perpendicular to the position vector $\vec{r}$ since $\vec{v}\cdot\vec{r} = 0$ as can be readily verified. The velocity vector is a free vector and can be moved anywhere and so it is placed at the end of the position vector, as illustrated in the figure 6-13 to show that the velocity is tangent to the circle. The magnitude of the velocity $\vec{v}$ is the speed v given by

$$|\vec{v}| = v = \sqrt{r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t} = r\omega$$

One can define an **angular velocity vector** $\vec{\omega}$ as follows. Use the right-hand rule and point the fingers of your right-hand in the direction of the position vector $\vec{r}$ and then rotate your fingers in the direction of motion of the particle. Your thumb then points in the direction of the **angular velocity vector**. For circular motion counterclockwise in the x, y-plane, one can define the angular velocity vector $\vec{\omega} = \omega\ \hat{\mathbf{e}}_3$. By defining an angular velocity vector one can express the velocity vector of a rotating particle by

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & \omega \\ r\cos\theta & r\sin\theta & 0 \end{vmatrix} = \hat{\mathbf{e}}_1 \left(-\omega r \sin\theta\right) - \hat{\mathbf{e}}_2 \left(-\omega r \cos\theta\right), \qquad \theta = \omega t \tag{6.57}$$

and this equation can be compared with equation (6.56).

The **acceleration of the rotating particle** is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = -r\omega^2 \cos\omega t\ \hat{\mathbf{e}}_1 - r\omega^2 \sin\omega t\ \hat{\mathbf{e}}_2 = -\omega^2 \vec{r}$$

This shows the acceleration is directed toward the origin. It is therefore called **a centripetal acceleration**.[8] The magnitude of the centripetal acceleration is

$$|\vec{a}| = \omega^2 r = \frac{v^2}{r} = v\omega$$

The acceleration can also be obtained by differentiating the vector velocity given by equation (6.57) to obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \omega \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

and since ω is a constant, then $\frac{d\vec{\omega}}{dt} = 0$ so that the above reduces to

$$\vec{a} = \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2 \vec{r}$$

[8] Centripetal means "center-seeking".

where the last simplification was obtained using **the vector identity** given by equation (6.32) and the result $\omega \cdot \vec{r} = 0$. The above results are derived under the assumption that the angular velocity $\omega = \frac{d\theta}{dt}$ was a constant.

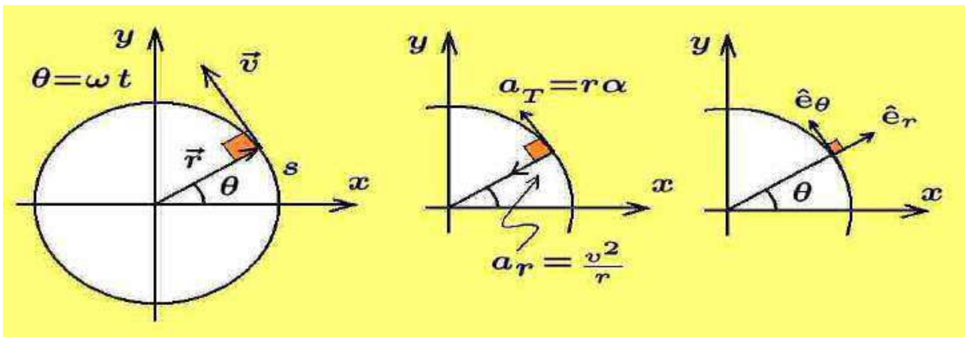


Figure 6-13. Particle moving in circular motion.

In contrast, let us examine what happens if **the angular velocity is not a constant**. The position vector to a particle undergoing circular motion is given by

$$\vec{r} = r\cos\theta\ \hat{\mathbf{e}}_1 + r\sin\theta\ \hat{\mathbf{e}}_2$$

where $\theta = \theta(t)$ is the angular displacement as a function of time. The velocity of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = -r\sin\theta\,\frac{d\theta}{dt}\ \hat{\mathbf{e}}_1 + r\cos\theta\,\frac{d\theta}{dt}\ \hat{\mathbf{e}}_2$$

Let $\frac{d\theta}{dt} = \omega(t)$ denote the angular speed which is a function of time t and express the velocity as

$$\vec{v} = -r\omega(t)\sin\theta\ \hat{\mathbf{e}}_1 + r\omega(t)\cos\theta\ \hat{\mathbf{e}}_2$$

The acceleration is obtained by taking the derivative of the velocity to obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = -\,r\left[\omega\cos\theta\,\frac{d\theta}{dt} + \frac{d\omega}{dt}\sin\theta\right]\ \hat{\mathbf{e}}_1 + r\left[-\omega(t)\sin\theta\,\frac{d\theta}{dt} + \frac{d\omega}{dt}\cos\theta\right]\ \hat{\mathbf{e}}_2$$
$$\vec{a} = -\,r\omega^2\cos\theta\ \hat{\mathbf{e}}_1 - r\alpha\sin\theta\ \hat{\mathbf{e}}_1 - r\omega^2\sin\theta\ \hat{\mathbf{e}}_2 + r\alpha\cos\theta\ \ \hat{\mathbf{e}}_2$$

where $\alpha = \alpha(t) = \dfrac{d\omega}{dt}$ is the angular acceleration. The acceleration vector can be broken up into two components by writing

$$\vec{a} = -r\omega^2\left[\cos\theta\ \hat{\mathbf{e}}_1 + \sin\theta\ \hat{\mathbf{e}}_2\right] + r\alpha\left[-\sin\theta\ \hat{\mathbf{e}}_1 + \cos\theta\ \hat{\mathbf{e}}_2\right]$$

The physical interpretation applied to the acceleration vector is as follows. Observe that the vectors

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \qquad \text{and} \qquad \hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2$$

are unit vectors and that these vectors are perpendicular to one another since they satisfy the dot product relation $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$. The vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ represent unit vectors in polar coordinates and are illustrated in the figure 6-13. The acceleration vector can then be expressed in the form

$$\vec{a} = -r\omega^2\,\hat{\mathbf{e}}_r + r\alpha\,\hat{\mathbf{e}}_\theta = \vec{a}_r + \vec{a}_t$$

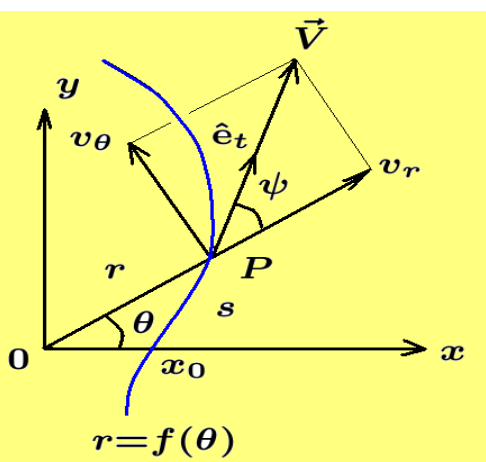
where $\vec{a}_r = -r\omega^2\,\hat{\mathbf{e}}_r$ is called **the radial component of the acceleration** or **centripetal acceleration** and $\vec{a}_t = r\alpha\,\hat{\mathbf{e}}_\theta$ is called **the tangential component of the acceleration**. These components have the magnitudes

$$|\vec{a}_r| = r\omega^2 \qquad \text{and} \qquad |\vec{a}_t| = r\alpha$$

Note that if ω is a constant, then $\alpha = 0$ and consequently the tangential component of the acceleration will always be zero leaving only the radial component of acceleration.

■

Example 6-21. Transverse and Radial Components of Velocity



Consider the motion of a particle which is described in polar coordinates by an equation of the form $r = f(\theta)$, where θ is measured in radians. Select a point P with coordinates (r, θ) on the curve and construct the radius vector $\vec{r}$ from the origin to the point P. Construct the tangent to the curve at the point P and define the angle ψ between the radius vector and the tangent. Label a fixed point on the curve, say the fixed point x_0 where the curve intersects the x-axis. Let s denote the arc length along the curve measured from x_0 to the point P. The velocity of the particle P as it moves along the curve is given by the change in distance with respect to time t and can be written $v = \dfrac{ds}{dt}$.

The velocity vector is in the direction of the tangent to the curve and **the component of the velocity along the direction** $0P$ is called the **radial component of the velocity** and denoted v_r. At the point P construct a line perpendicular to the line segment $0P$, then the component of the velocity projected onto this perpendicular line segment is called the **transverse component of the velocity** and denoted by v_θ. These projections of the velocity vector give the **radial and transverse components**

$$v_r = v\cos\psi \qquad \text{and} \qquad v_\theta = v\sin\psi$$

where $v = \dfrac{ds}{dt} = \sqrt{v_r^2 + v_\theta^2}$ is the magnitude of the velocity called the **speed of the particle**. The unit tangent vector to the curve is given by

$$\hat{\mathbf{e}}_t = \cos\psi\,\hat{\mathbf{e}}_r + \sin\psi\,\hat{\mathbf{e}}_\theta$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the radial direction and $\hat{\mathbf{e}}_\theta$ is a unit vector in the transverse direction. The derivative of the position vector with respect to the time t can be written as

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\;\hat{\mathbf{e}}_t = v\cos\psi\,\hat{\mathbf{e}}_r + v\sin\psi\,\hat{\mathbf{e}}_\theta = v_r\,\hat{\mathbf{e}}_r + v_\theta\,\hat{\mathbf{e}}_\theta$$

Therefore, when the position vector to the point P is written in the form

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 \quad \text{or} \quad \vec{r} = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 = r\,\hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is the unit vector in the radial direction given by $\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$, one finds the derivative of this unit vector with respect to θ produces the vector

$$\frac{d\,\hat{\mathbf{e}}_r}{d\theta} = \hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2$$

The derivative of the position vector with respect to arc length is a unit vector so that

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\,\hat{\mathbf{e}}_1 + \frac{dy}{ds}\,\hat{\mathbf{e}}_2 = r\frac{d\,\hat{\mathbf{e}}_r}{ds} + \frac{dr}{ds}\,\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_t = \cos\psi\,\hat{\mathbf{e}}_r + \sin\psi\,\hat{\mathbf{e}}_\theta$$

$$\text{where} \quad \frac{d\,\hat{\mathbf{e}}_r}{ds} = -\sin\theta\frac{d\theta}{ds}\,\hat{\mathbf{e}}_1 + \cos\theta\frac{d\theta}{ds}\,\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta\,\frac{d\theta}{ds}$$

Therefore, $\dfrac{d\vec{r}}{ds} = r\dfrac{d\theta}{ds}\,\hat{\mathbf{e}}_\theta + \dfrac{dr}{ds}\,\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_t = \cos\psi\,\hat{\mathbf{e}}_r + \sin\psi\,\hat{\mathbf{e}}_\theta$

Equating like components produces the result that

$$r\frac{d\theta}{ds} = \sin\psi \quad \text{and} \quad \frac{dr}{ds} = \cos\psi$$

The derivative of the position vector $\vec{r} = r\,\hat{\mathbf{e}}_r$ with respect to time t takes on the form

$$\frac{d\vec{r}}{dt} = r\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = r\frac{d\,\hat{\mathbf{e}}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = r\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = v_r\,\hat{\mathbf{e}}_r + v_\theta\,\hat{\mathbf{e}}_\theta = \vec{v}$$

where

$$\begin{aligned} v_r =& \frac{dr}{dt} = v\cos\psi \quad \text{is the radial component of the velocity} \\ v_\theta =& r\frac{d\theta}{dt} = v\sin\psi \quad \text{is the transverse component of the velocity} \\ \hat{\mathbf{e}}_r =& \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \quad \text{is a unit vector in the radial direction} \\ \hat{\mathbf{e}}_\theta =& -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2 \quad \text{is a unit vector in the transverse direction} \\ \frac{d\vec{r}}{dt} =& \vec{v} = v\cos\psi\,\hat{\mathbf{e}}_r + v\sin\psi\,\hat{\mathbf{e}}_\theta \quad \text{is alternative form for the velocity vector} \end{aligned}$$

Note also that if $\frac{d\theta}{dt} = \omega$ is the angular velocity, then one can write $v_\theta = r\omega$.

■

Example 6-22. Angular Momentum

Recall that a moment causes a rotational motion. Let us investigate what happens when Newton's second law is applied to rotational motion. The **angular momentum of a particle is defined as the moment of the linear momentum**. Let $\vec{H}$ denote the angular momentum; $m\vec{v}$, the linear momentum; and $\vec{r}$, the position vector of the particle, then by definition the moment of the linear momentum is expressed

$$\vec{H} = \vec{r} \times (m\vec{v}) = \vec{r} \times \left(m\frac{d\vec{r}}{dt}\right). \tag{6.58}$$

Differentiating this relation produces

$$\frac{d\vec{H}}{dt} = \vec{r} \times \left(m\frac{d^2\vec{r}}{dt^2}\right) + \frac{d\vec{r}}{dt} \times \left(m\frac{d\vec{r}}{dt}\right).$$

Observe that the second cross product term is zero because the vectors are parallel. Also note that by using Newton's second law, involving a constant mass, one can write

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}.$$

Comparing these last two equations it is found that the time rate of change of angular momentum is expressible in terms of the force $\vec{F}$ acting upon the particle. In particular, one can write

$$\frac{d\vec{H}}{dt} = \vec{r} \times \vec{F} = \vec{M}.$$

One of the many marvelous things introduced by the early Greek mathematicians was that symbols represent ideas and concepts. The symbols in our last equation tell us about a fundamental principal in Newtonian dynamics, that *the time rate of change of angular momentum equals the moment of the force acting on the particle.*

■

Angular Velocity

A **rigid body** is one where **any two distinct points remain a constant distance apart for all time**. A rigid body in motion can be studied by considering **both translational and rotational motion of the points within the body**. Assume there is no translational motion but only rotational motion of the rigid body. A simple rotation of every point in the rigid body, about a line through the body, can be described by (a) an axis of rotation L and (b) an angular velocity vector $\vec{\omega}$. If the axis of rotation remains fixed in space, then all points in the rigid body must move in circular arcs about the line L. Consider a point P revolving about L in a circular path of radius a as illustrated in figure 6-14.

The **average angular speed of the point** P is given by $\frac{\Delta\phi}{\Delta t}$, where $\Delta\phi$ is the angle swept out by P in a time interval Δt. The instantaneous angular speed is a scalar quantity ω determined by

$$\omega = \frac{d\phi}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}.$$

There is **a direction associated with the angular motion of** P **about the line** L and thus **an angular velocity vector** $\vec{\omega}$ **is introduced and defined so that**

(i) $\vec{\omega}$ **has a magnitude or length equal to the angular speed** ω,

(ii) $\vec{\omega}$ **is perpendicular to the plane of the circular path.**

(iii) The direction of $\vec{\omega}$ **is in the direction of advance of a right-hand screw when turned in the direction of rotation.**

Choose any point O on the line L and construct the position vector $\vec{r}$ from O to an arbitrary point P inside the rigid body. The arc length s swept out as P moves through the angle ϕ is given by $s = a\phi$. The magnitude of the linear speed v, of the point P, is given by

$$v = \frac{ds}{dt} = a\frac{d\phi}{dt} = a\omega = |\vec{v}|.$$

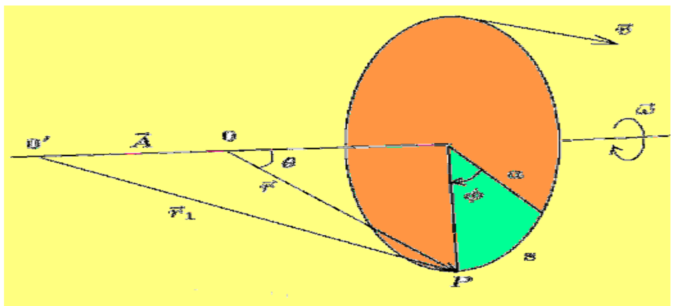


Figure 6-14. Rotation of a rigid body about a line.

The geometry in figure 6-14, is investigated and indicates that $a = |\vec{r}|\sin\theta$, and hence the magnitude of the velocity can be represented as

$$\frac{ds}{dt} = |\vec{v}| = |\vec{\omega}||\vec{r}|\sin\theta.$$

The velocity vector is always normal to the plane containing the position vector and the angular velocity vector. Therefore the velocity vector can be expressed as

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r} = |\vec{\omega}||\vec{r}|\sin\theta\,\hat{\mathbf{e}}_n,$$

where $\hat{\mathbf{e}}_n$ is a unit vector perpendicular to the plane containing the vectors $\vec{\omega}$ and $\vec{r}$. The above arguments demonstrate that the expression for the velocity of a rotating vector is independent of the orientation of the cartesian x-,y-,z-axes as long as the origin of the coordinate system lies on the axis of rotation. To prove this result let O' denote the origin of some new x', y', z' cartesian reference frame with its origin on the axis of rotation. If $\vec{r}_1$ is the position vector from this origin to the same point P considered earlier, one finds that

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1 = \vec{\omega} \times \vec{r}_1.$$

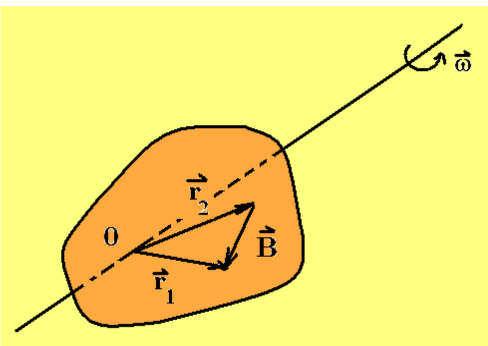
It therefore remains to show that $\vec{v}_1 = \vec{v}$. The geometry of figure 6-14, provides an aid in demonstrating that the vectors $\vec{r}_1$ and $\vec{r}$ are related by the vector equation

$$\vec{r}_1 = \vec{A} + \vec{r},$$

where $\vec{A}$ is a vector from the origin of one system to the origin of the other system and lying along the axis of rotation and in the same direction as $\vec{\omega}$. These results demonstrate that $\omega \times \vec{A} = \vec{0}$ and

$$\frac{d\vec{r}_1}{dt} = \vec{v}_1 = \vec{\omega} \times \vec{r}_1 = \vec{\omega} \times (\vec{A} + \vec{r}) = \vec{\omega} \times \vec{A} + \vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{r} = \vec{v} = \frac{d\vec{r}}{dt},$$

Here the distributive law for cross products has been employed and the fact that both $\vec{\omega}$ and $\vec{A}$ have the same direction produced a cross product of zero.



Let $\vec{B}$ denote any vector connecting two fixed points within a rigid body which is rotating about a line with angular velocity $\vec{\omega}$. Let $\vec{r}_1$ denote a vector to the terminus of $\vec{B}$ and let $\vec{r}_2$ denote a vector to the origin of $\vec{B}$, as measured from **some origin on the axis of rotation**. One can then write

$$\frac{d\vec{r}_1}{dt} = \omega \times \vec{r}_1 \qquad \text{and} \qquad \frac{d\vec{r}_2}{dt} = \omega \times \vec{r}_2$$

Observe that by vector addition $\vec{r}_2 + \vec{B} = \vec{r}_1$ so that

$$\frac{d\vec{B}}{dt} = \frac{d\vec{r}_1}{dt} - \frac{d\vec{r}_2}{dt} = \omega \times \vec{r}_1 - \omega \times \vec{r}_2 = \omega \times (\vec{r}_1 - \vec{r}_2) = \omega \times \vec{B}$$

Therefore one can state that in general, if $\vec{B}$ **is any fixed vector lying within a rigid body which is rotating, then with respect to any origin on the axis of rotation, one can state that**

$$\frac{d\vec{B}}{dt} = \vec{\omega} \times \vec{B} \tag{6.59}$$

This is an important result used in the study of rotating bodies.

Two-Dimensional Curves

The graphical representation of a function $y = f(x)$ in a rectangular cartesian coordinate system can also be presented in a vector language. A graph of the function $y = f(x)$ can be represented by a position vector $\vec{r}$, measured from the origin, which sweeps out the curve as the parameter x varies. In figure 6-15, the position vector $\vec{r}$ is illustrated. This position vector has the representation

$$\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2. \tag{6.60}$$

As the parameter x varies, the position vector $\vec{r}$ represents the distance and direction of the point $(x, f(x))$ with respect to the origin. The derivative

$$\frac{d\vec{r}}{dx} = \vec{r}\,'(x) = \hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2 \tag{6.61}$$

is also illustrated in figure 6-15. Observe that the derivative represents the tangent vector to the curve at the point $(x, f(x))$. There can be two tangent vectors to the curve at $(x, f(x))$, namely $\frac{d\vec{r}}{dx} = \vec{r}\,'(x)$ and $-\frac{d\vec{r}}{dx} = -\vec{r}\,'(x)$. Unless otherwise stated, the tangent vector in the sense of increasing parameter x is to be understood.

The cross product of the unit vector $\hat{\mathbf{e}}_3$, out of the plane of the curve, and the tangent vector $\frac{d\vec{r}}{dx} = \vec{r}\,'(x)$ to the curve, gives a normal vector $\vec{N}$ to the curve at the point $(x, f(x))$. One calculates this normal vector using the cross product

$$\vec{N} = \hat{\mathbf{e}}_3 \times \frac{d\vec{r}}{dx} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & 1 \\ 1 & f'(x) & 0 \end{vmatrix} = -f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \tag{6.62}$$

Note **there can be two normal vectors to the curve at the point** $(x, f(x))$, namely $\vec{N}$ and $-\vec{N}$. To verify that $\vec{N}$ is normal to the tangent vector at a general point $(x, f(x))$ one can examine the dot product of the normal and tangent vector $\vec{N} \cdot \frac{d\vec{r}}{dx}$ and show

$$\vec{N} \cdot \frac{d\vec{r}}{dx} = [-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2] \cdot [\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2] = 0$$

which demonstrates these two vectors are perpendicular to one another. Further, the magnitudes of the normal vector $\vec{N}$ and the tangent vector $\frac{d\vec{r}}{dx}$ are equal and can be represented

$$|\vec{N}| = \left|\frac{d\vec{r}}{dx}\right| = \sqrt{1 + [f'(x)]^2}. \tag{6.63}$$

One can use the magnitudes of the tangent and normal vectors to construct unit vectors in the tangent and normal directions at each point $(x, f(x))$ on the plane curve. One finds these unit vectors have the form

$$\hat{\mathbf{e}}_t = \frac{\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}} \qquad \text{and} \qquad \hat{\mathbf{e}}_n = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}. \tag{6.64}$$

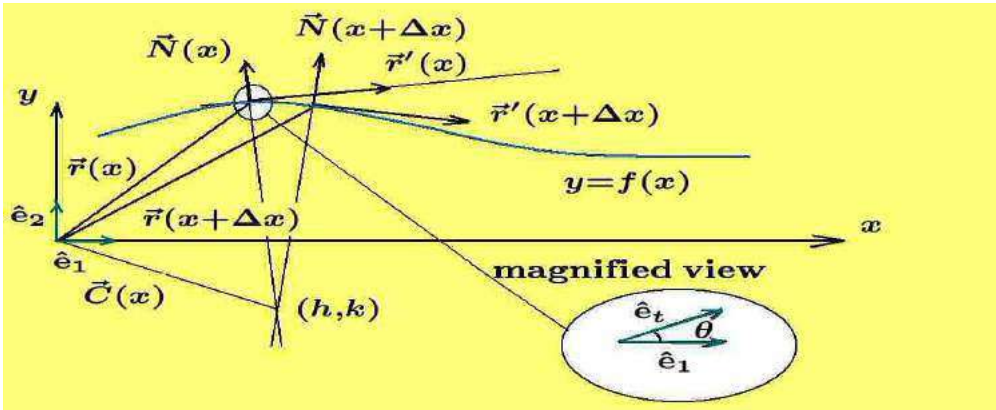


Figure 6-15.
Tangent line and normal line to plane curve change with position.

Recall from our earlier study of calculus that the arc length s measured along a curve from some fixed point $(x_0, f(x_0))$ is given by

$$s = \int_{x_0}^{x} \sqrt{1 + [f'(x)]^2}\, dx \tag{6.65}$$

and the derivative of this arc length with respect to the parameter x is

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2}. \tag{6.66}$$

Using chain rule differentiation one finds

$$\frac{d\vec{r}}{ds}\frac{ds}{dx} = \frac{d\vec{r}}{dx} = \frac{d\vec{r}}{ds}\sqrt{1 + [f'(x)]^2}$$

or

$$\hat{\mathbf{e}}_t = \frac{d\vec{r}}{ds} = \frac{1}{\sqrt{1 + [f'(x)]^2}}\frac{d\vec{r}}{dx}$$

which shows the unit tangent vector to the curve is the derivative of the position vector with respect to arc length. The choice of the sign on the square root determines the direction of the unit tangent vector.

At each point on the plane curve the unit tangent vector $\hat{\mathbf{e}}_t$ makes an angle θ with the constant unit vector $\hat{\mathbf{e}}_1$. The **absolute value of the rate of change of this angle with respect to arc length is called the curvature** and is denoted by the Greek letter κ. The curvature is thus represented by

$$\kappa = \left|\frac{d\theta}{ds}\right|.$$

By using the results $\tan\theta = \dfrac{dy}{dx}$ and $ds^2 = dx^2 + dy^2$, one can calculate the derivatives

$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} \qquad \text{and} \qquad \frac{ds}{dx} = \pm\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

The chain rule for differentiation can be employed to calculate the curvature

$$\kappa = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\theta}{dx}\frac{dx}{ds}\right| = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{6.67}$$

The unit tangent vector $\hat{\mathbf{e}}_t$ satisfies $\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t = 1$. Differentiating this relation with respect to arc length s and simplifying produces

$$\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} + \frac{d\,\hat{\mathbf{e}}_t}{ds} \cdot \hat{\mathbf{e}}_t = 2\,\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} = 0. \tag{6.68}$$

When the dot product of two nonzero vectors is zero, the two vectors are perpendicular to one another. Hence, the vector $\dfrac{d\,\hat{\mathbf{e}}_t}{ds}$ is perpendicular to the tangent vector $\hat{\mathbf{e}}_t$ when evaluated at a common point on the curve. It is known that the vector $\hat{\mathbf{e}}_n$ is perpendicular to the tangent vector. The vectors $\hat{\mathbf{e}}_n$ and $\dfrac{d\,\hat{\mathbf{e}}_t}{ds}$ are therefore colinear. Consequently there exists a suitable constant c such that

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = c\,\hat{\mathbf{e}}_n.$$

It is now demonstrated that $c = \kappa$, the curvature associated with the curve. To solve for the constant c differentiate $\hat{\mathbf{e}}_t$ with respect to the arc length s. From the expression

$$\frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dx} = \frac{d\,\hat{\mathbf{e}}_t}{dx} = \frac{\sqrt{1+[f'(x)]^2}f''(x)\,\hat{\mathbf{e}}_2 - [\,\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2][1+[f'(x)]^2]^{-\frac{1}{2}}f'(x)f''(x)}{1+[f'(x)]^2},$$

the derivative of the unit tangent vector with respect to arc length is given by

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{f''(x)}{[1+[f'(x)]^2]^{\frac{3}{2}}}\left[\frac{-f'(x)\,\hat{\mathbf{e}}_1 + \,\hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}}\right] = \frac{f''(x)}{[1+[f'(x)]^2]^{\frac{3}{2}}}\,\hat{\mathbf{e}}_n.$$

Taking the absolute value of both sides of this equation shows that the scalar curvature κ is a function of position and is given by

$$\kappa = \frac{|f''(x)|}{[1+[f'(x)]^2]^{\frac{3}{2}}}.$$

The **reciprocal of the curvature κ is called the radius of curvature ρ.** Note that **straight lines have a constant angle θ between a unit tangent vector and the x-axis and hence the curvature of straight lines is zero** since the curvature is a measure of how fast the tangent vector is changing with respect to arc length.

To understand the meaning of the radius of curvature, consider the vectors $\vec{N}(x)$ and $\vec{N}(x+\Delta x)$ which are normal to the curve $y=f(x)$ at the points $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$. These vectors are illustrated in figure 6-15. For appropriate scalars α and β, the vector equations

$$\vec{C}(x) = \vec{r}(x) + \alpha\vec{N}(x) \quad \text{and} \quad \vec{C}(x) = \vec{r}(x+\Delta x) + \beta\vec{N}(x+\Delta x)$$

depict the common point of intersection (h,k) of these normal lines to the plane curve, provided these normal lines are not parallel. The scalars α and β are related by the vector equation

$$\vec{r}(x) + \alpha\vec{N}(x) = \vec{r}(x+\Delta x) + \beta\vec{N}(x+\Delta x). \tag{6.69}$$

If in the limit as $\Delta x \to 0$, the point of intersection (h,k) approaches a specific value, this limit point is called **the center of curvature.** To find the center of curvature (h,k), the scalar α (or β) must be determined. This is accomplished by expanding the above equations relating α and β. When the vector equation (6.69) is expanded, one finds have

$$x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2 - \alpha f'(x)\,\hat{\mathbf{e}}_1 + \alpha\,\hat{\mathbf{e}}_2 = (x+\Delta x)\,\hat{\mathbf{e}}_1 + f(x+\Delta x)\,\hat{\mathbf{e}}_2 - \beta f'(x+\Delta x)\,\hat{\mathbf{e}}_1 + \beta\,\hat{\mathbf{e}}_2.$$

Equate like components the two scalar equations and show

$$\begin{aligned} x - \alpha f'(x) - (x+\Delta x) + \beta f'(x+\Delta x) = 0 \qquad \text{and} \\ f(x) + \alpha - f(x+\Delta x) - \beta = 0. \end{aligned} \tag{6.70}$$

By eliminating β from these two equations one finds

$$\alpha\left[\frac{f'(x+\Delta x) - f'(x)}{\Delta x}\right] = 1 + \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] f'(x+\Delta x). \tag{6.71}$$

In this equation let $\Delta x \to 0$ and solve for α and find

$$\alpha = \frac{1 + [f'(x)]^2}{f''(x)}, \qquad f''(x) \neq 0. \tag{6.72}$$

From this result the center of curvature is found to have the position vector

$$\vec{C}(x) = \vec{r}(x) + \alpha \vec{N}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2 + \frac{1 + [f'(x)]^2}{f''(x)}\left[-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\right]. \tag{6.73}$$

Note the position vector can also be expressed in the form

$$\vec{C}(x) = \vec{r}(x) + \rho\,\hat{\mathbf{e}}_n, \qquad \text{where} \qquad \rho = \frac{1}{\kappa}, \tag{6.74}$$

and consequently the coordinates of the center of curvature can be determined. These coordinates are given by

$$h = x - \frac{f'(x)}{f''(x)}(1 + [f'(x)]^2) \quad \text{and} \quad k = f(x) + \frac{1}{f''(x)}(1 + [f'(x)]^2), \tag{6.75}$$

provided that $f''(x) \neq 0$. For $f''(x) = 0$, there is a point of inflection, and the circle of curvature degenerates into a straight line which is the tangent line to the point of inflection of the curve. Consider the set of all circles which have their centers on the normal line to the curve and which pass through the point where the normal line intersects the curve (i.e., circles are tangent to the tangent vectors). Of all the circles, **there is only one which has a contact of the second order** and this circle has its center at the center of curvature (h, k). A contact of second order means that not only does the circle and curve have a common point of intersection and a common first derivative but also that they have a common second derivative. A proof of these statements is now offered. Let the equation of the circle be denoted by

$$(\xi - h)^2 + (\eta - k)^2 = \rho^2, \tag{6.76}$$

where the (ξ, η) axes coincide with the (x, y) axes and h, k, ρ are the functions of x derived above. If one considers x as being held constant and treats η as a function of ξ, then by differentiating the equation of the circle (6.76) twice one produces the derivatives

$$(\xi - h) + (\eta - k)\frac{d\eta}{d\xi} = 0 \qquad \text{and} \qquad 1 + \left(\frac{d\eta}{d\xi}\right)^2 + (\eta - k)\frac{d^2\eta}{d\xi^2} = 0 \tag{6.77}$$

At the common point of intersection where $(\xi, \eta) = (x, f(x))$ one finds

$$\xi - h = \frac{f'(x)}{f''(x)}(1 + [f'(x)]^2) \qquad \text{and} \qquad \eta - k = -\frac{1}{f''(x)}(1 + [f'(x)]^2)$$

so that

$$\frac{d\eta}{d\xi} = -\frac{\xi - h}{\eta - k} = f'(x) \qquad \text{and} \qquad \frac{d^2\eta}{d\xi^2} = -\frac{1 + \left(\frac{d\eta}{d\xi}\right)^2}{\eta - k} = f''(x)$$

This shows that the first and second derivatives at the common point of intersection of the curve and circle are the same and so this intersection is called **a contact of order two**.

Scalar and Vector Fields

Of extreme importance in science and engineering are the concepts of a **scalar field** and a **vector field**.

Scalar and vector fields

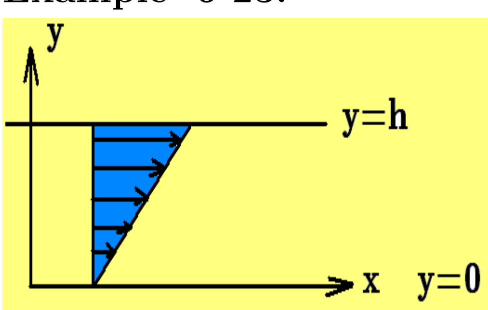
Let R denote a region of space in a cartesian coordinate system. If corresponding to each point (x, y, z) of the region R there corresponds a scalar function $\phi = \phi(x, y, z)$, then a scalar field is said to exist over the region R. If to each point (x, y, z) of a region R there corresponds a vector function

$$\vec{F} = \vec{F}(x, y, z) = F_1(x, y, z)\,\hat{e}_1 + F_2(x, y, z)\,\hat{e}_2 + F_3(x, y, z)\,\hat{e}_3,$$

then a vector field is said to exist in the region R.

That is, a scalar field is a one-to-one correspondence between points in space and scalar quantities and a vector field is a one-to-one correspondence between points in space and vector quantities. The functions which occur in the representation of a vector or scalar fields are assumed to be single valued, continuous, and differentiable everywhere within their region of definition.

Example 6-23.



An example of a vector field is the velocity of a fluid. In such a velocity field, at each point in some specified region a velocity vector exists which describes the fluid velocity. The velocity vector is a function of position within the specified region. Consider water flowing in a channel having a depth h as illustrated. Construct a set of x, y axes with $y = 0$ representing the bottom of the channel and $y = h$ representing the top of the channel. If the velocity of the fluid in the channel is given by the one-dimensional vector field $\vec{v} = \alpha y \,\hat{\mathbf{e}}_1$, for $0 \le y \le h$ and α is some proportionality constant, then the vector field associated with this flow can be graphically illustrated by sketching the vectors $\vec{v}$ at various depths in the channel. The resulting images represent one way of illustrating a vector field. The resulting sketch is called a vector field plot.

■

Example 6-24.

Consider the two-dimensional vector field $\vec{v} = \vec{v}(x, y) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2$. There are computer programs that can graphically illustrate this vector field by plotting vectors at selected points within a specified region. The resulting images of all the vectors illustrated at a finite set of points is called a vector field plot. The figure 6-16 illustrates a vector field plot for the above vector $\vec{v}$ sketched at selected points over the region $R = \{(x, y) \mid -5 \le x \le 5,\ -5 \le y \le 5\ \}$.

An alternative method of illustrating a vector field is to define a set of curves, called field lines, where each curve has the property that at each point (x, y) on any curve, the tangent to the curve at (x, y) has **the same direction as the vector field at that point**. If $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ is a position vector to a point (x, y) on a field line, then $d\vec{r}$ gives the direction of the tangent line and if this direction is to have the same direction as $\vec{v}$, then the two directions must be proportional and requires that

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 = k\vec{v}(x, y) = k\,[y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2] = ky\,\hat{\mathbf{e}}_1 + kx\,\hat{\mathbf{e}}_2 \tag{6.78}$$

where k is some proportionality constant.

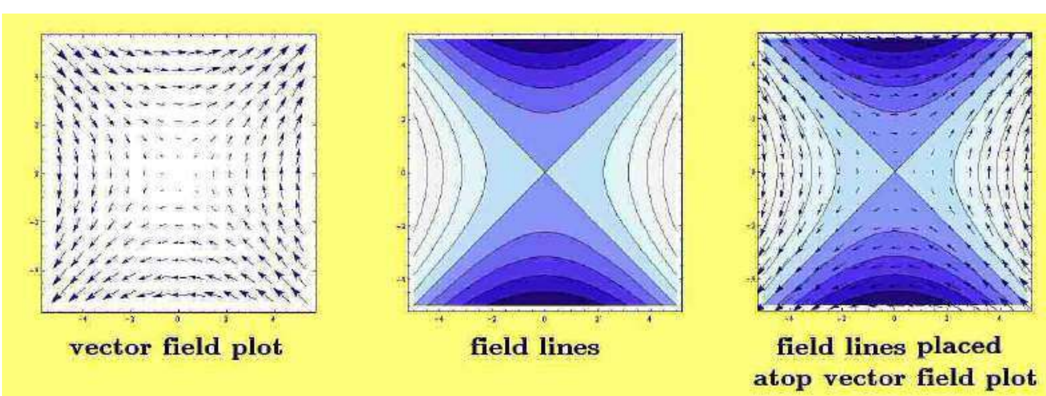


Figure 6-16. Vector field plot for $\vec{v} = \vec{v}(x, y) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2$

If these direction are the same, then by equating like components one must have

$$dx = ky \quad \text{and} \quad dy = kx \qquad \text{or} \qquad \frac{dx}{y} = \frac{dy}{x} = k \tag{6.79}$$

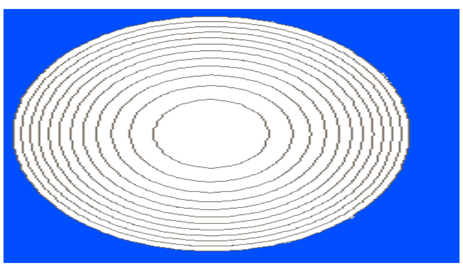
The equation (6.79) requires that $x\,dx = y\,dy$ and if one integrates both sides of this equation one obtains the family of field lines

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{C}{2} \qquad \text{or} \qquad x^2 - y^2 = C \tag{6.80}$$

where $C/2$ is selected as the constant of integration to make all terms have a factor of 2 in the denominator. Plotting these curves over the region R for various values of the constant C gives the field lines illustrated in the figure 6-16. The final figure in figure 6-16 illustrates the field lines atop the vector field plot in order that you can get a comparison of the two techniques.

■

Example 6-25.



An example of a two-dimensional scalar field is a scalar function $\phi = \phi(x,y)$ representing the temperature at each point (x,y) inside some specified region. The scalar field can be visualized by plotting the family of curves $\phi(x,y) = C$ for various values of the constant C.

The resulting family of curves are called level curves and represent curves where the temperature has a constant value. If the scalar field $\phi = \phi(x,y)$ represented height of the water above some reference point, then one can think of say an island where at different times the level of the water makes a contour of the island shape. In this case the family of curves $\phi(x,y) = C$, for various values of the constant C, are called level curves or contour plots since at various heights C the contour of the island is given. Example contour plots are illustrated in figures 6-16 and 6-17.

Note that there are many computer programs capable of drawing contour plots or level curves associated with a given scalar function. The figure 6-17 illustrates contour plots or level curves for several different two-dimensional scalar functions as the level C changes.

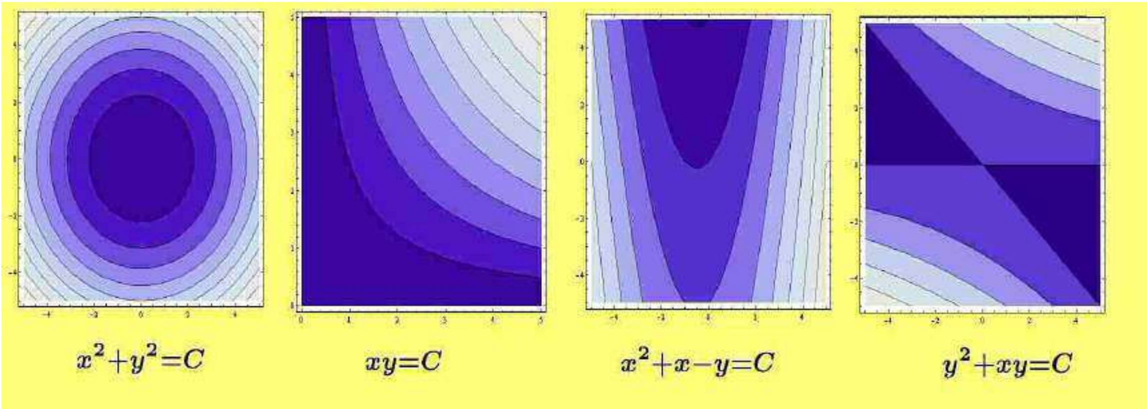


Figure 6-17. Contour plots of selected two-dimensional scalar functions.

■

A vector field is a one–to–one correspondence between points in space and vector quantities, whereas a scalar field is a one–to–one correspondence between points in space and scalar quantities. The concept of scalar and vector fields has many generalizations. A scalar field assigns a single number $\phi(x, y, z)$ to each point of space. A two-dimensional vector field would assign two numbers $(F_1(x, y, z),\ F_2(x, y, z))$ to each point of space, and a three-dimensional vector field would assign three numbers $(F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ to each point of space. An immediate generalization would be that an n-dimensional vector field would assign an n-tuple of numbers $(F_1, F_2, \ldots, F_n)$ to each point of space. Here each component F_i is a function of position, and one can write

$$F_i = F_i(x, y, z),\ i = 1, \ldots, n.$$

Other immediate ideas that come to mind are the concepts of assigning n^2 numbers to each point in space or n^3 numbers to each point in space. These higher dimensional correspondences lead to the study of matrices and tensor fields which are functions of position. In science and engineering, there is great interest in how such scalar and vector fields change with position and time.

Partial Derivatives

If a vector field $\vec{F} = \vec{F}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{e}}_1 + F_2(x, y, z)\,\hat{\mathbf{e}}_2 + F_3(x, y, z)\,\hat{\mathbf{e}}_3$ is referenced with respect to a fixed set of cartesian axes, then the partial derivatives of this vector field are given by:

$$\begin{aligned}\frac{\partial \vec{F}}{\partial x} &= \frac{\partial F_1}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial F_2}{\partial x}\,\hat{\mathbf{e}}_2 + \frac{\partial F_3}{\partial x}\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{F}}{\partial y} &= \frac{\partial F_1}{\partial y}\,\hat{\mathbf{e}}_1 + \frac{\partial F_2}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial F_3}{\partial y}\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{F}}{\partial z} &= \frac{\partial F_1}{\partial z}\,\hat{\mathbf{e}}_1 + \frac{\partial F_2}{\partial z}\,\hat{\mathbf{e}}_2 + \frac{\partial F_3}{\partial z}\,\hat{\mathbf{e}}_3.\end{aligned} \tag{6.81}$$

Observe that each component of the vector field $\vec{F}$ must be differentiated.

The higher partial derivatives are defined as derivatives of derivatives. For example, the second order partial derivatives are given by the expressions

$$\frac{\partial^2 \vec{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial x}\right), \quad \frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial y}\right), \quad \frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial y}\right), \tag{6.82}$$

where each component of the vectors are differentiated. This is analogous to the definitions of higher derivatives previously considered.

Total Derivative

The total differential of a vector field $\vec{F} = \vec{F}(x, y, z)$ is given by

$$d\vec{F} = \frac{\partial \vec{F}}{\partial x}dx + \frac{\partial \vec{F}}{\partial y}dy + \frac{\partial \vec{F}}{\partial z}dz$$

or

$$\begin{aligned} d\vec{F} = &\left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right)\hat{\mathbf{e}}_1\\ &+\left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right)\hat{\mathbf{e}}_2\\ &+\left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right)\hat{\mathbf{e}}_3.\end{aligned} \tag{6.83}$$

Example 6-26.

For the vector field

$$\vec{F} = \vec{F}(x, y, z) = (x^2 y - z)\,\hat{\mathbf{e}}_1 + (yz^2 - x)\,\hat{\mathbf{e}}_2 + xyz\,\hat{\mathbf{e}}_3$$

calculate the partial derivatives

$$\frac{\partial \vec{F}}{\partial x}, \quad \frac{\partial \vec{F}}{\partial y}, \quad \frac{\partial \vec{F}}{\partial z}, \qquad \frac{\partial^2 \vec{F}}{\partial x \partial y}$$

Solution: Using the above definitions produces the results

$$\begin{aligned}\frac{\partial \vec{F}}{\partial x} &= 2xy\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + yz\,\hat{\mathbf{e}}_3, & \frac{\partial \vec{F}}{\partial z} &= -\,\hat{\mathbf{e}}_1 + 2yz\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{F}}{\partial y} &= x^2\,\hat{\mathbf{e}}_1 + z^2\,\hat{\mathbf{e}}_2 + xz\,\hat{\mathbf{e}}_3, & \frac{\partial^2 \vec{F}}{\partial x \partial y} &= \frac{\partial^2 \vec{F}}{\partial y \partial x} = 2x\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_3.\end{aligned}$$

■

Notation

The position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ is sometimes represented in matrix notation as a row vector $\vec{r} = (x, y, z)$ or a column vector $\vec{r} = \text{col}(x, y, z)$. Sometimes the substitution $x = x_1$, $y = x_2$ and $z = x_3$ is made and these vectors are represented as $\vec{x} = (x_1, x_2, x_3)$ or $\vec{x} = \text{col}(x_1, x_2, x_3)$ and a vector function

$$\vec{F}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{e}}_1 + F_2(x, y, z)\,\hat{\mathbf{e}}_2 + F_3(x, y, z)\,\hat{\mathbf{e}}_3$$

is represented in the form of either a row vector or column vector

$$\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x})) \quad \text{or} \quad \vec{F}(\vec{x}) = \text{col}(F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x}))$$

where the representation of the basis vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ is to be understood and col is used to denote a column vector.

This change in notation is made in order that scalar and vector concepts can be extended to represent scalars and vectors in higher dimensions. For example, the representation $\vec{x} = (x_1, x_2, x_2, \ldots, x_n)$ would represent an $n-$dimensional vector, The scalar function $\phi = \phi(\vec{x}) = \phi(x_1, x_2, \ldots, x_n)$ would represent a scalar function of n-variables and the vector $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \ldots, F_n(\vec{x}))$ would represent an n-dimensional vector function of position.

Gradient, Divergence and Curl

The **gradient of a scalar function** $\phi = \phi(x, y, z)$ is the vector function

$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3$$

If the scalar function is represented in the form $\phi = \phi(x_1, x_2, x_3)$, then the gradient vector is sometimes expressed in the form

$$\text{grad}\,\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right)$$

where it is to be understood that the partial derivatives are to be evaluated at the point $(x_1, x_2, x_3) = (x, y, z)$. The vector operator

$$\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3$$

called the "del operator" or "nabla", is sometimes used to represent the gradient as an operator operating upon a scalar function to produce

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3$$

The **divergence of a vector function** $\vec{F}(x,y,z)$ is **a scalar function** defined by

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

If one uses the notation $\vec{x} = (x_1, x_2, x_3)$ the divergence is expressed

$$\operatorname{div}\vec{F}(\vec{x}) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

The del operator can be used to represent the divergence using the dot product operation

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\cdot\left(\vec{F}_1\,\hat{\mathbf{e}}_1 + \vec{F}_2\,\hat{\mathbf{e}}_2 + \vec{F}_3\,\hat{\mathbf{e}}_3\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The **curl of a vector function** $\vec{F}(x,y,z)$ is defined by the determinant operation[9]

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{e}}_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{e}}_3$$

If the notation $\vec{F} = \vec{F}(x_1, x_2, x_3)$ is used, then the curl is sometimes represented in the form

$$\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)$$

where the unit base vectors are to be understood. The operations of gradient, divergence and curl will be investigated in more detail in the next chapter.

Taylor Series for Vector Functions

Consider a vector function

$$\vec{F} = \vec{F}(\vec{x}) = \vec{F}(x_1, x_2) = F_1(x_1, x_2)\,\hat{\mathbf{e}}_1 + F_2(x_1, x_2)\,\hat{\mathbf{e}}_2$$

which is continuous and possesses $(n+1)$ partial derivatives. The Taylor series expansion for this function is just applying the Taylor series expansion to each of the scalar functions F_1, F_2. Associated with the vector $\vec{h} = (h_1, h_2)$ is the vector operator

$$\vec{h}\cdot\nabla = h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2}$$

9 See chapter 10 for properties of determinants.

so that if ϕ represents either of the components F_1 or F_2 one can write

$$(\vec{h}\cdot\nabla)\phi = h_1\frac{\partial\phi}{\partial x_1} + h_2\frac{\partial\phi}{\partial x_2}$$

Observe that the operator

$$\begin{aligned}(\vec{h}\cdot\nabla)^2\phi =&(\vec{h}\cdot\nabla)(\vec{h}\cdot\nabla)\phi\\ =&h_1\frac{\partial}{\partial x_1}\left(h_1\frac{\partial\phi}{\partial x_1} + h_2\frac{\partial\phi}{\partial x_2}\right) + h_2\frac{\partial}{\partial y}\left(h_1\frac{\partial\phi}{\partial x_1} + h_2\frac{\partial\phi}{\partial x_2}\right)\\ =&h_1^2\frac{\partial^2\phi}{\partial {x_1}^2} + 2h_1h_2\frac{\partial^2\phi}{\partial x_1\,\partial x_2} + h_2^2\frac{\partial^2\phi}{\partial {x_2}^2}\end{aligned}$$

In a similar fashion one can show

$$\begin{aligned}(\vec{h}\cdot\nabla)^3\phi =&(\vec{h}\cdot\nabla)(\vec{h}\cdot\nabla)^2\phi\\ =&h_1^3\frac{\partial^3\phi}{\partial x_1^3} + 3h_1^2h_2\frac{\partial^3\phi}{\partial x_1^2\partial x_2} + 3h_1h_2^2\frac{\partial^3\phi}{\partial x_1\partial x_2^2} + h_2^3\frac{\partial^3\phi}{\partial x_2^3}\end{aligned}$$

and in general for any positive integer n one can use the binomial expansion to calculate the operator

$$(\vec{h}\cdot\nabla)^n\,\phi = \left(h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2}\right)^n\phi$$

This operator can be used to represent the Taylor series expansion of a function $F = F(\vec{x})$ where $\vec{x} = (x_1, x_2)$. If $\vec{x}_0 = (x_1^0, x_2^0)$ is a constant and $\vec{h} = (h_1, h_2)$ denotes a small vector displacement from the point $\vec{x}_0$, then the Taylor series expansion can be written

$$F(\vec{x}_0 + \vec{h}) = \sum_{m=1}^{n}\frac{1}{m!}(\vec{h}\cdot\nabla)^m F(\vec{x})\Big|_{\vec{x}=\vec{x}_0} + \frac{1}{(n+1)!}(\vec{h}\cdot\nabla)^{n+1}F(\vec{x})\Big|_{\vec{x}=\vec{x}_0} \tag{6.84}$$

where all derivatives are to be evaluated at the point $\vec{x}_0$.

In three dimensions vectors of the form

$$\vec{F} = \vec{F}(\vec{x}) = \vec{F}(x_1, x_2, x_3) = F_1(x_1, x_2, x_3)\,\hat{\mathbf{e}}_1 + F_2(x_1, x_2, x_3)\,\hat{\mathbf{e}}_2 + F_3(x_1, x_2, x_3)\,\hat{\mathbf{e}}_3$$

which have $(n+1)$ partial derivatives can be expanded in a Taylor series by expanding each of the components in a Taylor series. Associated with the vector displacement $\vec{h} = (h_1, h_2, h_3)$ one can define the operator

$$(\vec{h}\cdot\nabla) = h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + h_3\frac{\partial}{\partial x_3}$$

and find that the Taylor series expansion has the same form as equation (6.84)

Differentiation of Composite Functions

Let $\phi = \phi(x, y, z)$ define a scalar field and consider a curve passing through the region where the scalar field is defined. Express the curve through the scalar field in the parametric form

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

with parameter t. The value of the scalar ϕ, at the points (x, y, z) along the curve, is a function of the coordinates on the curve. By substituting into ϕ the position of a general point on the curve, one can write

$$\phi = \phi(x(t), y(t), z(t)).$$

By substituting the time-varying coordinates of the curve into the function ϕ, one creates a composite function. The time rate of change of this composite function ϕ, as one moves along the curve, is derived from chain rule differentiation and

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}. \tag{6.85}$$

The equation (6.85) gives us the general rule

$$\frac{d[\ ]}{dt} = \frac{\partial [\ ]}{\partial x}\frac{dx}{dt} + \frac{\partial [\ ]}{\partial y}\frac{dy}{dt} + \frac{\partial [\ ]}{\partial z}\frac{dz}{dt} \tag{6.86}$$

where the quantity inside the brackets can be any scalar function of x, y and z. The second derivative of ϕ can be calculated by using the product rule and

$$\begin{aligned}\frac{d^2\phi}{dt^2} = \ & \frac{\partial \phi}{\partial x}\frac{d^2 x}{dt^2} + \frac{dx}{dt}\frac{d}{dt}\left[\frac{\partial \phi}{\partial x}\right] \\ & + \frac{\partial \phi}{\partial y}\frac{d^2 y}{dt^2} + \frac{dy}{dt}\frac{d}{dt}\left[\frac{\partial \phi}{\partial y}\right] \\ & + \frac{\partial \phi}{\partial z}\frac{d^2 z}{dt^2} + \frac{dz}{dt}\frac{d}{dt}\left[\frac{\partial \phi}{\partial z}\right].\end{aligned} \tag{6.87}$$

To evaluate the derivatives of the terms inside the brackets of equation (6.87) use the general differentiation rule given by equation (6.86). This produces a second derivative having the form

$$\begin{aligned}\frac{d^2\phi}{dt^2} = \ & \frac{\partial \phi}{\partial x}\frac{d^2 x}{dt^2} + \frac{dx}{dt}\left[\frac{\partial^2 \phi}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 \phi}{\partial x\, \partial y}\frac{dy}{dt} + \frac{\partial^2 \phi}{\partial x\, \partial z}\frac{dz}{dt}\right] \\ & + \frac{\partial \phi}{\partial y}\frac{d^2 y}{dt^2} + \frac{dy}{dt}\left[\frac{\partial^2 \phi}{\partial y\, \partial x}\frac{dx}{dt} + \frac{\partial^2 \phi}{\partial y^2}\frac{dy}{dt} + \frac{\partial^2 \phi}{\partial y\, \partial z}\frac{dz}{dt}\right] \\ & + \frac{\partial \phi}{\partial z}\frac{d^2 z}{dt^2} + \frac{dz}{dt}\left[\frac{\partial^2 \phi}{\partial z\, \partial x}\frac{dx}{dt} + \frac{\partial^2 \phi}{\partial z\, \partial y}\frac{dy}{dt} + \frac{\partial^2 \phi}{\partial z^2}\frac{dz}{dt}\right].\end{aligned} \tag{6.88}$$

Higher derivatives can be calculated by using the product rule for differentiation together with the rule for differentiating a composite function.

Integration of Vectors

Let $\vec{u}(s) = u_1(s)\,\hat{\mathbf{e}}_1 + u_2(s)\,\hat{\mathbf{e}}_2 + u_3(s)\,\hat{\mathbf{e}}_3$ denote a vector function of arc length, where the components $u_i(s)$, $i = 1,2,3$ are continuous functions. The indefinite integral of $\vec{u}(s)$ is defined as the indefinite integral of each component of the vector. This is expressed in the form

$$\begin{aligned}\int \vec{u}(s)\,ds =& \int u_1(s)\,ds\,\hat{\mathbf{e}}_1 + \int \vec{u}_2(s)\,ds\,\hat{\mathbf{e}}_2 + \int \vec{u}_3(s)\,ds\,\hat{\mathbf{e}}_3 + \vec{C},\\ =& \vec{U}(s) + \vec{C}.\end{aligned} \tag{6.89}$$

where $\vec{U}(s)$ is a vector such that $\frac{d\vec{U}}{ds} = \vec{u}(s)$ and $\vec{C}$ is a vector constant of integration.

The definite integral of $\vec{u}$ is defined as

$$\int_a^b \vec{u}(s)\,ds = \vec{U}(s)\bigg|_a^b = \vec{U}(b) - \vec{U}(a), \quad \text{where} \quad \frac{d\vec{U}(s)}{ds} = \vec{u}(s). \tag{6.90}$$

The following are some properties associated with the integration of vector functions. These properties are stated without proof.

1. For $\vec{c}$ a constant vector

$$\int \vec{c}\cdot\vec{u}(s)\,ds = \vec{c}\cdot\int \vec{u}(s)\,ds \quad \text{and} \quad \int \vec{c}\times\vec{u}(s)\,ds = \vec{c}\times\int \vec{u}(s)\,ds$$

2. For $\vec{c}_1$ and $\vec{c}_2$ constant vectors, the integral of a sum equals the sum of the integrals

$$\int \left[\vec{c}_1\cdot\vec{u}(s) + \vec{c}_2\cdot\vec{v}(s)\right]\,ds = \vec{c}_1\cdot\int \vec{u}(s)\,ds + \vec{c}_2\cdot\int \vec{v}(s)\,ds,$$

3. Integration by parts takes on the form

$$\int_a^b f(s)\vec{u}(s)\,ds = f(s)\vec{U}(s)\bigg|_a^b - \int_a^b f'(s)\vec{U}(s)\,ds, \tag{6.91}$$

where $f(s)$ is a scalar function and $\dfrac{d\vec{U}(s)}{ds} = \vec{u}(s)$.

Example 6-27. The acceleration of a particle is given by

$$\vec{a} = \sin t\,\hat{\mathbf{e}}_1 + \cos t\,\hat{\mathbf{e}}_2.$$

If at time $t = 0$ the position and velocity of the particle are given by

$$\vec{r}(0) = 6\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3 \qquad \text{and} \qquad \vec{v}(0) = 7\,\hat{\mathbf{e}}_1 - 6\,\hat{\mathbf{e}}_2 - 5\,\hat{\mathbf{e}}_3,$$

find the position and velocity as a function of time.

Solution: An integration of the acceleration with respect to time produces the velocity and

$$\int \vec{a}(t)\,dt = \vec{v} = \vec{v}(t) = -\cos t\,\hat{\mathbf{e}}_1 + \sin t\,\hat{\mathbf{e}}_2 + \vec{c}_1,$$

where $\vec{c}_1$ is a vector constant of integration. From the above initial condition for the velocity, the constant $\vec{c}_1$ can be determined. One finds

$$\vec{v}(0) = -\,\hat{\mathbf{e}}_1 + \vec{c}_1 = 7\,\hat{\mathbf{e}}_1 - 6\,\hat{\mathbf{e}}_2 - 5\,\hat{\mathbf{e}}_3 \qquad \text{or} \qquad \vec{c}_1 = 8\,\hat{\mathbf{e}}_1 - 6\,\hat{\mathbf{e}}_2 - 5\,\hat{\mathbf{e}}_3.$$

Consequently, the velocity can be expressed as a function of time in the form

$$\vec{v} = \vec{v}(t) = \frac{d\vec{r}}{dt} = (-\cos t + 8)\,\hat{\mathbf{e}}_1 + (\sin t - 6)\,\hat{\mathbf{e}}_2 - 5\,\hat{\mathbf{e}}_3.$$

An integration of the velocity with respect to time produces the position vector as a function of time and

$$\begin{gathered}\int \vec{v}(t)\,dt = \int \frac{d\vec{r}}{dt}\,dt = \int (-\cos t + 8)\,dt\;\hat{\mathbf{e}}_1 + \int (\sin t - 6)\,dt\;\hat{\mathbf{e}}_2 - 5\int dt\;\hat{\mathbf{e}}_3 + \vec{c}_2 \\ \vec{r}(t) = (-\sin t + 8t)\,\hat{\mathbf{e}}_1 + (-\cos t - 6t)\,\hat{\mathbf{e}}_2 - 5t\,\hat{\mathbf{e}}_3 + \vec{c}_2,\end{gathered}$$

where $\vec{c}_2$ is a vector constant of integration. From the above initial conditions, at time $t = 0$, one can determine this vector constant of integration and

$$\vec{r}(0) = -\,\hat{\mathbf{e}}_2 + \vec{c}_2 = 6\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3 \qquad \text{or} \qquad \vec{c}_2 = 6\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3.$$

The position vector as a function of time can be expressed as

$$\vec{r} = \vec{r}(t) = (-\sin t + 8t + 6)\,\hat{\mathbf{e}}_1 + (-\cos t - 6t - 2)\,\hat{\mathbf{e}}_2 + (-5t + 4)\,\hat{\mathbf{e}}_3.$$

■

Example 6-28. A particle in a force field $\vec{F} = \vec{F}(x, y, z)$ having a position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ moves according to Newton's second law such that

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \qquad \text{or} \qquad \vec{F}\,dt = m\,d\vec{v}.$$

An integration over the time interval t_1 to t_2 produces

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\,\vec{v}(t_2) - m\,\vec{v}(t_1).$$

The quantity $\int_{t_1}^{t_2} \vec{F}\,dt$ is called the linear impulse on the particle over the time interval (t_1, t_2). The quantity $m\vec{v}$ is called the linear momentum of the particle. The above equation tells us that the linear impulse equals the change in linear momentum.

■

Example 6-29. In 10 seconds a particle with a mass of 1 gram changes velocity from

$$\vec{v}_1 = 6\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 7\,\hat{\mathbf{e}}_3 \text{ cm/s} \qquad \text{to} \qquad \vec{v}_2 = -2\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3 \text{ cm/s}.$$

What average force produces this change?

Solution: The average force over a time interval (t_1, t_2) is given by

$$\vec{F}\text{avg} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \vec{F}\,dt.$$

But the integral $\int_{t_1}^{t_2} \vec{F}\,dt$ is the linear impulse and equals the change in linear momentum given by $m\vec{v}_2 - m\vec{v}_1$. The average force is therefore

$$\begin{aligned}\vec{F}\text{avg} &= \frac{1}{10}\left[(-2\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3) - (6\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 7\,\hat{\mathbf{e}}_3)\right] \\ &= \frac{1}{5}\left[-4\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3\right] \text{ dynes.}\end{aligned}$$

■

Line Integrals of Scalar and Vector Functions.

An important type of vector integration is **integration by line integrals.** Let C be a curve defined by a position vector

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3,$$

where x, y, z define some parametric representation of the curve C. The element of arc length along the curve, when squared, is given by

$$ds^2 = d\vec{r} \cdot d\vec{r} = dx^2 + dy^2 + dz^2.$$

An integration (summation) produces the following formulas for the arc length s.

1. If $y = y(x)$ and $z = z(x)$ are known in terms of the parameter x, the arc length between two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ on the curve can be represented in the form
$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx. \tag{6.92}$$
2. If the parametric equations of the curve are given by $x = x(t)$, $y = y(t)$ and $z = z(t)$, the arc length between two points P_0 and P_1 on the curve is given by
$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt, \tag{6.93}$$
where the parametric values $t = t_0$ and $t = t_1$ correspond to the points P_0 and P_1 and
$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0$$
$$x(t_1) = x_1, \quad y(t_1) = y_1, \quad z(t_1) = z_1.$$

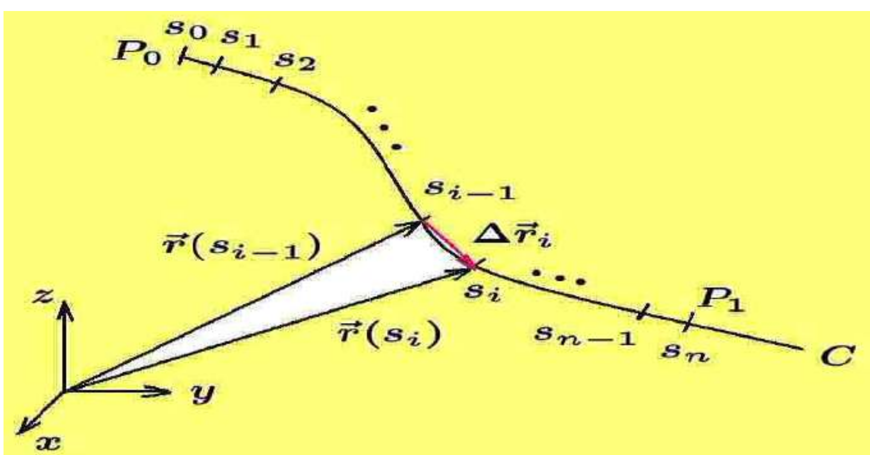


Figure 6-18. Curve C partitioned into $n-$segments between P_0 and P_1.

The above formulas result indirectly from the following limiting process. On that part of the curve between the given points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$, the arc length along the curve is divided into n segments by a set of numbers

$$s_0 < s_1 < \ldots < s_n,$$

where corresponding to each value of the arc length parameter s_i there is a position vector $\vec{r}(s_i) = x(s_i)\,\hat{\mathbf{e}}_1 + y(s_i)\,\hat{\mathbf{e}}_2 + z(s_i)\,\hat{\mathbf{e}}_3$, for $i = 1, \ldots, n$, as illustrated in figure 6-18.

A change in the element of arc length from $\vec{r}(s_{i-1})$ to $\vec{r}(s_i)$ is defined as

$$\Delta s_i = |\vec{r}(s_i) - \vec{r}(s_{i-1})| = |\Delta\vec{r}_i|\,.$$

The total arc length is obtained from the sum of these elements of arc length as the number of these lengths increase without bound and the partition gets finer and finer. In symbols, this limit is denoted as

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \int_{s_0}^{s_n} ds.$$

The above definition for arc length along the curve suggests how values of a scalar field can be summed as one moves through the scalar field along a curve C.

Definition (Line integral of a scalar function along a curve C.) ***Let $f = f(x,y,z)$ denote a scalar function of position. The line integral of f along a curve C is defined as***

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i, \tag{6.94}$$

where (x_i^*, y_i^*, z_i^*) is a point on the curve in the ith subinterval Δs_i and where the symbol $\int_C$ denotes an integral taken along the given curve C. This type of integral is called a line integral along the curve.

Similarly, define the summation of a vector field as one moves through the field along a curve C. This produces the following definition of a line integral of a vector function along a curve C.

Definition (Line integral along a curve C involving a dot product.) ***Let***

$$\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathrm{e}}_1 + F_2(x,y,z)\,\hat{\mathrm{e}}_2 + F_3(x,y,z)\,\hat{\mathrm{e}}_3$$

denote a vector function of position. The line integral of $\vec{F}$ along a given curve C, defined by a position vector $\vec{r} = \vec{r}(s) = x(s)\,\hat{\mathrm{e}}_1 + y(s)\,\hat{\mathrm{e}}_2 + z(s)\,\hat{\mathrm{e}}_3$, is defined as

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}(x_i^*,y_i^*,z_i^*)\cdot\frac{\vec{\Delta} r_i}{\Delta s_i}\Delta s_i\\ &=\int_C\left(F_1\frac{dx}{ds}+F_2\frac{dy}{ds}+F_3\frac{dz}{ds}\right)ds,\\ &=\int_C \vec{F}\cdot d\vec{r}=\int_C \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds\end{aligned}\tag{6.95}$$

***where** (x_i^*,y_i^*,z_i^*) **is a point inside the ith subinterval of the arc length** Δs_i.*

In the above definition the dot product $\vec{F}\cdot\frac{\vec{\Delta} r_i}{\Delta s_i}$ represents the projection of the vector $\vec{F}$ or component of $\vec{F}$ in the direction of the tangent vector to the curve C. The line integral of the vector function may be thought of as representing a summation of the tangential components of the vector $\vec{F}$ along the curve C between the points P_0 and P_1. Line integrals of this type arise in the calculation of the work done in moving through a force field along a curve. Here the work is given by a summation of force times distance traveled.

In particular, the above line integral can be expressed in the form

$$\int_C \vec{F}\cdot d\vec{r}=\int_C \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds=\int_C \vec{F}\cdot\hat{\mathbf{e}}_t\,ds=\int_C F_1dx+F_2dy+F_3dz,\tag{6.96}$$

where at each point on the curve C, the dot product $\vec{F}\cdot\hat{\mathbf{e}}_t$ is a scalar function of position and represents the projection of $\vec{F}$ on the unit tangent vector to the curve.

Summations of cross products along a curve produce another type of line integral.

Definition (Line integral along a curve C involving cross products.)
The line integral

$$\int_C \vec{F}\times d\vec{r}$$

is defined by the limiting process

$$\int_C \vec{F}\times d\vec{r}=\lim_{n\to\infty}\sum_{i=1}^{n}\vec{F}(x_i^*,y_i^*,z_i^*)\times\Delta\vec{r}_i,\tag{6.97}$$

***where** $\vec{F}=\vec{F}(x_i^*,y_i^*,z_i^*)$ **is the value of** $\vec{F}$ **at a point** (x_i^*,y_i^*,z_i^*) **in the ith subinterval of arc length on the curve** C.*

Integrals of this type arise in the calculation of magnetic dipole moments associated with current loops.

Note that each of the line integrals requires knowing the values of x, y and z along a given curve C and these values must be substituted into the integrand and after this substitution the summation process reduces to an ordinary integration.

Work Done.

Consider a particle moving from a point P_O to a point P_1 along a curve C which lies in a force field $\vec{F} = \vec{F}(x, y, z)$. At each point (x, y, z) on the curve there are force vectors acting on the particle as illustrated in figure 6-19.

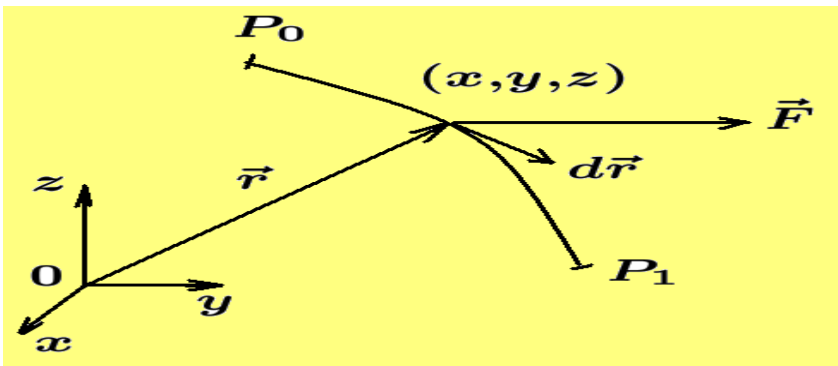


Figure 6-19. Moving along a curve C in a force field $\vec{F}$.

Examine the particle at a general point (x, y, z) on the given curve C. Construct the position vector $\vec{r}$, the force vector $\vec{F}$, and the tangent vector $d\vec{r}$ acting at this general point on the curve. The line integral

$$W_{P_0P_1} = \int_C \vec{F} \cdot d\vec{r} = \int_{P_0}^{P_1} \vec{F} \cdot \frac{d\vec{r}}{ds} ds = \int_{P_1}^{P_1} \vec{F} \cdot \hat{\mathbf{e}}_t \, ds$$

is a summation of the tangential component of the force times distance traveled along the curve C. Consequently, the above integral represents the work done in moving through the force field from point P_0 to P_1 along the curve C.

Example 6-30. Let a particle with constant mass m move along a curve C which lies in a vector force field $\vec{F} = \vec{F}(x, y, z)$. Also, let $\vec{r}$ denote the position vector of the particle in the force field and on the curve C. As the particle moves along the curve, at each point (x, y, z) of the curve, the particle experiences a force $\vec{F}(x, y, z)$

which is determined by the vector force field. Newton's second law of motion is expressed

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt}.$$

The work done in moving along the curve C between two points A and B can then be expressed as

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_A^B \vec{F}\cdot\vec{v}\,dt = \int_A^B m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = \int_A^B m\vec{v}\cdot\frac{d\vec{v}}{dt}\,dt.$$

Now utilize the vector identity

$$\frac{1}{2}\frac{d}{dt}\left(v^2\right) = \frac{1}{2}\frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right) = \vec{v}\cdot\frac{d\vec{v}}{dt},$$

so that the above line integral can be expressed in the form

$$\int_A^B \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_A^B \vec{F}\cdot\vec{v}\,dt = \int_A^B \frac{m}{2}\frac{d}{dt}\left(v^2\right)\,dt,$$

which is easily integrated. One finds

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{r} = \frac{m}{2}v^2\Big|_A^B = \frac{m}{2}\left(v_B^2 - v_A^2\right) = E_k(v_B) - E_k(v_A).$$

In this equation the line integral $W_{AB} = \int_A^B \vec{F}\cdot d\vec{r}$ is called the work done in moving the particle from A to B through the force field $\vec{F}$. The quantity $E_k(v) = \frac{m}{2}v^2$ is called the kinetic energy of the particle. The above equation tells us that the work done in moving a particle from A to B in a force field $\vec{F}$ must equal the change in the kinetic energy of the particle between the points A and B.

■

Representation of Line Integrals

The line integral $\int \vec{F}\cdot d\vec{r}$ can be expressed in many different forms:

1.
$$\int_A^B \vec{F}\cdot d\vec{r} = \int_{t_A}^{t_B} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_{t_A}^{t_B} \vec{F}\cdot\vec{v}\,dt$$

Integrals of this form are used if $\vec{F} = \vec{F}(t)$ and $\vec{v} = \vec{V}(t)$ are known functions of the parameter t.

2.
$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot\frac{d\vec{r}}{ds}\,ds = \int_{s_A}^{s_B} \vec{F}\cdot\hat{\mathbf{e}}_t\,ds$$

Here $\vec{F} \cdot \hat{\mathbf{e}}_t$ is the tangential component of the force $\vec{F}$ along the given curve C. This form of the line integral is used if $\vec{F} = \vec{F}(s)$ and $\hat{\mathbf{e}}_t$ are known functions of the arc length s.

3. For a force field given by

$$\vec{F} = \vec{F}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{e}}_1 + F_2(x, y, z)\,\hat{\mathbf{e}}_2 + F_3(x, y, z)\,\hat{\mathbf{e}}_3$$

and the position vector of a point (x, y, z) on a curve C given by

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \quad \text{with} \quad d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3,$$

Here the work done is represented in the form

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B F_1 dx + F_2 dy + F_3 dz.$$

Line integrals are written in this form when a parametric representation of the curve is known. In the special case where $\vec{r} = x\,\hat{\mathbf{e}}_1 + 0\,\hat{\mathbf{e}}_2 + 0\,\hat{\mathbf{e}}_3$, the above line integral reduces to an ordinary integral.

4. The line integral $\int_C \vec{F} \cdot d\vec{r}$ may be broken up into a sum of line integrals along different portions of the curve C. If the curve C is comprised of n separate curves $C_1, C_2, \ldots, C_n$, one can write

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \cdots + \int_{C_n} \vec{F} \cdot d\vec{r}.$$

5. When the curve C is a simple closed curve (i.e., the curve does not intersect itself), the line integral is represented by

$$\circlearrowleft\!\!\oint_C \vec{F} \cdot d\vec{r} \qquad \text{or} \qquad \circlearrowright\!\!\oint_C \vec{F} \cdot d\vec{r} \tag{6.98}$$

where the direction of integration is either in the counterclockwise sense or clockwise sense. Whenever the line integral is represented in the form $\oint_C \vec{F} \cdot d\vec{r}$ then it is to be understood that the integration direction is in the counterclockwise sense which is known as the positive sense. Note that when the curve is a simple closed curve, there is no need to specify a beginning and end point for the integration. One need only specify a direction to the integration. The integration is said to be in the positive sense if the integration is in a counterclockwise direction or it is said to be in the negative sense if the direction of integration is clockwise. The sense of integration is the same as that for angular measure. The situation is illustrated in figure 6-20.

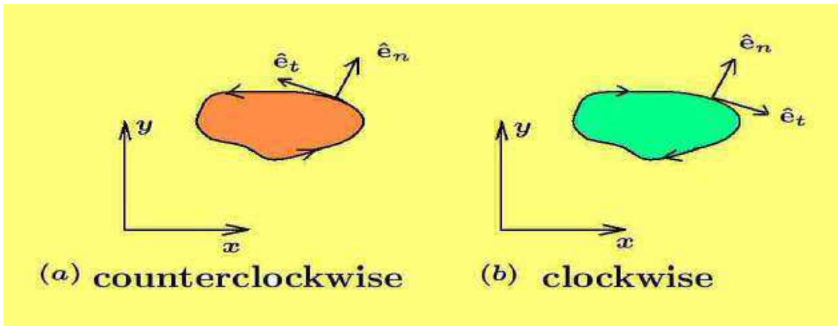


Figure 6-20. Direction of integration for line integrals.

The direction of integration around a simple closed curve can be referenced with respect to the unit outward normal $\hat{\mathbf{e}}_n$ and to the unit tangent vector $\hat{\mathbf{e}}_t$ to the simple close curve as the direction of the unit tangent produces an oriented simple closed curve.

6. If the direction of integration is reversed, then the sign of the line integral changes so that one can write

$$\oint_{C} \vec{F}\cdot d\vec{r} = -\oint_{C} \vec{F}\cdot d\vec{r}$$

Example 6-31. Consider a particle moving in a two-dimensional force field, where at any point (x, y) the force in pounds acting on the particle is given by

$$\vec{F} = \vec{F}(x,y) = (x^2+y)\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2$$

Find the work done in moving the particle from the origin to the point B along the path illustrated in figure 6-21, where distance traveled is measured in units of feet.

Solution: Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ denote the position vector of a point on the path OAB illustrated in figure 21. The work done is obtained by evaluating the line integral

$$W = \int_0^B \vec{F}\cdot d\vec{r}$$

Using the property that line integrals may be broken up into integration along separate curves, one can write $W = \int_O^B \vec{F}\cdot d\vec{r} = \int_O^A \vec{F}\cdot d\vec{r} + \int_A^B \vec{F}\cdot d\vec{r}$
where $\vec{F}\cdot d\vec{r} = (x^2+y)\,dx + xy\,dy$.

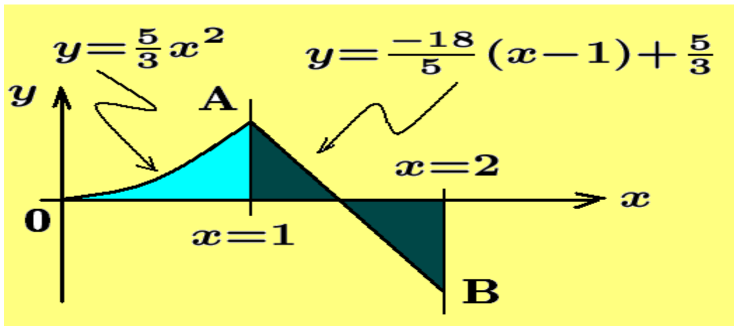


Figure 6-21. Find the work done in moving particle from origin to point B.

The portion of the work done in moving along the parabola from 0 to A, where $y=\frac{5}{3}x^2$ and $dy = \frac{5}{3}(2x\,dx)$, is

$$\int_O^A \vec{F}\cdot d\vec{r} = \int_0^1 [x^2 + (\frac{5}{3}x^2)]\,dx + x(\frac{5}{3}x^2)\,\frac{5}{3}(2x\,dx) = 2$$

The portion of the work done in moving along the straight-line from A to B, where $y = \frac{-18}{5}(x-1)+\frac{5}{3}$ and $dy = \frac{-18}{5}dx$, is expressed as

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_1^2 [x^2 + (\frac{-18}{5}(x-1)+\frac{5}{3}]\,dx + x(\frac{-18}{5}(x-1)+\frac{5}{3})(\frac{-18}{5}dx) = 4$$

The total work done is therefore given by the sum $W = 2+4 = 6$ ft-lbs. Here the unit of work is the unit of force times unit of distance traveled.

■

Example 6-32. Compute the value of the line integral

$$\oint_C \vec{F}\cdot d\vec{r},$$

where $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ and C is the circle $x^2+y^2=1$.

Solution: Let the circular path be represented in the parametric form

$$x = \cos t \qquad y = \sin t,$$

then the above line integral can be written

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C (x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2) \cdot (dx\;\hat{\mathbf{e}}_1 + dy\;\hat{\mathbf{e}}_2)$$
$$= \int_C x\,dx + y\,dy$$
$$= \int_0^{2\pi} (\cos t)(-\sin t)\,dt + (\sin t)(\cos t)\,dt = 0.$$

Here the direction of integration is in the positive sense as the parameter t varies from 0 to 2π.

■

Example 6-33. Compute the value of the line integral $\oint_C \vec{F} \times d\vec{r}$, where $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ and C is the circle $x^2 + y^2 = 1$

Solution: Write

$$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ x & y & 0 \\ dx & dy & 0 \end{vmatrix} = \hat{\mathbf{e}}_3(x\,dy - y\,dx)$$

and therefore

$$\oint_C \vec{F} \times d\vec{r} = \oint_C \hat{\mathbf{e}}_3(x\,dy - y\,dx).$$

If the circular path of integration is represented in the parametric form

$$x = \cos t \qquad y = \sin t$$

one finds

$$\oint_C \vec{F} \times d\vec{r} = \hat{\mathbf{e}}_3 \int_0^{2\pi} (\cos t)(\cos t)\,dt - (\sin t)(-\sin t)\,dt = \hat{\mathbf{e}}_3 \int_0^{2\pi} dt = 2\pi\,\hat{\mathbf{e}}_3.$$

■

Example 6-34.

Examine the work done in moving a particle through the force field

$$\vec{F} = (x+z)\,\hat{\mathbf{e}}_1 + (y+z)\,\hat{\mathbf{e}}_2 + 2z\,\hat{\mathbf{e}}_3$$

as the particle moves along the curve C described by the position vector

$$\vec{r} = t\,\hat{\mathbf{e}}_1 + t^2\,\hat{\mathbf{e}}_2 + (-3t+1)\,\hat{\mathbf{e}}_3$$

as the parameter t ranges from 0 to 2.

Solution: The work done is determined by evaluating the line integral $\int_C \vec{F}\cdot d\vec{r}$ where

$$\vec{F}\cdot d\vec{r} = (x+z)\,dx + (y+z)\,dy + 2z\,dz$$

On the given curve C use $x=t$, $y=t^2$ and $z=-3t+1$ with $dx=dt$, $dy=2t\,dt$ and $dz=-3\,dt$ and substitute these values into the line integral describing the work done. This produces the result

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^2 \big[(t-3t+1)(dt) + (t^2-3t+1)(2t\,dt) + 2(-3t+1)(-3\,dt)\big] = 18$$

where work has the units of force times units of distance traveled.

■

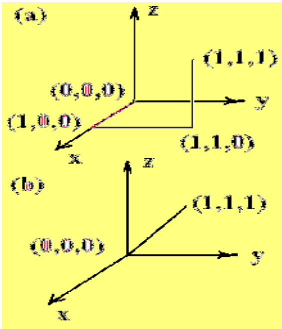
Example 6-35.

For $\vec{F} = x(y+1)\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x+1)\,\hat{\mathbf{e}}_3$ evaluate the line integral $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r}$

(a) Along the line segments illustrated.

(b) Along the straight line path from $(0,0,0)$ to $(1,1,1)$.

Solution $\vec{F}\cdot d\vec{r} = x(y+1)\,dx + y\,dy + z(x+1)\,dz$



(a) Along $(0,0,0)$ to $(1,0,0)$, $x=1$, $z=0$, $0\le x\le 1$

$$\int_0^1 x\,dx = \frac{1}{2}$$

Along $(1,0,0)$ to $(1,1,0)$, $x=1$, $z=0$, $0\le y\le 1$

$$\int_0^1 y\,dy = \frac{1}{2}$$

Along $(1,1,0)$ to $(1,1,1)$, $x=1$, $y=1$, $0\le z\le 1$

$$\int_0^1 2z\,dz = 1$$

therefore $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = \frac{1}{2}+\frac{1}{2}+1 = 2$

(b) The straight line path from $(0,0,0)$ to $(1,1,1)$ is represented by the parametric equation

$$x=t,\qquad y=t,\qquad z=t$$

for $0\le t\le 1$. Therefore

$$\int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 [t(t+1)+t+t(t+1)]\,dt = \frac{13}{6}$$

The work done in moving from $(0,0,0)$ to $(1,1,1)$ is path dependent.

■

Exercises

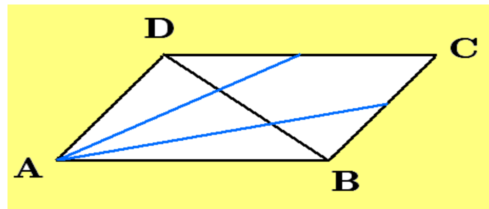
▶ **6-1.** For the vectors $\vec{A} = 3\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and $\vec{B} = 6\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$ calculate

$$(a)\quad \vec{A}+\vec{B} \qquad (b)\quad 6\vec{A}-3\vec{B} \qquad (c)\quad \vec{A}+2\vec{B}$$

▶ **6-2.** Use vectors to show that the diagonals of a parallelogram bisect one another.

▶ **6-3.** Use vectors to show that the line segment connecting the midpoints of two sides of a triangle is parallel to the third side and has one half the magnitude of the third side.

▶ **6-4.** In the parallelogram $ABCD$ illustrated, construct lines from the vertex A to the midpoints of the sides DC and BC. Show that these lines trisect the diagonal BD.



▶ **6-5.** Are the given vectors linearly dependent or linearly independent?

$$(a)\ \vec{A} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\,\hat{\mathbf{e}}_3 \quad (b)\ \vec{A} = 2\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3 \quad (c)\ \vec{A} = 3\,\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$$

$$\vec{B} = -4\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2 \qquad \vec{B} = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 \qquad \vec{B} = -\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3$$

$$\vec{C} = 7\,\hat{\mathbf{e}}_1 + 6\,\hat{\mathbf{e}}_2 - 6\,\hat{\mathbf{e}}_3 \qquad \vec{C} = 3\,\hat{\mathbf{e}}_3 \qquad \vec{C} = 14\,\hat{\mathbf{e}}_1 - 4\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3.$$

▶ **6-6.** If $\vec{A}, \vec{B}, \vec{C}$ are nonzero vectors and $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$, then determine if the following statements are true or false.

(i) The vectors $\vec{A}, \vec{B}, \vec{C}$ are linearly independent.

(ii) The vectors $\vec{A}, \vec{B}, \vec{C}$ are linearly dependent.

Justify your answers.

▶ **6-7.** Let $\vec{A} = \vec{A}(t)$ denote a vector which has a constant length C for all values of the parameter t.

(a) Show that $\vec{A}\cdot\vec{A} = C^2$

(b) Show that the derivative vector $\dfrac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

▶ **6-8.** Show that for $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ and $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ the distance d of an arbitrary point (x_0, y_0, z_0) from the line $\vec{r} = \vec{r}_1 + t\vec{A}$, is given by

$$d = |(\vec{r}_0 - \vec{r}_1)\times\hat{\mathbf{e}}_A|$$

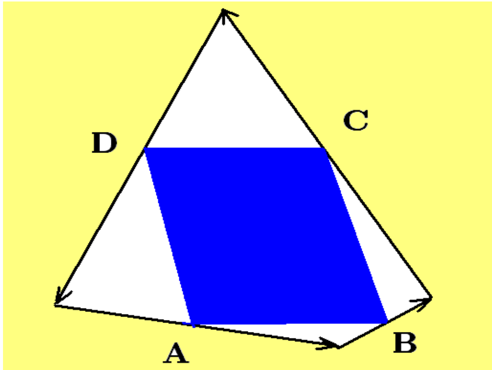
where $\hat{\mathbf{e}}_A$ is a unit vector in the direction of $\vec{A}$ and $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ is a position vector to the arbitrary point.

▶ **6-9.** Consider the triangle defined by the three vertices $(6,0,0)$, $(0,6,0)$ and $(0,0,12)$. Use vector methods to find the area of the triangle.

▶ **6-10.** Let the sides of a quadrilateral be denoted by the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ such that

$$\vec{A}+\vec{B}+\vec{C}+\vec{D}=\vec{0}.$$

Use vectors to show that the lines joining the midpoints of the sides of this quadrilateral form a parallelogram.



▶ **6-11.** Let $\vec{r}_0$ represent the position vector of the center of a sphere of radius ρ and let $\vec{r}$ represent the position vector of a variable point on the surface of the sphere. Find the equation of the sphere in a vector form. Simplify your result to a scalar form.

▶ **6-12.** For $\vec{A}=\hat{\mathbf{e}}_1+2\,\hat{\mathbf{e}}_2+2\,\hat{\mathbf{e}}_3$ and $\vec{B}=7\,\hat{\mathbf{e}}_1+4\,\hat{\mathbf{e}}_2+4\,\hat{\mathbf{e}}_3$

(a) Find a unit vector in the direction of $\vec{B}$.

(b) Find a unit vector in the direction of $\vec{A}$.

(c) Find the projection of $\vec{A}$ on $\vec{B}$.

(d) Find the projection of $\vec{B}$ on $\vec{A}$.

▶ **6-13.** (a) Find a unit vector perpendicular to the vectors

$$\vec{A}=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3 \quad \text{and} \quad \vec{B}=\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2-\hat{\mathbf{e}}_3$$

(b) Find the projection of $\vec{B}$ on $\vec{A}$.

▶ **6-14.** For $\vec{A}=-\hat{\mathbf{e}}_1+\sqrt{3}\,\hat{\mathbf{e}}_2+\sqrt{5}\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_\alpha=\cos\alpha\,\hat{\mathbf{e}}_1+\sin\alpha\,\hat{\mathbf{e}}_2$

(a) Verify that $\hat{\mathbf{e}}_\alpha$ is a unit vector for all α.

(b) Find the projection of $\vec{A}$ on $\hat{\mathbf{e}}_\alpha$.

(c) For what angle α is the projection equal to zero?

(d) For what angle α is the projection a maximum?

▶ **6-15.** Assume $\vec{A}(t)$ has derivatives of all orders. Find the constant vectors $\vec{A}_0,\vec{A}_1,\ldots,\vec{A}_n,\ldots$ if

$$\vec{A}(t)=\vec{A}_0+\vec{A}_1\frac{(t-t_0)}{1!}+\vec{A}_2\frac{(t-t_0)^2}{2!}+\cdots+\vec{A}_n\frac{(t-t_0)^n}{n!}+\cdots$$

Hint: Evaluate $\vec{A}(t)$ at $t=t_0$, then differentiate $\vec{A}(t)$ and evaluate result at $t=t_0$.

▶ **6-16.** Given the vectors $\vec{A} = \hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$ and $\vec{B} = 3\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3$
Evaluate the following quantities:

(a)	$\vec{A} \times \vec{B}$	(d)	$(\vec{A} + \vec{B}) \times \vec{A}$	(g)	$(\vec{A} + 3\vec{B}) \times \vec{B}$
(b)	$\vec{B} \times \vec{A}$	(e)	The angle between $\vec{A}$ and $\vec{B}$	(h)	$(\vec{B} - \vec{A}) \times (\vec{B} + \vec{A})$
(c)	$\vec{A} \cdot \vec{B}$	(f)	$3\vec{A} \times 2\vec{B}$	(i)	$\vec{A} \cdot (\vec{A} + \vec{B})$

▶ **6-17.** The sides of a parallelogram are $\vec{A} = \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$ and $\vec{B} = 2\,\hat{\mathbf{e}}_1 + 9\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$.
(a) Find the vectors which represent the diagonals of this parallelogram.
(b) Find the area of the parallelogram.

▶ **6-18.** Determine the direction cosines of the vector $\vec{\rho} = \sqrt{2}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3$.

▶ **6-19.** Explain why two vectors are said to be linearly dependent if their vector cross product is the zero vector.

▶ **6-20.** Three noncolinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ determine a plane. Let $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ denote the position vectors from the origin to each of these points, respectively, and let $\vec{r}$ denote the position vector of any variable point (x, y, z) in the plane.
(a) Describe and illustrate the vector $\vec{r}_3 - \vec{r}_1$.
(b) Describe and illustrate the vector $\vec{r}_2 - \vec{r}_1$.
(c) Describe and illustrate the vector $(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)$.
(d) Explain the geometrical significance $(\vec{r} - \vec{r}_1) \cdot [(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] = 0$.

▶ **6-21.** Find the parametric equations of the given line. Also find the tangent vector to the given line $\vec{r} = 3\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 + \lambda(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2)$.

▶ **6-22.** (a) Find the area of the triangle having vertices at the points

$$P_1(0, 0, 0) \quad P_2(0, 3, 4) \quad P_3(4, 3, 0).$$

(b) Find a unit normal vector to the plane passing through the above three points.
(c) Find the equation of the plane in part (b).

▶ **6-23.** **Distance between two skew lines** Let line ℓ_1 pass through points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$. Let line ℓ_2 pass through the points $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$.
(a) Show $\vec{N} = \overline{P_0P_1} \times \overline{P_2P_3}$ is perpendicular to both lines. (b) Show the projection of $\overline{P_2P_1}$ onto $\vec{N}$ gives the distance between the lines.

▶ **6-24.** Is the point $(6, 13, 12)$ on the line which passes through the points $P_1(1, 0, 1)$ and $P_2(3, 5, 2)$? Find the equation of the line.

▶ **6-25.**

(a) Derive the vector equations of a line in the following forms.

$$(\vec{r} - \vec{r}_1) \times (\vec{r}_2 - \vec{r}_1) = \vec{0} \qquad \text{and} \qquad \vec{r} = \vec{r}_1 + \lambda(\vec{r}_2 - \vec{r}_1)$$

for a line passing through the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

(b) Show these vector equations produce the same scalar equations for determining points on the line.

▶ **6-26.** Sketch the vectors $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_\beta = \cos\beta\,\hat{\mathbf{e}}_1 + \sin\beta\,\hat{\mathbf{e}}_2$ assuming α and β are acute constant angles.

(a) Show $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\beta$ are unit vectors.

(b) From the dot product $\hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\beta$ derive the addition formula for $\cos(\beta \pm \alpha)$

(c) From the cross product $\hat{\mathbf{e}}_\alpha \times \hat{\mathbf{e}}_\beta$, derive the addition formula for $\sin(\beta \pm \alpha)$

▶ **6-27.** Verify that $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \pm\,\hat{\mathbf{e}}_k$ where the + sign is used if (ijk) is an even permutation of (123) and the – sign is used if (ijk) is an odd permutation of (123).

(a) Verify the above by taking three consecutive numbers from the set $\{1, 2, 3, 1, 2, 3\}$ for the values of i, j, k. These are called the even permutations of the numbers (123).

(b) Verify the above by taking three consecutive numbers from the set $\{3, 2, 1, 3, 2, 1\}$ for the values of i, j, k. These are called the odd permutations of the numbers (123).

▶ **6-28.** Let $\alpha_1,\ \beta_1,\ \gamma_1$ and $\alpha_2,\ \beta_2,\ \gamma_2$ be the direction angles of two lines. Move each line parallel to itself until it passes through the origin. The angle between two lines is defined as the angle between the shifted lines, which pass through the origin.

(a) Show that the angle θ between two lines can be expressed in terms of the direction cosine of the lines and

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$$

(b) Find the angle between the lines defined by the equations

$$\vec{r} = (1 + 2t)\,\hat{\mathbf{e}}_1 + (1 + t)\,\hat{\mathbf{e}}_2 + (1 + 2t)\,\hat{\mathbf{e}}_3 \quad \text{and}$$

$$\vec{r} = (1 + 2t)\,\hat{\mathbf{e}}_1 + (2 + 2t)\,\hat{\mathbf{e}}_2 + (6 + t)\,\hat{\mathbf{e}}_3.$$

▶ **6-29.** Find the shortest distance from the point $(-1, 17, 7)$ to the line which passes through the points $P_1(2, 5, 4)$ and $P_2(3, 7, 6)$. Hint: See problem 6-8.

▶ **6-30.** If $\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$ and $\vec{B} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$ show that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

▶ **6-31.** If $\vec{A} \times \vec{B} = \vec{0}$ and $\vec{B} \times \vec{C} = \vec{0}$, then calculate $\vec{A} \times \vec{C}$. Justify your answer.

▶ **6-32.** (a) Find the equation of the plane which passes through the points

$$P_1(3, 10, 13) \qquad P_2(0, 11, 12) \qquad P_3(5, 12, 14).$$

(b) Find the perpendicular distance from the origin to this plane.
(c) Find the perpendicular distance from the point $(6, 3, 18)$ to this plane.

▶ **6-33.** Show that the rules for calculating the moment of a force about a line L can be altered as follows: If $\vec{r}$ is the position vector from a point P on the line L to any point on the line of action of the force $\vec{F}$, then $\vec{M} = \vec{r} \times \vec{F}$ is the moment about point P on the line L and $\vec{M} \cdot \hat{\mathbf{e}}_L$ is the moment about the line L, where $\hat{\mathbf{e}}_L$ is a unit vector in the direction of L.

▶ **6-34.** A force $\vec{F} = 100(\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 - 2\,\hat{\mathbf{e}}_3)$ lbs acts at the point $P_1(2, 2, 4)$.
(a) Find the moment of $\vec{F}$ about the origin.
(b) Find the moment of $\vec{F}$ about the point $P_2(-1, 3, -4)$.
(c) Find the moment about the line passing through the origin and the point P_2.

▶ **6-35.** Find the indefinite integral of the following vector functions

$$(a) \quad \vec{u}(t) = t\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - t^2\,\hat{\mathbf{e}}_3 \qquad (b) \quad \vec{u}(t) = t\,\hat{\mathbf{e}}_1 + \sin t\;\hat{\mathbf{e}}_2 + \cos t\;\hat{\mathbf{e}}_3$$

▶ **6-36.** Find the position vector and velocity of a particle which has an acceleration given by $\vec{a} = \cos t\;\hat{\mathbf{e}}_1 + \sin t\;\hat{\mathbf{e}}_2$ if at time $t = 0$ the position and velocity are given by $\vec{r}(0) = \vec{0}$ and $\vec{v}(0) = 2\,\hat{\mathbf{e}}_3$.

▶ **6-37.** The acceleration of a particle is given by $\vec{a} = \hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2$. If at time $t = 0$ the velocity is $\vec{v} = \vec{v}(0) = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_3$ and its position vector is $\vec{r} = \vec{r}(0) = \hat{\mathbf{e}}_2$, then find the velocity and position as a function of time.

▶ **6-38.** **Distance between parallel planes** If $(\vec{r} - \vec{r}_0) \cdot \vec{N} = 0$ and $(\vec{r} - \vec{r}_1) \cdot \vec{N} = 0$ are the equations of parallel planes, then show the distance between the planes is given by the projection of $\vec{r}_1 - \vec{r}_0$ onto the normal vector $\vec{N}$.

▶ **6-39.** In a rectangular coordinate system a particle moves around a unit circle in the plane $z=0$ with a constant angular velocity of $\omega = 5\,\text{rad/sec}$

(a) What is the angular velocity vector for this system?

(b) What is the velocity of the particle at any time t if the position of the particle is

$$\vec{r} = \cos 5t\ \hat{\mathbf{e}}_1 + \sin 5t\ \hat{\mathbf{e}}_2?$$

▶ **6-40.** A particle moves along a curve having the parametric equations

$$x = e^t, \qquad y = \cos t, \qquad z = \sin t.$$

(a) Find the velocity and acceleration vectors at any time t.

(b) Find the magnitude of the velocity and acceleration when $t = 0$.

▶ **6-41.** Let $x = x(t), \quad y = y(t)$ denote the parametric representation of a curve in two-dimensions. Using chain rule differentiation, show that the center of curvature vector, at any parameter value t, can be represented by

$$\vec{c}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\,(-\dot{y}\,\hat{\mathbf{e}}_1 + \dot{x}\,\hat{\mathbf{e}}_2)$$

provided $\dot{x}\ddot{y} - \dot{y}\ddot{x}$ is different from zero. Here the notation $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$ has been employed.

▶ **6-42.** Find the center and radius of curvature as a function of x for the given curves.

$$(a) \quad (x-2)^2 + (y-3)^2 = 16 \qquad\qquad (b) \quad y = e^x$$

▶ **6-43.** Let $\vec{e}$ denote a unit vector and let $\vec{A}$ denote a nonzero vector. In what direction $\vec{e}$ will the projection $\vec{A}\cdot\vec{e}$ be a maximum?

▶ **6-44.** Assume $\vec{A} = \vec{A}(t)$ and $\vec{B} = \vec{B}(t)$.

(a) Show that $\frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\cdot\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{B}$

(b) Show that $\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$

▶ **6-45.** Given $\vec{A} = t^2\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 + t^3\,\hat{\mathbf{e}}_3$ and $\vec{B} = \sin t\,\hat{\mathbf{e}}_1 + \cos t\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$.

Find $\quad (a)\ \frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) \qquad (b)\ \frac{d}{dt}\left(\vec{A}\times\vec{B}\right) \qquad (c)\ \frac{d}{dt}\left(\vec{B}\cdot\vec{B}\right)$

▶ **6-46.** For $\vec{u} = \vec{u}(t) = t^2\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 + 2t\,\hat{\mathbf{e}}_3$ and $\vec{v} = \vec{v}(t) = t^3\,\hat{\mathbf{e}}_1 + t^2\,\hat{\mathbf{e}}_2 + t^6\,\hat{\mathbf{e}}_3$ find the derivatives

$$(a) \quad \frac{d}{dt}(\vec{u}\cdot\vec{v}) \qquad\qquad (b) \quad \frac{d}{dt}(\vec{u}\times\vec{v})$$

▶ **6-47.** If $\vec{U} = \vec{U}(x,y) = (2x^2y + y^2x)\,\hat{\mathbf{e}}_1 + (xy + 3x^2y)\,\hat{\mathbf{e}}_2$,

then find $\dfrac{\partial \vec{U}}{\partial x}, \dfrac{\partial \vec{U}}{\partial y}, \dfrac{\partial^2 \vec{U}}{\partial x^2}, \dfrac{\partial^2 \vec{U}}{\partial y^2}, \dfrac{\partial^2 \vec{U}}{\partial x\,\partial y}$

▶ **6-48.** Consider a rigid body in pure rotation with angular velocity given by $\vec{\omega} = \omega_1\,\hat{\mathbf{e}}_1 + \omega_2\,\hat{\mathbf{e}}_2 + \omega_3\,\hat{\mathbf{e}}_3$.. For 0 an origin on the axis of rotation and the vector $\vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ denoting the position vector of a particle P in the rigid body, show that the components x, y, z must satisfy the differential equations

$$\frac{dx}{dt} = \omega_2 z - \omega_3 y, \qquad \frac{dy}{dt} = \omega_3 x - \omega_1 z, \qquad \frac{dz}{dt} = \omega_1 y - \omega_2 x$$

▶ **6-49.** For the space curve $\vec{r} = \vec{r}(t) = t^2\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 + t^2\,\hat{\mathbf{e}}_3$ find

(a) $\dfrac{d\vec{r}}{dt}$ and $\dfrac{ds}{dt} = \left|\dfrac{d\vec{r}}{dt}\right|$

(b) The unit tangent vector to the curve at any time t.

▶ **6-50.** For $\vec{A},\ \vec{B},\ \vec{C}$ functions of time t show

$$\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right] = \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \frac{d\vec{A}}{dt}\times\left(\vec{B}\times\vec{C}\right)$$

▶ **6-51.** Letting $x = r\cos\theta, \quad y = r\sin\theta$ the position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ becomes a function of r and θ which can be denoted $\vec{r} = \vec{r}(r,\theta)$.

(a) Show that $\frac{\partial\vec{r}}{\partial r}$ is perpendicular to the vector $\frac{\partial\vec{r}}{\partial\theta}$ and assign a physical interpretation to your results.

(b) Find unit vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ in the directions $\frac{\partial\vec{r}}{\partial r}$ and $\frac{\partial\vec{r}}{\partial\theta}$ and sketch these unit vectors.

▶ **6-52.** Evaluate the given line integrals along the curve $y = 3x$ from $(1,3)$ to $(2,6)$ using $\vec{F} = \vec{F}(x,y) = xy\,\hat{\mathbf{e}}_1 + (y-x)\,\hat{\mathbf{e}}_2$.

$$(a) \quad \int_C \vec{F}\cdot d\vec{r} \qquad\qquad (b) \quad \int_C \vec{F}\times d\vec{r}$$

▶ **6-53.** For $\vec{F} = (xy+1)\,\hat{\mathbf{e}}_1 + (x+z+1)\,\hat{\mathbf{e}}_2 + (z+1)\,\hat{\mathbf{e}}_3$, evaluate the line integral $I = \int_C \vec{F}\cdot d\vec{r}$, where C is the curve consisting of the straight-line segments $\overline{OA} + \overline{AB} + \overline{BC}$, where O is the origin $(0,0,0)$, and A, B, C are, respectively, the points $(1,0,0), \quad (1,1,0), \quad (1,1,1)$.

► **6-54.** Evaluate the line integral $I = \int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = 3(x+y)\,\hat{\mathbf{e}}_1 + 5xy\,\hat{\mathbf{e}}_2$ and C is the curve $y = x^2$ between the points $(0,0)$ and $(2,4)$.

► **6-55.** For $\vec{F} = x\,\hat{\mathbf{e}}_1 + 2xy\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$ evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve consisting of the straight-line segments $\overline{OA} + \overline{AB}$, where O is the origin and $A,\ B$ are respectively the points $(1,1,0),\quad (1,1,2)$.

► **6-56.** For $P_1 = (1,1,1)$ and $P_2 = (2,3,5)$, evaluate the line integral

$$I = \int_{P_1}^{P_2} \vec{A}\cdot d\vec{r}, \qquad \text{where} \qquad \vec{A} = yz\,\hat{\mathbf{e}}_1 + xz\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$$

and the integration is

(a) Along the straight-line joining P_1 and P_2.

(b) Along any other path joining P_1 to P_2.

► **6-57.** Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = yz\,\hat{\mathbf{e}}_1 + 2x\,\hat{\mathbf{e}}_2 + y\,\hat{\mathbf{e}}_3$ and C is the unit circle $x^2 + y^2 = 1$ lying in the plane $z = 2$.

► **6-58.** Find the work done in moving a particle in the force field $\vec{F} = x\,\hat{\mathbf{e}}_1 - z\,\hat{\mathbf{e}}_2 + 2y\,\hat{\mathbf{e}}_3$ along the parabola $y = x^2,\ z = 2$ between the points $(0,0,2)$ and $(1,2,2)$.

► **6-59.** Find the work done in moving a particle in the force field $\vec{F} = y\,\hat{\mathbf{e}}_1 - x\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ along the straight-line path joining the points $(1,1,1)$ and $(2,3,5)$.

► **6-60.** Sketch some level curves $\phi(x,y) = k$ for the values of k indicated.

$(a)\quad \phi = 4x - 2y, \quad k = -2,-1,0,1,2 \qquad (c)\quad \phi = x^2 + y^2, \quad k = 0,1,9,25$

$(b)\quad \phi = xy, \quad k = -2,-1,0,1,2 \qquad (d)\quad \phi = 9x^2 + 4y^2, \quad k = 16,36,64$

Give a physical interpretation to your results.

► **6-61.** Sketch the two-dimensional vector fields or their associated field lines.

$(a)\quad \vec{F} = x\,\hat{\mathbf{e}}_1 - y\,\hat{\mathbf{e}}_2 \qquad (b)\quad \vec{F} = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2 \qquad (c)\quad \vec{2}y\,\hat{\mathbf{e}}_1 + 2x\,\hat{\mathbf{e}}_2$

► **6-62.**

(a) Show line through (x_0,y_0,z_0) and parallel to vector $\vec{A}$ is $(\vec{r} - \vec{r}_0) \times \vec{A} = \vec{0}$

(b) Show line through (x_0,y_0,z_0) and (x_1,y_1,z_1) is given by $(\vec{r} - \vec{r}_0) \times (\vec{r} - \vec{r}_1) = \vec{0}$

(c) Show line through (x_0,y_0,z_0) and perpendicular to the vectors $\vec{A}$ and $\vec{B}$ is given by $(\vec{r} - \vec{r}_0) \times (\vec{A} \times \vec{B}) = \vec{0}$

(d) Find equation of line through (x_0,y_0,z_0) and perpendicular to plane through the noncolinear points P_1,P_2 and P_3.

▶ **6-63.** For the curves defined by the given parametric equations, find the position vector, velocity vector and acceleration vector at the given time.

$$(a)\quad x=t,\quad y=2t,\quad z=3t,\qquad t_0=1$$
$$(b)\quad x=\cos 2t,\quad y=\sin 2t,\quad z=0,\qquad t_0=0$$
$$(c)\quad x=\cos 2t,\quad y=\sin 2t,\quad z=3t,\qquad t_0=\pi$$

▶ **6-64.** Show for $\vec{A}=\vec{A}(t)$, $\vec{B}=\vec{B}(t)$, and $\vec{C}=\vec{C}(t)$ that

$$\frac{d}{dt}\left[\vec{A}\cdot(\vec{B}\times\vec{C})\right]=\vec{A}\cdot\vec{B}\times\frac{d\vec{C}}{dt}+\vec{A}\cdot\frac{d\vec{B}}{dt}\times\vec{C}+\frac{d\vec{A}}{dt}\cdot\vec{B}\times\vec{C}$$

▶ **6-65.** If $\vec{F}=(x^2+z)\,\hat{\mathbf{e}}_1+xyz\,\hat{\mathbf{e}}_2+x^2y^2z^2\,\hat{\mathbf{e}}_3$ find the partial derivatives

$$(a)\ \frac{\partial\vec{F}}{\partial x},\quad (b)\ \frac{\partial\vec{F}}{\partial y},\quad (c)\ \frac{\partial\vec{F}}{\partial z},\quad (d)\ \frac{\partial^2\vec{F}}{\partial x^2},\quad (e)\ \frac{\partial^2\vec{F}}{\partial y^2},\quad (f)\ \frac{\partial^2\vec{F}}{\partial z^2}$$

▶ **6-66.** Find the partial derivatives

$$(a)\ \frac{\partial\Phi}{\partial x},\quad (b)\ \frac{\partial\Phi}{\partial y},\quad (c)\ \frac{\partial^2\Phi}{\partial x^2},\quad (d)\ \frac{\partial^2\Phi}{\partial y^2},\quad (e)\ \frac{\partial^2\Phi}{\partial x\,\partial y}$$

in each of the following cases.

$$(i)\quad \Phi=u^2+v^2\quad\text{with } u=xy \text{ and } v=x+y$$
$$(ii)\quad \Phi=uv \text{ with } u=xy \text{ and } v=x+y$$
$$(iii)\quad \Phi=v^2+2v \text{ with } v=x+y$$

▶ **6-67.** Let $\Phi=\Phi(r,\theta)$ denote a scalar function of position in polar coordinates. If the coordinates are changed to cartesian, where $x=r\cos\theta \qquad y=r\sin\theta$,

(a) Show that

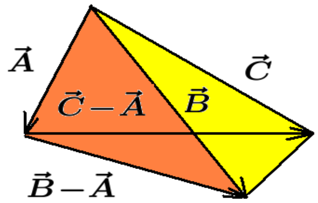
$$\frac{\partial\Phi}{\partial y}=\frac{\partial\Phi}{\partial r}\sin\theta+\frac{\partial\Phi}{\partial\theta}\frac{\cos\theta}{r}$$
$$\frac{\partial^2\Phi}{\partial y^2}=\frac{\partial\Phi}{\partial r}\frac{\cos^2\theta}{r}+\frac{\partial^2\Phi}{\partial r^2}\sin^2\theta+2\frac{\partial^2\Phi}{\partial r\,\partial\theta}\frac{\sin\theta\cos\theta}{r}-2\frac{\partial\Phi}{\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial^2\Phi}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

(b) Show that $\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}=\dfrac{\partial^2\Phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\Phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2}{}^2$

▶ **6-68.** Show the equation of the tangent plane to point (x_1,y_1,z_1) on the surface of sphere centered at (x_0,y_0,z_0), having radius a, is given by $(\vec{r}-\vec{r}_1)\cdot(\vec{r}_1-\vec{r}_0)=0$ Sketch a diagram illustrating these vectors.

▶ **6-69.** For the scalar function of position $F=F(u,v)$, where $u=u(x,y),\ v=v(x,y)$ calculate the quantities $\dfrac{\partial F}{\partial x},\quad \dfrac{\partial F}{\partial y},\quad \dfrac{\partial^2 F}{\partial x^2},\quad \dfrac{\partial^2 F}{\partial x\,\partial y},\quad \dfrac{\partial^2 F}{\partial y^2}$

▶ **6-70.** Consider the tetrahedron defined by the vectors $\vec{A}, \vec{B}, \vec{C}$ illustrated.



(a) Show the vectors $\vec{n}_1 = \frac{1}{2}\vec{A} \times \vec{B},\ \vec{n}_2 = \frac{1}{2}\vec{B} \times \vec{C},$ $\vec{n}_3 = \frac{1}{2}\vec{C} \times \vec{A},\ \vec{n}_4 = \frac{1}{2}(\vec{C} - \vec{A}) \times (\vec{B} - \vec{A})$ are normal to the faces of the tetrahedron with magnitudes equal to the area of the faces. (b) Show $\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0}$

▶ **6-71.** Find the work done in moving a particle in a counterclockwise direction around a unit circle in the $z = 0$ plane if the particle moves in the force field

$$\vec{F} = \vec{F}(x, y, z) = (x + y + z)\,\hat{\mathbf{e}}_1 + (2x - y + 3z)\,\hat{\mathbf{e}}_2 + (3x - y - z)\,\hat{\mathbf{e}}_3.$$

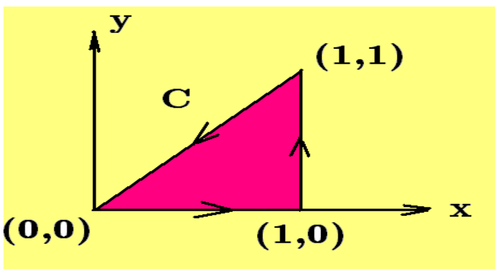
▶ **6-72.** The straight-line defined by the parametric equations

$$x = 2 + \lambda, \qquad y = 3 + 2\lambda, \qquad z = 4 - 2\lambda$$

with parameter λ, is drawn through the force field $\vec{F} = \vec{F}(x, y, z) = xy\,\hat{\mathbf{e}}_1 + yz\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$. Evaluate the given line integrals along this line from the point $P_1(2, 3, 4)$ to the point $P_2(4, 7, 0)$

$$(a) \quad \int_{P_1}^{P_2} (x^2 + y^2)ds \qquad (b) \quad \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \qquad (c) \quad \int_{P_1}^{P_2} \vec{F} \times d\vec{r}$$

▶ **6-73.** A particle moves around the closed curve C illustrated in figure. It moves in a vector field $\vec{F}$ defined by



$$\vec{F} = \vec{F}(x, y) = 6\,(y^2 - x)\,\hat{\mathbf{e}}_1 + 6\,x\,\hat{\mathbf{e}}_2.$$

Evaluate the line integrals in parts (a) and (b).

(a) $\circlearrowleft\!\oint \vec{F} \cdot d\vec{r}$ $\quad (b)$ $\circlearrowleft\!\oint \vec{F} \times d\vec{r}$ $\quad (c)$ Show that $\circlearrowleft\!\oint \vec{F} \cdot d\vec{r} = -\circlearrowright\!\oint \vec{F} \cdot d\vec{r}$

▶ **6-74.** Evaluate the given line integrals along the path
$C = \{\, (x, y) \mid x = 2t,\ y = 1 + t + t^2 \,\}$ from $t = 0$ to $t = 3$.

$$(a) \quad \int_C y\,dx + (x + y)\,dy \qquad (b) \quad \int_C y\,dx - x\,dy \qquad (c) \quad \int_C 2xy\,dx + x^2\,dy$$

▶ **6-75.** Evaluate the given line integrals around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, both clockwise and counterclockwise.

$$(a) \quad \oint_C x(y + 1)\,dx + (x + 1)y\,dy \qquad (b) \quad \oint_C (x^2 - y^2)\,dx + (x^2 + y^2)\,dy \qquad (c) \quad \oint_C y\,dx + x\,dy$$

Chapter 7

Vector Calculus I

One aspect of vector calculus can be described as taking many of the concepts from scalar calculus, generalizing these concepts and representing them in a vector format. These alternative vector representations have many applications in representing two-dimensional and three-dimensional physical problems. Let us begin by examining the representation of curves using vectors.

Curves

A **two-dimensional curve** can be defined

(*i*) Explicitly $y = f(x)$

(*ii*) Implicitly $F(x,y) = 0$

(*iii*) Parametrically $x = x(t), \quad y = y(t)$

(*iv*) As a vector $\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2$ or $\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$

A **three-dimensional curve** can be defined

(*i*) Parametrically $x = x(t), \quad y = y(t), \quad z = z(t)$

(*ii*) As a vector $\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$

(*iii*) A curve in space is sometimes defined as the intersection of two surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ and in this special case the curve is defined by a set of (x,y,z) values which are common to both surfaces.

It is assumed that the functions used to define these curves are continuous single-valued functions which are everywhere differentiable. Also note that **the parametric and vector representations of a curve are not unique.**

In two-dimensions a parametric curve $\{x(t), y(t)\}$, for $a \le t \le b$ has end points $(x(a), y(a))$ and $(x(b), y(b))$. A curve is called a **closed curve** if its **end points** coincide and $x(a) = x(b)$ and $y(a) = y(b)$. If (x_0, y_0) is a point on the given curve, which is not an end point, such that there exists more than one value of the parameter t such that $(x(t), y(t)) = (x_0, y_0)$, then the point (x_0, y_0) is called **a multiple point** or **a point where the curve crosses itself**. A curve is called a **simple closed curve** if it has no multiple points and the end points coincide. Simple closed curves are defined by one-to-one mappings. The above definitions of end points, closed curve, simple closed curve and multiple points apply to parametric curves $\{x(t), y(t), z(t)\}$ in three-dimensions and to n-dimensional parametric curves defined by $\{x_1(t), x_2(t), \ldots, x_n(t)\}$ as the parameter t ranges from a to b.

A curve is called an **oriented curve** if

(i) The curve is piecewise smooth.

(ii) The position vector $\vec{r} = \vec{r}(t)$, when expressed in terms of a parameter t, determines the direction of the tangent vector to each point on the curve.

(iii) The direction of the tangent vector is said to determine the orientation of the curve.

(iv) A plane curve which is a simple closed curve which does not cross itself is said to have either a clockwise or counterclockwise orientation which depends upon the directions of the tangent vector at each point on the closed curve.

Tangents to Space Curve

In three-dimensions the derivative vector $\frac{d\vec{r}}{dt} = x'(t)\,\hat{\mathbf{e}}_1 + y'(t)\,\hat{\mathbf{e}}_2 + z'(t)\,\hat{\mathbf{e}}_3$ is tangent to the point $(x(t), y(t), z(t))$ on the curve $\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ for any fixed value of the parameter t. The tangent line to the curve $\vec{r} = \vec{r}(t)$ at the point where the parameter has the value $t = t^*$ is given by

$$\vec{R} = \vec{R}(\lambda) = \vec{r}(t^*) + \lambda \frac{d\vec{r}}{dt}\bigg|_{t=t^*} \qquad -\infty < \lambda < \infty$$

where λ is a parameter. The tangent line defined by the vector $\vec{R}$ can also be expressed in the expanded form

$$\vec{R} = \vec{R}(\lambda) = (x(t^*) + \lambda x'(t*))\,\hat{\mathbf{e}}_1 + (y(t^*) + \lambda y'(t^*))\,\hat{\mathbf{e}}_2 + (z(t^*) + \lambda z'(t^*))\,\hat{\mathbf{e}}_3$$

where t^* represents some fixed value of the parameter t. The element of arc length ds along the curve $\vec{r} = \vec{r}(t)$ is obtained from the relation

$$ds^2 = d\vec{r} \cdot d\vec{r} = (dx)^2 + (dy)^2 + (dz)^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt)^2$$

$$\text{and} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \tag{7.1}$$

$$\text{so that one can write} \quad \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The total arc length for the curve $\vec{r} = \vec{r}(t)$ for $t_0 \le t \le t_1$ is given by

$$\text{arc length of curve} = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

The unit tangent vector to the curve can therefore be expressed by

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{1}{\frac{ds}{dt}}\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \tag{7.2}$$

which shows that the derivative of the position vector with respect to arc length s produces a **unit tangent vector to the curve.**

Example 7-1. **Reflection property for the parabola.**

The parabola $y^2 = 4px$ with focus F having coordinates $(p, 0)$ can be represented parametrically. One parametric representation for the position vector is

$$\vec{r} = \vec{r}(t) = \frac{t^2}{4p}\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2, \qquad -\infty < t < \infty \tag{7.3}$$

and the resulting parabola is illustrated in the figure 7-1. In this figure assume the surface of the parabola is a mirrored surface.

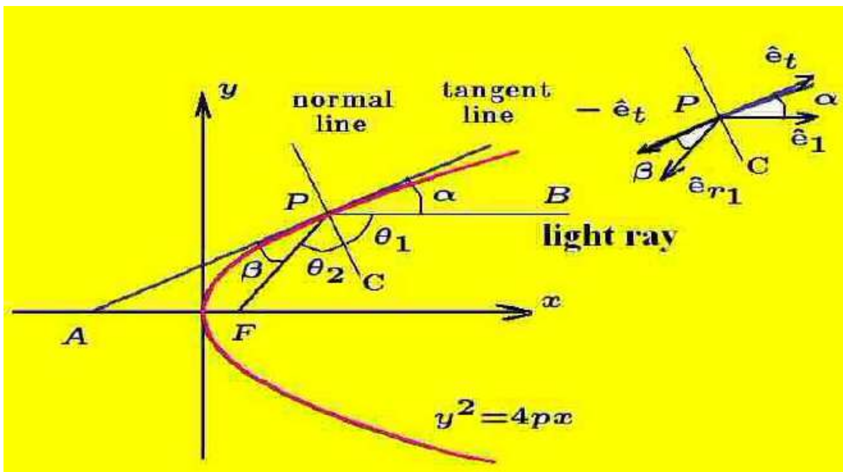


Figure 7-1. Light ray PB gets reflected to ray through focus.

The derivative vector

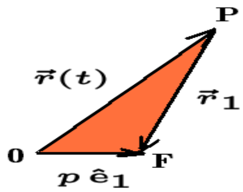
$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \frac{t}{2p}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$$

produces a tangent vector to the curve and the vector

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{\frac{t}{2p}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1 + t^2/4p^2}} = \frac{t\,\hat{\mathbf{e}}_1 + 2p\,\hat{\mathbf{e}}_2}{\sqrt{4p^2 + t^2}}$$

is a unit tangent vector to the curve.

Consider a general point P on the parabola where a light ray PB parallel to the $x-$axis hits the parabola. Construct the normal to the parabola and label the angle $\angle BPC$ the angle θ_1 and then label the angle $\angle FPC$ the angle θ_2. The angle θ_1 is called the **angle of incidence** and the angle θ_2 is called the **angle of reflection.** Also in figure 7-1 are the complementary angles to θ_1 and θ_2. These angles are labeled as α and β.



Construct the vector $\vec{r}_1$ from point P to the focus F and by using vector addition show with the aid of equation (7.3) that

$$\vec{r}(t) + \vec{r}_1 = p\,\hat{\mathbf{e}}_1 \quad \text{or} \quad \vec{r}_1 = (p - t^2/4p)\,\hat{\mathbf{e}}_1 - t\,\hat{\mathbf{e}}_2$$

A unit vector in the direction of $\vec{r}_1$ is

$$\hat{\mathbf{e}}_{r_1} = \frac{(p - t^2/4p)\,\hat{\mathbf{e}}_1 - t\,\hat{\mathbf{e}}_2}{\sqrt{(p - t^2/4p)^2 + t^2}} = \frac{(4p^2 - t^2)\,\hat{\mathbf{e}}_1 - 4pt\,\hat{\mathbf{e}}_2}{\sqrt{(4p^2 - t^2)^2 + 16p^2t^2}}$$

Using the definition of the dot product one can show

$$\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_1 = \cos\alpha = \frac{t}{\sqrt{4p^2 + t^2}}$$

$$(-\,\hat{\mathbf{e}}_t) \cdot \hat{\mathbf{e}}_{r_1} = \cos\beta = \frac{-t(4p^2 - t^2) + 8p^2t}{\sqrt{4p^2 + t^2}\sqrt{(4p^2 + t^2)^2 + 16p^2t^2}} = \frac{t(t^2 + 4p^2)}{\sqrt{4p^2 + t^2}\sqrt{(4p^2 + t^2)^2 + 16p^2t^2}}$$

If $\cos\alpha = \cos\beta$ for all values of the parameter t, then one must show that

$$\frac{t}{\sqrt{4p^2 + t^2}} = \frac{t(t^2 + 4p^2)}{\sqrt{4p^2 + t^2}\sqrt{(4p^2 + t^2)^2 + 16p^2t^2}} \tag{7.4}$$

Using algebra one can establish that equation (7.4) is indeed true and so the angles α and β are equal. Simplify the equation (7.4) to the form

$$\sqrt{(4p^2 + t^2)^2 + 16p^2t^2} = t^2 + 4p^2$$

and then square both sides to show

$$16p^4 - 8p^2t^2 + t^4 + 16p^2t^2 = (t^2 + 4p^2)^2$$

which reduces to the identity $(t^2 + 4p^2)^2 = (t^2 + 4p^2)^2$. The equality of the angles α and β implies $\theta_1 = \theta_2$ or the angle of incidence is equal to the angle of reflection. One can also show that the distances AF=FP which shows the triangle PFA is an isosceles triangle with angle $\angle FAP$ equal to angle $\angle APF$ implying the complementary angles θ_1 and θ_2 are equal. These results show that all light coming in parallel to the $x-$axis will be reflected by the mirrored parabolic surface and pass through the focus. Conversely, if a light source is placed at the focus, than rays of light from the focus are reflected parallel to the $x-$axis.

■

Example 7-2. **Reflection property of the ellipse.**

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having eccentricity $e < 1$ and foci at the points F_1, F_2 with coordinates $(c, 0)$ and$(-c, 0)$ respectively. For this ellipse $b^2 = a^2 - c^2$ and $c = ae$. If P represents an arbitrary point (x_0, y_0) on the ellipse, then one can construct the vector $\vec{r}_1$ from P to F_1 and also construct the vector $\vec{r}_2$ from point P to F_2. The magnitude of these vectors when summed gives

$$|\vec{r}_1| + |\vec{r}_2| = 2a \tag{7.5}$$

The vectors $\vec{r}_1$, $\vec{r}_2$ and the ellipse are illustrated in the figure 7-2. If the ellipse is mirrored, then a ray of light from the focus F_1 will reflect from an arbitrary point P on the ellipse to the focus at F_2.

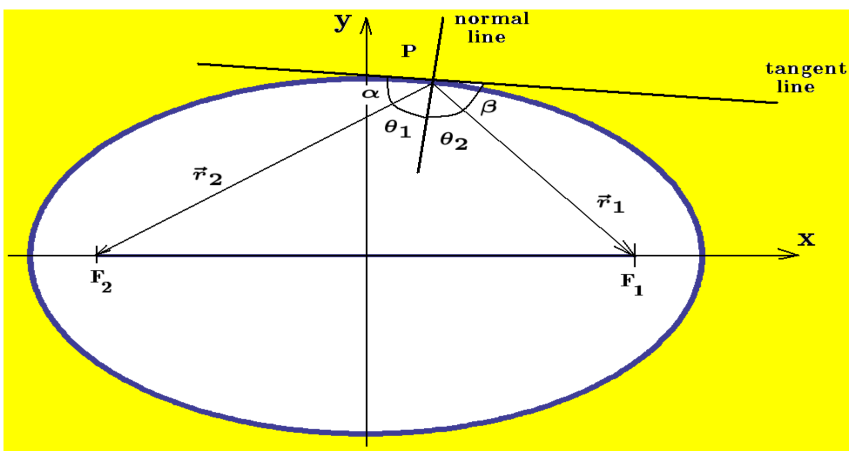


Figure 7-2. Light ray from one focus passes through other focus.

The position vector of a general point on the ellipse can be represented in the parametric form

$$\vec{r} = \vec{r}(t) = a\cos t\,\hat{\mathbf{e}}_1 + b\sin t\,\hat{\mathbf{e}}_2, \qquad 0 \le t \le 2\pi \tag{7.6}$$

A point P on the ellipse with coordinates (x_0, y_0) is described by equation (7.6) by assigning the proper value for the parameter t. The proper value for the parameter t, call it t_0, is determined by solving the equations

$$x_0 = a\cos t \qquad \text{and} \qquad y_0 = b\sin t$$

simultaneously, to obtain $t_0 = \tan^{-1}\left(\frac{ay_0}{bx_0}\right)$. The derivative vector

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{\mathbf{e}}_1 + b\cos t\,\hat{\mathbf{e}}_2$$

evaluated at the value t_0, represents a tangent vector to the ellipse at the point P. A unit vector in the direction of the tangent line at the point P is then given by

$$\hat{\mathbf{e}}_t = \frac{-a\sin t\,\hat{\mathbf{e}}_1 + b\cos t\,\hat{\mathbf{e}}_2}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

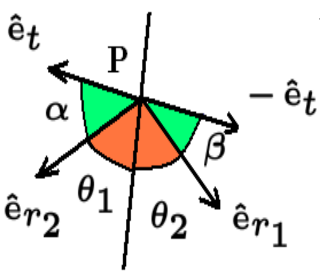
where everything is understood to be evaluated at $t = t_0$. Using vector addition one can show the vectors $\vec{r}_1$ and $\vec{r}_2$ must satisfy

$$\vec{r}(t) + \vec{r}_1 = c\,\hat{\mathbf{e}}_1 \qquad \text{and} \qquad \vec{r}(t) + \vec{r}_2 + c\,\hat{\mathbf{e}}_1 = \vec{0}$$

These equations allow one to express the vectors $\vec{r}_1$ and $\vec{r}_2$ in the form

$$\begin{aligned}\vec{r}_1 =& (c - a\cos t)\,\hat{\mathbf{e}}_1 - b\sin t\,\hat{\mathbf{e}}_2\\ \vec{r}_2 =& (-c - a\cos t)\,\hat{\mathbf{e}}_2 - b\sin t\,\hat{\mathbf{e}}_2\end{aligned}$$

where again, these vectors are to be evaluated at the parameter value t_0.



Unit vectors in the directions of $\vec{r}_1$ and $\vec{r}_2$ can be expressed

$$\begin{aligned}\hat{\mathbf{e}}_{r_1} =& \frac{(c - a\cos t)\,\hat{\mathbf{e}}_1 - b\sin t\,\hat{\mathbf{e}}_2}{\sqrt{(c - a\cos t)^2 + b^2\sin^2 t}}\\ \hat{\mathbf{e}}_{r_2} =& \frac{-(c + a\cos t)\,\hat{\mathbf{e}}_1 - b\sin t\,\hat{\mathbf{e}}_2}{\sqrt{(c + a\cos t)^2 + b^2\sin^2 t}}\end{aligned}$$

By employing the definition of the dot product of unit vectors one can verify that

$$\begin{aligned}\hat{\mathbf{e}}_{r_1}\cdot(-\,\hat{\mathbf{e}}_t) = \cos\beta =& \frac{a\sin t(c - a\cos t) + b^2\sin t\cos t}{r_1\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\\ \hat{\mathbf{e}}_{r_2}\cdot(\,\hat{\mathbf{e}}_t) = \cos\alpha =& \frac{a\sin t(c + a\cos t) - b^2\sin t\cos t}{(2a - r_1)\sqrt{a^2\sin^2 t + b^2\cos^2 t}}\end{aligned}$$

where $r_1 = |r_1| = \sqrt{(c - a\cos t)^2 + b^2\sin^2 t}$. If $\cos\alpha = \cos\beta$ for all values of the parameter t, then one must show that

$$\frac{a\sin t(c - a\cos t) + b^2\sin t\cos t}{r_1\sqrt{a^2\sin^2 t + b^2\cos^2 t}} = \frac{a\sin t(c + a\cos t) - b^2\sin t\cos t}{(2a - r_1)\sqrt{a^2\sin^2 t + b^2\cos^2 t}} \tag{7.7}$$

Using algebra one can verify that the equation (7.7) reduces down to an identity so that the angles α and β are equal. This in turn implies $\theta_1 = \theta_2$ which states that the angle of incidence equals the angle of reflection.

Sound waves also are reflected in the same way. Elliptically shaped rooms or domes have the property that someone whispering at one focus of the ellipse can easily be heard at the other focus of the ellipse. This gives rise to the phrase "whispering galleries". Constructions which make use of this reflection property of the ellipse can be found at Statuary Hall in the United States capital, St Paul's Cathedral in London, the Grand Central Terminal in New York City and in certain museums throughout the world.

■

Example 7-3. **Reflection property of the hyperbola.**

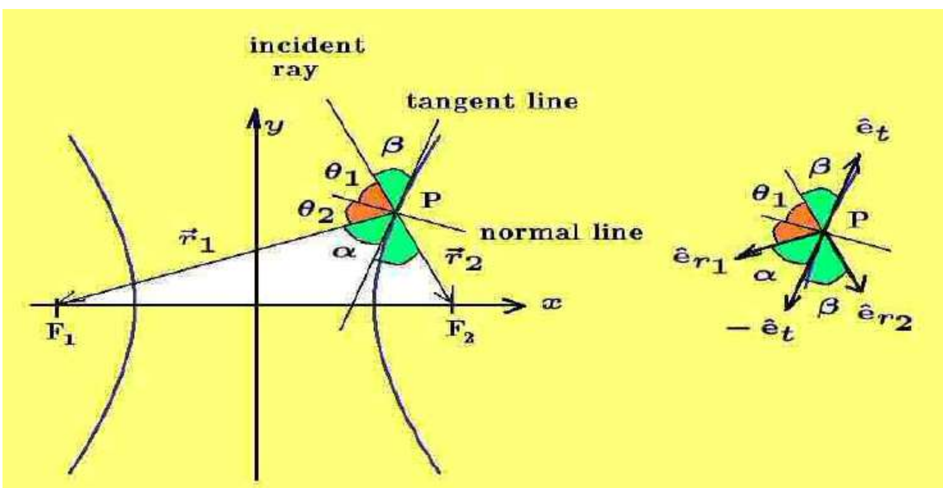


Figure 7-3.
Light ray directed toward one focus reflects and goes through other focus.

Sketch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci F_1, F_2 having coordinates $(-c, 0)$ and $(c, 0)$ respectively, where $c = ae$ and $e > 1$ is the eccentricity of the hyperbola. Construct an incident ray aimed at the focus F_2 which passes through a known point (x_0, y_0) on the hyperbola. Label the point where this ray intersects the hyperbola as point P. Sketch the tangent line and normal line to the hyperbola at the point P and label the angle of incidence as θ_1 and the angle of reflection as θ_2. The complementary angles associated with these angles are labeled α and β respectively. Next construct the vector $\vec{r}_1$ running from the point P on the hyperbola to the focus F_1 and then construct the vector $\vec{r}_2$ running from the P to the other focus F_2 as illustrated in the figure 7-3.

The position vector to a general point on the right branch of the hyperbola can be expressed

$$\vec{r} = \vec{r}(t) = a\cosh t\,\hat{\mathbf{e}}_1 + b\sinh t\,\hat{\mathbf{e}}_2$$

where t is a parameter. The value of the parameter t, call it t_0, corresponding to the point P having coordinates (x_0, y_0) is obtained by solving the equations

$$x_0 = a\cosh t \qquad \text{and} \qquad y_0 = b\sinh t$$

simultaneously to obtain $t_0 = \tanh^{-1}\left(\dfrac{ay_0}{bx_0}\right)$. The derivative of the position vector is

$$\frac{d\vec{r}}{dt} = a\sinh t\,\hat{\mathbf{e}}_1 + b\cosh t\,\hat{\mathbf{e}}_2$$

and this vector, when evaluated using the parameter value t_0 is a tangent vector to the hyperbola at the point P. The vector

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{a\sinh t\,\hat{\mathbf{e}}_1 + b\cosh t\,\hat{\mathbf{e}}_2}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}$$

also evaluated at t_0, is a unit tangent vector to the hyperbola at the point P. Using vector addition one can show that the vectors $\vec{r}_1$ and $\vec{r}_2$ are given by

$$\vec{r}_1 = -\,(a\cosh t + c)\,\hat{\mathbf{e}}_1 - b\sinh t$$
$$\vec{r}_2 = -\,(a\cosh t - c)\,\hat{\mathbf{e}}_1 - b\sinh t\,\hat{\mathbf{e}}_2$$

all to be evaluated at the parameter value t_0. Unit vectors in the directions of $\vec{r}_1$ and $\vec{r}_2$ are

$$\hat{\mathbf{e}}_{r_1} = \frac{-(a\cosh t + c)\,\hat{\mathbf{e}}_1 - b\sinh t\,\hat{\mathbf{e}}_2}{\sqrt{(a\cosh t + c)^2 + b^2\sinh^2 t}}$$
$$\hat{\mathbf{e}}_{r_2} = \frac{-(a\cosh t - c)\,\hat{\mathbf{e}}_1 - b\sinh t\,\hat{\mathbf{e}}_2}{\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t}}$$

also to be evaluated at $t = t_0$. The given hyperbola satisfies the properties that $a^2 + b^2 = c^2$ and $|\vec{r}_1| - |\vec{r}_2| = 2a$ or

$$\sqrt{(a\cosh t + c)^2 + b^2\sinh^2 t} - \sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} = 2a$$

for all values of the parameter t. The angles α and β constructed at the point P can be calculated from the dot products

$$(-\hat{\mathbf{e}}_t)\cdot\hat{\mathbf{e}}_{r_2} = \cos\alpha = \frac{a\sinh t\,(a\cosh t + c) + b^2\sinh t\cosh t}{(\sqrt{a^2\sinh^2 t + b^2\cosh^2 t})(\sqrt{(a\cosh t + c)^2 + b^2\sinh^2 t})}$$
$$= \frac{a\sinh t\,(a\cosh t + c) + b^2\sinh t\cosh t}{(\sqrt{a^2\sinh^2 t + b^2\cosh^2 t})(\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a)} \tag{7.8}$$

$$(-\hat{\mathbf{e}}_t)\cdot\hat{\mathbf{e}}_{r_1} = \cos\beta = \frac{a\sinh t\,(a\cosh t - c) + b^2\sinh t\cosh t}{(\sqrt{a^2\sinh^2 t + b^2\cosh^2 t})(\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t})} \tag{7.9}$$

If $\cos\alpha = \cos\beta$ then one must show that the right-hand sides of equations (7.8) and (7.9) are equal. Setting the right-hand sides equal to one another and simplifying produces

$$\frac{a^2\sinh t\cosh t + ac\sinh t + b^2\sinh t\cosh t}{\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a} = \frac{a^2\sinh t\cosh t - ac\sinh t + b^2\sinh t\cosh t}{\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t}} \tag{7.10}$$

To show equation (7.10) reduces to an identity, first show equation (7.10) can be written

$$|\vec{r}_2| = \frac{c\,\cosh t - a}{c\,\cosh t + a}\sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a$$

and then use the fact that $c^2 = a^2 + b^2$ and $|\vec{r}_2| = c\,\cosh t - a$ to simplify the above equation to the form

$$c\,\cosh t + a = \sqrt{(a\cosh t - c)^2 + b^2\sinh^2 t} + 2a \tag{7.11}$$

It is now an easy exercise to show equation (7.11) reduces to an identity.

All this algebra shows that the angles α and β are equal and consequently the complementary angles θ_1 and θ_2 are also equal, showing the hyperbola has the property that the angle of incidence equals the angle of reflection. The above results imply that a ray of light aimed at the focus F_2 will be reflected and pass through the other focus. This reflection property of the hyperbola is **one of the basic principles used in the construction of a reflecting telescope.**

■

Normal and Binormal to Space Curve

Recall that the **unit tangent vector to a space curve** $\vec{r} = \vec{r}(t)$, for any value of the parameter t, is given by the equation

$$\hat{\mathbf{e}}_t = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \tag{7.12}$$

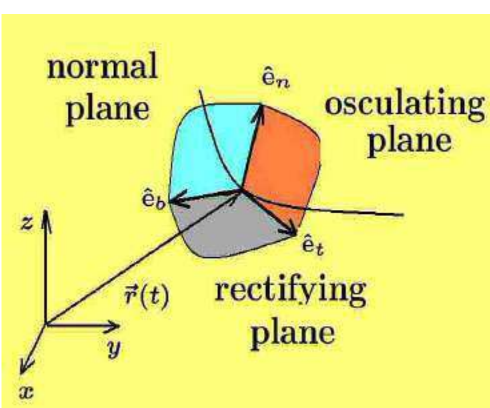
and satisfies $\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t = 1$. Differentiating this relation with respect to the **arc length parameter** s one finds

$$\frac{d}{ds}\left[\hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t\right] = \hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} + \frac{d\,\hat{\mathbf{e}}_t}{ds} \cdot \hat{\mathbf{e}}_t = 0 \quad \text{or} \qquad 2\,\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_t}{ds} = 0 \tag{7.13}$$

The zero dot product in equation (7.13) demonstrates that the vector $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ is perpendicular to the unit tangent vector $\hat{\mathbf{e}}_t$. Note that this vector can be calculated using the chain rule for differentiation $\frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \frac{d\,\hat{\mathbf{e}}_t}{dt}$ where $\frac{ds}{dt}$ is calculated using the equation (7.1). Observe that there are an infinite number of vectors which are perpendicular to the unit tangent vector $\hat{\mathbf{e}}_t$. The **unit vector with the same direction as the vector** $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ is called **the principal unit normal vector to the curve** $\vec{r}(t)$ for each value of the parameter t. The **principal unit normal vector** in the direction of the derivative vector $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ is given the label $\hat{\mathbf{e}}_n$. The vector $\frac{d\,\hat{\mathbf{e}}_t}{ds}$ has the same direction as $\hat{\mathbf{e}}_n$ and so one can write

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n \tag{7.14}$$

where κ is a scaling constant called **the curvature of the curve** $\vec{r}(t)$. The curvature κ will vary as the parameter t changes. The quantity $\rho = \frac{1}{\kappa}$ is called the **radius of curvature at the point associated with the parameter value of** t. The unit vector $\hat{\mathbf{e}}_b$ calculated from the **cross product of** $\hat{\mathbf{e}}_t$ **and** $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n$, is perpendicular to both the unit tangent $\hat{\mathbf{e}}_t$ and unit normal $\hat{\mathbf{e}}_n$ and is called **the unit binormal vector to the curve** as the parameter t changes.



The vectors $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_b$ are called **a moving triad along the curve** $\vec{r}(t)$ because the unit vectors $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_n$, $\hat{\mathbf{e}}_b$ generated a localized **right-handed coordinate system** which changes as the parameter t changes. The plane which contains the unit tangent $\hat{\mathbf{e}}_t$ and principal normal $\hat{\mathbf{e}}_n$ is called the **osculating plane**. The plane containing the unit binormal $\hat{\mathbf{e}}_b$ and unit normal $\hat{\mathbf{e}}_n$ is called **the normal plane.** The plane which is perpendicular to the principal normal $\hat{\mathbf{e}}_n$ is called **the rectifying plane.** Let $\vec{r}(t^*)$ denote the position vector to a fixed point on the given curve and let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a variable point in one of these planes. One can then show

$$\begin{aligned}&\text{The osculating plane can be written}\quad (\vec{r} - \vec{r}(t^*)) \cdot \hat{\mathbf{e}}_b = 0\\ &\text{The normal plane can be written}\quad (\vec{r} - r(t^*)) \cdot \hat{\mathbf{e}}_t = 0\\ &\text{The rectifying plane can be written}\quad (\vec{r} - \vec{r}(t^*)) \cdot \hat{\mathbf{e}}_n = 0\end{aligned} \tag{7.15}$$

The equations of the straight lines through the fixed point $\vec{r}(t^*)$ and having the directions of $\hat{\mathbf{e}}_t, \hat{\mathbf{e}}_n$ or $\hat{\mathbf{e}}_b$ are given by

$$\begin{aligned}(\vec{r} - \vec{r}(t^*)) \times \hat{\mathbf{e}}_t =&\vec{0} \qquad \text{Tangent line}\\ (\vec{r} - \vec{r}(t^*)) \times \hat{\mathbf{e}}_n =&\vec{0} \qquad \text{Line normal to curve}\\ (\vec{r} - \vec{r}(t^*)) \times \hat{\mathbf{e}}_b =&\vec{0} \qquad \text{Line in binormal direction}\end{aligned} \tag{7.16}$$

Let us examine the three unit vectors $\hat{\mathbf{e}}_t, \hat{\mathbf{e}}_b$ and $\hat{\mathbf{e}}_n$ and their derivatives with respect to the arc length parameter s. One can calculate the derivatives $\frac{d\,\hat{\mathbf{e}}_t}{ds}$, $\frac{d\,\hat{\mathbf{e}}_b}{ds}$ and $\frac{d\,\hat{\mathbf{e}}_n}{ds}$ with the aid of the triple scalar product relations

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

It has been demonstrated that $\frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$ and $\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n$, consequently one finds that

$$\frac{d\,\hat{\mathbf{e}}_b}{ds} = \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} + \frac{d\,\hat{\mathbf{e}}_t}{ds} \times \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} + \kappa\,\hat{\mathbf{e}}_n \times \hat{\mathbf{e}}_n = \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} \tag{7.17}$$

Take the dot product of both sides of equation (7.17) with the vector $\hat{\mathbf{e}}_t$ and use the above triple scalar product result to show

$$\hat{\mathbf{e}}_t \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} = \hat{\mathbf{e}}_t \cdot \hat{\mathbf{e}}_t \times \frac{d\,\hat{\mathbf{e}}_n}{ds} = 0$$

This result shows that the vector $\hat{\mathbf{e}}_t$ is perpendicular to the vector $\frac{d\,\hat{\mathbf{e}}_b}{ds}$. By differentiating the relation $\hat{\mathbf{e}}_b \cdot \hat{\mathbf{e}}_b = 1$ one finds that

$$\hat{\mathbf{e}}_b \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} + \frac{d\,\hat{\mathbf{e}}_b}{ds} \cdot \hat{\mathbf{e}}_b = 2\,\hat{\mathbf{e}}_b \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} = 0$$

which implies that the vector $\frac{d\,\hat{\mathbf{e}}_b}{ds}$ is also perpendicular to the vector $\hat{\mathbf{e}}_b$. These two results show that the derivative vector $\frac{d\,\hat{\mathbf{e}}_b}{ds}$ must be in the direction of the normal vector $\hat{\mathbf{e}}_n$. Hence, there exists a constant K such that

$$\frac{d\,\hat{\mathbf{e}}_b}{ds} = K\,\hat{\mathbf{e}}_n \tag{7.18}$$

where K is a constant. By convention, the constant K is selected as $-\tau$, where τ is called **the torsion** and the reciprocal $\sigma = \frac{1}{\tau}$ is called **the radius of torsion.** Taking the dot product of both sides of equation (7.18) with the unit vector $\hat{\mathbf{e}}_n$ gives

$$\tau = \tau(s) = -\,\hat{\mathbf{e}}_n \cdot \frac{d\,\hat{\mathbf{e}}_b}{ds} \tag{7.19}$$

The torsion is a measure of the **twisting of a curve out of a plane** and is a measure of how the osculating plane changes with respect to arc length. The torsion can be positive or negative and if the torsion is zero, then the curve must be **a plane curve**. The three vectors $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_b$, $\hat{\mathbf{e}}_n$ form a right-handed system of unit vectors and so one can write $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_b \times \hat{\mathbf{e}}_t$. Differentiating this relation with respect to arc length gives

$$\frac{d\,\hat{\mathbf{e}}_n}{ds} = \hat{\mathbf{e}}_b \times \frac{d\,\hat{\mathbf{e}}_t}{ds} + \frac{d\,\hat{\mathbf{e}}_b}{ds} \times \hat{\mathbf{e}}_t = \hat{\mathbf{e}}_b \times \kappa\,\hat{\mathbf{e}}_n - \tau\,\hat{\mathbf{e}}_n \times \hat{\mathbf{e}}_t = -\kappa\,\hat{\mathbf{e}}_t + \tau\,\hat{\mathbf{e}}_b \tag{7.20}$$

These results give the Frenet[1]-Serret[2] formulas

$$\begin{aligned} \frac{d\,\hat{\mathbf{e}}_t}{ds} &= \kappa\,\hat{\mathbf{e}}_n \\ \frac{d\,\hat{\mathbf{e}}_b}{ds} &= -\,\tau\,\hat{\mathbf{e}}_n \\ \frac{d\,\hat{\mathbf{e}}_n}{ds} &= \tau\,\hat{\mathbf{e}}_b - \kappa\,\hat{\mathbf{e}}_t \end{aligned} \tag{7.21}$$

Using matrix notation[3], the Frenet-Serret formulas can be written as

$$\begin{bmatrix} \frac{d\,\hat{\mathbf{e}}_t}{ds} \\ \frac{d\,\hat{\mathbf{e}}_b}{ds} \\ \frac{d\,\hat{\mathbf{e}}_n}{ds} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & -\tau \\ -\kappa & \tau & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{e}}_t \\ \hat{\mathbf{e}}_b \\ \hat{\mathbf{e}}_n \end{bmatrix} \tag{7.22}$$

Recall that if $\vec{B}$ is a vector which rotates about a line with angular velocity $\vec{\omega}$, then $\frac{d\vec{B}}{dt} = \vec{\omega} \times \vec{B}$. One can use this result to give a physical interpretation to the Frenet-Serret formulas. One can write

$$\frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \kappa\,\hat{\mathbf{e}}_n\frac{ds}{dt} = \kappa\frac{ds}{dt}\,\hat{\mathbf{e}}_b \times \hat{\mathbf{e}}_t = \vec{\omega} \times \hat{\mathbf{e}}_t \qquad \text{where} \quad \vec{\omega} = \kappa\frac{ds}{dt}\,\hat{\mathbf{e}}_b$$

[1] Jean Frédéric Frenet (1816-1900) A French mathematician.
[2] Joseph Alfred Serret (1819-1885) A French mathematician.
[3] See chapter 10 for a description of the matrix notation.

This is interpreted as showing the vector $\hat{\mathbf{e}}_t$ rotates about a line through $\hat{\mathbf{e}}_b$ with angular velocity $\vec{\omega}$. If the curvature $\kappa = 0$, then $\vec{\omega}$ is also zero and so the tangent vector $\hat{\mathbf{e}}_t$ is not rotating and consequently **the curve is a straight line.** Similarly, one can write

$$\frac{d\,\hat{\mathbf{e}}_b}{dt} = \frac{d\,\hat{\mathbf{e}}_b}{ds}\frac{ds}{dt} = -\tau\,\hat{\mathbf{e}}_n\frac{ds}{dt} = -\tau\frac{ds}{dt}\,\hat{\mathbf{e}}_b \times \hat{\mathbf{e}}_t = \tau\frac{ds}{dt}\,\hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_b = \vec{\omega} \times \hat{\mathbf{e}}_b \quad \text{where} \quad \vec{\omega} = \tau\frac{ds}{dt}\,\hat{\mathbf{e}}_t$$

and this result is interpreted as meaning the vector $\hat{\mathbf{e}}_b$ is rotating about the $\hat{\mathbf{e}}_t$ direction with angular velocity $\vec{\omega}$. If the torsion $\tau = 0$, then there is no rotation of the binormal vector and so the curve **remains a plane curve** in the plane of the normal and tangent vectors.

It is left as an exercise to give a physical interpretation to the derivative $\frac{d\,\hat{\mathbf{e}}_n}{dt}$.

Example 7-4. Determine how to calculate the curvature κ of a space curve.

Solution Use the fact $\frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$ so that $|\frac{d\,\hat{\mathbf{e}}_t}{ds}| = \kappa$, since $\hat{\mathbf{e}}_n$ is a unit vector. Let $\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ denote the position vector to a point on the space curve where t represents some parameter. If the arc length parameter s is used, then

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{1}{|\frac{d\vec{r}}{dt}|}\frac{d\vec{r}}{dt} = \hat{\mathbf{e}}_t$$

is a unit tangent vector to the curve and the derivative of this vector with respect to arc length s gives

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$$

Taking the dot product of this vector with itself gives

$$\frac{d^2\vec{r}}{ds^2} \cdot \frac{d^2\vec{r}}{ds^2} = (\kappa\,\hat{\mathbf{e}}_n\,) \cdot (\kappa\,\hat{\mathbf{e}}_n\,) = \kappa^2 = [x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2$$

where

$$\begin{aligned} x'(t) =& \frac{dx}{ds}\frac{ds}{dt} = x'(s)\frac{ds}{dt} \qquad \Longrightarrow \qquad x'(s) = \frac{x'(t)}{\frac{ds}{dt}} \\ x''(t) =& x'(s)\frac{d^2s}{dt^2} + \frac{d}{dt}x'(s)\frac{ds}{dt} \\ x''(t) =& x'(s)\frac{d^2s}{dt^2} + x''(s)\left(\frac{ds}{dt}\right)^2 \end{aligned}$$

and solving for $x''(s)$ one finds $x''(s) = \dfrac{x''(t) - x'(s)\frac{d^2s}{dt^2}}{\left(\frac{ds}{dt}\right)^2}$ The derivatives for $y''(s)$ and $z''(s)$ are calculate in a similar fashion.

∎

Example 7-5. Determine how to calculate the torsion τ of a space curve.

Solution The derivative $\frac{d\vec{r}}{ds} = \hat{\mathbf{e}}_t$ is a unit tangent vector to the space curve and $\frac{d^2\vec{r}}{ds^2} = \frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n$. Calculating the third derivative of the position vector with respect to the arc length parameter gives

$$\frac{d^3\vec{r}}{ds^3} = k\frac{d\,\hat{\mathbf{e}}_n}{ds} + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n = \kappa(\tau\,\hat{\mathbf{e}}_b - \kappa\,\hat{\mathbf{e}}_t) + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n = \kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n$$

Use the properties of the triple scalar product with

$$\frac{d\vec{r}}{ds}\cdot\left(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right) = \hat{\mathbf{e}}_t\cdot\left(\kappa\,\hat{\mathbf{e}}_n\times[\kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n]\right) \tag{7.23}$$

together with the cross products

$$\hat{\mathbf{e}}_n\times\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t, \qquad \hat{\mathbf{e}}_n\times\hat{\mathbf{e}}_t = -\,\hat{\mathbf{e}}_b, \qquad \hat{\mathbf{e}}_n\times\hat{\mathbf{e}}_n = \vec{0}$$

and show equation (7.23) simplifies to $\frac{d\vec{r}}{ds}\cdot\left(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right) = \hat{\mathbf{e}}_t\cdot[k^2\tau\,\hat{\mathbf{e}}_t + \kappa^3\,\hat{\mathbf{e}}_b] = \kappa^2\tau$

Using the result from the previous example that $\kappa^2 = [x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2$ one can write

$$\tau = \frac{\frac{d\vec{r}}{ds}\cdot\left(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right)}{[x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2}$$

which can also be expressed as the determinant

$$\tau = \frac{1}{[x''(s)]^2 + [y''(s)]^2 + [z''(s)]^2}\begin{vmatrix} x'(s) & y'(s) & z'(s) \\ x''(s) & y''(s) & z''(s) \\ x'''(s) & y'''(s) & z'''(s) \end{vmatrix}$$

■

Example 7-6. **Velocity and Acceleration.**

A physical example illustrating the use of the unit tangent and normal vectors is found in determining the normal and tangential components of the velocity and acceleration vectors as a particle moves along a curve. If $\vec{r}$ denotes the position vector of the particle, $\vec{v}$ its velocity, and $\vec{a}$ its acceleration, then

$$\vec{v} = \frac{d\vec{r}}{dt}, \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}, \qquad v^2 = \vec{v}\cdot\vec{v} = \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = \left(\frac{ds}{dt}\right)^2, \tag{7.24}$$

where s is the arc length along the curve. Using chain rule differentiation gives

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{\mathbf{e}}_t v.$$

Analysis of this equation demonstrates that the velocity vector $\vec{v}$ is directed along the tangent vector at any time t and has the magnitude given by $v = \frac{ds}{dt}$ which represents the speed of the particle.

The derivative of the velocity vector with respect to time t is the acceleration and

$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{\mathbf{e}}_t \frac{dv}{dt} + \frac{d\,\hat{\mathbf{e}}_t}{dt} v.$$

From the Frenet-Serret formula and using chain rule differentiation, it can be shown that the time rate of change of the unit tangent vector is

$$\frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \frac{v}{\rho}\,\hat{\mathbf{e}}_n.$$

Substituting this result into the acceleration vector gives

$$\vec{a} = \frac{dv}{dt}\,\hat{\mathbf{e}}_t + \frac{v^2}{\rho}\,\hat{\mathbf{e}}_n.$$

The resulting acceleration vector lies in the osculating plane. The tangential component of the acceleration is given by $\frac{dv}{dt}$, and the normal component of the acceleration is given by $\frac{v^2}{\rho}$. ■

Surfaces

A surface can be defined

(*i*) Explicitly $\quad z = f(x, y)$

(*ii*) Implicitly $\quad F(x, y, z) = 0$

(*iii*) Parametrically $\quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$

(*iv*) As a vector $\quad \vec{r} = \vec{r}(u, v) = x(u, v)\,\hat{\mathbf{e}}_1 + y(u, v)\,\hat{\mathbf{e}}_2 + z(u, v)\,\hat{\mathbf{e}}_3$

or $\quad \vec{r} = \vec{r}(x, y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + f(x, y)\,\hat{\mathbf{e}}_3$

(*v*) By rotating a curve about a line.

Here again it should be noted that **the parametric representation of a surface is not unique.**

If the functions used to define the above surfaces are continuous and differentiable functions and are such that the functions defining the surface and their partial derivatives are all well defined at points on the surface, then the surfaces are called **smooth surfaces.** If the surface is defined implicitly by an equation of the form $F(x, y, z) = 0$, then those points on the surface where at least one of the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ is different from zero are called **regular points on the surface.** If all of these partial derivatives are zero at a point on the surface, then that point is called **a singular point of the surface.**

To represent **a curve on a given surface** defined in terms of two parameters u and v, one can specify how these parameters change. For example, if

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3 \tag{7.25}$$

defines a given surface, then

(i) One can specify that the parameters u and v change as as a function of time t and write $u = u(t)$ and $v = v(t)$, then the position vector $\vec{r} = \vec{r}(u,v)$ becomes a function of a single variable t

$$\vec{r} = \vec{r}(t) = x(u(t),v(t))\,\hat{\mathbf{e}}_1 + y(u(t),v(t))\,\hat{\mathbf{e}}_2 + z(u(t),v(t))\,\hat{\mathbf{e}}_3, \qquad a \le t \le b$$

which sweeps out a curve lying on the surface.

(ii) If one specifies that v is a function of u, say $v = f(u)$, this reduces the vector $\vec{r}(u,v)$ to a function of a single variable which defines the curve on the surface. This surface curve is given by

$$\vec{r} = \vec{r}(u) = x(u,f(u))\,\hat{\mathbf{e}}_1 + y(u,f(u))\,\hat{\mathbf{e}}_2 + z(u,f(u))\,\hat{\mathbf{e}}_3$$

(iii) An equation of the form $g(u,v) = 0$ implicitly defines u as a function of v or v as a function of u and can be used to define a curve on the surface. The equation $g(u,v) = 0$, together with the equation (7.25), is said to define the surface curve implicitly.

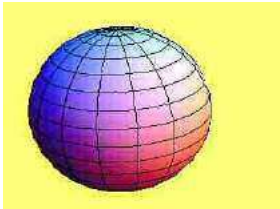
(iv) Consider the special curves

$$\begin{aligned} \vec{r} =& \vec{r}(u,v_0) \qquad v_0 \text{ constant} \\ \vec{r} =& \vec{r}(u_0,v) \qquad u_0 \text{ constant} \end{aligned}$$

sketched on the surface for the values

$$\begin{aligned} u_0 &\in \{\ \alpha, \alpha+h, \alpha+2h, \alpha+3h, \dots\ \} \\ v_0 &\in \{\ \ \beta, \beta+k, \beta+2k, \beta+3k, \dots\ \} \end{aligned}$$

where α, β, h and k have fixed constant values. These special curves are called **coordinate curves on the surface.** The partial derivatives $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$ evaluated at a common point (u_0,v_0) are **tangent vectors to the coordinate curves** and the cross product $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$ produces **a normal to the surface.**



Coordinate curves on sphere.

For example, consider the unit sphere

$$\vec{r}(u,v) = \cos u \sin v\, \hat{\mathbf{e}}_1 + \sin u \sin v\, \hat{\mathbf{e}}_2 + \cos v\, \hat{\mathbf{e}}_3$$

where $0 \le u \le 2\pi$ and $0 \le v \le \pi$. The curves $\vec{r}(u_0, v)$ for equi-spaced constants u_0 gives the coordinate curves called lines of longitude on the sphere. The curves $\vec{r}(u, v_0)$ for equi-spaced constants v_0 give the curves called lines of latitude on the sphere.

A surface is called **an oriented surface** if

(i) each nonboundary point on the surface has two unit normals $\hat{\mathbf{e}}_n$ and $-\hat{\mathbf{e}}_n$. By selecting one of these unit normals one is said to give an **orientation to the surface.** Thus, an oriented surface will always have two orientations.

(ii) The unit normal selected defines a surface orientation and this unit normal must vary continuously over the surface.

(iii) Each nonboundary point on the oriented surface has a tangent plane.

(iv) If the surface is that of a solid, then the unit normal at each point on the surface which is directed outward from the surface is usually selected as the **preferred orientation for the closed surface.**

A surface S is said to be a **simple closed surface** if the surface divides all of three dimensional space into three regions defined by

(i) points interior to S, where the distance between any two points inside S is finite.

(ii) points on the surface S.

(iii) points exterior to the surface S.

A **smooth surface** is one where a **normal vector can be constructed at each point of the surface**.

The sphere

The general equation of a sphere is

$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0$$

where α, β, γ and δ are constants. This is a simple closed surface with outward normal defining its orientation. It is customary to complete the square on the x, y and z terms and express this equation in the form

$$\left(x + \frac{\alpha}{2}\right)^2 + \left(y + \frac{\beta}{2}\right)^2 + \left(z + \frac{\gamma}{2}\right)^2 = \frac{\alpha^2 + \beta^2 + \gamma^2}{4} - \delta \tag{7.26}$$

After completing the square on the x, y and z terms, the following cases can arise.

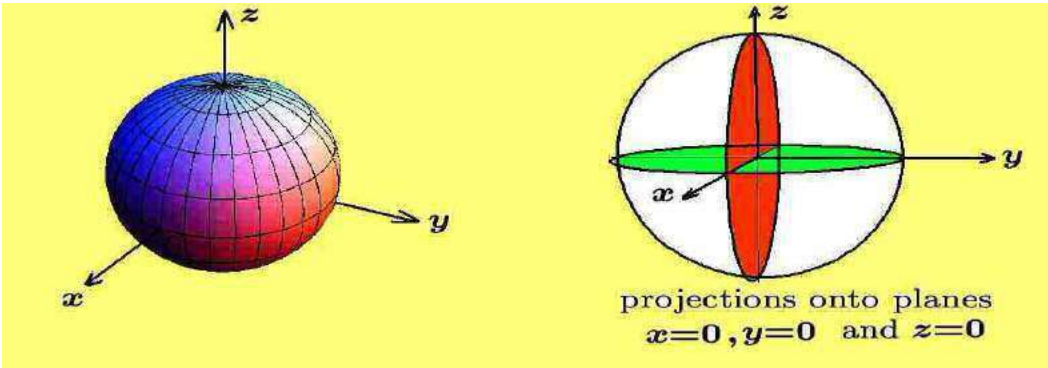


Figure 7-4.

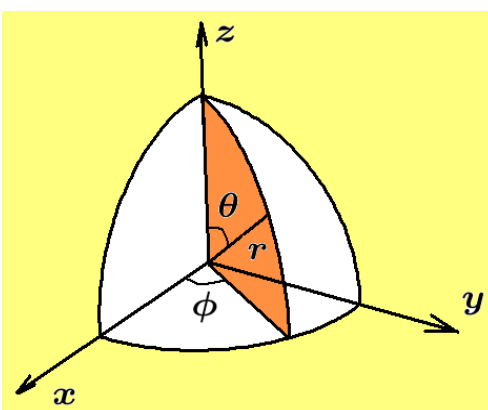
Sphere centered at origin and projections onto planes $x = 0$, $y = 0$ and $z = 0$.

$$\text{If} \quad \frac{\alpha^2+\beta^2+\gamma^2}{4} - \delta = \begin{cases} r^2 > 0, & \text{then } r \text{ is radius of sphere centered at } \left(-\frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}\right) \\ 0, & \text{then } 0 \text{ is radius of sphere centered at } \left(-\frac{\alpha}{2}, -\frac{\beta}{2}, -\frac{\gamma}{2}\right) \\ -r^2 < 0, & \text{then no real sphere exists} \end{cases}$$

In the case the right-hand side of equation (7.26) is negative, then a virtual sphere is said to exist. A sphere centered at the point (x_0, y_0, z_0) with radius $r > 0$ has the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{7.27}$$

The figure 7-4 illustrates a sphere and projections of the sphere onto the $x = 0$, $y = 0$ and $z = 0$ planes.



A sphere with constant radius $r > 0$ and centered at the origin can also be represented in the parametric form

$$\begin{aligned} x &= x(\phi, \theta) = r \sin\theta \cos\phi, \\ y &= y(\phi, \theta) = r \sin\theta \sin\phi, \\ z &= z(\phi, \theta) = r \cos\theta \end{aligned} \tag{7.28}$$

where $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$. These parameters are illustrated in the accompanying figure. Note that when θ is held constant, one obtains a coordinate curve representing a line of latitude on the sphere given by $\lambda = \frac{\pi}{2} - \theta$ and when ϕ is held constant, one obtains a coordinate curve representing some line of longitude on the sphere.

The representation $\vec{r} = \vec{r}(\phi, \theta) = r \sin\theta \cos\phi \, \hat{\mathbf{e}}_1 + r \sin\theta \sin\phi \, \hat{\mathbf{e}}_2 + r \cos\theta \, \hat{\mathbf{e}}_3$ is a vector representation for points on the sphere of radius r with $\vec{r}(\phi_0, \theta)$ a curve of longitude and $\vec{r}(\phi, \theta_0)$ a line of latitude and these curves are called **coordinate curves** on the surface of the sphere. The vectors $\frac{\partial \vec{r}}{\partial \phi}$ and $\frac{\partial \vec{r}}{\partial \theta}$ are tangent vectors to the coordinate curves. The cross product $\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}$ produces **a normal vector to the surface of the sphere**.

The Ellipsoid

The ellipsoid centered at the point (x_0, y_0, z_0) is represented by the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1 \tag{7.29}$$

and if

$a = b > c$ it is called **an oblate spheroid**.

$a = b < c$ it is called **a prolate spheroid**.

$a = b = c$ it is called **a sphere of radius** a.

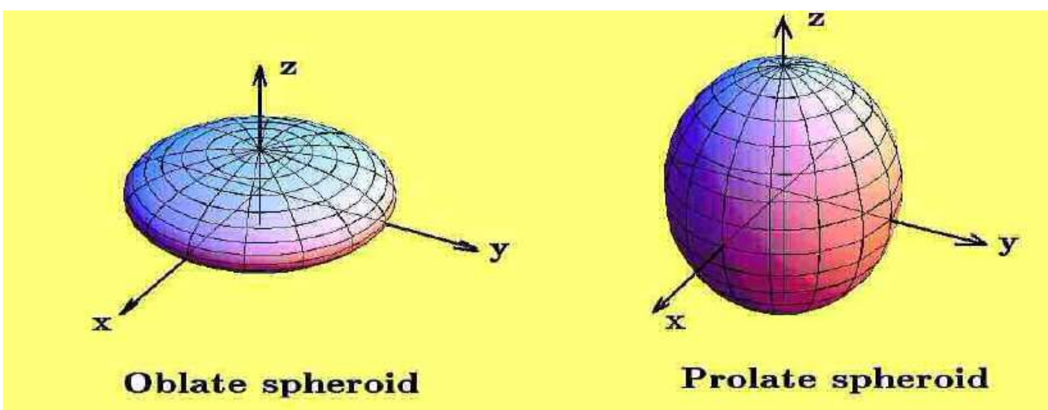


Figure 7-5. Oblate and prolate spheroids.

The ellipsoid can also be represented by the parametric equations

$$x - x_0 = a \cos\theta \cos\phi, \qquad y - y_0 = b \cos\theta \sin\phi, \qquad z - z_0 = c \sin\theta \tag{7.30}$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $-\pi \leq \phi \leq \pi$. The figure 7-5 illustrates the oblate and prolate spheroids centered at the origin.

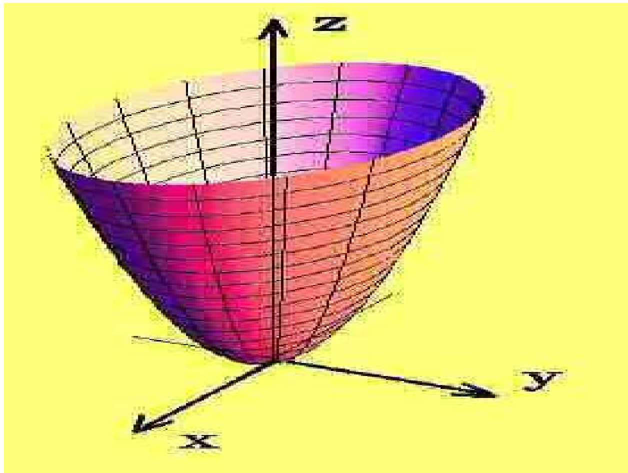


Figure 7-6. Elliptic paraboloid

The Elliptic Paraboloid

The elliptic paraboloid centered at the point (x_0, y_0, z_0) is described by the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = \frac{z-z_0}{c} \tag{7.31}$$

It can also be represented by the parametric equations

$$x - x_0 = a\sqrt{u}\cos v, \qquad y - y_0 = b\sqrt{u}\sin v, \qquad z - z_0 = cu \tag{7.32}$$

where $0 \le v \le 2\pi$ and $0 \le u \le h$. The elliptic paraboloid centered at the origin is illustrated in the figure 7-6.

The Elliptic Cone

The elliptic cone centered at the point (x_0, y_0, z_0) is represented by an equation having the form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = \frac{(z-z_0)^2}{c^2} \tag{7.33}$$

A parametric representation for the elliptic cone is given by

$$x - x_0 = au\cos v, \qquad y - y_0 = bu\sin v, \qquad z - z_0 = cu$$

for $0 \le v \le 2\pi$ and $-h \le u \le h$. The elliptic cone centered at the origin is illustrated in the figure 7-7.

The position vector $\vec{r} = \vec{r}(u,v) = au\cos v\,\hat{\mathbf{e}}_1 + bu\sin v\,\hat{\mathbf{e}}_2 + cu\,\hat{\mathbf{e}}_3$ describes a point on the surface centered at the origin and the curves $\vec{r}(u_0,v)$, $\vec{r}(u,v_0)$ define the coordinate curves. The partial derivatives of $\vec{r}$ with respect to u and v are tangent vectors to the coordinate curves and these vectors can be used to construct a normal vector to the surface.

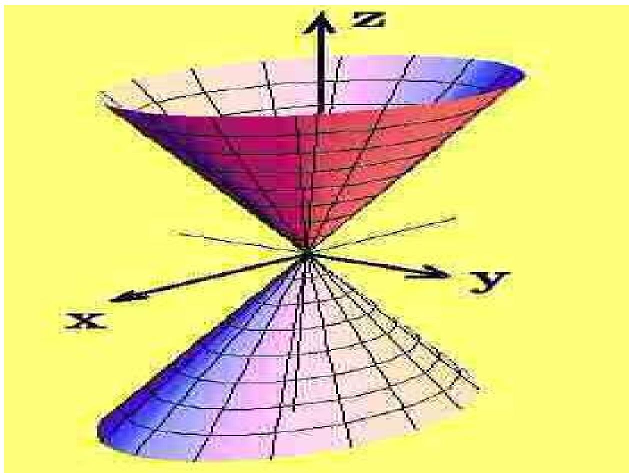


Figure 7-7. Elliptic cone

The Hyperboloid of One Sheet

The hyperboloid of one sheet centered at the point (x_0,y_0,z_0) and symmetric about the $z-$axis is given by the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{(z-z_0)^2}{c^2} = 1 \tag{7.34}$$

It can also be represented using the parametric equations

$$x - x_0 = a\cos u\cosh v, \qquad y - y_0 = b\sin u\cosh v, \qquad z - z_0 = c\sinh v \tag{7.35}$$

where $0 \le u \le 2\pi$ and $-h < v < h$. Here h is usually selected as a small number, say $h = 1$ as the selection of h as a large number gives a scaling difference between the parameters and distorts the final image.

The Hyperboloid of Two Sheets

The hyperboloid of two sheets centered at the point (x_0,y_0,z_0) and symmetric about the $z-$axis is describe by the equation

$$-\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1 \tag{7.36}$$

It can also be represented by the parametric equations

$$x - x_0 = a \cos v \sinh u, \qquad y - y_0 = b \sin v \sinh u, \qquad z - z_0 = c \cosh u \tag{7.37}$$

where $0 \le v \le 2\pi$ and $0 \le u \le h$, with both $c > 0$ and $c < 0$ producing a surface with two parts. Here again the selection of h should be of the same magnitude or less than v or else the final image gets distorted. The hyperboloid of one sheet and some hyperboloids of two sheets are illustrated in the figure 7-8. Note in this figure that the axes x, y and z have undergone various permutations. These permutations show that the axis of symmetry for the hyperboloid of two sheets is always associated with the term which has the positive sign. In a similar fashion one can do a permutation of the symbols x, y and z in the equation describing the hyperboloid of one sheet to obtain different axes of symmetry.

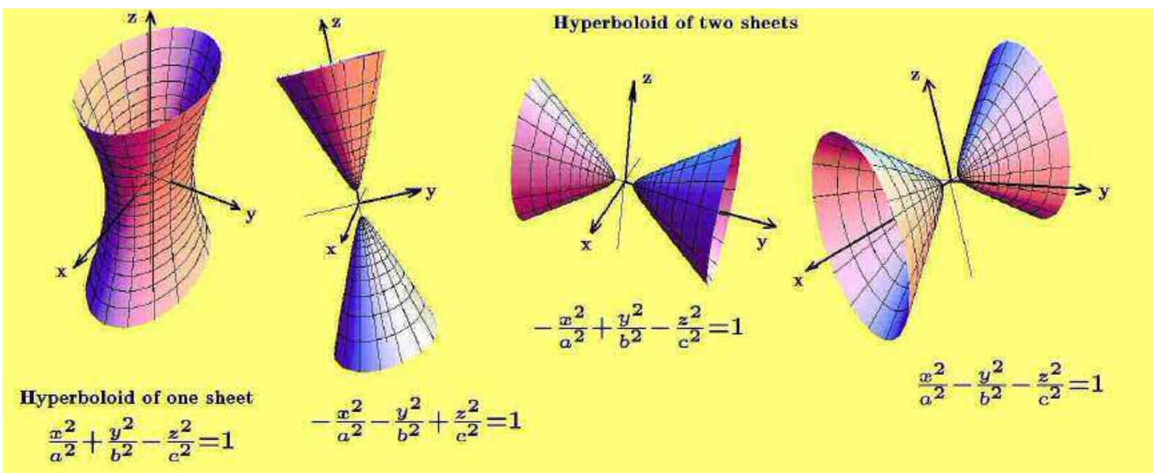


Figure 7-8. Hyperboloid of one sheet and several hyperboloids of two sheets.

In a similar fashion one can perform a permutation of the symbols x, y and z to give alternative representations of any of the surfaces previously defined.

One can use the parametric equations to define a position vector $\vec{r} = \vec{r}(u, v)$ from which the coordinate curves $\vec{r}(u_0, v)$ and $\vec{r}(u, v_0)$ can be constructed. The partial derivatives of $\vec{r}(u, v)$ with respect to u and v produce tangent vectors to the coordinate curves and these tangent vectors can be used to construct a normal vector to each point on the surface.

The Hyperbolic Paraboloid

The hyperbolic paraboloid centered at the the point (x_0, y_0, z_0) is described by the equation

$$\frac{z-z_0}{c} = -\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}, \tag{7.38}$$

This surface is saddle shaped and can also be described using the parametric equations

$$x - x_0 = u, \qquad y - y_0 = v, \qquad z - z_0 = c\left(-\frac{u^2}{a^2} + \frac{v^2}{b^2}\right) \tag{7.39}$$

where $-h \le u \le h$ and $-k \le v \le k$ for selected constants h and k. These parametric equations can be used to construct the two-parameter surface $\vec{r}(u,v)$ from which the coordinate curves and normal vector can be constructed.

It is left as an exercise to show that under a rotation of axes and scaling using the equations

$$\frac{x-x_0}{a} = \bar{x}\cos\theta - \bar{y}\sin\theta, \qquad \frac{y-y_0}{b} = \bar{x}\sin\theta + \bar{y}\cos\theta, \qquad \frac{z-z_0}{c} = \bar{z}$$

with $\theta = \pi/4$, the hyperbolic paraboloid can be represented $\bar{z} = \bar{x}\bar{y}$.

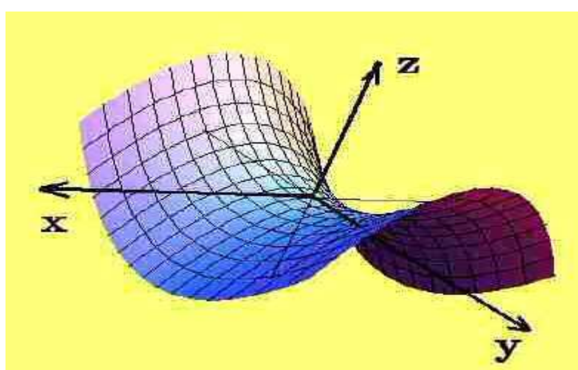


Figure 7-9. Hyperbolic paraboloid.

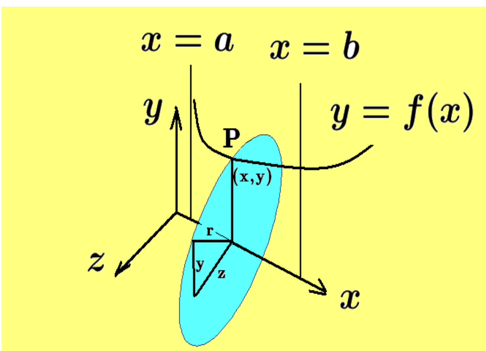
Surfaces of Revolution

Any surface which can be created by rotating a curve about a fixed line is called a surface of revolution. The fixed line about which the curve is rotated is called the axis of revolution. Some examples of surfaces of revolution are the sphere which is created by rotating the semi-circle $x^2 + y^2 = r^2$, $-r \le x \le r$ and $y \ge 0$ about the $y = 0$

axis. A paraboloid is obtained by rotating the parabola $y = x^2$, $0 \le x \le x_0$ about the $x = 0$ axis.

The general procedure for determining the equation for representing a surface of revolution is as follows. First select a general point P on the given curve and then rotate the point P about the axis of revolution to form a circle. This usually involves some parameter used to describe the general point. One can then determine the equation of the surface by eliminating the parameter from the resulting equations.

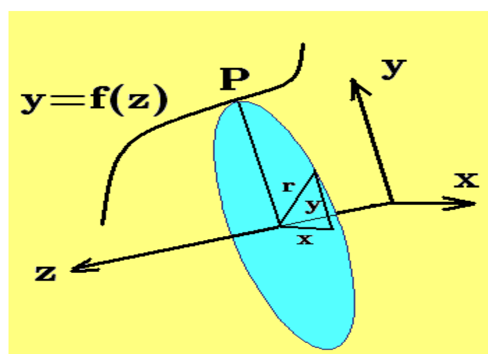
Example 7-7. A curve $y = f(x)$ for $a \le x \le b$ is rotated about the x–axis. Find the equation describing the surface of revolution.



Solution A general point P on the given curve, when rotated about the x–axis produces the circle $y^2 + z^2 = r^2$, where $r = f(x)$ is the radius of the circle. Eliminating the parameter r gives the equation of the surface of revolution as $y^2 + z^2 = [f(x)]^2$ ■

Example 7-8. The curve $y = f(z)$ for $a \le z \le b$ is rotated about the z–axis. Find the equation describing the surface of revolution.

Solution A general point P on the given curve is rotated about the z–axis to form the circle $x^2 + y^2 = r^2$ where $r = f(z)$ is the radius of the circle. Eliminating r between these two equations gives the equation for the surface of revolution as $x^2 + y^2 = [f(z)]^2$



■

Example 7-9. A curve described by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $t_0 \le t \le t_1$, is rotated about the line

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}$$

where $\vec{b} = b_1\,\hat{\mathbf{e}}_1 + b_2\,\hat{\mathbf{e}}_2 + b_3\,\hat{\mathbf{e}}_3$ is the direction vector of the line. Find the equation of the surface of revolution.

Solution In the figure 7-10 the point P represents a general point on the space curve. Let the coordinates of this point be denoted by $(x(t^*), y(t^*), z(t^*))$ where $t_0 \le t^* \le t_1$ and t^* is held constant. Construct the vector $\vec{r}_P$ from the origin to the point P and construct the position vector $\vec{r}_0$ from the origin to the fixed point (x_0, y_0, z_0) on the axis of rotation. A **unit vector** in the direction of the axis or rotation is described by

$$\hat{\mathbf{e}}_B = \frac{1}{|\vec{b}|}\vec{b} = \frac{b_1\,\hat{\mathbf{e}}_1 + b_2\,\hat{\mathbf{e}}_2 + b_3\,\hat{\mathbf{e}}_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}} = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3 \tag{7.40}$$

where $\hat{\mathbf{e}}_B \cdot \hat{\mathbf{e}}_B = B_1^2 + B_2^2 + B_3^2 = 1$. Also construct the vector $\vec{r}_P - \vec{r}_0$ from the point (x_0, y_0, z_0) to the point P as illustrated in the figure 7-10.

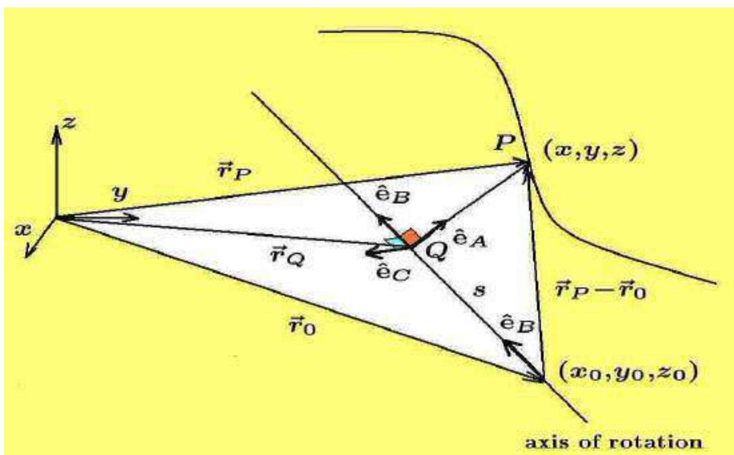


Figure 7-10. Space curve revolved about line to form surface of revolution.

Consider a line perpendicular to the axis of rotation and passing through the point P. Denote by Q the point where this line intersects the axis of rotation. The distance s from the point (x_0, y_0, z_0) to the point Q is given by the projection of the vector $\vec{r}_P - \vec{r}_0$ onto the unit vector $\hat{\mathbf{e}}_B$. This projection gives the distance

$$s = \hat{\mathbf{e}}_B \cdot (\vec{r}_P - \vec{r}_0) \tag{7.41}$$

The point Q can be described by the position vector

$$\vec{r}_Q = \vec{r}_0 + s\,\hat{\mathbf{e}}_B \tag{7.42}$$

The distance from P to Q represents the radius of the circle of revolution when the point P is revolved about the axis of rotation. This distance, call it R, is given by the magnitude of the vector $\vec{r}_P - \vec{r}_Q$ and one can determine this distance from the dot product relation

$$R^2 = (\vec{r}_P - \vec{r}_Q) \cdot (\vec{r}_P - \vec{r}_Q) \tag{7.43}$$

Construct the unit vector $\hat{\mathbf{e}}_A$ pointing from the point Q to the point P by expanding the equation

$$\hat{\mathbf{e}}_A = \frac{\vec{r}_P - \vec{r}_Q}{|\vec{r}_P - \vec{r}_Q|} = \frac{\vec{r}_P - \vec{r}_Q}{R} \tag{7.44}$$

The unit vector $\hat{\mathbf{e}}_C$ which is perpendicular the unit vectors $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_B$ can be constructed using the cross product

$$\hat{\mathbf{e}}_C = \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B \tag{7.45}$$

Note that when the point P is revolved about the axis of rotation, the circle generated lies in the plane of the vectors $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_C$ and a point on this circle can be described using the equation

$$\vec{r} = \vec{r}_Q + R\cos\theta\,\hat{\mathbf{e}}_A + R\sin\theta\,\hat{\mathbf{e}}_C, \qquad 0 \le \theta \le 2\pi \tag{7.46}$$

Recall that the point P represents a general point on the space curve and so the vector in equation (7.46) is really a function of the two variables t and θ and one can express the equation (7.46) as the two parameter surface described by

$$\vec{r} = \vec{r}(t,\theta) = \vec{r}_Q + R\cos\theta\,\hat{\mathbf{e}}_A + R\sin\theta\,\hat{\mathbf{e}}_C, \qquad 0 \le \theta \le 2\pi \tag{7.47}$$

where the vectors $\vec{r}_Q$, $\hat{\mathbf{e}}_A$ and $\hat{\mathbf{e}}_C$ are all calculated in terms of the parameter t and can be constructed using the equations (7.40), (7.41), (7.42), (7.43), (7.44), (7.45). That is, to construct the surface of revolution, one can construct a circle for each value of the space parameter t varying between two fixed values, say $t_0 \le t \le t_1$.

An alternative method for constructing the circles of revolution of each point P on the space curve for $t_0 \le t \le t_1$ is as follows. First, assume P is fixed and construct the sphere centered at the point (x_0, y_0, z_0) which passes through the point P. This sphere has a radius given by $r = |\vec{r}_P - \vec{r}_0|$ and this radius is a function of the parameter t^* used to describe the point P. The equation of the sphere centered at (x_0, y_0, z_0) and passing through the point P is given by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{7.48}$$

Next construct the plane which passes through the point P and is perpendicular to the axis of rotation. The equation of this plane is given by

$$(\vec{r} - \vec{r}_P) \cdot \hat{\mathbf{e}}_B = 0 \qquad \text{or} \qquad (x - x(t^*))B_1 + (y - y(t^*))B_2 + (z - z(t^*))B_3 = 0 \tag{7.49}$$

This plane is the plane of rotation of the point P and it intersects the sphere in the circle described by the point P as it moves around the axis of rotation. To obtain the equation for the surface of revolution one must eliminated the parameter t^* from the equations (7.48) and (7.49). This elimination is not always an easy task to perform.

■

Ruled Surfaces

A surface $\vec{r} = \vec{r}(u, v)$ or $z = f(x, y)$ or $F(x, y, z) = 0$ is called a ruled surface if it has the following property. *Through each point on the surface it is possible to draw a straight line which lies entirely on the surface.* For example, consider the set of all straight lines which pass through a fixed point V and which intersect a fixed curve C, which is not a straight line through V. The surface generated is called a general cone with the point V called the vertex of the cone, the curve C being called the directrix of the cone and the lines on the surface of the cone are called the generating lines. Some example cones are illustrated in the figure 7-11.

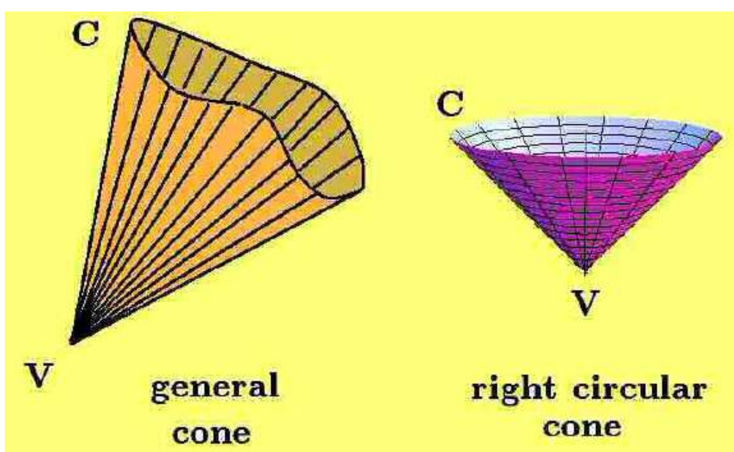


Figure 7-11. A cone is an example of a ruled surface.

Another example of a ruled surface are general cylindrical surfaces which can be described as a collection of straight lines all parallel to a given direction.

In general, a ruled surface can be thought of as set of points created by moving a straight line. One way of creating the equation of a ruled surface is to consider two curves where both curves are defined in terms of a parameter t and represented by the position vectors $\vec{r}_1(t)$ and $\vec{r}_2(t)$ as illustrated in the figure 7-12.

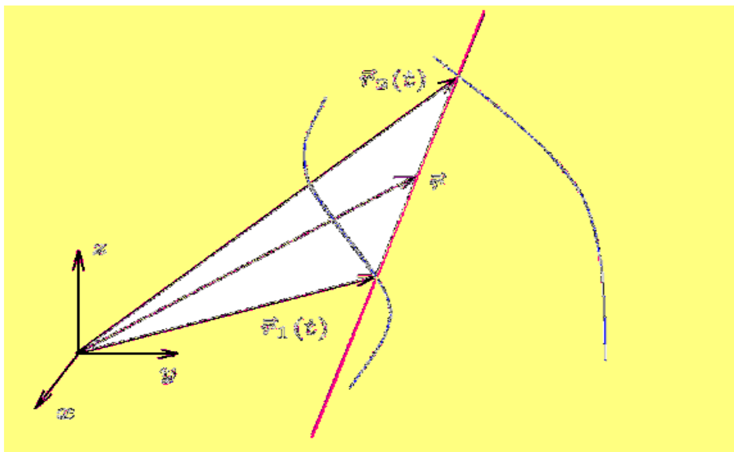


Figure 7-12. Generating a ruled surface using two curves.

For a fixed value of the parameter t, one can draw a straight line between the two points $\vec{r}_1(t)$ and $\vec{r}_2(t)$ as illustrated in the figure 7-12. If $\vec{r}$ is the position vector to a general point on this line it can be represented by the equations

$$\vec{r} = \vec{r}(t,u) = (1-u)\vec{r}_1(t) + u\vec{r}_2(t) \qquad -u_0 \le u \le u_0 \tag{7.50}$$

where u is a parameter and u_0 is some specified constant . Note that when $u=0$, then $\vec{r} = \vec{r}_1$ and when $u=1$, $\vec{r} = \vec{r}_2$. As the parameter t changes the line sweeps out a surface.

Ruled surfaces can be observed on cylinders, cones, hyperboloids of one sheet, as well as elliptic and hyperbolic paraboloids. Ruled surfaces have been studied since the time of the early Greeks and many architectural structures can be described as ruled surfaces.

Surface Area

The position vector

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3 \qquad \alpha \le u \le \beta, \quad \gamma \le v \le \delta \tag{7.51}$$

defines a surface in terms of two parameters u and v. The family of curves $\vec{r}(u, v_0)$, with v_0 taking on selected constant values, defines **a set of coordinate curves on the surface**. Similarly, the family of curves $\vec{r}(u_0, v)$, with u_0 taking on selected constant values, defines another set of coordinate curves. The vector $\frac{\partial \vec{r}}{\partial u}$ is **a tangent vector to the coordinate curve** $\vec{r}(u, v_0)$ and the vector $\frac{\partial \vec{r}}{\partial v}$ is **a tangent vector to the coordinate curve** $\vec{r}(u_0, v)$. If at every common point of intersection of the coordinate curves $\vec{r}(u_0, v)$ and $\vec{r}(u, v_0)$ one finds that $\frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}\Big|_{(u_0,v_0)} = 0$, then the coordinate curves are said to form **an orthogonal net on the surface**.

The vector

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv \tag{7.52}$$

lies in the tangent plane to the point $\vec{r}(u,v)$ on the surface and one can say that the vector element $d\vec{r}$ **defines a parallelogram with vector sides** $\frac{\partial \vec{r}}{\partial u}\,du$ **and** $\frac{\partial \vec{r}}{\partial v}\,dv$ as illustrated in the figure 7-13.

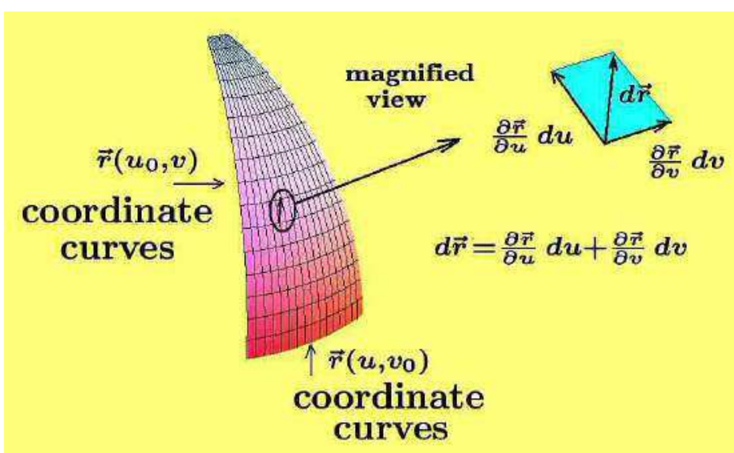


Figure 7-13. Defining an element of area on a surface.

Define **the element of surface area** dS on a given surface as the area of the elemental parallelogram formed using the vector components of $d\vec{r}$. Recall that the magnitude of the cross product of the sides of a parallelogram gives the area of the parallelogram and consequently one can express the element of surface area as

$$dS = \left| \frac{\partial \vec{r}}{\partial u}\, du \times \frac{\partial \vec{r}}{\partial v}\, dv \right| = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \, dudv \tag{7.53}$$

Using the vector identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

with $\vec{A} = \vec{C} = \frac{\partial \vec{r}}{\partial u}$ and $\vec{B} = \vec{D} = \frac{\partial \vec{r}}{\partial v}$ one finds

$$\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) = \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}\right)\left(\frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}\right) - \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}\right) \cdot \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}\right)$$

Define the quantities

$$\begin{aligned} E =& \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ F =& \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ G =& \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned} \tag{7.54}$$

then one can write

$$dS = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \, dudv = \sqrt{EG - F^2}\, dudv \tag{7.55}$$

Alternatively one can write

$$\begin{aligned} \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} =& \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \\ =& \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\hat{\mathbf{e}}_1 - \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\hat{\mathbf{e}}_3 \end{aligned}$$

and the magnitude of this cross product is given by

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(-\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2} \tag{7.56}$$

Expanding the equation (7.56) one finds that the element of surface can be represented by the equation (7.55). To find the area of the surface one need only evaluate the double integral

$$\text{Surface Area} = \int_\alpha^\beta \int_\gamma^\delta \sqrt{EG - F^2}\, dv\, du \tag{7.57}$$

which represents a summation of the elements of surface area over the surface between appropriate limits assigned to the parameters u and v.

Note that the vectors $\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ and $-\vec{N} = \frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}$ are both normal vectors to the surface $\vec{r} = \vec{r}(u,v)$ and

$$\hat{\mathbf{e}}_n = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|} = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\sqrt{EG - F^2}}$$

are unit normals to the surface.

In the special case the surface is defined by $\vec{r} = \vec{r}(x,y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x,y)\,\hat{\mathbf{e}}_3$ one can show the element of surface area is given by

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \frac{dx\,dy}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

Here the surface element dS is projected onto the xy-plane to determine the limits of integration.

In the special case the surface is defined by $\vec{r} = \vec{r}(x,z) = x\,\hat{\mathbf{e}}_1 + y(x,z)\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ one can show the element of surface area is given by

$$dS = \frac{dx\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1|} = \frac{dx\,dz}{\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}}$$

Here the surface element dS is projected onto the xz-plane to determine the limits of integration.

In the special case the surface is defined by $\vec{r} = \vec{r}(y,z) = x(y,z)\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ the element of surface area is found to be given by

$$dS = \frac{dy\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|} = \frac{dy\,dz}{\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}}$$

In this case the surface element dS is projected onto the yz-plane to determine the limits of integration.

Arc Length

Consider a curve $u = u(t)$, $v = v(t)$ on a surface $\vec{r} = \vec{r}(u,v)$ for $t_0 \le t \le t_1$. The **element of arc length** ds associated with this curve can be determined from the vector element

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv$$

using

$$\begin{aligned} ds^2 =& d\vec{r}\cdot d\vec{r} = \left(\frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv\right)\cdot\left(\frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv\right) \\ ds^2 =& E\,du^2 + 2F\,du\,dv + G\,du^2 \end{aligned} \tag{7.58}$$

where E, F, G are given by the equations (7.54). The length of the curve is then given by the integral

$$s = \text{arc length} = \int_{t_0}^{t_1} \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt \tag{7.59}$$

where the limits of integration t_0 and t_1 correspond to the endpoints associated with the curve as determined by the parameter t.

The Gradient, Divergence and Curl

The **gradient** is a field characteristic that describes **the spatial rate of change of a scalar field**. Let $\phi = \phi(x,y,z)$ represent a scalar field, then the gradient of ϕ is a vector and is written

$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3. \tag{7.60}$$

Here it is assumed that the scalar field $\phi = \phi(x,y,z)$ possesses first partial derivatives throughout some region R of space in order that the gradient vector exists. The operator

$$\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3 \tag{7.61}$$

is called the "del "operator or nabla operator and can be used to express the gradient in the operator form

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\phi. \tag{7.62}$$

Note that the operator is not commutative and $\nabla\phi \ne \phi\nabla$.

If $\vec{v} = \vec{v}(x,y,z) = v_1(x,y,z)\,\hat{\mathbf{e}}_1 + v_2(x,y,z)\,\hat{\mathbf{e}}_2 + v_3(x,y,z)\,\hat{\mathbf{e}}_3$ denotes a vector field with components which are well defined, continuous and everywhere differential, then **the divergence of** $\vec{v}$ is defined

$$\text{div}\,\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{7.63}$$

Using the del operator ∇ the divergence can be represented

$$\begin{aligned}\operatorname{div}\vec{v} = \nabla\cdot\vec{v} =&(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3)\cdot(v_1\,\hat{\mathbf{e}}_1 + v_2\,\hat{\mathbf{e}}_2 + v_3\,\hat{\mathbf{e}}_3)\\ \operatorname{div}\vec{v} = \nabla\cdot\vec{v} =&\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\end{aligned} \tag{7.64}$$

Again make note of the fact that the del operator is not commutative and $\nabla\cdot\vec{v} \neq \vec{v}\cdot\nabla$. **If the divergence of a vector field is zero, $\nabla\cdot\vec{v} = 0$, then the vector field is called solenoidal.**

If $\vec{v} = \vec{v}(x,y,z) = v_1(x,y,z)\,\hat{\mathbf{e}}_1 + v_2(x,y,z)\,\hat{\mathbf{e}}_2 + v_3(x,y,z)\,\hat{\mathbf{e}}_3$ denotes a vector field with components which are well defined, continuous and everywhere differential, then **the curl or rotation of $\vec{v}$** is written[4] $\operatorname{curl}\vec{v} = \nabla\times\vec{v} = \operatorname{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}$ which can be expressed in the expanded determinant form[5]

$$\begin{aligned}\operatorname{curl}\vec{v} = \nabla\times\vec{v} =&\operatorname{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}\\ =&\,\hat{\mathbf{e}}_1\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_2 & v_3\end{vmatrix} - \hat{\mathbf{e}}_2\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ v_1 & v_3\end{vmatrix} + \hat{\mathbf{e}}_3\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ v_1 & v_2\end{vmatrix}\\ \operatorname{curl}\vec{v} = \nabla\times\vec{v} =&\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\,\hat{\mathbf{e}}_1 - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\,\hat{\mathbf{e}}_2 + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\,\hat{\mathbf{e}}_3\end{aligned} \tag{7.65}$$

If the curl of a vector field is zero, curl $\vec{v} = \nabla\times\vec{v} = \vec{0}$, then the vector field is said to be irrotational.

Example 7-10. Find the gradient of the scalar field $\phi = x^2y + zxy^2$ at the point $(1,1,2)$.

Solution Using the above definition show that

$$\text{grad}\ \phi = \nabla\phi = (2xy + zy^2)\,\hat{\mathbf{e}}_1 + (x^2 + 2xyz)\,\hat{\mathbf{e}}_2 + xy^2\,\hat{\mathbf{e}}_3$$

$$\text{and}\quad \text{grad}\ \phi\Big|_{(1,1,2)} = 4\,\hat{\mathbf{e}}_1 + 5\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3.$$

■

[4] The curl of $\vec{v}$ is sometimes referred to as the rotation of $\vec{v}$ and written $\operatorname{rot}\vec{v}$.

[5] See chapter 10 for properties of determinants.

Example 7-11. Find the divergence of the vector field given by

$$\vec{v} = xyz\,\hat{\mathbf{e}}_1 + yz^2\,\hat{\mathbf{e}}_2 + zxy^2\,\hat{\mathbf{e}}_3$$

Solution By definition

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\cdot\left(xyz\,\hat{\mathbf{e}}_1 + yz^2\,\hat{\mathbf{e}}_2 + zxy^2\,\hat{\mathbf{e}}_3\right)$$

$$\operatorname{div}\vec{v} = \nabla\cdot\vec{v} = yz + z^2 + xy^2$$

■

Example 7-12. Find the curl of the vector field $\vec{v} = xyz\,\hat{\mathbf{e}}_1 + yz^2\,\hat{\mathbf{e}}_2 + zxy^2\,\hat{\mathbf{e}}_3$

Solution By definition

$$\operatorname{curl}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & yz^2 & zxy^2 \end{vmatrix} = \hat{\mathbf{e}}_1 \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & zxy^2 \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xyz & zxy^2 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xyz & yz^2 \end{vmatrix}$$

$$\operatorname{curl}\vec{v} = \nabla\times\vec{v} = (2xyz - 2yz)\,\hat{\mathbf{e}}_1 - (zy^2 - xy)\,\hat{\mathbf{e}}_2 - xz\,\hat{\mathbf{e}}_3$$

■

Properties of the Gradient, Divergence and Curl

Let $u = u(x,y,z)$ and $v = v(x,y,z)$ denote scalar functions which are continuous and differentiable everywhere and let $\vec{A} = \vec{A}(x,y,z)$ and $\vec{B} = \vec{B}(x,y,z)$ denote vector functions which are continuous and differentiable everywhere. One can then verify that the del or nabla operator has the following properties.

(i) $\operatorname{grad}(u+v) = \operatorname{grad}u + \operatorname{grad}v$ or $\nabla(u+v) = \nabla u + \nabla v$

(ii) $\operatorname{grad}(uv) = u\operatorname{grad}v + v\operatorname{grad}u$ or $\nabla(uv) = u\,\nabla v + v\,\nabla u$

(iii) $\operatorname{grad}f(u) = f'(u)\operatorname{grad}u$ or $\nabla(f(u)) = f'(u)\,\nabla u$

(iv) $|\operatorname{grad}u| = |\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}$

(v) If a vector field is irrotational $\operatorname{curl}\vec{F} = \vec{0}$, then it is **derivable from a scalar function by taking the gradient**, then one can write $\vec{F} = \vec{F}(x,y,z) = \operatorname{grad}u(x,y,z)$, or $\vec{F} = \nabla u$. The vector field $\vec{F}$ is called **a conservative vector field**. The function u from which the vector field is derivable is called the **scalar potential.**

(vi) $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$ or $\operatorname{div}(\vec{A}+\vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}$

(vii) $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$ or $\operatorname{curl}(\vec{A}+\vec{B}) = \operatorname{curl}\vec{A} + \operatorname{curl}\vec{B}$

(viii) $\nabla(u\vec{A}) = (\nabla u)\cdot\vec{A} + u(\nabla\cdot\vec{A})$

(ix) $\nabla\times(u\vec{A}) = (\nabla u)\times\vec{A} + u(\nabla\times\vec{A})$

(x) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(xi) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - \vec{B}(\nabla\cdot\vec{A}) - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B})$

(xii) $\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$

(xiii) $\nabla\cdot(\nabla u) = \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

The operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the **Laplacian operator**.

(xiv) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

(xv) $\nabla\times(\nabla u) = \text{curl}\,(\text{grad}\, u) = \vec{0}$ **The curl of the gradient of u is the zero vector.**

(xvi) $\nabla\cdot(\nabla\times\vec{A}) = \text{div}\,(\text{curl}\,\vec{A}) = 0$ **The divergence of the curl of $\vec{A}$ is the scalar zero.**

(xvii) If a vector field $\vec{F}(x,y,z)$ is solenoidal, then it is **derivable from a vector function $\vec{A} = \vec{A}(x,y,z)$ by taking the curl**. One can then write $\vec{F} = \text{curl}\,\vec{A}$ and hence $\text{div}\,\vec{F} = 0$. The vector function $\vec{A}$ is called **the vector potential** from which $\vec{F}$ is derivable.

(xviii) If f is a function of $u_1, u_2, \ldots, u_n$ where $u_i = u_i(x,y,z)$ for $i = 1, 2, \ldots, n$, then

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \cdots + \frac{\partial f}{\partial u_n}\nabla u_n$$

Many properties and physical interpretations associated with the operations of gradient, divergence and curl are given in the next chapter.

Example 7-13. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x,y,z). Show that

$$\text{grad}\,(r) = \text{grad}\,|\vec{r}| = \frac{1}{r}\vec{r} = \hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the direction of $\vec{r}$.

Solution Let $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, then

$$\text{grad}\,(r) = \text{grad}\,|\vec{r}| = \frac{\partial r}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r}{\partial z}\,\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2x = \frac{x}{r}\\
\frac{\partial r}{\partial y} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2y = \frac{y}{r}\\
\frac{\partial r}{\partial z} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2z = \frac{z}{r}
\end{aligned}$$

Substituting for the partial derivatives in the gradient gives

$$\text{grad}\,(r) = \text{grad}\,|\vec{r}| = \frac{1}{r}\vec{r} = \hat{\mathbf{e}}_r$$

■

Example 7-14. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x, y, z). Show that

$$\text{grad}\,(\frac{1}{r}) = \text{grad}\,\frac{1}{|\vec{r}|} = -\frac{1}{r^2}\text{grad}\,(r) = -\frac{1}{r^3}\vec{r} = -\frac{1}{r^2}\,\hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the direction of $\vec{r}$.

Solution Let $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ so that $\frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$. By definition

$$\text{grad}\,(\frac{1}{r}) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = \frac{-x}{r^3} \\
\frac{\partial}{\partial y}\left(\frac{1}{r}\right) &= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y) = \frac{-y}{r^3} \\
\frac{\partial}{\partial z}\left(\frac{1}{r}\right) &= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z) = \frac{-z}{r^3}
\end{aligned}$$

Substituting for the partial derivatives in the gradient gives

$$\text{grad}\,(\frac{1}{r}) = \text{grad}\,\frac{1}{|\vec{r}|} = -\frac{\vec{r}}{r^3} = -\frac{1}{r^2}\left(\frac{\vec{r}}{r}\right) == \frac{1}{r^2}\text{grad}\,r = -\frac{1}{r^2}\,\hat{\mathbf{e}}_r$$

■

Example 7-15. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x, y, z) and let $r = |\vec{r}|$. Find $\text{grad}\,(r^n)$.

Solution By definition

$$\text{grad}\,(r^n) = \nabla(r^n) = \frac{\partial r^n}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r^n}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r^n}{\partial z}\,\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial r^n}{\partial x} &= nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = r^{n-2}x \\
\frac{\partial r^n}{\partial y} &= nr^{n-1}\frac{\partial r}{\partial y} = nr^{n-1}\frac{y}{r} = nr^{n-2}y \\
\frac{\partial r^n}{\partial z} &= r^{n-1}\frac{\partial r}{\partial z} = nr^{n-1}\frac{z}{r} = nr^{n-2}z
\end{aligned}$$

so that

$$\text{grad}\,(r^n) = nr^{n-2}\vec{r} = nr^{n-1}\,\hat{\mathbf{e}}_r$$

■

Example 7-16. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to a general point (x, y, z) and let $r = |\vec{r}|$. Find $\text{grad}\, f(r)$ where $f = f(r)$ is any continuous differentiable function of r.

Solution By definition

$$\text{grad}\, f(r) = \frac{\partial f}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_3$$

where

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{df}{dr}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}\\
\frac{\partial f}{\partial y} &= \frac{df}{dr}\frac{\partial r}{\partial y} = f'(r)\frac{y}{r}\\
\frac{\partial f}{\partial z} &= \frac{df}{dr}\frac{\partial r}{\partial z} = f'(r)\frac{z}{r}
\end{aligned}$$

so that

$$\text{grad}\, f(r) = f'(r)\frac{1}{r}\vec{r} = f'(r)\,\hat{\mathbf{e}}_r$$

Compare this result with the result from the previous example.

■

Example 7-17. If $\phi = \phi(x, y, z)$ is continuous and possess derivatives which are also continuous, show that the curl of the gradient of ϕ produces the zero vector. That is, show

$$\text{curl}(\text{grad}\,\phi) = \nabla \times (\nabla\phi) = \vec{0}$$

Solution The function ϕ is differentiable so that

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3$$

and the curl of this vector is represented

$$\begin{aligned}
\text{curl}(\text{grad}\,\phi) =& \nabla \times (\nabla\phi) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}\\
\text{curl}(\text{grad}\,\phi) =& \hat{\mathbf{e}}_1\left(\frac{\partial^2 \phi}{\partial y\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial y}\right) - \hat{\mathbf{e}}_2\left(\frac{\partial^2 \phi}{\partial x\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial x}\right) + \hat{\mathbf{e}}_3\left(\frac{\partial^2 \phi}{\partial x\,\partial y} - \frac{\partial^2 \phi}{\partial y\,\partial x}\right) = \vec{0}
\end{aligned}$$

because the mixed partial derivatives inside the parenthesis are equal to one another.

■

Example 7-18. Show that $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Solution Calculate $\nabla \times \vec{A}$ using determinants to obtain

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) \hat{\mathbf{e}}_1 - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) \hat{\mathbf{e}}_2 + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \hat{\mathbf{e}}_3$$

One can then calculate the curl of the curl as

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \end{vmatrix} \tag{7.66}$$

The $\hat{\mathbf{e}}_1$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\hat{\mathbf{e}}_1 \left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right] = \hat{\mathbf{e}}_1 \left[\frac{\partial^2 A_2}{\partial x\, \partial y} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial x\, \partial z}\right]$$

By adding and subtracting the term $\dfrac{\partial^2 A_1}{\partial x^2}$ to the above result one finds the $\hat{\mathbf{e}}_1$ component can be expressed in the form

$$\hat{\mathbf{e}}_1 \left\{\left[-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right] + \frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\right\} \tag{7.67}$$

In a similar fashion it can be verified that the $\hat{\mathbf{e}}_2$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\hat{\mathbf{e}}_2 \left\{\left[-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right] + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\right\} \tag{7.68}$$

and the $\hat{\mathbf{e}}_3$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\hat{\mathbf{e}}_3 \left\{\left[-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right] + \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\right\} \tag{7.69}$$

Adding the results from the equations (7.67), (7.68), (7.69) one obtains the result

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{7.70}$$

■

Directional Derivatives

Let $\vec{r} = \vec{r}(s)$ denote **an arbitrary space curve which passes through the point** $P(x, y, z)$ **of the region** R, where the scalar function $\phi = \phi(x, y, z)$ exists and has all first-order partial derivatives which are continuous. Here the space curve is expressed in terms of the arc length parameter s, where s is measured from some fixed point on the curve. In general, the scalar field $\phi = \phi(x, y, z)$ varies with position and has different values when evaluated at different points in space. Let us evaluate ϕ at points along the curve $\vec{r}$ to **determine how ϕ changes with position along the curve**. The rate of change of ϕ with respect to arc length along the curve is given by

$$
\begin{aligned}
\frac{d\phi}{ds} &= \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} \\
\frac{d\phi}{ds} &= \left(\frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3\right) \cdot \left(\frac{dx}{ds}\,\hat{\mathbf{e}}_1 + \frac{dy}{ds}\,\hat{\mathbf{e}}_2 + \frac{dz}{ds}\,\hat{\mathbf{e}}_3\right) \\
\frac{d\phi}{ds} &= \text{grad }\phi \cdot \frac{d\vec{r}}{ds} = \nabla\,\phi \cdot \hat{\mathbf{e}}_t,
\end{aligned}
$$

where the right-hand side is to be evaluated at a point P on the arbitrary curve $\vec{r}(s)$ in R. The right-hand side of this equation is the dot product of the gradient vector with the unit tangent vector to the curve at the point P and physically represents **the projection of the vector** grad ϕ **in the direction of this tangent vector.** Note that the curve $\vec{r}(s)$ represents **an arbitrary curve through the point** P, and hence, the unit tangent vector represents **an arbitrary direction.** Therefore, one may interpret the derivative $\frac{d\phi}{ds} = \text{grad }\phi \cdot \vec{e}$ as representing **the rate of change of ϕ as one moves in the direction** $\vec{e}$. Here the derivative equals **the projection of the vector** grad ϕ **in the direction** $\vec{e}$. Such derivatives are called **directional derivatives.**

(Directional derivative) ***The component of the gradient $\phi = \phi(x, y, z)$ in the direction of a unit vector $\hat{\mathrm{e}} = \cos\alpha\,\hat{\mathrm{e}}_1 + \cos\beta\,\hat{\mathrm{e}}_2 + \cos\gamma\,\hat{\mathrm{e}}_3$ is equal to the projection $\nabla\phi \cdot \hat{\mathrm{e}}$ and is called the directional derivative of ϕ in the direction $\hat{\mathrm{e}}$. The directional derivative is written as***

$$
\begin{aligned}
\frac{d\phi}{ds} &= \textbf{grad }\phi \cdot \vec{e} = \nabla\phi \cdot \hat{\mathrm{e}} \\
&= \left(\frac{\partial \phi}{\partial x}\;\hat{\mathrm{e}}_1 + \frac{\partial \phi}{\partial y}\;\hat{\mathrm{e}}_2 + \frac{\partial \phi}{\partial z}\;\hat{\mathrm{e}}_3\right) \cdot (\cos\alpha\,\hat{\mathrm{e}}_1 + \cos\beta\,\hat{\mathrm{e}}_2 + \cos\gamma\,\hat{\mathrm{e}}_3)
\end{aligned}
\tag{7.71}
$$

where s denotes distance in the direction $\vec{e}$. If $\hat{\mathrm{e}} = \hat{\mathrm{e}}_n$ is a unit normal vector to a surface, the notation $\frac{\partial \phi}{\partial n} = \textbf{grad}\,\phi \cdot \hat{\mathrm{e}}_n$ is used to denote **a normal derivative to the surface.**

The directional derivative is a measure of how the scalar field ϕ changes as you move in a certain direction. Since **the maximum projection of a vector is the magnitude of the vector itself**, the gradient of ϕ is **a vector which points in the direction of**

the greatest rate of change of ϕ. The length of the gradient vector is $|\text{grad}\ \phi|$ and represents the magnitude of this greatest rate of change.

In other words, the gradient of a scalar field is a vector field which represents the direction and magnitude of the greatest rate of change of the scalar field.

Example 7-19. Show **the gradient of $\boldsymbol{\phi}$ is a normal vector to the surface $\boldsymbol{\phi = \phi(x,y,z) = c}$ = constant.**

Solution: Let $\vec{r}(s)$, where s is arc length, represent any curve lying in the surface $\phi(x,y,z) = c$. Along this curve the scalar field has the value $\phi = \phi(x(s), y(s), z(s)) = c$ and the rate of change of ϕ along this curve is given by

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} = \frac{dc}{ds} = 0$$

$$\text{or} \qquad \frac{d\phi}{ds} = \text{grad}\ \phi \cdot \frac{d\vec{r}}{ds} = \text{grad}\ \phi \cdot \hat{\mathbf{e}}_t = 0.$$

The resulting equation tells us that the vector grad ϕ is perpendicular to the unit tangent vector to the curve on the surface. But this unit tangent vector lies in the tangent plane to the surface at the point of evaluation for the gradient. **Thus, grad $\boldsymbol{\phi}$ is normal to the surface $\boldsymbol{\phi(x,y,z) = c}$.** The family of surfaces $\phi = \phi(x,y,z) = c$, for various values of c, are called **level surfaces**. In two-dimensions, the family of curves $\phi = \phi(x,y) = c$, for various values of c, are called **level curves**. **The gradient of $\boldsymbol{\phi}$ is a vector perpendicular to these level surfaces or level curves.** ■

Example 7-20. Find the unit tangent vector at a point on the curve defined by the intersection of the two surfaces

$$F(x,y,z) = c_1 \quad \text{and} \quad G(x,y,z) = c_2,$$

where c_1 and c_2 are constants.

Solution: If two surfaces $F = c_1$ and $G = c_2$ intersect in a curve, then at a point (x_0, y_0, z_0) common to both surfaces and on the curve one can calculate the normal vectors to both surfaces. These normal vectors are

$$\nabla F = \text{grad}\ F \quad \text{and} \quad \nabla G = \text{grad}\ G$$

which are evaluated at the point (x_0, y_0, z_0) common to both surfaces and on the curve of intersection of the surfaces. The cross product

$$(\nabla F) \times (\nabla G)$$

is a vector tangent to the curve of intersection and perpendicular to both of the normal vectors ∇F and ∇G. A unit tangent vector to the curve of intersection is constructed having the form form

$$\hat{\mathbf{e}}_t = \frac{\nabla F \times \nabla G}{|\nabla F \times \nabla G|}.$$

■

Example 7-21. In two-dimensions a curve $y = f(x)$ or $\vec{r} = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$ can be represented in the implicit form $\phi = \phi(x, y) = y - f(x) = 0$ so that

$$\text{grad }\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = -f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 = \vec{N}$$

is a vector normal[6] to the curve at the point $(x, f(x))$. A unit normal to this curve is given by

$$\hat{\mathbf{e}}_n = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}$$

The vector $\vec{T} = \dfrac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2$ and unit vector $\hat{\mathbf{e}}_t = \dfrac{\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2}{\sqrt{1 + [f'(x)]^2}}$ are tangent to the curve and one can verify that $\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_t = 0$ showing these vectors are orthogonal.

■

Applications for the Gradient

In two-dimensions, let $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$ denote a unit vector in an arbitrary, but constant, direction α and let $\phi = \phi(x, y)$ denote any scalar function of position. At a point (x_0, y_0), the directional derivative of ϕ in the direction α becomes

$$\frac{d\phi}{ds} = \text{grad }\phi \cdot \hat{\mathbf{e}}_\alpha = \frac{\partial \phi}{\partial x}\cos\alpha + \frac{\partial \phi}{\partial y}\sin\alpha$$

and the magnitude of this directional derivative changes as the angle α changes. As the angle α varies, the maximum and minimum directional derivatives, at the point (x_0, y_0), occur in those directions α which satisfy

$$\frac{d}{d\alpha}\left[\frac{d\phi}{ds}\right] = -\frac{\partial \phi}{\partial x}\sin\alpha + \frac{\partial \phi}{\partial y}\cos\alpha = 0. \tag{7.72}$$

Note there exists two angles α lying in the region between 0 and 2π radians which satisfy the above equation. These directions must be tested to see which corresponds to a maximum and which corresponds to a minimum directional derivative. These

[6] Always remember that there are two normals to a curve, namely $\hat{\mathbf{e}}_n$ and $-\hat{\mathbf{e}}_n$,

angles specify the directions one should travel in order to achieve the maximum (or minimum) rate of change of the scalar ϕ.

A physical example illustrating this idea is heat flow. Heat always flows from regions of higher temperature to regions of lower temperature. Let $T(x,y)$ denote a scalar field which represents the temperature T at any point (x,y) in some region R within a material medium. The level curves $T(x,y)=T_0$ are called isothermal curves and represent the constant "levels"of temperature. The vector $\operatorname{grad} T$, evaluated at a point on an isothermal curve, points in the direction of greatest temperature change. The vector is also normal to the isothermal curve. Fourier's law of heat conduction states that the heat flow $\vec{q}$ [joules/cm^2 sec] is in a direction opposite to this greatest rate of change and

$$\vec{q} = -k \operatorname{grad} T,$$

where k [joules/cm – sec – deg C] is the thermal conductivity of the material in which the heat is flowing.

Example 7-22. In two-dimensions a curve $y=f(x)$ can be represented in the implicit form $\phi=\phi(x,y)=y-f(x)=0$ so that

$$\operatorname{grad}\,\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = -f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 = \vec{N}$$

is a vector normal to the curve at the point $(x, f(x))$. A unit normal vector to the curve is given by

$$\hat{\mathbf{e}}_n = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}}$$

Another way to construct this normal vector is as follows. The position vector $\vec{r}$ describing the curve $y=f(x)$ is given by $\vec{r} = x\,\hat{\mathbf{e}}_1 + f(x)\,\hat{\mathbf{e}}_2$ with tangent $\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2$. The unit tangent vector to the curve is given by $\hat{\mathbf{e}}_t = \frac{\hat{\mathbf{e}}_1 + f'(x)\,\hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}}$. The vector $\hat{\mathbf{e}}_3$ is perpendicular to the planar surface containing the curve and consequently the vector $\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_t$ is normal to the curve. This cross product is given by

$$\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_t = \frac{-f'(x)\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2}{\sqrt{1+[f'(x)]^2}} = \hat{\mathbf{e}}_n$$

and produces a unit normal vector to the curve. Note that there are always two normals to every curve or surface. It is important to observe that if $\vec{N}$ is normal to a point on the surface, then the vector $-\vec{N}$ is also a normal to the same point on the surface. If the surface is a closed surface, then one normal is called an **inward normal** and the other an **outward normal**.

■

Maximum and Minimum Values

The directional derivative of a scalar field ϕ in the direction of a unit vector $\vec{e}$ has been defined by the projection

$$\frac{d\phi}{ds} = \text{grad } \phi \cdot \vec{e}.$$

Define a second directional derivative of ϕ in the direction $\vec{e}$ as the directional derivative of a directional derivative. The second directional derivative is written

$$\frac{d^2\phi}{ds^2} = \text{grad } \left[\frac{d\phi}{ds}\right] \cdot \vec{e} = \text{grad } [\text{grad } \phi \cdot \vec{e}] \cdot \vec{e}. \tag{7.73}$$

Higher directional derivatives are defined in a similar manner.

Example 7-23. Let $\phi(x, y)$ define a two-dimensional scalar field and let

$$\hat{\mathbf{e}}_\alpha = \cos\alpha\, \hat{\mathbf{e}}_1 + \sin\alpha\, \hat{\mathbf{e}}_2$$

represent a unit vector in an arbitrary direction α. The directional derivative at a point (x_0, y_0) in the direction $\hat{\mathbf{e}}_\alpha$ is given by

$$\frac{d\phi}{ds} = \text{grad } \phi \cdot \hat{\mathbf{e}}_\alpha = \frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha,$$

where it is to be understood that the derivatives are evaluated at the point (x_0, y_0). The second directional derivative is given by

$$\begin{aligned}
\frac{d^2\phi}{ds^2} &= \text{grad } \left(\frac{d\phi}{ds}\right) \cdot \hat{\mathbf{e}}_\alpha \\
\frac{d^2\phi}{ds^2} &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha\right)\cos\alpha + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha\right)\sin\alpha \\
\frac{d^2\phi}{ds^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2\phi}{\partial x\partial y}\sin\alpha\cos\alpha + \frac{\partial^2\phi}{\partial y^2}\sin^2\alpha.
\end{aligned}$$

■

Directional derivatives can be used to determine the maximum and minimum values of functions of several variables. Recall from calculus a function of a single variable $y = f(x)$ has a relative maximum (or relative minimum) at a point x_0 if for any x in a neighborhood of x_0 and different from x_0, the inequality $f(x) < f(x_0)$ (or

$f(x) > f(x_0)$) holds. The determination of relative maximum and minimum values of a differential function $y = f(x)$ over an interval (a, b) consists of

1. Determining the critical points where $f'(x) = 0$ and then testing these critical points.
2. Testing the boundary points $x = a$ and $x = b$.

The second derivative test for relative maximum and minimum values states that if x_0 is a critical point, then

1. $f(x)$ has the maximum value $f(x_0)$ if $f''(x_0) < 0$ (i.e., curve is concave downward if the second derivative is negative).
2. $f(x)$ has a minimum value $f(x_0)$ if $f''(x_0) > 0$ (i.e., the curve is concave upward if the second derivative is positive).

The above concepts for the relative maximum and minimum values of a function of one variable can be extended to higher dimensions when one must deal with functions of more than one variable. The extension of these concepts can be accomplished by utilizing the gradient and directional derivatives.

In the following discussion, it is assumed that the given surface is in an explicit form. If the surface is given in the implicit form $F(x, y, z) = 0$, then it is assumed that one can solve for z in terms of x and y to obtain $z = z(x, y)$. By a **delta neighborhood of a point** (x_0, y_0) in two-dimensions is meant the set of all points inside the circular disk

$$N_0(\delta) = \{x, y \mid (x - x_0)^2 + (y - y_0)^2 < \delta^2\}.$$

The function $z(x, y)$, which is continuous and whose derivatives exist, has a relative maximum at a point (x_0, y_0) if $z(x, y) < z(x_0, y_0)$ for all x, y in a some δ neighborhood of (x_0, y_0). Similarly, the function $z(x, y)$ has a relative minimum at a point (x_0, y_0) if $z(x, y) > z(x_0, y_0)$ for all x, y in some δ neighborhood of the point (x_0, y_0). Points where the surface $z = z(x, y)$ has a relative maximum or minimum are called critical points and at these points one must have

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0$$

simultaneously. Critical points are those points where the tangent plane to the surface $z = z(x, y)$ is parallel to the x, y plane. If the points (x, y) are restricted to a region R of the plane $z = 0$, then the boundary points of R must be tested separately for the determination of any local maximum or minimum values on the surface.

The problem of determining the relative maximum and minimum values of a function of two variables is now considered. In the discussions that follow, note that the problem of determining the maximum and minimum for a function of two variables is reduced to the simpler problem of finding the maximum and minimum of a function of a single variable.

If (x_0, y_0) is a critical point associated with the surface $z = z(x, y)$, then one can slide the free vector given by $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$ to the critical point and construct a plane normal to the plane $z = 0$, such that this plane contains the vector $\hat{\mathbf{e}}_\alpha$. This plane intersects the surface in a curve. The situation is depicted graphically in the figure 7-14

At a critical point where $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, the directional derivative satisfies

$$\frac{dz}{ds} = \text{grad } z \cdot \hat{\mathbf{e}}_\alpha = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha = 0$$

for all directions α.

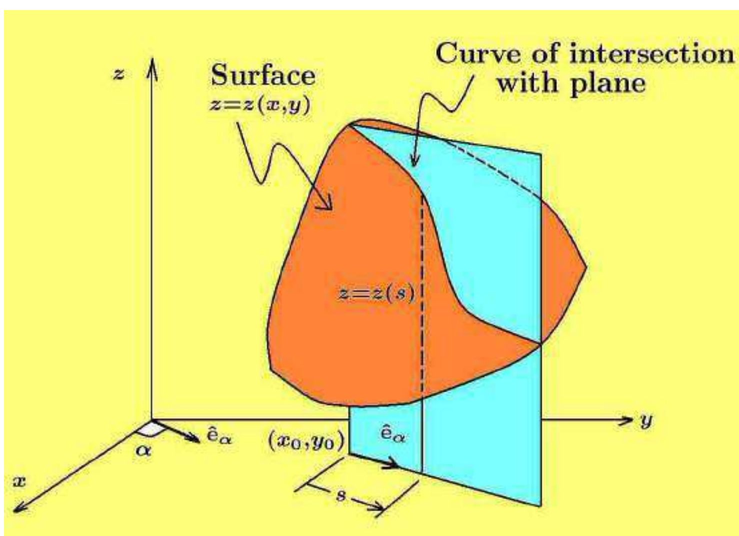


Figure 7-14. Curve of intersection with plane containing $\hat{\mathbf{e}}_\alpha$.

Here the directional derivative represents the variation of the surface height z with respect to a distance s in the $\hat{\mathbf{e}}_\alpha$ direction. (i.e., measure the rate of change of the scalar field z which represents the height of the curve.) To picture what the above equations are describing, let

$$x = x_0 + s\cos\alpha \quad \text{and} \quad y = y_0 + s\sin\alpha$$

represent the equation of the line of intersection of the plane $z=0$ with the plane normal to $z=0$ containing $\hat{\mathbf{e}}_\alpha$. The plane containing $\hat{\mathbf{e}}_\alpha$ and the normal to the plane $z=0$ intersects the surface $z(x,y)$ in a curve given by

$$z = z(x,y) = z(x_0 + s\cos\alpha, y_0 + s\sin\alpha) = z(s).$$

The directional derivative of the scalar field $z(x,y)$ in the direction α is then

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha.$$

Observe that the curve of intersection $z = z(s)$ is a two-dimensional curve, and the methods of calculus may be applied to determine the relative maximum and minimum values along this curve. However, one must test this curve of intersection corresponding to all directions α.

One can conclude that at a critical point (x_0, y_0) one must have $\frac{dz}{ds} = 0$ for all α. If in addition $\frac{d^2z}{ds^2} > 0$ for all directions α, then $z_0 = z(x_0, y_0)$ corresponds to a relative minimum. If the second derivative $\frac{d^2z}{ds^2} < 0$ for all directions α, then $z_0 = z(x_0, y_0)$ corresponds to a relative maximum.

Calculate the second directional derivative and show

$$\frac{d^2z}{ds^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 z}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y^2}\sin^2\alpha.$$

The sign of the second directional derivative determines whether a maximum or minimum value for z exists, and hence one must be able to analyze this derivative for all directions α. Let

$$A = \frac{\partial^2 z}{\partial x^2} \qquad B = \frac{\partial^2 z}{\partial x\,\partial y} \qquad C = \frac{\partial^2 z}{\partial y^2}$$

represent the values of the second partial derivatives evaluated at a critical point (x_0, y_0). One can then express the second directional derivative in a form which is more tractable for analysis. Factor out the leading term and then complete the square on the first two terms to obtain

$$\begin{aligned} \frac{d^2z}{ds^2} &= A\cos^2\alpha + 2B\cos\alpha\sin\alpha + C\sin^2\alpha \\ &= A\left[\cos^2\alpha + 2\frac{B}{A}\cos\alpha\sin\alpha + \frac{C}{A}\sin^2\alpha\right] \\ &= A\left[\left(\cos\alpha + \frac{B}{A}\sin\alpha\right)^2 + \frac{(AC-B^2)}{A^2}\sin^2\alpha\right]. \end{aligned} \tag{7.74}$$

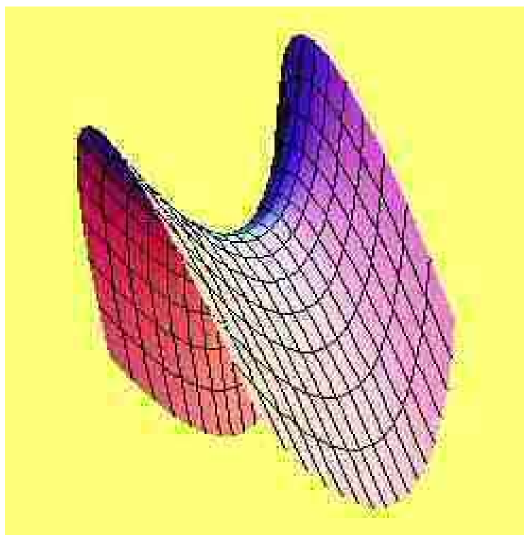


Figure 7-15. Saddle point for $z(x_0, y_0)$

One can now make the following observations:

1. If $AC - B^2 = z_{xx}z_{yy} - (z_{xy})^2 = 0$, then in those directions α which satisfy $\cos\alpha + \frac{B}{A}\sin\alpha = 0$, the second derivative vanishes. For all other values of α, the second derivative is of constant sign, which is the same sign as A. If the above conditions are satisfied, then the second derivative test for a maximum or minimum fails.
2. If $AC - B^2 = z_{xx}z_{yy} - (z_{xy})^2 < 0$, then the second derivative is not of constant sign, but assumes different signs in different directions α. In particular, for the special case $\alpha = 0$ one finds $\frac{d^2z}{ds^2} = A$ and for α satisfying $\cos\alpha + \frac{B}{A}\sin\alpha = 0$ there results

$$\frac{d^2z}{ds^2} = \frac{A(AC-B^2)}{A^2}\sin^2\alpha.$$

Hence, if $A > 0$, then $A(AC - B^2)$ is negative and alternatively if $A < 0$, then $A(AC - B^2)$ is positive. In either case, the second derivative has a nonconstant sign value and in this situation the critical point (x_0, y_0) is said to correspond to a saddle point. Such a critical point is illustrated in figure 7-15.

3. If $AC - B^2 = z_{xx}z_{yy} - (z_{xy})^2 > 0$, the second derivative is of constant sign, which is the sign of A.
   (a) If $A > 0$, $\frac{d^2z}{ds^2} > 0$, the curve $z = z(s)$ is concave upward for all α, and hence the critical point corresponds to a relative minimum.
   (b) If $A < 0$, $\frac{d^2z}{ds^2} < 0$, the curve $z = z(s)$ is concave downward for all α, and therefore the critical point corresponds to a relative maximum.

Example 7-24. Find the maximum and minimum values of

$$z = z(x, y) = x^2 + y^2 - 2x + 4y$$

Solution: The first and second partial derivatives of z are

$$\frac{\partial z}{\partial x} = 2x - 2, \quad \frac{\partial z}{\partial y} = 2y + 4, \quad A = \frac{\partial^2 z}{\partial x^2} = 2, \quad C = \frac{\partial^2 z}{\partial y^2} = 2, \quad B = \frac{\partial^2 z}{\partial x\, \partial y} = 0$$

Setting the first partial derivatives equal to zero and solving for x and y gives the critical points. For this example there is only one critical point which occurs at $(x_0, y_0) = (1, -2)$. From the second derivatives of z one finds $A = 2$, $B = 0$, $C = 2$ and $AC - B^2 = 4 > 0$, and consequently the critical point $(1, -2)$ corresponds to a relative minimum of the function.

The use of level curves to analyze complicated surfaces is sometimes helpful. For example, the level curves of the above function can be expressed in the form

$$z = z(x, y) = (x - 1)^2 + (y + 2)^2 - 5 = k = \text{constant}.$$

By assigning values to the constant k one can determine the general character of the surface. It is left as an exercise to show these level curves are circles which are cross section of the surface known as a paraboloid.

■

Lagrange Multipliers

Consider the problem of finding stationary values associated with a function $f = f(x, y)$ subject to a constraint condition that $g = g(x, y) = 0$. Recall that a necessary condition for $f = f(x,y)$ to have an extremum value at a point (a,b) requires that the differential $df = 0$ or

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0. \tag{7.75}$$

Whenever the small changes dx and dy are independent, one obtains the necessary conditions that

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

at a critical point. Whenever a constraint condition is required to be satisfied, then the small changes dx and dy are no longer independent and one must find the relationship between the small changes dx and dy as the point (x,y) moves along the constraint curve. From the differential relation $dg = 0$ one finds that

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy = 0$$

must be satisfied. Assume that $\frac{\partial g}{\partial y} \neq 0$, then one can obtain

$$dy = \frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}\,dx \tag{7.76}$$

as the dependent relationship between the small changes dx and dy.

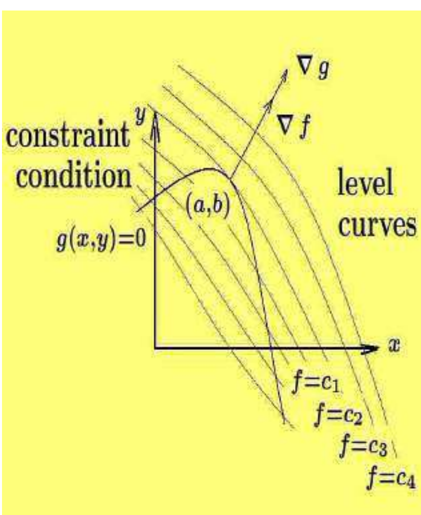


Figure 7-17.
Maximum-minimum problem with constraint.

Substitute the dy from equation (7.76) into the equation (7.75) to produce the result

$$df = \frac{1}{\frac{\partial g}{\partial y}}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)dx = 0 \tag{7.77}$$

that must hold for an arbitrary change dx. This gives the following necessary condition. The critical points (x,y) of the function f, subject to the constraint equation $g(x,y) = 0$, must satisfy the equations

$$\begin{aligned} \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} =&0 \\ g(x,y) =&0 \end{aligned} \tag{7.78}$$

simultaneously.

The equations (7.78) can be interpreted that when a member of the family of curves $f(x,y)=c=constant$ is tangent to the constraint curve $g(x,y)=0$, there results the common values of

$$\frac{dy}{dx}=\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}=\frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} \qquad \Rightarrow \qquad \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=0.$$

One can give a physical picture of the problem. Think of the constraint condition given by $g=g(x,y)=0$ as defining a curve in the x,y-plane and then consider the family of level curves $f=f(x,y)=c$, where c is some constant. A representative sketch of the curve $g(x,y)=0$, together with several level curves from the family, $f=c$ are illustrated in the figure 7-17. Among all the level curves that intersect the constraint condition curve $g(x,y)=0$ select that curve for which c has the largest or smallest value. Here it is assumed that the constraint curve $g(x,y)=0$ is a smooth curve without singular points.

If (a,b) denotes a point of tangency between a curve of the family $f=c$ and the constraint curve $g(x,y)=0$, then at this point both curves will have gradient vectors that are collinear and so one can write $\nabla f+\lambda\nabla g=\vec{0}$ for some constant λ called a Lagrange multiplier. This relationship together with the constraint equation produces the three scalar equations

$$\begin{aligned}\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=&0\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=&0\\ g(x,y)=&0.\end{aligned} \qquad \Rightarrow \qquad \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=0 \tag{7.79}$$

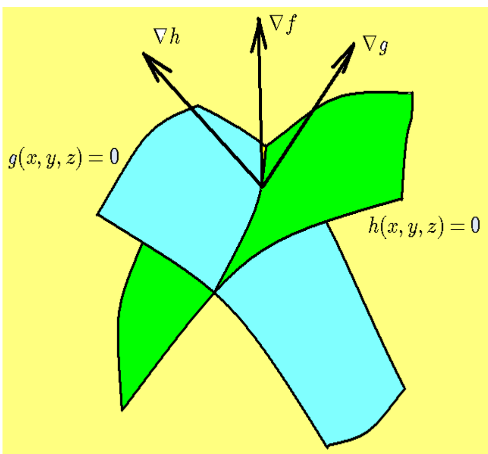
Lagrange viewed the above problem in the following way. Define the function

$$F(x,y,\lambda)=f(x,y)+\lambda g(x,y) \tag{7.80}$$

where $f(x,y)$ is called an objective function and represents the function to be maximized or minimized. The parameter λ is called a Lagrange multiplier and the function $g(x,y)$ is obtained from the constraint condition. Lagrange observed that a stationary value of the function F, without constraints, is equivalent to the problem of stationary values of f with a constraint condition because one would have at a stationary value of F the conditions

$$\begin{aligned}\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} =&0\\ \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} =&0\\ \frac{\partial F}{\partial \lambda} = g(x,y) =&0 \qquad \text{The constraint condition.}\end{aligned} \tag{7.81}$$

These represent three equations in the three unknowns x, y, λ that must be solved. The equations (7.80) and (7.81) are known as **the Lagrange rule** for the method of Lagrange multipliers.



The method of Lagrange multipliers can be applied in higher dimensions. For example, consider the problem of finding maximum and minimum values associated with a function $f = f(x,y,z)$ subject to the constraint conditions $g(x,y,z) = 0$ and $h(x,y,z) = 0$. Here the equations $g(x,y,z) = 0$ and $h(x,y,z) = 0$ describe two surfaces that may or may not intersect. Assume the surfaces intersect to give a space curve.

The problem is to find an extremal value of $f = f(x,y,z)$ as (x,y,z) varies along the curve of intersection of surfaces $g = 0$ and $h = 0$. At a critical point where a stationary value exists, the directional derivative of f along this curve must be zero. Here the directional derivative is given by $\frac{df}{ds} = \nabla f \cdot \hat{\mathbf{e}}_t$, where $\hat{\mathbf{e}}_t$ is a unit tangent vector to the space curve and $\nabla f = \text{grad}\, f$ denotes the gradient of f. Note that if the directional derivative is zero, then ∇f must lie in a plane normal to the curve of intersection.

Another way to view the problem, and also suggest that the concepts can be extended to higher dimensional spaces, is to introduce the notation $\bar{x} = (x_1, x_2, x_3) = (x,y,z)$ to denote a vector to a point on the curve of intersection of the two surfaces $g(x_1,x_2,x_3) = 0$ and $h(x_1,x_2,x_3) = 0$. At a stationary value of f one must have

$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \frac{\partial f}{\partial x_2}\, dx_2 + \frac{\partial f}{\partial x_3}\, dx_3 = \text{grad}\, f \cdot d\bar{x} = 0$$

This implies that grad f is normal to the curve of intersection since it is perpendicular to the tangent vector $d\bar{x}$ to the curve of intersection. At a stationary point, the normal plane containing the vector grad f also contains the vectors ∇g and ∇h since $dg = \text{grad}\, g \cdot d\bar{x} = 0$ and $dh = \text{grad}\, h \cdot d\vec{x} = 0$ at the stationary point. Hence, if these three vectors are noncollinear, then there will exist scalars λ_1 and λ_2 such that

$$\nabla f + \lambda_1 \nabla g + \lambda_2 \nabla h = 0 \tag{7.82}$$

at a stationary point. The equation (7.82) is a vector equation and is equivalent to the three scalar equations

$$\begin{aligned} \frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x} =&0 \\ \frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y} =&0 \\ \frac{\partial f}{\partial z} + \lambda_1 \frac{\partial g}{\partial z} + \lambda_2 \frac{\partial h}{\partial z} =&0. \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g}{\partial x_1} + \lambda_2 \frac{\partial h}{\partial x_1} =&0 \\ \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g}{\partial x_2} + \lambda_2 \frac{\partial h}{\partial x_2} =&0 \\ \frac{\partial f}{\partial x_3} + \lambda_1 \frac{\partial g}{\partial x_3} + \lambda_2 \frac{\partial h}{\partial x_3} =&0. \end{aligned}$$

depending upon the notation you are using. These three equations together with the constraint equations $g = 0$ and $h = 0$ gives us five equations in the five unknowns $x, y, z, \lambda_1, \lambda_2$ that must be satisfied at a stationary point.

By the Lagrangian rule one can form the function

$$F = F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1 g(x, y, z) + \lambda_2 h(x, y, z)$$

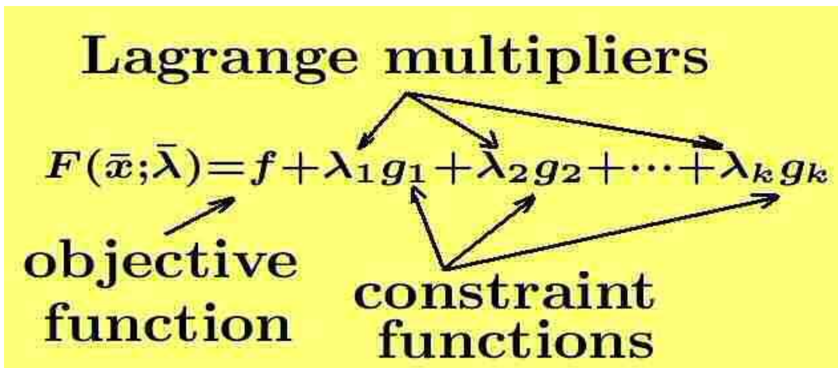
where $f(x, y, z)$ is the objective function, $g(x, y, z)$ and $h(x, y, z)$ are the constraint functions and λ_1, λ_2 are the Lagrange multipliers. Observe that F has a stationary value where

$$\begin{aligned} \frac{\partial F}{\partial x} =& \frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} + \lambda_2 \frac{\partial h}{\partial x} = 0 \\ \frac{\partial F}{\partial y} =& \frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g}{\partial y} + \lambda_2 \frac{\partial h}{\partial y} = 0 \\ \frac{\partial F}{\partial z} =& \frac{\partial f}{\partial z} + \lambda_1 \frac{\partial g}{\partial z} + \lambda_2 \frac{\partial h}{\partial z} = 0 \\ \frac{\partial F}{\partial \lambda_1} =& g(x, y, z) = 0 \\ \frac{\partial F}{\partial \lambda_2} =& h(x, y, z) = 0 \end{aligned} \tag{7.83}$$

These are the same five equations, with unknowns $x, y, z, \lambda_1, \lambda_2$, for determining the stationary points as previously noted.

Generalization of Lagrange Multipliers

In general, to find an extremal value associated with a n-dimensional function given by $f = f(\bar{x}) = f(x_1, x_2, \dots, x_n)$ subject to k constraint conditions that can be written in the form $g_i(\bar{x}) = g_i(x_1, x_2, \dots, x_n) = 0$, for $i = 1, 2, \dots, k$, where k is less than n. It is required that the gradient vectors $\nabla g_1, \nabla g_2, \dots, \nabla g_k$ be linearly independent vectors, then one can employ the method of Lagrange multipliers as follows. The Lagrangian rule requires that the function $F = f + \sum_{i=1}^{k} \lambda_i g_i$ can be written in the expanded form



(7.84)

which contains the objective function f, summed with each of the constraint functions g_i, multiplied by a Lagrange multiplier λ_i, for the index i having the values $i = 1, \dots, k$. Here the function F and consequently the function f has stationary values at those points where the following equations are satisfied

$$\begin{aligned} \frac{\partial F}{\partial x_i} &= 0, \quad \text{for } i = 1, \dots, n \\ \frac{\partial F}{\partial \lambda_j} &= 0, \quad \text{for } j = 1, \dots, k \end{aligned} \tag{7.85}$$

The equations (7.85) represent a system of $(n+k)$ equations in the $(n+k)$ unknowns $x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_k$ for determining the stationary points. In general, the stationary points will be found in terms of the λ_i values. The vector $(\bar{x}_0, \bar{\lambda}_0)$ where $\bar{x}_0$ and $\bar{\lambda}_0$ are solutions of the system of equations (7.85) can be thought of as critical points associated with the Lagrangian function $F(\bar{x}, \bar{\lambda})$ given by equation (7.84). The resulting stationary points must then be tested to determine whether they correspond to a relative maximum value, minimum value or saddle point. One can form the Hessian[7] matrix associated with the function $F(\bar{x}; \bar{\lambda})$ and analyze this matrix at the

[7] See page 318 for definition of Hessian matrix.

critical points. Whenever the determinant of the Hessian matrix is zero at a critical point, then the critical point $(\bar{x}_0, \bar{\lambda}_0)$ is said to be degenerate and one must seek an alternative method to test for an extremum.

Vector Field and Field Lines

A vector field is a vector-valued function representing a mapping from R^n to a vector $\vec{V}$. Any vector which varies as a function of position in space is said to represent a vector field. The vector field $\vec{V} = \vec{V}(x, y, z)$ is a one-to-one correspondence between points in space (x, y, z) and a vector quantity $\vec{V}$. This correspondence is assumed to be continuous and differentiable within some region R. Examples of vector fields are velocity, electric force, mechanical force, etc. Vector fields can be represented graphically by plotting vectors at selected points within a region. These kind of graphical representations are called vector field plots. Alternative to plotting many vectors at selected points to visualize a vector field, it is sometimes easier to use the concept of field lines associated with a vector field. A field line is a curve where at each point (x, y, z) of the curve, the tangent vector to the curve has the same direction as the vector field at that point. If $\vec{r} = x(t)\,\hat{\mathbf{e}}_1 + y(t)\,\hat{\mathbf{e}}_2 + z(t)\,\hat{\mathbf{e}}_3$ is the position vector describing a field line, then by definition of a field line the tangent vector $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3$ evaluated at a point t_0 must be in the same direction as the vector $\vec{V}_0 = \vec{V}(x(t_0), y(t_0), z(t_0))$. If this relation is true for all values of the parameter t, then one can state that the vectors $\frac{d\vec{r}}{dt}$ and $\vec{V}$ must be colinear at each point on the curve representing the field line. This requires

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 = k\left[V_1(x, y, z)\,\hat{\mathbf{e}}_1 + V_2(x, y, z)\,\hat{\mathbf{e}}_2 + V_3(x, y, z)\,\hat{\mathbf{e}}_3\right]$$

where k is some proportionality constant. Equating like components in the above equation one obtains the system of differential equations

$$\frac{dx}{dt} = kV_1(x, y, z), \qquad \frac{dy}{dt} = kV_2(x, y, z), \qquad \frac{dz}{dt} = kV_3(x, y, z)$$

which must be solved to obtain the equations of the field lines.

Surface Integrals

In this section various types of surface integrals are introduced. In particular, surface integrals of the form

$$\iint\limits_R f(x, y, z)\,d\vec{S}, \qquad \iint\limits_R \vec{F}\cdot d\vec{S}, \qquad \iiint\limits_R \vec{F}\times d\vec{S},$$

are defined and illustrated. Throughout the following discussion all surfaces are considered to be oriented (two-sided) surfaces.

Consider a surface in space with an element of surface area dS constructed at some general point on the surface as is illustrated in figure 7-16.

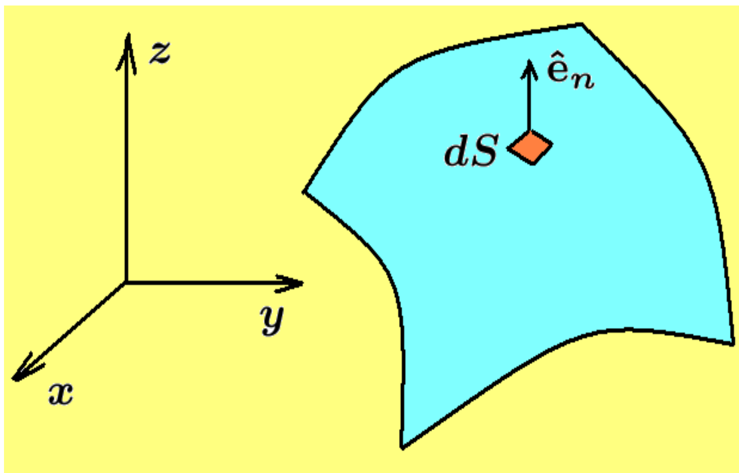


Figure 7-16. Element of surface area.

In the representation of various vector integrals, it is convenient to **define vector elements of surface area $d\vec{S}$ whose magnitude is dS and whose direction is the same as the unit outward normal $\hat{\mathbf{e}}_n$ to the surface.** Define this vector element of surface area as $d\vec{S} = \hat{\mathbf{e}}_n\, dS$ which can be considered as the limit associated with the area $\vec{\Delta}S = \hat{\mathbf{e}}_n\, \Delta S$.

Normal to a Surface

If $\hat{\mathbf{e}}_n$ is a normal to a smooth surface, then $-\hat{\mathbf{e}}_n$ is also normal to the surface. That is, all smooth orientated surfaces possess two normals. If the surface is a closed surface, there is an inside surface and an outside surface. The outside surface is called the positive side of the surface. The unit normal to the positive side of a surface is called the positive normal or outward normal. If the surface is not closed, then one can arbitrarily select one side of the surface and call it the positive side, therefore, the normal drawn to this positive side is also called the outward normal.

If the surface is expressed in an implicit form $F(x,y,z)=0$, then a unit normal to the surface can be obtained from the relation:

$$\hat{\mathbf{e}}_n = \frac{\text{grad } F}{|\text{grad } F|}.$$

If the surface is expressed in the explicit form $z=z(x,y)$, then a unit normal to the surface can be found from the relation

$$\hat{\mathbf{e}}_n = \frac{\text{grad } [z(x,y)-z]}{|\text{ grad } [z(x,y)-z]|} = \frac{\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 - \,\hat{\mathbf{e}}_3}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} \tag{7.86}$$

Surfaces can also be expressed in the parametric form

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v),$$

where u and v are parameters. The functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ must be such that one and only one point (u,v) maps to any given point on the surface. These functions are also assumed to be continuous and differentiable. In this case, the position vector to a point on the surface can be represented as

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3.$$

The curves

$$\vec{r}(u,v)\Big|_{v=\text{Constant}} \qquad \text{and} \qquad \vec{r}(u,v)\Big|_{u=\text{Constant}}$$

sweep out coordinate curves on the surface and the vectors

$$\frac{\partial \vec{r}}{\partial u}, \qquad \frac{\partial \vec{r}}{\partial v}$$

are tangent vectors to these coordinate curves. A unit normal to the surface at a point P on the surface can then be calculated from the cross product of the tangent vectors tangent vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ evaluated at the point P. One can calculate the unit normal

$$\hat{\mathbf{e}}_n = \frac{\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|}.$$

It should be noted that if the cross product

$$\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u} \neq \vec{0},$$

the surface is called a smooth surface. If at a point with surface coordinates (u_0, v_0) this cross product equals the zero vector, the point on the surface is called a singular point of the surface.

Example 7-25. The parametric equations

$$x = a\cos u \sin v, \qquad y = a\sin u \sin v, \qquad z = a\cos v$$

with $0 \le u \le 2\pi$ and $0 \le v \le \pi$, represent the surface of a sphere of radius a. These parametric equations were obtained from the geometry of figure 7-18.

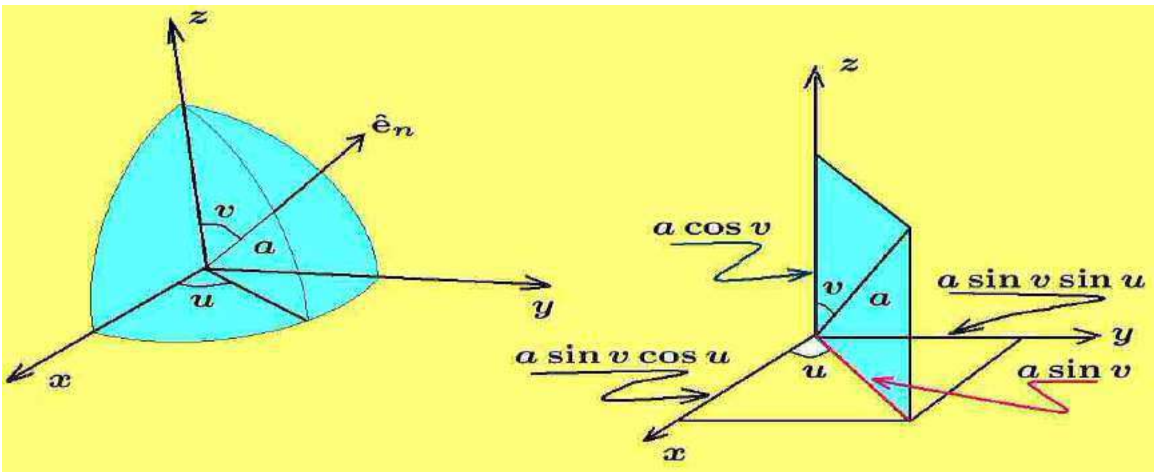


Figure 7-18. Surface of a sphere of radius a.

The position vector of a point on the surface of this sphere can be represented by the vector

$$\vec{r}(u,v) = a\cos u \sin v\,\hat{\mathbf{e}}_1 + a\sin u \sin v\,\hat{\mathbf{e}}_2 + a\cos v\,\hat{\mathbf{e}}_3.$$

For u_0 and v_0 constants, the curves $\vec{r}(u_0, v)$, $\quad 0 \le v \le \pi$, are meridian lines on the sphere while the curves $\vec{r}(u, v_0)$, $\quad 0 \le u \le 2\pi$, are circles of constant latitude. The tangent vectors to these curves are found by taking the derivatives

$$\frac{\partial \vec{r}}{\partial u} = -a\sin u \sin v\,\hat{\mathbf{e}}_1 + a\cos u \sin v\,\hat{\mathbf{e}}_2$$
$$\frac{\partial \vec{r}}{\partial v} = a\cos u \cos v\,\hat{\mathbf{e}}_1 + a\sin u \cos v\,\hat{\mathbf{e}}_2 - a\sin v\,\hat{\mathbf{e}}_3.$$

From these tangent vectors, a normal vector to the surface is constructed by taking a cross product and

$$\vec{N} = \frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}.$$

It can be verified that a unit normal to this surface is

$$\hat{\mathbf{e}}_n = \frac{\vec{N}}{|\vec{N}|} = \frac{1}{a}\vec{r}.$$

That is, the unit outer normal to a point P on the surface of the sphere has the same direction as the position vector $\vec{r}$ to the point P.

■

Example 7-26. If the surface is described in the explicit form $z = z(x,y)$ the position vector to a point on the surface can be represented

$$\vec{r} = \vec{r}(x,y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x,y)\,\hat{\mathbf{e}}_3$$

This vector has the partial derivatives

$$\frac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1 + \frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_3 \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\mathbf{e}}_2 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_3$$

so that a normal to the surface can be calculated from the cross product

$$\vec{N} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$

A unit normal to the surface is

$$\hat{\mathbf{e}}_n = \frac{\vec{N}}{|\vec{N}|} = \frac{-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} = n_x\,\hat{\mathbf{e}}_1 + n_y\,\hat{\mathbf{e}}_2 + n_z\,\hat{\mathbf{e}}_3$$

where n_x, n_y, n_z are the direction cosines of the unit normal. Note also that the vector

$$\hat{\mathbf{e}}_{n^*} = \frac{\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

is also normal to the surface.

Example 7-27. Consider a surface described by the implicit form $F(x,y,z)=0$. Recall that equations of this form define z as a function of x and y and the derivatives of z with respect to x and y are given by

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Substituting these derivatives into the equation (7.86) and simplifying one finds that the direction cosines of the unit normal to the surface are given by

$$n_x = \frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}, \quad n_y = \frac{\frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}, \quad n_z = \frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

Tangent Plane to Surface

Consider a smooth surface defined by the equation

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3$$

In order to construct a tangent plane to a regular point $\vec{r} = \vec{r}(u_0,v_0)$ where the surface coordinates have the values (u_0,v_0), one must first construct the normal to the surface at this point. One such normal is

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = N_1\,\hat{\mathbf{e}}_1 + N_2\,\hat{\mathbf{e}}_2 + N_3\,\hat{\mathbf{e}}_3$$

The point on the surface is

$$x_0 = x(u_0,v_0), \qquad y_0 = y(u_0,v_0), \qquad z_0 = z(u_0,v_0)$$

which can be described by the position vector $\vec{r}_0 = \vec{r}(u_0,v_0)$. If $\vec{r}$ represents the variable point

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

which varies over the plane through the point (x_0,y_0,z_0), then the vector $\vec{r}-\vec{r}_0$ must lie in the tangent plane and consequently is perpendicular to the normal vector $\vec{N}$. One can then write

$$(\vec{r}-\vec{r}_0)\cdot\vec{N} = 0 \tag{7.87}$$

as the equation of the plane through the point (x_0,y_0,z_0) which is perpendicular to $\vec{N}$ and consequently tangent to the surface. In equation (7.87) one can substitute any of the normal vectors calculated in the previous examples.

The equation of the line through the point (x_0, y_0, z_0) which is perpendicular to the tangent plane is given by

$$\vec{r} = \vec{r}_0 + \lambda \vec{N} \tag{7.88}$$

where λ is a scalar. The equation of the line can also be expressed by the parametric equations

$$x = x_0 + \lambda N_1, \qquad y = y_0 + \lambda N_2, \qquad z = z_0 + \lambda N_3 \tag{7.89}$$

where again, the normal vector $\vec{N}$ can be replaced by any of the normals previously calculated.

■

Element of Surface Area

Consider the case where the surface is given in the explicit form $z = z(x, y)$. In this case, the position vector of a point on the surface is given by

$$\vec{r} = \vec{r}(x, y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x, y)\,\hat{\mathbf{e}}_3. \tag{7.90}$$

The curves

$$\vec{r}(x, y)\Big|_{y=\text{ Constant}} \qquad \text{and} \qquad \vec{r}(x, y)\Big|_{x=\text{Constant}}$$

are coordinate curves lying in the surface which intersect at a common point (x, y, z). The vectors

$$\frac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1 + \frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_3 \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\mathbf{e}}_2 + \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_3$$

are tangent to these coordinate curves, and consequently the differential of the position vector

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x}\,dx + \frac{\partial \vec{r}}{\partial y}\,dy$$

lies in the tangent plane to the surface at the common point of intersection of the coordinate curves. This differential is illustrated in figure 7-19.

Consider an element of area $\Delta A = \Delta x\,\Delta y$ in the xy plane of figure 7-19. When this element of area is projected onto the surface $z = z(x, y)$, it intersects the surface in an element of surface area ΔS. When projected onto the tangent plane to the surface it intersects the tangent plane in an element of surface area ΔR. These projections are illustrated in figure 7-19(c).

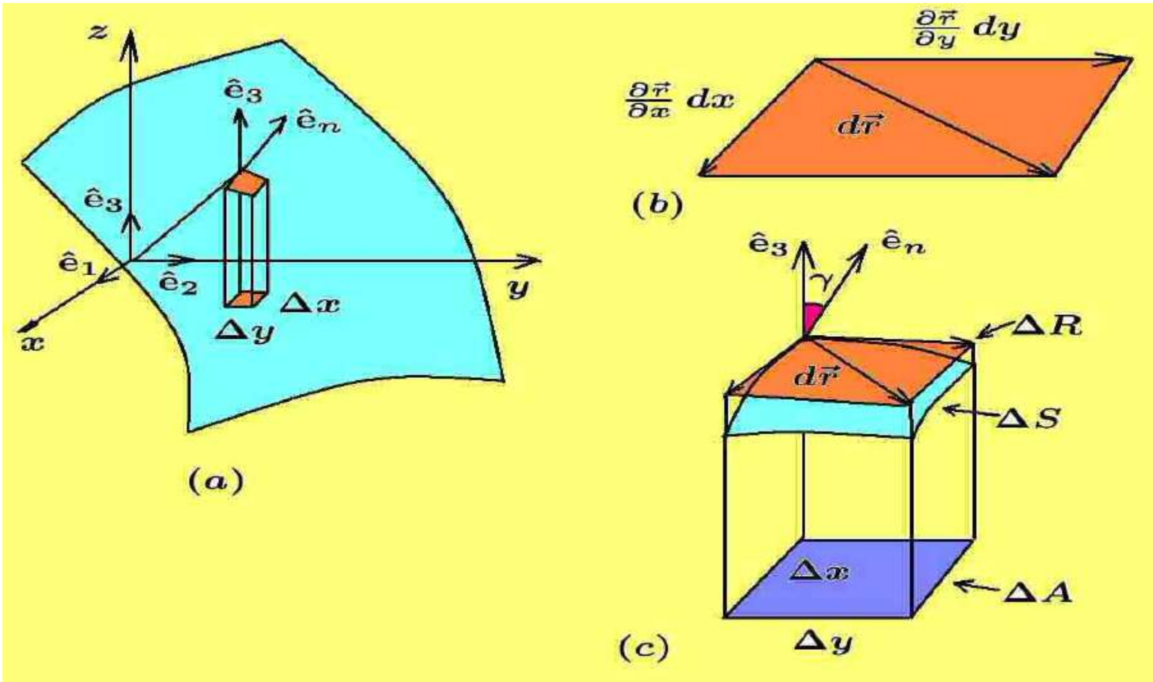


Figure 7-19. Element of surface area and element parallelogram.

In the limit as Δx and Δy tend toward zero, ΔR approaches ΔS and one can define $d\vec{R} = d\vec{S}$, where the element of area dR lies in the tangent plane to the surface at the point (x, y, z). **In the limit as Δx and Δy approach zero, the element of area is defined as the area of the elemental parallelogram defined by the vector $d\vec{r}$ and illustrated in figure 7-19(b). The area of this elemental parallelogram can be calculated from the cross product relation**

$$\left(\frac{\partial \vec{r}}{\partial x}\, dx\right) \times \left(\frac{\partial \vec{r}}{\partial y}\, dy\right) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ dx & 0 & \frac{\partial z}{\partial x}\, dx \\ 0 & dy & \frac{\partial z}{\partial y}\, dy \end{vmatrix} = \left(-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right) dx\, dy. \tag{7.91}$$

The area of the elemental parallelogram is the magnitude of the above cross product, and can be expressed

$$dS = dR = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy. \tag{7.92}$$

Given a surface in the explicit form $z = z(x, y)$, define the outward normal to the surface $\phi(x, y, z) = z - z(x, y) = 0$ by

$$\hat{\mathbf{e}}_n = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}}. \tag{7.93}$$

From equations (7.92) and (7.93), obtain the vector element of surface area

$$d\vec{S} = \hat{\mathbf{e}}_n\, dS = \left(-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\right) dx\, dy.$$

Taking the dot product of both sides of the above equation with the unit vector $\hat{\mathbf{e}}_3$ gives

$$|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|\, dS = dx\, dy \qquad \text{or} \qquad dS = \frac{dx\, dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = \frac{dx\, dy}{\cos\gamma} \tag{7.94}$$

with an absolute value placed upon the dot product to ensure that the surface area is positive (i.e., recall that there are two normals to the surface which differ in sign). In equation (7.94), the element of surface area has been expressed in terms of its projection onto the xy plane. The angle $\gamma = \gamma(x, y)$ is the angle between the outward normal to the surface and the unit vector $\hat{\mathbf{e}}_3$. This representation of the element of surface area is valid provided that $\cos\gamma \neq 0$; That is, it is assumed that the surface is such that the normal to the surface is nowhere parallel to the xy plane.

We have previously shown that for surfaces which have a normal parallel to the xy plane, the element of surface area can be projected onto either of the planes $x = 0$ or $y = 0$. If the surface element is projected onto the plane $x = 0$, then the element of surface area takes the form

$$dS = \frac{dy\, dz}{|\,\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n|} \tag{7.95}$$

and if projected onto the plane $y = 0$ it has the form

$$dS = \frac{dx\, dz}{|\,\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n|}. \tag{7.96}$$

If the element of surface area dS is projected onto the $z = 0$ plane, the total surface area is then

$$S = \iint_R dS = \iint_R \frac{dx\, dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|},$$

where the integration extends over the region R, where the surface is projected onto the $z = 0$ plane. Similar integrals result for the other representations of surface area.

Example 7-28. Find the surface area of that part of the plane

$$\phi(x,y,z) = 2x + 2y + z - 12 = 0$$

which lies in the first octant.

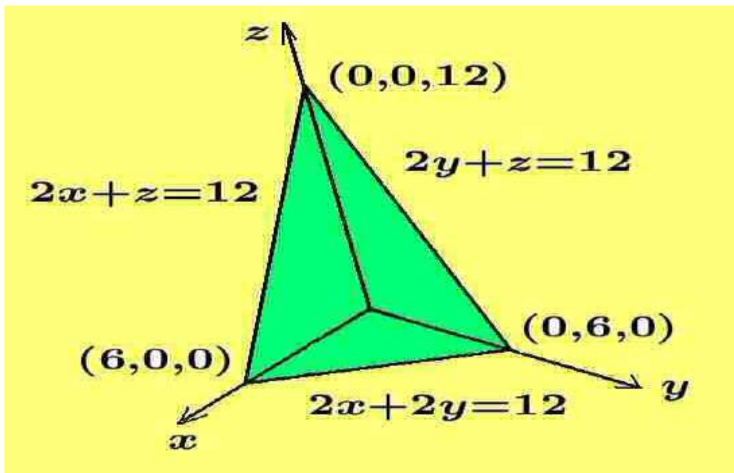


Figure 7-20. Surface area of plane in first octant.

Solution The given plane is sketched as in figure 7-20.

The unit normal to the plane is

$$\hat{\mathbf{e}}_n = \frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3.$$

The projection of the surface element dS onto the $z = 0$ plane produces

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = 3dx\,dy$$

By summing $dx\,dy$ over the region where $x > 0,\ y > 0$, and $x + y \le 6$, one obtains the limits of integration for the surface area. The surface area is determined from the integral

$$S = \int_{x=0}^{x=6}\int_{y=0}^{y=6-x} 3dx\,dy = \int_0^6 3(6-x)\,dx = -\frac{3}{2}(6-x)^2\Big|_0^6 = 54.$$

If the element of surface area is projected onto the plane $y = 0$, there results

$$dS = \frac{dx\,dz}{|\,\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n|} = \frac{3}{2}dx\,dz$$

and the limits of summation are determined as dx and dz range over the region $x > 0,\ z > 0$, and $2x + z \le 12$. This produces the surface integral

$$S = \int_{x=0}^{x=6} \int_{z=0}^{z=12-2x} \frac{3}{2}\, dx\, dz = \int_0^6 \frac{3}{2}(2)(6-x)\, dx = 54.$$

Similarly, if the element dS is projected onto the plane $x = 0$, it can be verified that

$$dS = \frac{dy\, dz}{|\,\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n|} = \frac{3}{2} dydz$$

and the surface area is given by

$$S = \int_{y=0}^{y=6} \int_{z=0}^{12-2y} \frac{3}{2} dz\, dy = 54.$$

■

Element of Volume

In a general (u,v,w) curvilinear coordinate system the (x,y,z) rectangular coordinates of a point are given as functions of (u,v,w) and written

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w)$$

so that the position vector to a point P can be written

$$\vec{r} = \vec{r}(u,v,w) = x(u,v,w)\,\hat{\mathbf{e}}_1 + y(u,v,w)\,\hat{\mathbf{e}}_2 + z(u,v,w)\,\hat{\mathbf{e}}_3$$

The vector $\frac{\partial \vec{r}}{\partial u}$ is tangent to the coordinate curve $\vec{r} = \vec{r}(u, v_0, w_0)$, the vector $\frac{\partial \vec{r}}{\partial v}$ is tangent to the coordinate curve $\vec{r} = \vec{r}(u_0, v, w_0)$ and the vector $\frac{\partial \vec{r}}{\partial w}$ is tangent to the coordinate curve $\vec{r} = \vec{r}(u_0, v_0, w)$. Unit vectors to the coordinates curves are

$$\hat{\mathbf{e}}_u = \frac{\frac{\partial \vec{r}}{\partial u}}{|\frac{\partial \vec{r}}{\partial u}|}, \qquad \hat{\mathbf{e}}_v = \frac{\frac{\partial \vec{r}}{\partial v}}{|\frac{\partial \vec{r}}{\partial v}|}, \qquad \hat{\mathbf{e}}_w = \frac{\frac{\partial \vec{r}}{\partial w}}{|\frac{\partial \vec{r}}{\partial w}}$$

The magnitudes h_u, h_v, h_w defined by

$$h_u = |\frac{\partial \vec{r}}{\partial u}|, \qquad h_v = |\frac{\partial \vec{r}}{\partial v}|, \qquad h_w = |\frac{\partial \vec{r}}{\partial w}|$$

are called scaled factors. The vector change

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv + \frac{\partial \vec{r}}{\partial w}\, dw = h_u\, du\ \hat{\mathbf{e}}_1 + h_v\, dv\ \hat{\mathbf{e}}_2 + h_w\, dw\ \hat{\mathbf{e}}_3$$

can be thought of as defining **an element of volume** dV in the shape of a parallelepiped with vector sides $\vec{A} = h_u\,du\,\hat{\mathbf{e}}_1$, $\vec{B} = h_v\,dv\,\hat{\mathbf{e}}_2$ and $\vec{C} = h_w\,dw\,\hat{\mathbf{e}}_3$. The volume of this parallelepiped is given by

$$dV = |\vec{A}\cdot(\vec{B}\times\vec{C})| = |(h_u\,du\,\hat{\mathbf{e}}_1)\cdot((h_v\,dv\,\hat{\mathbf{e}}_2)\times(h_w\,dw\,\hat{\mathbf{e}}_3)| = h_u h_v h_w\,du\,dv\,dw$$

In rectangular coordinates (x, y, z) one finds $h_x = 1$, $h_y = 1$, and $h_z = 1$ and the element of volume is $dV = dxdydz$.

In cylindrical coordinates (r, θ, z) where $x = r\cos\theta, y = r\sin\theta, z = z$ one finds $h_r = 1$, $h_\theta = r$ and $h_z = 1$ and the element of volume is $dV = r\,dr\,d\theta\,dz$

In spherical coordinates (ρ, θ, ϕ) where $x = \rho\sin\theta\cos\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\theta$, one finds $h_\rho = 1$, $h_\theta = \rho$, $h_\phi = \rho\sin\theta$ and the element of volume is $dV = \rho^2\sin\theta\,d\rho\,d\theta\,d\phi$

These elements of volume must be summed over appropriate regions of space in order to calculate volume integrals of the form

$$\iiint f(x,y,z)\,dxdydz, \qquad \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz, \qquad \iiint f(\rho,\theta,\phi)\,\rho^2\sin\theta\,d\rho\,d\theta\,d\phi$$

Surface Placed in a Scalar Field

If a surface is placed in a region of a scalar field $f(x, y, z)$, one can divide the surface into n small areas

$$\Delta S_1,\ \Delta S_2, \ldots, \Delta S_n.$$

For n large, define $f_i = f_i(x_i, y_i, z_i)$ as the value of the scalar field over the surface element $\vec{\Delta}S_i$ as i ranges from 1 to n. The summation of the elements $f_i\Delta\vec{S}_i$ over all i as n increases without bound defines the surface integral

$$\iint\limits_R f(x,y,z)\,\hat{\mathbf{e}}_n dS = \iint\limits_R f(x,y,z)d\vec{S} = \lim_{n\to\infty}\sum_{i=1}^n f_i(x_i,y_i,z_i)\Delta\vec{S}_i, \tag{7.97}$$

where the integration is determined by the way one represents the element of surface area $d\vec{S}$. The integral can be represented in different forms depending upon how the given surface is specified.

Surface Placed in a Vector Field

For a surface S in a region of a vector field $\vec{F} = \vec{F}(x, y, z)$ the integral

$$\iint\limits_R \vec{F}\cdot d\vec{S} = \iint\limits_R \vec{F}\cdot\,\hat{\mathbf{e}}_n dS \tag{7.98}$$

represents a scalar which is the sum of the projections of $\vec{F}$ onto the normals to the surface elements. If the surface is divided into n small surface elements ΔS_i, where $i = 1, \ldots, n$. Let $\vec{F}_i = \vec{F}(x_i, y_i, z_i)$ represent the value of the vector field over the ith surface element. The summation of the elements

$$\vec{F}_i \cdot \Delta\vec{S}_i = \vec{F}_i \cdot \hat{\mathbf{e}}_{n_i} \Delta S_i$$

over all surface elements represents the sum of the normal components of $\vec{F}_i$ multiplied by ΔS_i as i varies from 1 to n. A summation gives the surface integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}_i \cdot \Delta\vec{S}_i = \iint\limits_R \vec{F} \cdot d\vec{S}. \tag{7.99}$$

Again, the form of this integral depends upon how the given surface is represented. Integrals of this type arise when calculating the volume rate of change associated with velocity fields. It is called a flux integral and represents the amount of a substance moving across an imaginary surface placed within the vector field.

The vector integral

$$\iint\limits_R \vec{F} \times d\vec{S}$$

represents a vector which is obtained by summing the vector elements $\vec{F}_i \times \Delta\vec{S}_i$ over the given surface. The fundamental theorem of integral calculus enables such sums to be expressed as integrals and one can write

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}_i \times \Delta\vec{S}_i = \iint\limits_R \vec{F} \times d\vec{S}. \tag{7.100}$$

Integrals of this type arise as special cases of some integral theorems that are developed in the next chapter.

Each of the above surface integrals can be represented in different forms depending upon how the element of surface area is represented. The form in which the given surface is represented usually dictates the method used to calculate the surface area element. Sometimes the representation of a surface in a different form is helpful in determining the limits of integration to certain surface integrals.

Example 7-29. Evaluate the surface integral $\iint\limits_R \vec{F}\cdot d\vec{S}$, where S is the surface of the cube bounded by the planes

$$x=0, \quad x=1, \quad y=0, \quad y=1, \quad z=0, \quad z=1$$

and $\vec{F}$ is the vector field $\vec{F}=(x^2+z)\,\hat{\mathbf{e}}_1+(xy-z)\,\hat{\mathbf{e}}_2+(x+y)\,\hat{\mathbf{e}}_3$

Solution The given surface is illustrated in figure 7-21.

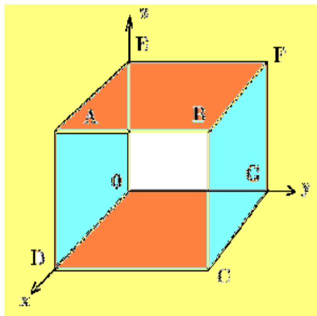


Figure 7-21. Surface of a cube.

The given surface is piecewise continuous and thus the surface integral can be broken up and written as the sum of the surface integrals over each face of the cube. The following calculations illustrates the mechanics involved in evaluating this type of surface integral.

(i) On face ABCD the unit normal to the surface is the vector $\vec{n}=\hat{\mathbf{e}}_1$ and x has the value 1 everywhere so that

$$\iint\limits_R \vec{F}\cdot d\vec{S}=\int_0^1\int_0^1 \vec{F}\cdot\vec{n}\,dS=\int_0^1\int_0^1(1+z)\,dydz=\int_0^1(1+z)\,dz=\frac{3}{2}$$

(ii) On face EFG0 the unit normal to the surface is the vector $\vec{n}=-\hat{\mathbf{e}}_1$ and x has the value 0 everywhere so that

$$\iint\limits_R \vec{F}\cdot d\vec{S}=\int_0^1\int_0^1 \vec{F}\cdot\vec{n}\,dS=\int_0^1\int_0^1 -z\,dydz=\int_0^1 -z\,dz=-\frac{1}{2}$$

(iii) On face BFGC the unit normal to the surface is the vector $\vec{n} = \hat{\mathbf{e}}_2$ and y has the value 1 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 (x - z)\, dxdz = \int_0^1 \frac{1}{2}\, dz - \int_0^1 z\, dz = 0$$

(iv) On face AE0D the unit normal to the surface is the vector $\vec{n} = -\hat{\mathbf{e}}_2$ and y has the value 0 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 z\, dzdx = \int_0^1 z\, dz = \frac{1}{2}$$

(v) On face ABFE the unit normal to the surface is the vector $\vec{n} = \hat{\mathbf{e}}_3$and z has the value 1 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 (x + y)\, dxdy = \int_0^1 \frac{1}{2}\, dy + \int_0^1 y\, dy = 1$$

(vi) On face DCG0 the unit normal to the surface is the vector $\vec{n} = -\hat{\mathbf{e}}_3$ and z has the value 0 everywhere so that

$$\iint_R \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \vec{F} \cdot \vec{n}\, dS = \int_0^1 \int_0^1 -(x + y)\, dxdy = -1$$

A summation of the surface integrals over each face gives

$$\iint_R \vec{F} \cdot d\vec{S} = \frac{3}{2} - \frac{1}{2} + 0 + \frac{1}{2} + 1 - 1 = \frac{3}{2}.$$

■

Example 7-30. Evaluate the surface integral $\iint_R f(x, y, z)\, d\vec{S}$, where S is the surface of the plane

$$G(x, y, z) = 2x + 2y + z - 1 = 0$$

which lies in the first octant and $f = f(x, y, z)$ is the scalar field given by $f = xyz$.

Solution The given surface is sketched in figure 7-22. The unit normal at any point on the surface is

$$\hat{\mathbf{e}}_n = \frac{\text{grad } G}{|\text{grad } G|} = \frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3.$$

The element of surface area dS is projected upon the xy plane giving

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = 3\,dx\,dy$$

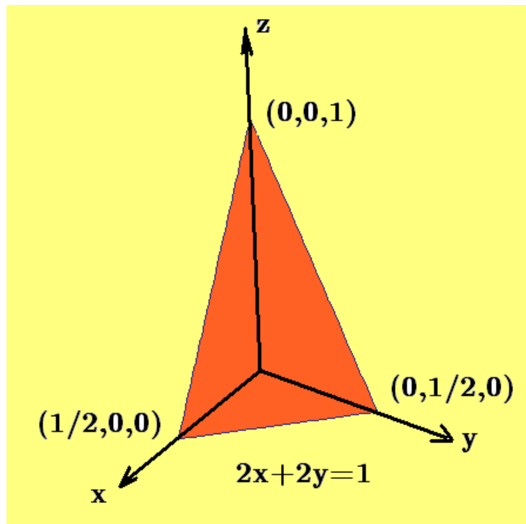


Figure 7-22. Plane $2x + 2y + z = 1$ in first octant.

On the surface $z = 1-2x-2y$, and therefore the surface integral can be represented in terms of only x and y. One finds

$$\begin{aligned}\iint_R f(x,y,z)\,d\vec{S} =& \iint_R xyz\;\hat{\mathbf{e}}_n dS \\ =& \int_{x=0}^{x=\frac{1}{2}} \int_{y=0}^{y=\frac{1}{2}-x} xy(1-2x-2y)\left[\frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3\right] 3dx\,dy \\ =& \,(2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \,\hat{\mathbf{e}}_3) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} (xy - 2x^2y - 2xy^2)\,dx\,dy\end{aligned}$$

Integrate with respect to y and show

$$\iint_R f(x,y,z)\,d\vec{S} = (2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \,\hat{\mathbf{e}}_3) \int_0^{\frac{1}{2}} \left[\frac{1}{2}x(\frac{1}{2}-x)^2 - x^2(\frac{1}{2}-x)^2 - \frac{2}{3}x(\frac{1}{2}-x)^3\right] dx$$

Now integrate with respect to x and simplify the result to obtain

$$\iint_R f(x,y,z)\,d\vec{S} = \frac{1}{1920}\,(2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + \,\hat{\mathbf{e}}_3)$$

■

Example 7-31. Evaluate the surface integral $\iint\limits_R \vec{F} \times d\vec{S}$, where S is the plane $2x + 2y + z - 1 = 0$ in the first octant and $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$.

Solution Here

$$\iint\limits_R \vec{F} \times d\vec{S} = \iint\limits_R \vec{F} \times \hat{\mathbf{e}}_n \, dS$$

and from the previous example

$$\hat{\mathbf{e}}_n = \frac{2}{3}\,\hat{\mathbf{e}}_1 + \frac{2}{3}\,\hat{\mathbf{e}}_2 + \frac{1}{3}\,\hat{\mathbf{e}}_3.$$

As in the previous example, the element of surface area is projected upon the xy plane to obtain $dS = 3dx\,dy$. Therefore,

$$\vec{F} \times \hat{\mathbf{e}}_n = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ x & y & z \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}(y - 2z)\,\hat{\mathbf{e}}_1 - \frac{1}{3}(x - 2z)\,\hat{\mathbf{e}}_2 + \frac{2}{3}(x - y)\,\hat{\mathbf{e}}_3$$

and the surface integral is

$$\iint\limits_R \vec{F} \times d\vec{S} = \hat{\mathbf{e}}_1 \iint\limits_R (y - 2z)\,dx\,dy - \hat{\mathbf{e}}_2 \iint\limits_R (x - 2z)\,dx\,dy + 2\,\hat{\mathbf{e}}_3 \iint\limits_R (x - y)\,dx\,dy.$$

Here the element of surface area has been projected upon the xy plane and all integrations are with respect to x and y. Consequently, one must express z in terms of x and y. From the equation of the plane, the value of z on the surface is given by $z = 1 - 2x - 2y$ and the surface integral becomes

$$\begin{aligned} \iint\limits_R \vec{F} \times d\vec{S} = \;& \hat{\mathbf{e}}_1 \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} (5y + 4x - 2)\,dy\,dx \\ & - \hat{\mathbf{e}}_2 \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} (5x + 4y - 2)\,dy\,dx \\ & + 2\,\hat{\mathbf{e}}_3 \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} (x - y)\,dy\,dx. \end{aligned}$$

These integrals are easily evaluated and the final result is

$$\iint\limits_R \vec{F} \times d\vec{S} = -\frac{1}{16}\,\hat{\mathbf{e}}_1 + \frac{1}{16}\,\hat{\mathbf{e}}_2 + \frac{1}{48}\,\hat{\mathbf{e}}_3.$$

■

Summary

When a surface is represented in parametric form, the position vector of a point on the surface can be represented as

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{\mathbf{e}}_1 + y(u,v)\,\hat{\mathbf{e}}_2 + z(u,v)\,\hat{\mathbf{e}}_3$$

where

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v)$$

is the parametric representation of the surface. The differential of the position vector $\vec{r} = \vec{r}(u,v)$ is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv = \vec{S}_1 + \vec{S}_2 \tag{7.101}$$

and this differential can be thought of as a vector addition of the component vectors $\vec{S}_1 = \frac{\partial \vec{r}}{\partial u}\,du$ and $\vec{S}_2 = \frac{\partial \vec{r}}{\partial v}\,dv$ which make up the sides on an elemental parallelogram having area dS lying on the surface. The vectors $\vec{S}_1$ and $\vec{S}_2$ are tangent vectors to the coordinate curves $\vec{r}(u, v_2)$ and $\vec{r}(u_1, v)$ where u_1 and v_2 are constants. A representation of coordinate curves on a surface and an element of surface area are illustrated in the figure 7-23.

The unit normal to the surface at a point having the surface coordinates (u,v), can be found from either of the cross product relations

$$\hat{\mathbf{e}}_n = \pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|} \quad \text{or} \quad \hat{\mathbf{e}}_n = \mp \frac{\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}}{\left|\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{u}}{\partial v}\right|} \tag{7.102}$$

The above results differing in sign. That is, $\hat{\mathbf{e}}_n$ and $-\hat{\mathbf{e}}_n$ are both normals to the surface and selecting one of these vectors gives an orientation to the surface.

Example 7-32. Find the unit normal to the sphere defined by

$$x = r\cos\phi\sin\theta, \;\; y = r\sin\phi\sin\theta, \;\; z = r\cos\theta$$

Solution Here

$$\begin{aligned}\frac{\partial \vec{r}}{\partial \theta} =& r\cos\phi\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\phi\cos\theta\,\hat{\mathbf{e}}_2 - r\sin\theta\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial \phi} =& -r\sin\phi\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\phi\sin\theta\,\hat{\mathbf{e}}_2\end{aligned}$$

and the cross product is

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \end{vmatrix} = \hat{\mathbf{e}}_1(r^2\sin^2\theta\cos\phi) + \hat{\mathbf{e}}_2(r^2\sin^2\theta\sin\phi) + \hat{\mathbf{e}}_3(r^2\sin\theta\cos\theta)$$

One finds $\left|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}\right| = r^2 \sin\theta$ so that a unit vector to the surface of the sphere is

$$\hat{\mathbf{e}}_n = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3$$

■

For $\vec{r}$ a position vector to a point on the surface, the element $d\vec{r}$ lies in the tangent plane to the surface at the point determined by the parameters u and v. The element of surface area dS is also determined from the differential element $d\vec{r}$ and is given by the magnitude of the cross products of the vectors $\vec{S}_1$ and $\vec{S}_2$ representing the sides of the elemental parallelogram which defines the element of surface area. This element of surface area is calculated from the cross product

$$dS = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv$$

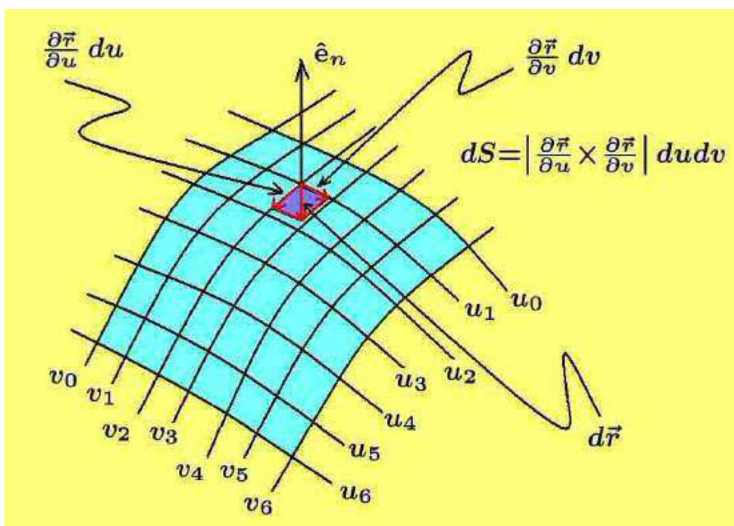


Figure 7-23.
Coordinate curves on surface and elemental parallelogram.

Using the dot product relation

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

one can readily verify that

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{EG - F^2}, \tag{7.103}$$

where

$$E = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$
$$F = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$
$$G = \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

Then the surface area can be represented in the form

$$S = \int_{R_{uv}} \sqrt{EG - F^2}\, du\, dv, \tag{7.104}$$

where the integration is over those parameter values u and v which define the surface.

The various surface integrals can also be represented in terms of the parameters u and v. These integrals have the forms

$$\iint_R f(x,y,z)\, d\vec{S} = \iint_{R_{uv}} f(x(u,v), y(u,v), z(u,v)) \sqrt{EG - F^2}\, \hat{\mathbf{e}}_n\, du\, dv$$

and

$$\iint_R \vec{F}(x,y,z) \cdot d\vec{S} = \iint_{R_{uv}} \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \hat{\mathbf{e}}_n \sqrt{EG - F^2}\, du\, dv. \tag{7.105}$$

Example 7-33. A cylinder of radius a and height h has the parametric representation $x = x(\theta, z) = a\cos\theta$, $y = y(\theta, z) = a\sin\theta$, $z = z(\theta, z) = z$, where the parameters θ and z, are illustrated in figure 7-24, and satisfy $0 \le \theta \le 2\pi$ and $0 \le z \le h$.

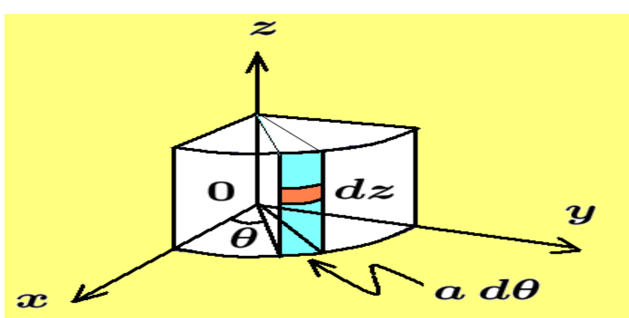


Figure 7-24.
Surface area in cylindrical coordinates.

A point on the surface of the cylinder can be represented by the position vector

$$\vec{r} = \vec{r}(\theta, z) = a\cos\theta\, \hat{\mathbf{e}}_1 + a\sin\theta\, \hat{\mathbf{e}}_2 + z\, \hat{\mathbf{e}}_3.$$

The coordinate curves are

$$\text{The straight-lines,} \quad \vec{r}(\theta_0, z),\ 0 \le z \le h$$
$$\text{and the circles,} \quad \vec{r}(\theta, z_0),\ 0 \le \theta \le 2\pi,$$

where θ_0 and z_0 are constants. The tangent vectors to the coordinate curves are given by

$$\frac{\partial \vec{r}}{\partial \theta} = -a\sin\theta\,\hat{\mathbf{e}}_1 + a\cos\theta\,\hat{\mathbf{e}}_2 \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3$$

Consequently, we have $E = a^2,\ F = 0$, and $G = 1$. The element of surface area is then $dS = \sqrt{EG - F^2}\,d\theta\,dz = a\,d\theta\,dz$. The surface area of the cylinder of height h is therefore

$$S = \int_0^h \int_0^{2\pi} a\,d\theta\,dz = 2\pi a h.$$

■

Volume Integrals

The summation of scalar and vector fields over a region of space can be expressed by volume integrals having the form

$$\iiint\limits_V f(x,y,z)\,dV \qquad \text{and} \qquad \iiint\limits_V \vec{F}(x,y,z)\,dV,$$

where $dV = dx\,dy\,dz$ is an element of volume and V is the region over which the integrations are to extend.

The integral of the scalar field is an ordinary triple integral. The triple integral of the vector function $\vec{F} = \vec{F}(x,y,z)$ can be expressed as

$$\iiint\limits_V \vec{F}\,dV = \hat{\mathbf{e}}_1 \iiint\limits_V F_1(x,y,z)\,dV + \hat{\mathbf{e}}_2 \iiint\limits_V F_2(x,y,z)\,dV + \hat{\mathbf{e}}_3 \iiint\limits_V F_3(x,y,z)\,dV, \tag{7.106}$$

where each component is a scalar triple integral.

Whenever appropriate, the above integrals are sometimes expressed

(i) in cylindrical coordinates (r,θ,z), where $x = r\cos\theta, y = r\sin\theta, z = z$ and the element of volume is represented $dV = r\,dr\,d\theta\,dz$

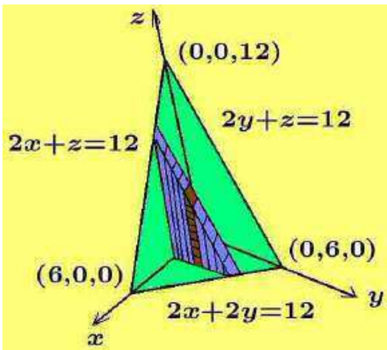
(ii) in spherical coordinates (ρ,θ,ϕ) where $x = \rho\sin\theta\cos\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\theta$ and the element of volume is $dV = \rho^2\sin\theta\,d\rho\,d\phi\,d\theta$.

(iii) in curvilinear coordinates (u,v,w) where $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$ and the element of volume is given by[8] $dV = \left|\frac{\partial \vec{r}}{\partial u}\cdot\left(\frac{\partial \vec{r}}{\partial v}\times\frac{\partial \vec{r}}{\partial w}\right)\right|\,du\,dv\,dw$ where $\vec{r} = x(u,v,w)\,\hat{\mathbf{e}}_1 + y(u,v,w)\,\hat{\mathbf{e}}_2 + z(u,v,w)\,\hat{\mathbf{e}}_3$

[8] See pages 143 and 156 for details.

Example 7-34. Evaluate the integral $\iiint_V f(x,y,z)\,dV$, where $f(x,y,z) = 6(x+y)$, $dV = dxdydz$ is an element of volume and V represents the volume enclosed by the planes $2x+2y+z=12$, $x=0$, $y=0$, and $z=0$. The integration is to be performed over this volume.

Solution



The figure on the left is a copy of the figure 7-20 with summations of the element dV illustrated. From this figure the limits of integration can be determined. The volume element is $dV = dxdydz$ is placed at the general point (x,y,z) within the volume. This volume element can be visualized as a cube inside the volume. Summation of these cubic elements aids in determining the limits of integration for the integral to be calculated. If this cube is summed in the z-direction, a parallelepiped is produced. This parallelepiped has lower limit $z=0$ and upper limit $z=12-2x-2y$. If the parallelepiped is summed in the y-direction, then a triangular slab is formed with lower limit $y=0$ and upper limit $y=6-x$. Summing the triangular slabs in the x-direction from $x=0$ to $x=6$ gives the limits of integration in the x-direction. At each stage of the summation process, the volume element is weighted by the scalar function $f(x,y,z)$ giving the integral $\iiint_V f(x,y,z)\,dV$. From all this summation one can verify the above integral can be expressed.

$$\begin{aligned}
\iiint_V f(x,y,z)\,dV &= \int_{x=0}^{x=6}\int_{y=0}^{y=6-x}\int_{z=0}^{z=12-2x-2y} 6(x+y)\,dz\,dy\,dx\\
&= \int_0^6 \left[\int_0^{6-x} 6(x+y)(12-2x-2y)\,dy\right] dx\\
&= \int_0^6 (432-36x^2+4x^3)\,dx = 1296
\end{aligned}$$

This integral is calculated by first integrating in the z-direction holding the other variables constant. This is followed by an integration in the y-direction holding x constant. The last integration is then in the x-direction.

■

Example 7-35. Evaluate the integral $\iiint_V \vec{F}(x,y,z)\,dV,$ where

$$\vec{F} = x\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$$

and $dV = dxdydz$ is a volume element. The limits of integration are determined from the volume bounded by the surfaces $y = x^2$, $y = 4$, $z = 0$, and $z = 4$.

Solution From figure 7-25 the limits of integration can be determined by sketching an element of volume $dV = dx\,dy,\,dz$ and then summing these elements in the $x-$direction from $x = 0$ to $x = \sqrt{y}$ to form a parallelepiped. Next sum the parallelepiped in the $z-$direction from $z = 0$ to $z = 4$ to form a slab. Finally, the slab can be summed in $y-$direction from $y = 0$ to $y = 4$ to fill up the volume.

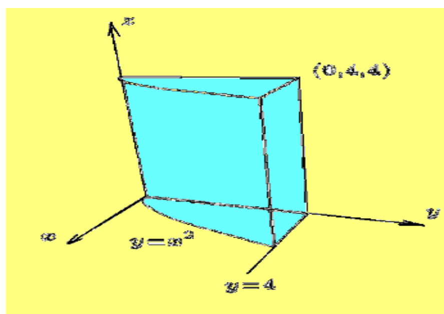


Figure 7-25.
Volume bounded by $y = x^2$, and the planes $y = 4$, $z = 0$ and $z = 4$

One then has

$$\iiint_V \vec{F}\cdot dV = \int_{y=0}^{y=4}\int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{y}} (x\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)\,dx\,dz\,dy$$
$$= \int_0^4\int_0^4\int_0^{\sqrt{y}} [x\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3]\,dxdzdy$$

Perform the integrations over each vector component and show that

$$\iint_V \vec{F}\cdot dV = 16\,\hat{\mathbf{e}}_1 + \frac{128}{3}\,\hat{\mathbf{e}}_2 + \frac{64}{3}\,\hat{\mathbf{e}}_3$$

■

Volume Elements Revisited

Consider **the volume element** $dV = dxdydz$ from cartesian coordinates and introduce a change of variables

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

from an x, y, z rectangular coordinate system to a u, v, w curvilinear coordinate system. One finds the vector

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\, \hat{\mathbf{e}}_1 + y(u, v, w)\, \hat{\mathbf{e}}_2 + z(u, v, w)\, \hat{\mathbf{e}}_3$$

is the position vector of a general point within a region determined by the restrictions placed upon the u, v, w variables. The surfaces

$$\vec{r}(u, v, w_0), \quad \vec{r}(u, v_0, w), \quad \vec{r}(u_0, v, w)$$

are called **coordinates surfaces** and the curves

$$\vec{r}(u_0, v_0, w), \quad \vec{r}(u_0, v, w_0), \quad \vec{r}(u, v_0, w_0)$$

are called **coordinate curves.** The coordinate curves represent intersections of the coordinate surfaces. The partial derivatives

$$\frac{\partial \vec{r}}{\partial u}, \qquad \frac{\partial \vec{r}}{\partial v}, \qquad \frac{\partial \vec{r}}{\partial w}$$

represent tangent vectors to the coordinate curves and the quantities

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$$

are called scale factors associated with the tangents to the coordinate curves. These scale factors are used to calculate unit vectors

$$\hat{\mathbf{e}}_u = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u}, \qquad \hat{\mathbf{e}}_v = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}, \qquad \hat{\mathbf{e}}_w = \frac{1}{h_w}\frac{\partial \vec{r}}{\partial w}$$

to the coordinate curves. If these unit vectors are all perpendicular to one another the coordinate system is called **an orthogonal coordinate system.** The differential

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv + \frac{\partial \vec{r}}{\partial w}\, dw$$

represents a small change in $\vec{r}$. One can think of the differential $d\vec{r}$ as the diagonal of a parallelepiped having the vector sides

$$\frac{\partial \vec{r}}{\partial u}\, du, \qquad \frac{\partial \vec{r}}{\partial v}\, dv, \quad \text{and} \quad \frac{\partial \vec{r}}{\partial w}\, dw.$$

The volume of this parallelepiped produces the volume element dV of the curvilinear coordinate system and this volume element is given by the formula

$$dV = \left| \frac{\partial \vec{r}}{\partial u} \cdot \left(\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w} \right) \right| \, du\, dv\, dw.$$

This result can be expressed in the alternate form

$$dV = \left| J \begin{pmatrix} x, y, z \\ u, v, w \end{pmatrix} \right| \, du\, dv\, dw,$$

where one can make use of the property of representing scalar triple products in terms of determinants to obtain

$$J \begin{pmatrix} x, y, z \\ u, v, w \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The quantity $J \begin{pmatrix} x, y, z \\ u, v, w \end{pmatrix}$ is called **the Jacobian of the transformation** from x, y, z coordinates to u, v, w coordinates. The absolute value signs are to insure the element of volume is positive.

As an example, the volume element $dV = dx\, dy\, dz$ under the change of variable to **cylindrical coordinates** (r, θ, z), with coordinate transformation

$$x = x(r,\theta,z) = r\cos\theta, \quad y = y(r,\theta,z) = r\sin\theta, \quad z = z(r,\theta,z) = z$$

has the Jacobian determinant $\left| J \begin{pmatrix} x, y, z \\ r, \theta, z \end{pmatrix} \right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$ which gives the new volume element $dV = r\, dr\, d\theta\, dz$.

As another example, the volume element $dV = dx\, dy\, dz$ under the change of variable to **spherical coordinates** (ρ, θ, ϕ), where

$$x = x(\rho,\theta,\phi) = \rho\sin\theta\cos\phi, \quad y = y(\rho,\theta,\phi) = \rho\sin\theta\sin\phi, \quad z = z(\rho,\theta,\phi) = \rho\cos\theta$$

one finds the Jacobian $\left| J \begin{pmatrix} x, y, z \\ \rho, \theta, \phi \end{pmatrix} \right| = \begin{vmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix} = \rho^2\sin\theta$ giving the new volume element $dV = \rho^2 sin\theta\, d\rho\, d\phi\, d\theta$.

Verification of the above results is left as an exercise.

Cylindrical Coordinates (r, θ, z)

The transformation from rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z) is given by

$$x = x(r, \theta, z) = r\cos\theta, \qquad y = y(r, \theta, z) = r\sin\theta, \qquad z = z(r, \theta, z) = z$$

so that a general position vector is given by $\vec{r} = \vec{r}(r, \theta, z) = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ In cylindrical coordinates the coordinate surfaces are

$$\vec{r}(r_0, \theta, z) = r_0\cos\theta\,\hat{\mathbf{e}}_1 + r_0\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \quad \text{a cylinder}$$
$$\vec{r}(r, \theta_0, z) = r\cos\theta_0\,\hat{\mathbf{e}}_1 + r\sin\theta_0\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \quad \text{a plane perpendicular to } z\text{−axis}$$
$$\vec{r}(r, \theta, z_0) = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3 \quad \text{a plane through the } z\text{−axis}$$

These surfaces are illustrated in the figure 7-26.

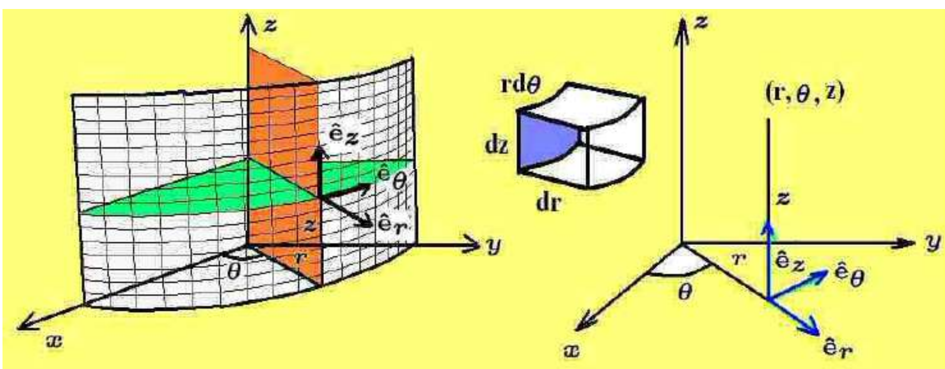


Figure 7-26.
Coordinate surfaces and coordinate curves for cylindrical coordinates (r, θ, z).

The coordinate curves are

$$\vec{r}(r_0, \theta_0, z), \quad \text{lines perpendicular to plane } z = 0$$
$$\vec{r}(\theta_0, z_0), \quad \text{lines emanating from the origin}$$
$$\vec{r}(r_0, \theta, z_0), \quad \text{circles of radius } r_0 \text{ in the plane } z = z_0$$

The vectors $\dfrac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$, $\dfrac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\theta\,\hat{\mathbf{e}}_2$, $\dfrac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3$ are tangent vectors to the coordinate curves and the vectors

$$\hat{\mathbf{e}}_r = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2, \qquad \hat{\mathbf{e}}_\theta = \frac{1}{r}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2, \qquad \hat{\mathbf{e}}_z = \frac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3 \quad (7.107)$$

are unit vectors tangent to the coordinate curves, where $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$, $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_z = 0$ and $\hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_z = 0$. The unit vector $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta$ produces the triad system $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$ so that the cylindrical coordinate system is a right-handed orthogonal coordinate system. The unit vectors are sometimes expressed in the matrix[9] form

$$\hat{\mathbf{e}}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{e}}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In the cylindrical coordinate system the element of volume is given by $dV = r\,dr\,d\theta\,dz$ and the element of surface area is $dS = r\,d\theta\,dz$. The direction $\hat{\mathbf{e}}_r$ is called the radial direction, the direction $\hat{\mathbf{e}}_\theta$ is called the azimuthal direction and the direction $\hat{\mathbf{e}}_z$ is called the vertical direction.

Spherical Coordinates (ρ, θ, ϕ)

The transformation from rectangular coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ) is given by the equations

$$x = x(\rho,\theta,\phi) = \rho\sin\theta\cos\phi, \qquad y = y(\rho,\theta,\phi) = \rho\sin\theta\sin\phi, \qquad z = z(\rho,\theta,\phi) = \rho\cos\theta$$

and the general position vector is given by

$$\vec{r} = \vec{r}(\rho,\theta,\phi) = \rho\sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \rho\sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \rho\cos\theta\,\hat{\mathbf{e}}_3$$

In this coordinate system the coordinate surfaces are

$$\begin{aligned} &\vec{r}(\rho_0,\theta,\phi), && \text{a sphere} \quad x^2+y^2+z^2=\rho_0^2 \\ &\vec{r}(\rho,\theta_0,\phi), && \text{a cone} \quad x^2+y^2=\tan^2\theta z^2 \\ &\vec{r}(\rho,\theta,\phi_0), && \text{a plane through the } z\text{−axis } y = x\tan\phi \end{aligned}$$

The coordinate curves in spherical coordinates are obtained from the intersection of the coordinate surfaces and can be represented by

$$\begin{aligned} &\vec{r}(\rho_0,\theta_0,\phi), && \text{circles of latitude} \\ &\vec{r}(\rho_0,\theta,\phi_0), && \text{meridian curve} \\ &\vec{r}(\rho,\theta_0,\phi_0), && \text{lines through the origin} \end{aligned}$$

These coordinate surfaces and coordinate lines are illustrated in the figure 7-27.

[9] See chapter 10 for a discussion of the matrix calculus.

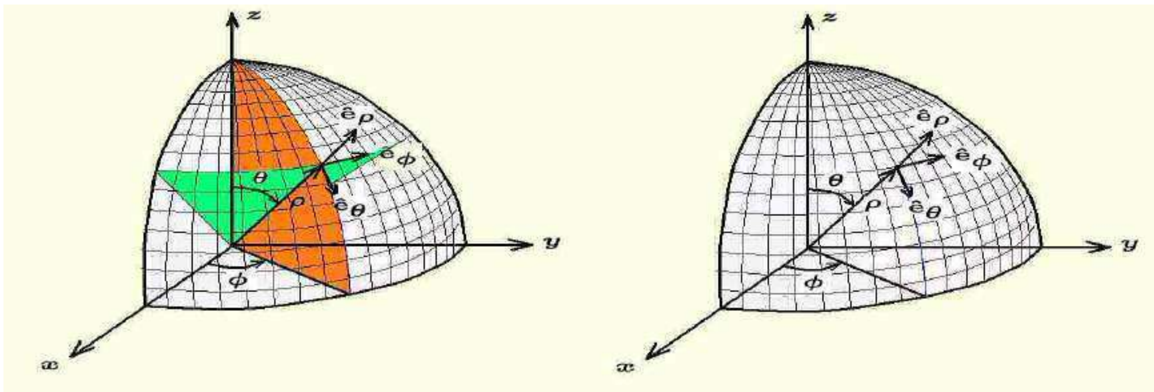


Figure 7-27.
Coordinate surfaces and coordinate curves for cylindrical coordinates.

The partial derivative vectors

$$\begin{aligned}\frac{\partial \vec{r}}{\partial \rho} =& \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial \theta} =& \rho\cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \rho\cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \rho\sin\theta\,\hat{\mathbf{e}}_3\\ \frac{\partial \vec{r}}{\partial \phi} =& -\rho\sin\theta\sin\phi\,\hat{\mathbf{e}}_1 + \rho\sin\theta\cos\phi\,\hat{\mathbf{e}}_2\end{aligned}$$

are tangent vectors to the coordinate curves and the scaled vectors

$$\begin{aligned}\hat{\mathbf{e}}_\rho =& \frac{\partial \vec{r}}{\partial \rho} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_\theta =& \frac{1}{\rho}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \sin\theta\,\hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_\phi =& \frac{1}{\rho\sin\theta} = -\sin\phi\,\hat{\mathbf{e}}_1 + \cos\phi\,\hat{\mathbf{e}}_2\end{aligned} \tag{7.108}$$

are unit vectors tangent to the coordinate curves. The spherical coordinate system is a right-handed orthogonal coordinate system because

$$\hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_\theta = 0, \qquad \hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_\phi = 0, \qquad \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\phi = 0, \qquad \hat{\mathbf{e}}_\rho \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\phi$$

The above unit vectors are sometimes expressed in the matrix form[10] as the column vectors.

$$\hat{\mathbf{e}}_\rho = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}, \qquad \hat{\mathbf{e}}_\theta = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}, \qquad \hat{\mathbf{e}}_\phi = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

[10] See chapter 10 for a discussion of matrices.

The element of volume in spherical coordinates is given by $dV = \rho^2 \sin\theta \, d\theta \, d\phi \, d\rho$ and the element of surface area is $dS = \rho^2 \sin\theta \, d\theta \, d\phi$, with ρ constant. The direction $\hat{\mathbf{e}}_\rho$ is called the radial direction, the vector $\hat{\mathbf{e}}_\theta$ is called the polar direction[11] and the direction $\hat{\mathbf{e}}_\phi$ is called the azimuthal direction.

Example 7-36. For $\vec{F} = (x - z)\,\hat{\mathbf{e}}_1 + (y - x)\,\hat{\mathbf{e}}_2 + (z + x + y)\,\hat{\mathbf{e}}_3$, let S denote the surface enclosing the volume V bounded by the hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0$, and the plane $z = 0$. Calculate (i) $I_1 = \iiint_V \nabla \cdot \vec{F}\, dV$ (ii) $I_2 = \iint_S \vec{F} \cdot \hat{\mathbf{e}}_n \, dS$

Solution Show $\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = 3$ and use spherical coordinates with $dV = \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$, and show

$$I_1 = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{1} 3\rho^2 \sin\theta \, d\rho \, d\phi \, d\theta = 2\pi$$

Break the surface integral I_2 into an integration I_{upper} over the hemisphere and an integral I_{lower} surface integral over the plane $z = 0$. On I_{upper} use $dS = \sin\theta \, d\theta \, d\phi$ and $\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ with $\vec{F} \cdot \hat{\mathbf{e}}_n = x^2 + y^2 + z^2 = 1$. One finds

$$I_{upper} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \, d\phi = 2\pi$$

On the plane $z = 0$, use $dS = dxdy$ and $\hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_3$ with $\vec{F} \cdot \hat{\mathbf{e}}_n = -(z + x + y)\Big|_{z=0} = -(x + y)$ so that

$$I_{lower} = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -(x + y)\, dydx = 0$$

and consequently $I_2 = I_{upper} + I_{lower} = 2\pi$.

Example 7-37. Let S denote the surface of the hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0$ and let C denote the curve $x^2 + y^2 = 1$ lying on the surface S. Calculate the integrals

$$(i) \quad I_3 = \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathbf{e}}_n \, dS \qquad (ii) \quad I_4 = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = y\,\hat{\mathbf{e}}_1 + (z^2 + 2x)\,\hat{\mathbf{e}}_2 + 2yz\,\hat{\mathbf{e}}_3$

Solution One finds $\operatorname{curl} \vec{F} = \hat{\mathbf{e}}_3$ and on the hemisphere $\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$, so that $\operatorname{curl} \vec{F} \cdot \hat{\mathbf{e}}_n = z$. Let $dS = \dfrac{dxdy}{|\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n|} = \dfrac{dxdy}{z}$ and show

$$I_3 = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \pi$$

[11] The angle θ is called the polar angle or zenith angle and the angle ϕ is called the azimuthal angle.

To evaluate the line integral let

$$x = \cos\theta, \;\; y = \sin\theta \text{ with } dx = -\sin\theta\, d\theta, \;\; dy = \cos\theta\, d\theta$$

and show

$$I_4 = \int_C \vec{F}\cdot d\vec{r} = \int_C (y-z)\,dx + (2x-y)\,dy = \int_0^{2\pi} [-\sin^2\theta + 2\cos^2\theta - \sin\theta\cos\theta]\,d\theta = \pi$$

Note that $z = 0$ on C.

■

Example 7-38. Evaluate the flux integral $I = \iint_S \vec{F}\cdot d\vec{S}$ where the vector field is given by $\vec{F} = \vec{F}(x,y,z) = z\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + x\,\hat{\mathbf{e}}_3$ and S is the surface of the unit sphere $x^2+y^2+z^2 = 1$.

Solution Transform to spherical coordinates where the position vector to a point of the unit sphere is

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 = \sin\theta\cos\phi\;\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\;\hat{\mathbf{e}}_2 + \cos\theta\;\hat{\mathbf{e}}_3$$

for $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$. An element of surface area in spherical coordinates is $dS = \sin\theta\, d\theta d\phi$ and a unit normal $\hat{\mathbf{e}}_n$ to the surface of the unit sphere is in the same direction as the vector $\vec{r}$ above so that one can write

$$\hat{\mathbf{e}}_n = \sin\theta\cos\phi\;\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\;\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3$$

Substituting these values into the flux integral one obtains

$$I = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} [\cos\theta(\sin\theta\cos\phi) + (\sin\theta\sin\phi)(\sin\theta\sin\phi) + (\sin\theta\cos\phi)\cos\theta]\;\sin\theta\,d\theta d\phi$$

Evaluating the inner integral, holding ϕ constant gives

$$I = \int_0^{2\pi} \frac{4}{3}\sin^2\phi\,d\phi$$

which can be integrated to obtain the value $I = \dfrac{4\pi}{3}$.

■

Exercises

► **7-1.** Sketch the given surfaces (i) $\dfrac{y^2}{a^2}+\dfrac{z^2}{b^2}=\dfrac{x^2}{c^2}$ $\quad(ii)$ $\dfrac{z^2}{a^2}+\dfrac{x^2}{b^2}=\dfrac{y^2}{c^2},\quad a>b>c$

► **7-2.** Sketch the given surfaces (i) $\dfrac{y^2}{a^2}+\dfrac{z^2}{b^2}=\dfrac{x}{c}$ $\quad(ii)$ $\dfrac{z^2}{a^2}+\dfrac{x^2}{b^2}=\dfrac{y}{c},\ a>b>c$

► **7-3.** Sketch the given surfaces defined by the parametric equations

$$(i)\qquad x-x_0=u,\qquad y-y_0=v,\qquad z-z_0=c\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)$$
$$(ii)\qquad x-x_0=u,\qquad y-y_0=v,\qquad z-z_0=c\left(\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)$$

► **7-4.** The curve $\vec{r}=\vec{r}(t)=\alpha\cos\omega t\,\hat{\mathbf{e}}_1+\alpha\sin\omega t\,\hat{\mathbf{e}}_2+\beta t\,\hat{\mathbf{e}}_3$, where α,β and ω are constants, describes a circular helix of radius α. For this space curve calculate the following quantities.

(a) The unit tangent vector $\hat{\mathbf{e}}_t$

(b) The unit normal vector $\hat{\mathbf{e}}_n$

(c) The unit binormal vector $\hat{\mathbf{e}}_b$

(d) The curvature κ

(e) The torsion V

► **7-5.** If $\vec{r}=\vec{r}(t)$ denotes a space curve, show that the curvature is given by

$$\kappa=\frac{\sqrt{(\vec{r}\,'\cdot\vec{r}\,')(\vec{r}\,''\cdot\vec{r}\,'')-(\vec{r}\,'\cdot\vec{r}\,'')^2}}{(\vec{r}\,'\cdot\vec{r}\,')^{3/2}}$$

where $'=\frac{d}{dt}$ denotes differentiation with respect to the argument of the function.

► **7-6.** If $\vec{r}=\vec{r}(x)=x\,\hat{\mathbf{e}}_1+y(x)\,\hat{\mathbf{e}}_2$ is the position vector describing a curve in the $x,y-$plane, show that the curvature is given by

$$\kappa=\frac{|y''|}{(1+(y')^2)^{3/2}}$$

where $'=\frac{d}{dx}$ denotes differentiation with respect to the argument of the function.

► **7-7.**

(a) For $\vec{r}=\vec{r}(s)$ the position vector of a curve, show that $\dfrac{d\vec{r}}{ds}\times\dfrac{d^2\vec{r}}{ds^2}=\kappa\,\hat{\mathbf{e}}_b$.

(b) For $\vec{r}=\vec{r}(s)$ the position vector of a curve, show that $\left|\dfrac{d\vec{r}}{ds}\times\dfrac{d^2\vec{r}}{ds^2}\right|=\kappa$.

▶ **7-8.** If $\vec{r} = \vec{r}(t)$ denotes a space curve, show that the torsion can be calculate from the relation

$$V = \frac{\vec{r}\,' \cdot (\vec{r}\,'' \times \vec{r}\,''')}{(\vec{r}\,' \cdot \vec{r}\,')(\vec{r}\,'' \cdot \vec{r}\,'') - (\vec{r}\,' \cdot \vec{r}\,'')^2}$$

where prime $' = \frac{d}{dt}$ always denotes differentiation with respect to the argument of the function. Hint: Show $\frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} \right) = \kappa^2 V$

▶ **7-9.**

(a) Find the curvature of the straight line $\vec{r} = \vec{r}(t) = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 + (7\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 - 3\,\hat{\mathbf{e}}_3)t$

(b) Find the torsion of the plane curve $\vec{r} = \vec{r}(x) = x\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2$

▶ **7-10.** For $\vec{r} = \vec{r}(t)$ the position vector of a curve, show that

$$|\vec{r}\,' \times \vec{r}\,''| = \kappa |\vec{r}\,'|^3, \quad \text{where} \quad ' = \frac{d}{dt}$$

▶ **7-11.** Find the directional derivative of ϕ in the specified direction, at the given point.

(i) $\phi = y^2x^2z + x^3z, \quad P(1,1,1), \quad 3\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3$

(ii) $\phi = xyz, \quad P(2,1,-1) \quad 5\,\hat{\mathbf{e}}_1 - 4\,\hat{\mathbf{e}}_2 + 20\,\hat{\mathbf{e}}_3$

(iii) $\phi = xy^2 + yz^3, \quad P(1,-1,0), \quad 2\,\hat{\mathbf{e}}_1 - 5\,\hat{\mathbf{e}}_2 - 14\,\hat{\mathbf{e}}_3$

(iv) $\phi = x^2y^2 + yz^3x, \quad P(1,1,1), \quad \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3$

▶ **7-12.**

(i) Let $\phi = x^2y$ define a two-dimensional scalar field. Find the directional derivative of ϕ at the point $(2, \sqrt{3})$ in the direction $\hat{\mathbf{e}}_\alpha = \cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2$

(ii) In what direction α is the directional derivative a maximum?

(iii) In what direction α is the directional derivative a minimum?

▶ **7-13.** Show that $\frac{d}{dt}\left[\vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right) \right] = \vec{r} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3} \right)$

▶ **7-14.** Prove that $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$

▶ **7-15.** Discuss the critical points of the function

$$z = z(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{2}x^2 - \frac{3}{2}y^2 - 2x + 2y$$

▶ **7-16.** Show that the Frenet-Serret formulas may be expressed in the form

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \vec{\omega} \times \hat{\mathbf{e}}_t, \qquad \frac{d\,\hat{\mathbf{e}}_b}{ds} = \vec{\omega} \times \hat{\mathbf{e}}_b, \qquad \frac{d\,\hat{\mathbf{e}}_n}{ds} = \vec{\omega} \times \hat{\mathbf{e}}_n$$

by finding the vector $\vec{\omega}$. Hint: Let $\vec{\omega} = \alpha\,\hat{\mathbf{e}}_t + \beta\,\hat{\mathbf{e}}_n + \gamma\,\hat{\mathbf{e}}_b$ and examine the above cross products to solve for α, β, and γ.

▶ **7-17.** Let $\vec{r}(s)$ denote the position vector of a space curve which is defined in terms of the arc length s.

(a) Show that the equation of the rectifying plane can be written as

$$(\vec{r}(s)-\vec{r}(s_0))\cdot\frac{d^2\vec{r}(s_0)}{dx^2}=0$$

(b) Show that the equation of the osculating plane can be written as

$$[\vec{r}(s)-\vec{r}(s_0)]\cdot\left[\frac{d\vec{r}(s_0)}{ds}\times\frac{d^2\vec{r}(s_0)}{ds^2}\right]=0$$

(c) Show that the equation of the normal plane can be written as

$$[\vec{r}(s)-\vec{r}(s_0)]\cdot\frac{d\vec{r}(s_0)}{ds}=0$$

▶ **7-18.** Show that the direction cosines (ℓ_1,ℓ_2,ℓ_3) of the normal to the surface $\vec{r}=\vec{r}(u,v)$ are given by

$$\ell_1=\frac{\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}}{D},\quad \ell_2=\frac{\begin{vmatrix}\frac{\partial z}{\partial u} & \frac{\partial x}{\partial u}\\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v}\end{vmatrix}}{D},\quad \ell_3=\frac{\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}}{D},$$

where

$$D=\sqrt{EG-F^2}.$$

▶ **7-19.** Show that the direction cosines (ℓ_1,ℓ_2,ℓ_3) of the normal to the surface $F(x,y,z)=0$ are given by

$$\ell_1=\frac{\frac{\partial F}{\partial x}}{H},\quad \ell_2=\frac{\frac{\partial F}{\partial y}}{H},\quad \ell_3=\frac{\frac{\partial F}{\partial z}}{H},$$

where

$$H^2=\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2.$$

▶ **7-20.** Show that the direction cosines (ℓ_1,ℓ_2,ℓ_3) of the normal to the surface $z=z(x,y)$ are given by

$$\ell_1=\frac{-\frac{\partial z}{\partial x}}{H},\quad \ell_2=\frac{-\frac{\partial z}{\partial y}}{H},\quad \ell_3=\frac{1}{H},$$

where

$$H^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1$$

▶ **7-21.** Find a unit normal vector to the cylinder $x^2 + y^2 = 1$

▶ **7-22.** Find a unit normal vector to the sphere $x^2 + y^2 + z^2 = 1$

▶ **7-23.** Find a unit normal vector to the plane $ax + by + cz = d$

▶ **7-24.** Evaluate the surface integral $\iint_R x^2yz\, d\vec{S}$ over the cylinder $x^2 + y^2 = 1$ lying in the first octant between the planes $z = 0$ and $z = 2$.

▶ **7-25.** Evaluate the surface integral $\iint_R xyz\, d\vec{S}$ where integration is over the upper half of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

▶ **7-26.** Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where $\vec{F} = x\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2$ and S is the surface of the cylinder $x^2 + y^2 = 1$ between the planes $z = 0$ and $z = 2$.

▶ **7-27.** Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where $\vec{F} = z\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ lying in the first octant.

▶ **7-28.** Evaluate the surface integral $\iint_R \vec{F} \cdot d\vec{S}$, where $\vec{F} = \hat{\mathbf{e}}_3$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

▶ **7-29.** Evaluate the surface integral $\iint_R \vec{F} \times d\vec{S}$, where $\vec{F} = (z+y)\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2 - y\,\hat{\mathbf{e}}_3$ and S is the surface of the plane $z = 1$, where $0 \le x \le 1$ and $0 \le y \le 1$.

▶ **7-30.** Show that any curve on a surface defined by the parametric equations

$$x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$$

has an element of arc length given by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where E, F, G are defined by the equations (7.54).

▶ **7-31.**

(a) Show that when the curve $z = f(x)$, $x_0 < x < x_1$, is rotated 360° about the z-axis, the surface formed has a surface area

$$S = \int_0^{2\pi} \int_{x_0}^{x_1} r\sqrt{1 + [f'(r)]^2}\, dr\, d\theta$$

(b) The curve $z = f(x) = \frac{H}{R}x$ for $0 \le x \le R$ is rotated 360° about the z axis. Find the surface area generated.

▶ **7-32.** Consider a circle of radius $\rho < a$ centered at $x = a > 0$ in the xz plane. The parametric equations of this circle are

$$x - a = \rho\cos\theta, \quad z = \rho\sin\theta, \qquad 0 \le \theta \le 2\pi, \quad a > \rho.$$

If the circle is rotated about the z-axis, a torus results.

(a) Show that the parametric equations of the torus are

$$x = (a + \rho\cos\theta)\cos\phi, \quad y = (a + \rho\cos\theta)\sin\phi, \quad z = \rho\sin\theta, \quad 0 \le \phi \le 2\pi$$

(b) Find the surface area of the torus.

(c) Find the volume of the torus.

▶ **7-33.** Calculate the arc length along the given curve between the points specified.

$$(a) \quad y = x, \qquad p_1(0,0), \qquad p_2(3,3)$$
$$(b) \quad x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$
$$(c) \quad x = t, \quad y = 2t, \quad z = 2t, \qquad p_1(0,0,0), \quad p_2(2,4,4)$$
$$(d) \quad y = x^2, \qquad p_1(0,0), \qquad p_2(2,4)$$

▶ **7-34.**

(a) Describe the surface $\vec{r} = u\,\hat{\mathbf{e}}_1 + v\,\hat{\mathbf{e}}_2$ and sketch some coordinate curves on the surface.

(b) Describe the surface $\vec{r} = v\cos u\,\hat{\mathbf{e}}_1 + v\sin u\,\hat{\mathbf{e}}_2$ and sketch some coordinate curves on the surface.

(c) Describe the surface $\vec{r} = \sin u\cos v\,\hat{\mathbf{e}}_1 + \sin u\sin v\,\hat{\mathbf{e}}_2 + \cos u\,\hat{\mathbf{e}}_3$ and sketch some coordinate curves on the surface.

(d) Construct a unit normal vector to each of the above surfaces.

▶ **7-35.** Evaluate the surface integral $I = \iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = 4y\,\hat{\mathbf{e}}_1 + 4(x+z)\,\hat{\mathbf{e}}_2$ and S is the surface of the plane $x + y + z = 1$ lying in the first octant.

▶ **7-36.** Evaluate the surface integral $I = \iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = x^2\,\hat{\mathbf{e}}_1 + y^2\,\hat{\mathbf{e}}_2 + z^2\,\hat{\mathbf{e}}_3$ and S is the surface of the unit cube bounded by the planes $x = 0,\ y = 0,\ z = 0$ and $x = 1,\ y = 1,\ z = 1$.

▶ **7-37.** Evaluate the surface integral $I = \iint_S f(x,y,z)\,dS$, where $f(x,y,z) = 2(x+1)y$ and S is the surface of the cylinder $x^2+y^2=1,\ 0 \le z \le 3$, in the first octant.

▶ **7-38.** Evaluate the integral $I = \iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = (x+z)\,\hat{\mathbf{e}}_1 + (y+z)\,\hat{\mathbf{e}}_2 - (x+y)\,\hat{\mathbf{e}}_3$ and S is the surface of the sphere $x^2+y^2+z^2=9$ where $z \ge 0$.

▶ **7-39.** Evaluate the surface integral $I = \iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = 4y\,\hat{\mathbf{e}}_1 + 4(x+z)\,\hat{\mathbf{e}}_2$ and S is the surface of the plane $x+y+z=1$ which lies in the first octant.

▶ **7-40.** **(Lagrange multipliers)**

Lagrange multipliers are used to help find the maximum or minimum values associated with functions of several variables when the variables are subject to certain constraint conditions. The following is a two-dimensional example of finding the minimum value of a function when the variables in the problem are subject to constraints. Let D denote the distance from the origin $(0,0)$ to a point (x,y) which lies on the line $x+y+2=0$. Let $F(x,y)=D^2=x^2+y^2$ denote the square of this distance. The mathematical problem is to find values for (x,y) which minimize $F(x,y)$ when (x,y) is constrained to move along the given line. Mathematically one writes

$$\begin{aligned} \text{Minimize} \quad & F(x,y) = x^2+y^2 \\ \text{subject to the constraint condition} \quad & G(x,y) = x+y-2 = 0. \end{aligned}$$

The point (x,y), where F has a minimum value, is called a critical point.

(a) Show that at a critical point ∇F is normal to the curve $F =$ constant and ∇G is normal to the line $G=0$.

(b) Show that at a critical point the vectors ∇F and ∇G are colinear. Consequently, one can write

$$\nabla F + \lambda \nabla G = \vec{0}$$

where λ is a scalar called a Lagrange multiplier.

(c) Show that at a critical point which minimizes F, the function $H = F + \lambda G$, satisfies the equations

$$\frac{\partial H}{\partial \lambda} = 0, \qquad \frac{\partial H}{\partial x} = 0, \qquad \frac{\partial H}{\partial y} = 0.$$

Calculate these equations and find the point (x,y) which minimizes F.

► **7-41.** **(Lagrange multipliers)**
Use Lagrange multipliers to

$$\text{Minimize} \qquad \omega = \omega(x,y,z) = x^2 + y^2 + z^2,$$

subject to the constraint conditions: $\quad g(x,y,z) = x + y + z - 6 = 0$

$$h(x,y,z) = 3x + 5y + 7z - 34 = 0$$

► **7-42.** Given the vector field $\vec{F} = (x^2 + y - 4)\,\hat{\mathbf{e}}_1 + 3xy\,\hat{\mathbf{e}}_2 + (2xz + z^2)\,\hat{\mathbf{e}}_3$. Evaluate the surface integral

$$I = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

over the upper half of the unit sphere centered at the origin.

► **7-43.** Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = x^2\,\hat{\mathbf{e}}_1 + (y+6)\,\hat{\mathbf{e}}_2 - z\,\hat{\mathbf{e}}_3$ and S is the surface of the unit cube bounded by the planes $x = 0,\ y = 0,\ z = 0$ and $x = 1,\ y = 1,\ z = 1$.

► **7-44.**

(a) Show in the special case the surface is defined by $\vec{r} = \vec{r}(x,y) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z(x,y)\,\hat{\mathbf{e}}_3$ the element of surface area is given by

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

(b) Show in the special case the surface is defined by $\vec{r} = \vec{r}(x,z) = x\,\hat{\mathbf{e}}_1 + y(x,z)\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ the element of surface area is given by

$$dS = \frac{dx\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$$

(c) Show in the special case the surface is defined by $\vec{r} = \vec{r}(y,z) = x(y,z)\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ the element of surface area is given by

$$dS = \frac{dy\,dz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1|} = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

► **7-45.** Given n particles having masses $m_1, m_2, \ldots, m_n$. Let $\vec{r}_i,\ i = 1, 2, \ldots, n$ denote the position vector describing the position of the ith particle. Find the vector describing the center of mass of the system of particles.

► **7-46.** Show that $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \begin{vmatrix} \vec{A}\cdot\vec{C} & \vec{A}\cdot\vec{D} \\ \vec{B}\cdot\vec{C} & \vec{B}\cdot\vec{D} \end{vmatrix}$

► **7-47.** Let $F = F(x,y,z)$ and $G = G(x,y,z)$ denote continuous functions which are everywhere differentiable. Show that

$$(a)\quad \nabla(F+G) = \nabla F + \nabla G \qquad (b)\quad \nabla(FG) = F\nabla G + G\nabla F \qquad (c)\quad \nabla\left(\frac{F}{G}\right) = \frac{G\nabla F - F\nabla G}{G^2}$$

► **7-48.** **(Least-squares)**

(a) Assume that $(x_1,y_1), (x_2,y_2), \ldots, (x_i,y_i), \ldots, (x_N,y_N)$ are N known distinct data points that are plotted in the x,y plane along with a sketch of the straight line

$$y = \alpha x + \beta$$

where α and β are constants to be determined. The situation is illustrated in figure 7-28.

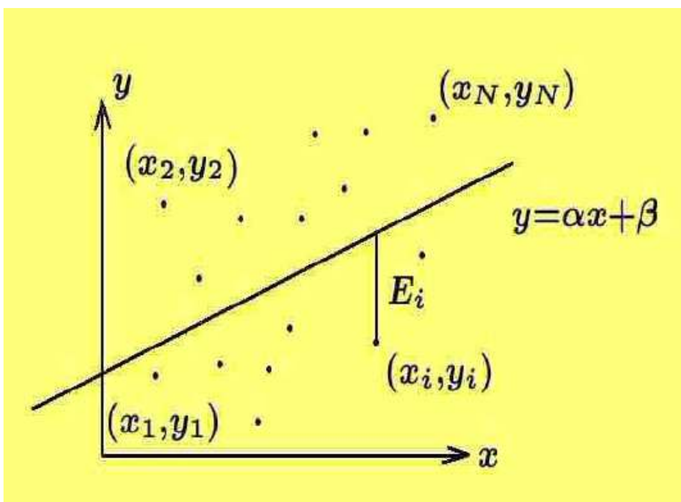


Figure 7-28. Linear least-squares fit.

Each data point (x_i,y_i) has associated with it an error E_i which is defined as the difference between the y value of the line and the y value of the data point. For example, the error associated with the data point (x_i,y_i) can be written

$$E_i = E_i(\alpha,\beta) = \{y \text{ of line}\} - \{y \text{ of data point}\}$$

$$E_i = E_i(\alpha,\beta) = (\alpha x_i + \beta) - y_i.$$

Find the error associated with each data point and then square these errors and sum them to obtain the quantity $\sum_{i=1}^{N} E_i^2$ called the sum of the errors squared. The "best "linear least squares fit to all the data points is defined as the line which minimizes the sum of the errors squared. This requires finding those values of α and β which minimize the sum of the errors squared given by

$$E(\alpha,\beta)=\sum_{i=1}^{N} E_i^2=\sum_{i=1}^{N}(\alpha x_i+\beta-y_i)^2, \quad \text{to have a minimum value}$$

(a) Show that the best linear least-squares fit requires that the coefficients α and β be chosen to satisfy the equations

$$\alpha=\frac{N\sum_{i=1}^{N} x_i y_i-\left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{\Delta}$$
$$\beta=\frac{\left(\sum_{i=1}^{N} x_i^2\right)\left(\sum_{i=1}^{N} y_i\right)-\left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i y_i\right)}{\Delta},$$

where $$\Delta=N\sum_{i=1}^{N} x_i^2-\left(\sum_{i=1}^{N} x_i\right)^2.$$

(b) Given the data points

$$(1,10),\ (2,4),\ (2.5,6),\ (3,12),\ (3.5,5),\ (4,10)$$

Find the best linear least-squares fit. Hint: Construct a table of values of the form

x_i	y_i	x_i^2	$x_i y_i$
$\sum_{i=1}^{4} x_i$	$\sum_{i=1}^{4} y_i$	$\sum_{i=1}^{4} x_i^2$	$\sum_{i=1}^{4} x_i y_i$

(c) Plot the least squares straight line and the data points.

Chapter 8

Vector Calculus II

In this chapter we examine in more detail the operations of gradient, divergence and curl as well as introducing other mathematical operators involving vectors. There are several important theorems dealing with the operations of divergence and curl which are extremely useful in modeling and representing physical problems. These theorems are developed along with some examples to illustrate how powerful these results are. Also considered is the representation of the many vector operations and their use when dealing with a general orthogonal coordinate system.

Vector Fields

Let $\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_3 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$ denote a continuous vector field with continuous partial derivatives in some region R of space. A vector field is **a one-to-one correspondence between points in space and vector quantities** so by selecting a discrete set of points $\{(x_i,y_i,z_i)\}\,|\,i=1,\ldots,n,\ (x_i,y_i,z_i)\in R\}$ one could sketch in tiny vectors each proportional to the given vector evaluated at the selected points. This would be one way of visualizing the vector field. Imagine a surface being placed in this vector field, then at each point (x,y,z) on the surface there is associated a vector $\vec{F}(x,y,z)$. This is another way of visualizing a vector field. One can think of the surface as being punctured by arrows of different lengths. These arrows then represent the direction and magnitude of the vectors in the vector field. The situation is illustrated in figure 8-1.

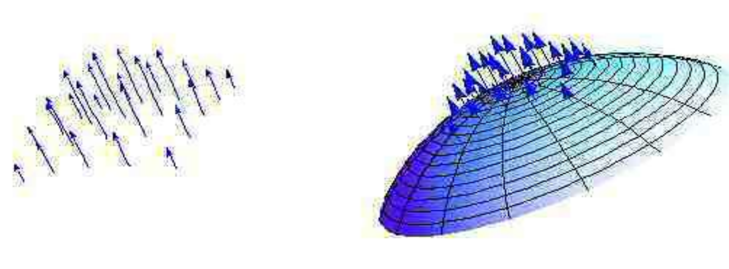


Figure 8-1. Representation of vector field $\vec{F}(x,y,z)$ at a select set of points.

Another way to visualize a vector field is to create a bundle of curves in space, where **each curve in the bundle** has the property that at every point (x,y,z) on any one curve, **the direction of the tangent vector to the curve is the same as the direction of the vector field $\vec{F}(x,y,z)$ at that point.** Curves with this property are called **field lines** associated with the vector field $\vec{F}(x,y,z)$. Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ be a position vector to a point on a field line curve, then $d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3$ is in the direction of the tangent to the curve. If the curve is a field line, then one can write $d\vec{r} = \alpha\vec{F}$, where α is a proportionality constant. That is, if the curve is a field line, then the vectors $d\vec{r}$ and $\vec{F}$ are colinear at all points along the curve and one can write

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3 = \alpha F_1(x,y,z)\,\hat{\mathbf{e}}_1 + \alpha F_2(x,y,z)\,\hat{\mathbf{e}}_2 + \alpha F_3(x,y,z)\,\hat{\mathbf{e}}_3$$

where α is some proportionality constant. By equating like components in the above equation, one obtains

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)} = \alpha \tag{8.1}$$

The equations (8.1) represent a system of differential equations to be solved to obtain the representation of the field lines.

Example 8-1. Find and sketch the field lines associated with the vector field

$$\vec{V} = \vec{V}(x,y) = (3-x)(4-y)\,\hat{\mathbf{e}}_1 + (6-x^2)(4+y^2)\,\hat{\mathbf{e}}_2$$

Solution If $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ describes a field line, then one can write

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 = \alpha\vec{V}(x,y) = \alpha(3-x)(4-y)\,\hat{\mathbf{e}}_1 + \alpha(6-x^2)(4+y^2)\,\hat{\mathbf{e}}_2$$

where α is a proportionality constant. Equating like components one can show that the field lines must satisfy

$$dx = \alpha(3-x)(4-y), \qquad dy = \alpha(6-x^2)(4+y^2)$$

or one could write

$$\frac{dx}{(3-x)(4-y)} = \frac{dy}{(6-x^2)(4+y^2)} = \alpha \tag{8.2}$$

Separate the variables in equation (8.2) to obtain

$$\frac{6-x^2}{3-x}\,dx = \frac{4-y}{4+y^2}\,dy$$

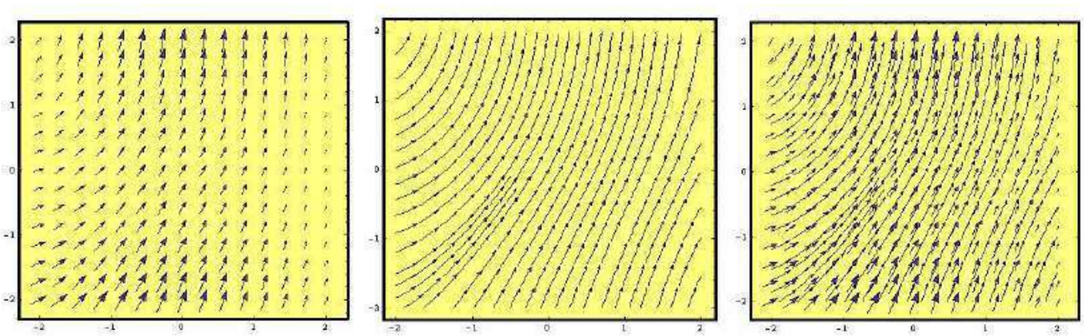
and then integrate both sides to obtain

$$\int \frac{6-x^2}{3-x}\,dx = \int \frac{4-y}{4+y^2}\,dy \tag{8.3}$$

Use a table of integrals and evaluate the integrals and then collect all the constants of integration and combine them into just one arbitrary constant C to obtain the result

$$3x + \frac{1}{2}x^2 + 3\ln|x-3| = 2\tan^{-1}(y/2) - \frac{1}{2}\ln(4+y^2) + C \tag{8.4}$$

The equation (8.4) represents **a one-parameter family of curves** which describe the field lines associated with the given vector field. Assign values to the constant C and sketch the corresponding field line. Place arrows on the curves to show the direction of the vector field.



(a) (b) (c)

Figure 8-2.

Vector plot and field lines for $\vec{V}(x,y) = (3-x)(4-y)\,\hat{\mathbf{e}}_1 + (6-x^2)(4+y^2)\,\hat{\mathbf{e}}_2$

The figure 8-2 illustrates three graphs created by a computer. The figure 8-2(a) represents a vector field plot over the region $-2 \le x \le 2$ and $-2 \le y \le 2$. The figure 8-2(b) is a graph of the field lines associated with the vector field with arrows placed on the field lines. The figure 8-2(c) is the vectors of figure 8-2(a) placed on top of the field lines of figure 8-2(b) to compare the different representations.

■

Divergence of a Vector Field

The study of field lines leads to the concept of **intensity of a vector field** or **the density of the field lines in a region.** To visualize this, place an imaginary surface in a vector field and try to determine how the vector field punctures this surface. Let the surface be divided into n small areas ΔS_i and let $\vec{F}_i = \vec{F}(x_i, y_i, z_i)$ denote the value of the vector field associated with each surface element. The dot product $\vec{F}_i \cdot \Delta\vec{S}_i = \vec{F}_i \cdot \hat{\mathbf{e}}_n \Delta S_i$ represents the projection of the vector $\vec{F}_i$ onto the normal to the element ΔS_i multiplied by the area of the element. Such a product is a measure of **the number of field lines which pass through the area** ΔS_i and is called a **flux across the surface boundary**. The total flux across the surface is denoted by the **flux integral**

$$\varphi = \lim_{\substack{n\to\infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} \vec{F}_i \cdot \Delta\vec{S}_i = \iint_R \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot \hat{\mathbf{e}}_n \, dS \tag{8.5}$$

The surface area over which the integration is performed can be part of a surface or it can be over all points of a closed surface. The evaluation of a flux integral over a closed surface measures the total contribution of the normal component of the vector field over the surface.

The term flux can mean flow in some instances. For example, let an imaginary plane surface of area 1 square centimeter be placed perpendicular to a uniform velocity flow of magnitude V_0, such that the velocity is the same at all points over the surface. In 1 second there results a column of fluid V_0 units long which passes through the unit of surface area. The dimension of the flux integral is volume per unit of time and can be interpreted as the rate of flow or flux of the velocity across the surface.

In the above example, the flux was a quantity which is recognized as volume rate of flow. In many other problems **the flux is only a definition** and does not readily have any physical meaning. For example, the electric flux over the surface of a sphere due to a point charge at its center is given by the flux integral $\iint_R \vec{E} \cdot d\vec{S}$, where $\vec{E}$ is the electrostatic intensity. The flux cannot be interpreted as flow because nothing is flowing. In this case the flux is considered as **a measure of the density of the field lines that pass through the surface** of the sphere.

The value for the flux depends upon the size of the surface that is placed in the vector field under consideration and therefore cannot be used to describe a characteristic of the vector field. However, if an arbitrary closed surface is placed in a vector field and the flux integral over this surface is evaluated and the result is divided by the volume enclosed by the surface, one obtains the ratio of $\frac{\text{Flux}}{\text{Volume}}$. By letting the volume and surface area of the arbitrary closed surface approach zero, the ratio of $\frac{\text{Flux}}{\text{Volume}}$ turns out to measure **a point characteristic of the vector field** called **the divergence**. Symbolically, the divergence is a scalar quantity and is defined by the limiting process

$$\text{div}\,\vec{F} = \lim_{\substack{\Delta V \to 0 \\ \Delta S \to 0}} \frac{\iint_R \vec{F} \cdot d\vec{S}}{\Delta V} = \lim_{\substack{\Delta V \to 0 \\ \Delta S \to 0}} \frac{\text{Flux}}{\text{Volume}} \tag{8.6}$$

Consider the evaluation of this limit in the special case where the closed surface is a sphere. Consider a sphere of radius $\epsilon > 0$ centered at a point $P_0(x_0, y_0, z_0)$ situated in a vector field

$$\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3.$$

Express the sphere in the parametric form as

$$\begin{aligned} x &= x_0 + \epsilon \sin\theta \cos\phi, \qquad && 0 \le \phi \le 2\pi \\ y &= y_0 + \epsilon \sin\theta \sin\phi, \qquad && 0 \le \theta \le \pi \\ z &= z_0 + \epsilon \cos\theta \end{aligned}$$

then the position vector to a point on this sphere is given by

$$\vec{r} = \vec{r}(\theta,\phi) = (x_0 + \epsilon \sin\theta\cos\phi)\,\hat{\mathbf{e}}_1 + (y_0 + \epsilon\sin\theta\sin\phi)\,\hat{\mathbf{e}}_2 + (z_0 + \epsilon\cos\theta)\,\hat{\mathbf{e}}_3$$

The coordinate curves on the surface of this sphere are

$$\vec{r}(\theta_0,\phi) \quad \text{and} \quad \vec{r}(\theta,\phi_0) \quad \theta_0, \phi_0 \text{ constants}$$

and one can show the element of surface area on the sphere is given by

$$dS = \left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right| \, d\theta\, d\phi = \sqrt{EG - F^2}\, d\theta\, d\phi = \epsilon^2 \sin\theta\, d\theta\, d\phi$$

A **unit normal** to the surface of the sphere is given by

$$\hat{\mathbf{e}}_n = \frac{\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right|} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3$$

The flux integral given by equation (8.6) and integrated over the surface of a sphere can then be expressed as

$$\begin{aligned} \varphi &= \iint_R \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot \hat{\mathbf{e}}_n\, dS \\ \varphi &= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \vec{F}(x_0 + \epsilon\sin\theta\cos\phi, y_0 + \epsilon\sin\theta\sin\phi, z_0 + \epsilon\cos\theta)\,\hat{\mathbf{e}}_n \epsilon^2 \sin\theta\, d\theta d\phi. \end{aligned}$$

Treating the vector function $\vec{F}$ as a function of ϵ, one can expand $\vec{F}$ in a Taylor's series about $\epsilon = 0$, to obtain

$$\vec{F} = \vec{F}(x_0, y_0, z_0) + \epsilon \frac{d\vec{F}}{d\epsilon} + \frac{\epsilon^2}{2!}\frac{d^2\vec{F}}{d\epsilon^2} + \frac{\epsilon^3}{3!}\frac{d^3\vec{F}}{d\epsilon^3} + \cdots \tag{8.7}$$

where all the derivatives are to be evaluated at $\epsilon = 0$. Substituting the expressions for the unit normal and the Taylor's series into the flux integral produces

$$\varphi = \epsilon^2 \mu_0 + \epsilon^3 \mu_1 + \epsilon^4 \mu_2 + \cdots,$$

where

$$\begin{aligned}
\mu_0 &= \int_0^{\pi}\int_0^{2\pi} \vec{F}(x_0, y_0, z_0) \cdot \hat{\mathbf{e}}_n \sin\theta \, d\phi \, d\theta \\
\mu_1 &= \int_0^{\pi}\int_0^{2\pi} \frac{d\vec{F}}{d\epsilon}\Big|_{\epsilon=0} \cdot \hat{\mathbf{e}}_n \sin\theta \, d\phi \, d\theta \\
\mu_2 &= \int_0^{\pi}\int_0^{2\pi} \frac{d^2\vec{F}}{d\epsilon^2}\Big|_{\epsilon=0} \cdot \hat{\mathbf{e}}_n \sin\theta \, d\phi \, d\theta,
\end{aligned}$$

plus higher order terms in ϵ. The vector $\vec{F}(x_0, y_0, z_0)$ is a constant and an evaluation of the integral defining μ_0 produces $\mu_0 = 0$. To calculate the second integral defining μ_1 observe that the chain rule for functions of more than one variable produces the result

$$\begin{aligned}
\frac{d\vec{F}}{d\epsilon} &= \frac{\partial\vec{F}}{\partial x}\frac{\partial x}{\partial\epsilon} + \frac{\partial\vec{F}}{\partial y}\frac{\partial y}{\partial\epsilon} + \frac{\partial\vec{F}}{\partial z}\frac{\partial z}{\partial\epsilon} \\
\frac{d\vec{F}}{d\epsilon} &= \frac{\partial\vec{F}}{\partial x}\sin\theta\cos\phi + \frac{\partial\vec{F}}{\partial y}\sin\theta\sin\phi + \frac{\partial\vec{F}}{\partial z}\cos\theta
\end{aligned}$$

This result can be expressed in the component form

$$\begin{aligned}
\frac{d\vec{F}}{d\epsilon}\Big|_{\epsilon=0} = & \left(\frac{\partial F_1}{\partial x}\sin\theta\cos\phi + \frac{\partial F_1}{\partial y}\sin\theta\sin\phi + \frac{\partial F_1}{\partial z}\cos\theta\right)\hat{\mathbf{e}}_1 \\
&+ \left(\frac{\partial F_2}{\partial x}\sin\theta\cos\phi + \frac{\partial F_2}{\partial y}\sin\theta\sin\phi + \frac{\partial F_2}{\partial z}\cos\theta\right)\hat{\mathbf{e}}_2 \\
&+ \left(\frac{\partial F_3}{\partial x}\sin\theta\cos\phi + \frac{\partial F_3}{\partial y}\sin\theta\sin\phi + \frac{\partial F_3}{\partial z}\cos\theta\right)\hat{\mathbf{e}}_3\Big|_{\epsilon=0}
\end{aligned}$$

where all the derivatives are to be evaluated at $\epsilon = 0$. Evaluating the integrals defining μ_1, produces

$$\mu_1 = \frac{4}{3}\pi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\Big|_{\epsilon=0}$$

The flux integral then has the form

$$\varphi = \frac{4}{3}\pi\epsilon^3\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \epsilon^4\mu_2 + \epsilon^5\mu_3 + \cdots$$

where the derivatives are to be evaluated at $\epsilon = 0$. The volume of the sphere of radius ϵ centered at the point (x_0, y_0, z_0) is given by $\frac{4}{3}\pi\epsilon^3$ and consequently the limit of the ratio of $\frac{\text{Flux}}{\text{Volume}}$ as ϵ tends toward zero produces the scalar relation

$$\operatorname{div} \vec{F} = \lim_{\substack{\Delta V \to 0 \\ \Delta S \to 0}} \frac{\iint_R \vec{F} \cdot d\vec{S}}{\Delta V} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big|_{\epsilon=0} \tag{8.8}$$

Recalling the definition of the operator ∇, the mathematical expression of the divergence may be represented

$$\begin{aligned} \operatorname{div} \vec{F} = \nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right) \cdot (F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \end{aligned} \tag{8.9}$$

Example 8-2. Find the divergence of the vector field

$$\vec{F}(x,y,z) = x^2 y\,\hat{\mathbf{e}}_1 + (x^2 + yz^2)\,\hat{\mathbf{e}}_2 + xyz\,\hat{\mathbf{e}}_3$$

Solution: By using the result from equation (8.9), the divergence can be expressed

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial(x^2 y)}{\partial x} + \frac{\partial(x^2 + yz^2)}{\partial y} + \frac{\partial(xyz)}{\partial z} = 2xy + z^2 + xy = 3xy + z^2$$

■

The Gauss Divergence Theorem

A relation known as **the Gauss divergence theorem** exists between **the flux and divergence of a vector field**. Let $\vec{F}(x,y,z)$ denote a vector field which is continuous with continuous derivatives. For **an arbitrary closed sectionally continuous surface** S **which encloses a volume** V, the Gauss' divergence theorem states

$$\iiint_V \operatorname{div} \vec{F}\,dV = \iiint \nabla \cdot \vec{F}\,dV = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{\mathbf{e}}_n\,dS. \tag{8.10}$$

which states that the surface integral of the normal component of $\vec{F}$ summed over a closed surface equals the integral of the divergence of $\vec{F}$ summed over the volume enclosed by S. This theorem can also be represented in the expanded form as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}_n\,dS, \tag{8.11}$$

where $\hat{\mathbf{e}}_n$ is the exterior or positive normal to the closed surface.

The proof of the Gauss divergence theorem begins by first verifying the integrals

$$\begin{aligned}\iiint_V \frac{\partial F_1}{\partial x}\,dV &= \iint_S F_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_n\,dS\\ \iiint_V \frac{\partial F_2}{\partial y}\,dV &= \iint_S F_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_n\,dS\\ \iiint_V \frac{\partial F_3}{\partial z}\,dV &= \iint_S F_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_n\,dS.\end{aligned} \tag{8.12}$$

The addition of these integrals then produces the desired proof. Note that the arguments used in proving each of the above integrals are essentially the same for each integral. For this reason, only the last integral is verified.

Let the closed surface S be composed of an upper half S_2 defined by $z = z_2(x, y)$ and a lower half S_1 defined by $z = z_1(x, y)$ as illustrated in figure 8-3. An element of volume $dV = dx\,dy\,dz$, when summed in the $z-$direction from zero to the upper surface, forms a parallelepiped which intersects both the lower surface and upper surface as illustrated in figure 8-3. Denote the unit normal to the lower surface by $\hat{\mathbf{e}}_{n_1}$ and the unit normal to the upper surface by $\hat{\mathbf{e}}_{n_2}$. The parallelepiped intersects the upper surface in an element of area dS_2 and it intersects the lower surface in an element of area dS_1. The projection of S for both the upper surface and lower surface onto the xy-plane is denoted by the region R.

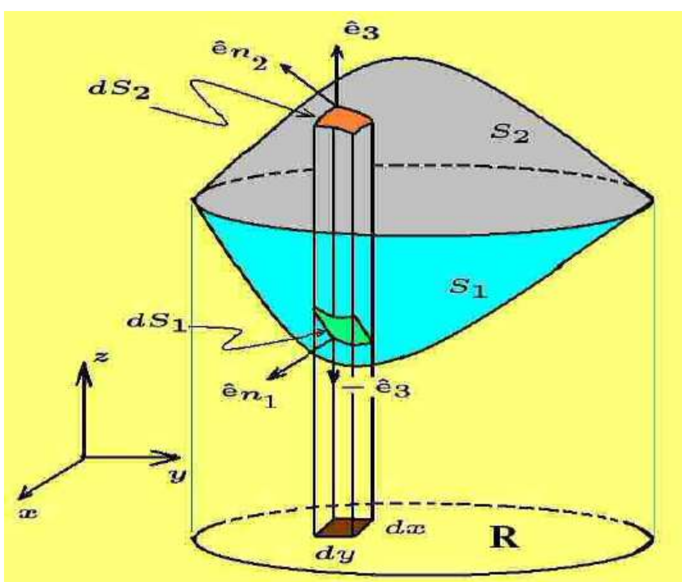


Figure 8-3. Integration over a simple closed surface.

An integration in the z-direction produces

$$\iiint_V \frac{\partial F_3}{\partial z}\, dzdx\, dy = \iint_R F_3(x,y,z)\Big|_{z_1(x,y)}^{z_2(x,y)} dx\, dy$$
$$= \iint_R F_3(x,y,z_2(x,y))\, dx\, dy - \iint_R F_3(x,y,z_1(x,y))\, dx\, dy.$$

The element of surface area on the upper and lower surfaces can be represented by

$$\text{On the surface } S_2, \qquad dS_2 = \frac{dx\, dy}{\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_2}}$$
$$\text{On the surface } S_1, \qquad dS_1 = \frac{dx\, dy}{-\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_1}}$$

so that the above integral can be expressed as

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iint_{S_2} F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_2}\, dS_2 + \iint_{S_1} F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_{n_1}\, dS_1 = \iint_S F_3\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n\, dS,$$

which establishes the desired result.

Similarly by dividing the surface into appropriate sections and projecting the surface elements of these sections onto appropriated planes, the remaining integrals may be verified.

Example 8-3. Verify the divergence theorem for the vector field

$$\vec{F}(x,y,z) = 2x\, \hat{\mathbf{e}}_1 - 3y\, \hat{\mathbf{e}}_2 + 4z\, \hat{\mathbf{e}}_3$$

over the region in the first octant bounded by the surfaces

$$z = x^2 + y^2, \qquad z = 4, \quad x = 0, \quad y = 0$$

Solution The given surfaces define a closed region over which the integrations are to be performed. This region is illustrated in figure 8-4. The divergence of the given vector field is given by

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2 - 3 + 4 = 3,$$

and thus the volume integral part of the Gauss divergence theorem can be determined by summing the element of volume $dV = dx\, dy\, dz$ first in the $z-$direction from the surface $z = x^2 + y^2$ to the plane $z = 4$. The resulting parallelepiped is then summed in the $y-$direction from $y = 0$ to the circle $y = \sqrt{4-x^2}$ to form a slab. The slab is then summed in the $x-$direction from $x = 0$ to $x = 2$.

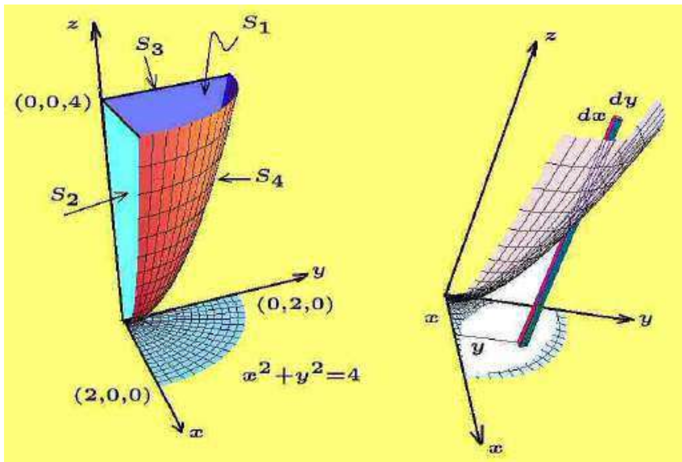


Figure 8-4. Integration over closed surface area defined by $S_1 \cup S_2 \cup S_3 \cup S_4$.

The resulting volume integral is then represented

$$\iiint_V \text{div}\, \vec{F}\, dV = \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} \int_{z=x^2+y^2}^{z=4} 3\, dz dy\, dx$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} 3[4-(x^2+y^2)]\, dy\, dx$$

$$= \int_0^2 3(4y - x^2 y - \frac{1}{3}y^3)\Big|_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 (8-2x^2)\sqrt{4-x^2}\, dx = 6\pi.$$

For the surface integral part of Gauss' divergence theorem, observe that the surface enclosing the volume is composed of four sections which can be labeled S_1, S_2, S_3, S_4 as illustrated in the figure 8-4. The surface integral can then be broken up and written as a summation of surface integrals. One can write

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S} + \iint_{S_4} \vec{F} \cdot d\vec{S}.$$

Each surface integral can be evaluated as follows.

On S_1, $\quad z=4,\ \ \hat{\mathbf{e}}_n=\hat{\mathbf{e}}_3, \quad dS=dx\,dy$ and

$$\iint\limits_{S_1} \vec{F}\cdot d\vec{S} = \int_0^2\int_0^{\sqrt{4-x^2}} 16\,dy\,dx = 16\int_0^2 \sqrt{4-x^2}\,dx = 16\pi.$$

On S_2, $\quad y=0,\ \ \hat{\mathbf{e}}_n=-\,\hat{\mathbf{e}}_2, \quad dS=dx\,dz$ and

$$\iint\limits_{S_2} \vec{F}\cdot d\vec{S} = \iint -3y\,dx\,dz = 0.$$

On S_3, $\quad x=0,\ \hat{\mathbf{e}}_n=-\,\hat{\mathbf{e}}_1, \quad dS=dy\,dz$ and

$$\iint\limits_{S_3} \vec{F}\cdot d\vec{S} = \iint -2x\,dy\,dz = 0.$$

On S_4 the surface is defined by $\phi = x^2+y^2-z=0$, and the normal is determined from

$$\hat{\mathbf{e}}_n = \frac{\text{grad}\ \phi}{|\text{grad}\,\phi|} = \frac{2x\,\hat{\mathbf{e}}_1+2y\,\hat{\mathbf{e}}_2-\,\hat{\mathbf{e}}_3}{\sqrt{4(x^2+y^2)+1}} = \frac{2x\,\hat{\mathbf{e}}_1+2y\,\hat{\mathbf{e}}_2-\,\hat{\mathbf{e}}_3}{\sqrt{4z+1}},$$

and consequently the element of surface area can be represented by

$$dS = \frac{dx\,dy}{|\,\hat{\mathbf{e}}_n\cdot\,\hat{\mathbf{e}}_3|} = \sqrt{4z+1}\,dx\,dy.$$

One can then write

$$\begin{aligned}\iint\limits_{S_4} \vec{F}\cdot d\vec{S} = \iint\limits_{S_4} \vec{F}\cdot\,\hat{\mathbf{e}}_n\,dS &= \iint\limits_{S_4} (4x^2-6y^2-4z)\,dx\,dy\\ &= \iint\limits_{S_4} (4x^2-6y^2-4(x^2+y^2))\,dx\,dy\\ &= \int_0^2\int_0^{\sqrt{4-x^2}} -10y^2\,dy\,dx = -\int_0^2 \frac{10}{3}y^3\,\Big|_0^{\sqrt{4-x^2}}\,dx\\ &= -\int_0^2 \frac{10}{3}\sqrt{(4-x^2)^3}\,dx = -10\pi.\end{aligned}$$

The total surface integral is the summation of the surface integrals over each section of the surface and produces the result 6π which agrees with our previous result.

Sometimes it is convenient to change the variables in a surface or volume integral. For example, the integral over the surface S_4 is not an integral which is easily evaluated. The geometry suggests a change to cylindrical coordinates. In cylindrical coordinates the following relations are satisfied:

$$
\begin{aligned}
x &= r\cos\theta, \quad y = r\sin\theta, \quad z = x^2+y^2 = r^2\\
\vec{r} &= x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + r^2\,\hat{\mathbf{e}}_3\\
E &= 1+4r^2, \quad F=0, \quad G=r^2\\
\hat{\mathbf{e}}_n &= \frac{2r\cos\theta\,\hat{\mathbf{e}}_1 + 2r\sin\theta\,\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3}{\sqrt{1+4r^2}}\\
\vec{F}\cdot\hat{\mathbf{e}}_n &= \frac{4r^2\cos^2\theta - 6r^2\sin^2\theta - 4r^2}{\sqrt{1+4r^2}}, \qquad dS = r\sqrt{1+4r^2}\,dr d\theta.
\end{aligned}
$$

The integral over the surface S_4 can then be expressed in the form

$$
\iint\limits_{S_4} \vec{F}\cdot d\vec{S} = \int_0^{\frac{\pi}{2}}\int_0^2 (4\cos^2\theta - 6\sin^2\theta - 4)r^3\,drd\theta = -10\pi.
$$

■

Physical Interpretation of Divergence

The divergence of a vector field is **a scalar field** which is interpreted as representing the flux per unit volume diverging from a small neighborhood of a point. In the limit as the volume of the neighborhood tends toward zero, the limit of the ratio of flux divided by volume is called **the instantaneous flux per unit volume at a point** or **the instantaneous flux density at a point.**

If $\vec{F}(x,y,z)$ defines a vector field which is continuous with continuous derivatives in a region R and if at some point P_0 of R, one finds that

div $\vec{F} > 0$, then a source is said to exist at point P_0.

div $\vec{F} < 0$, then a sink is said to exist at point P_0.

div $\vec{F} = 0$, then $\vec{F}$ is called solenoidal and no sources or sinks exist.

The Gauss divergence theorem states that if div $\vec{F} = 0$, then the flux $\varphi = \iint\limits_S \vec{F}\cdot d\vec{S}$ over the closed surface vanishes. When the flux vanishes the vector field is called **solenoidal**, and in this case, the flux of the vector field $\vec{F}$ into a volume exactly equals the flux of the field $\vec{F}$ out of the volume. Consider the field lines discussed earlier and visualize a bundle of these field lines forming a tube. Cut the tube by two plane areas S_1 and S_2 normal to the field lines as in figure 8-5.

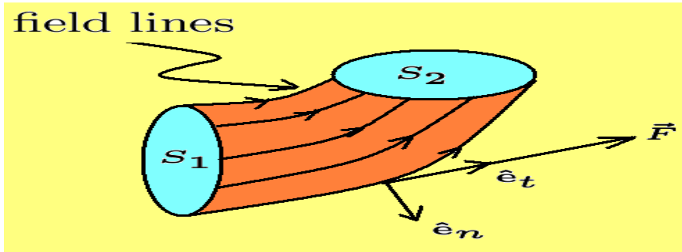


Figure 8-5. Tube of field lines.

The sides of the tube are composed of field lines, and at any point on a field line the direction of the tangents to the field lines are in the same direction as the vector field $\vec{F}$ at that point. The unit normal vector at a point on one of these field lines is perpendicular to the unit tangent vector and therefore perpendicular to the vector $\vec{F}$ so that the dot product $\vec{F}\cdot\hat{\mathbf{e}}_n = 0$ must be zero everywhere on the sides of the tube. The sides of the tube consist of field lines, and therefore there is no flux of the vector field across the sides of the tube and all the flux enters, through S_1, and leaves through S_2. In particular, if $\vec{F}$ is solenoidal and div $\vec{F}=0$, then

$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{Sides} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = 0 \quad \text{or} \quad \iint_{S_1} \vec{F}\cdot d\vec{S} = -\iint_{S_2} \vec{F}\cdot d\vec{S}$$

If the vector field is a velocity field then one can say that the flux or flow into S_1 must equal the flow leaving S_2.

Physically, the divergence assigns a number to each point of space where the vector field exists. The number assigned by the divergence is a scalar and represents the rate per unit volume at which the field issues (or enters) from (or toward) a point. In terms of figure 8-5, if more flux lines enter S_1 than leaves S_2, the divergence is negative and a sink is said to exist. If more flux lines leave S_2 than enter S_1, a source is said to exist.

Example 8-4. Consider the vector field

$$\vec{V} = \frac{kx}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_1 + \frac{ky}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_2 \qquad (x,y)\neq(0,0),\ k \text{ a constant}$$

A sketch of this vector field is illustrated in figure 8-6.

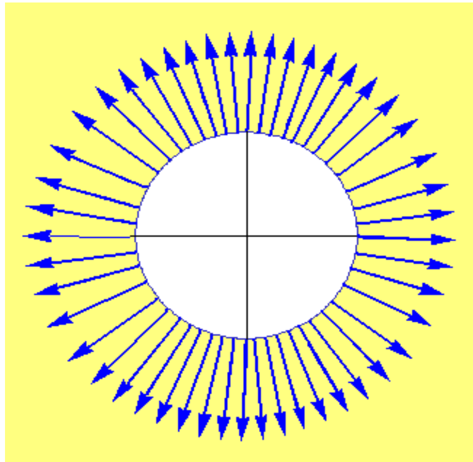


Figure 8-6. Vector field $\vec{V} = \frac{kx}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_1 + \frac{ky}{\sqrt{x^2+y^2}}\,\hat{\mathbf{e}}_2$, $k > 0$ constant.

Observe that the magnitude of the vector field at any point $(x, y) \neq (0, 0)$ is given by $|\vec{V}| = k$. In polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$, the vector field can be represented by

$$\vec{V} = k(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2).$$

Thus, on the circle $r =$ Constant, the vector field may be thought of as tiny needles of length $|k|$, which emanate outward (inward if k is negative) and are orthogonal to the circle $r =$ constant. The divergence of this vector field is

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V} = \frac{ky^2}{(x^2+y^2)^{3/2}} + \frac{kx^2}{(x^2+y^2)^{3/2}} = \frac{k}{r},$$

where $r = \sqrt{x^2+y^2}$. The divergence of this field is positive if $k > 0$ and negative if $k < 0$. If the vector field $\vec{V}$ represents a velocity field and $k > 0$, the flow is said to emanate from a source at the origin. If $k < 0$, the flow is said to have a sink at the origin.

■

Green's Theorem in the Plane

Let C denote a simple closed curve enclosing a region R of the xy plane. If $M(x, y)$ and $N(x, y)$ are continuous function with continuous derivatives in the region R, then **Green's theorem in the plane** can be written as

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy, \tag{8.13}$$

where the line integral is taken in **a counterclockwise direction** around the simple closed curve C which encloses the region R.

To prove this theorem, let $y = y_2(x)$ and $y = y_1(x)$ be single-valued continuous functions which describe the upper and lower portions C_2 and C_1 of the simple closed curve C in the interval $x_1 \le x \le x_2$ as illustrated in figure 8-7(a).

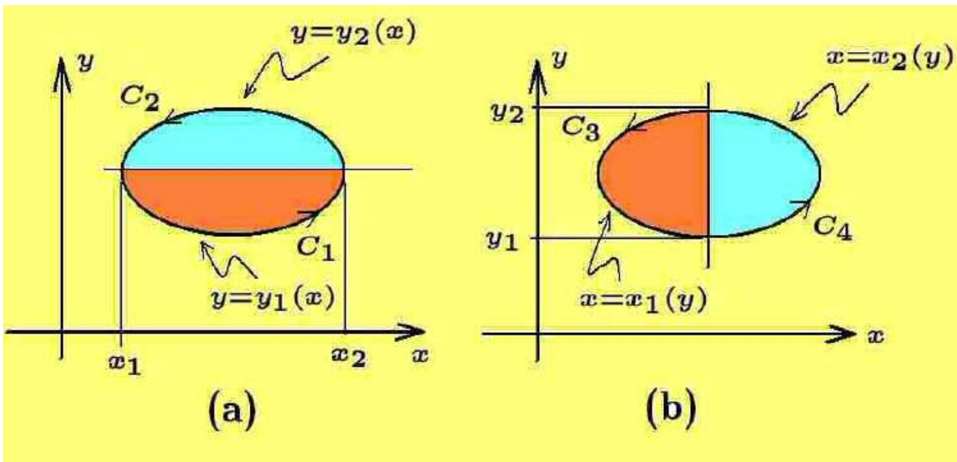


Figure 8-7. Simple closed curve for Green's theorem.

The right-hand side of equation (8.13) can be expressed

$$\begin{aligned}
-\iint_R \frac{\partial M}{\partial y}\,dx\,dy = -\int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} \frac{\partial M}{\partial y}\,dy\,dx = -\int_{x_1}^{x_2} M(x,y)\Big|_{y_1(x)}^{y_2(x)}\,dx \\
= \int_{x_1}^{x_2} [M(x,y_1(x)) - M(x,y_2(x))]\,dx \\
= \int_{x_1}^{x_2} M(x,y_1(x))\,dx + \int_{x_2}^{x_1} M(x,y_2(x))\,dx \\
= \int_{C_1} M(x,y_1(x))\,dx + \int_{C_2} M(x,y_2(x))\,dx = \oint_C M(x,y)\,dx.
\end{aligned} \tag{8.14}$$

Now let $x = x_1(y)$ and $x = x_2(y)$ be single-valued continuous functions which describe the left and right sections C_3 and C_4 of the curve C in the interval $y_1 \le y \le y_2$.

The remaining part of the right-hand side can then be expressed

$$
\begin{aligned}
\iint\limits_R \frac{\partial N}{\partial x}\,dx\,dy &= \int_{y_1}^{y_2}\int_{x_1(y)}^{x_2(y)} \frac{\partial N}{\partial x}\,dx\,dy = \int_{y_1}^{y_2} N(x,y)\Big|_{x_1(y)}^{x_2(y)}\,dy \\
&= \int_{y_1}^{y_2} [N(x_2(y),y) - N(x_1(y),y)]\,dy \\
&= \int_{y_1}^{y_2} N(x_2(y),y)\,dy + \int_{y_2}^{y_1} N(x_1(y),y)\,dy \\
&= \int_{C_3} N(x_1(y),y)\,dy + \int_{C_4} N(x_2(y),y)\,dy = \oint_C N(x,y)\,dy.
\end{aligned}
\tag{8.15}
$$

Adding the results of equations (8.14) and (8.15) produces the desired result.

Example 8-5. Verify Green's theorem in the plane in the special case

$$M(x,y) = x^2 + y^2 \qquad \text{and} \qquad N(x,y) = xy,$$

where C is the wedge shaped curve illustrated in figure 8-8.

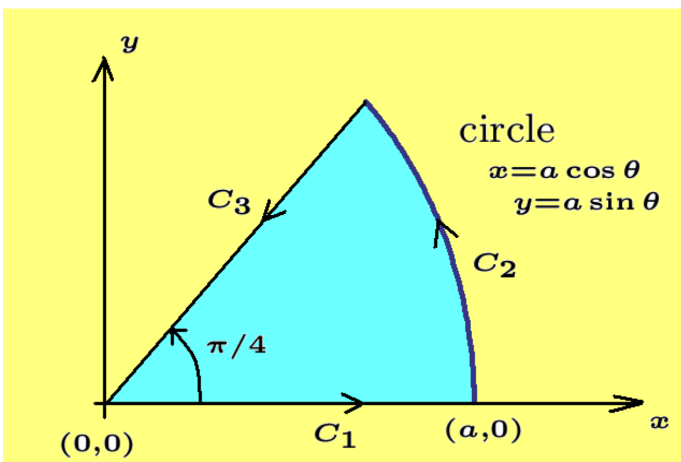


Figure 8-8. Wedge shaped path for Green's theorem example.

Solution The boundary curve can be broken up into three parts and the left-hand side of the Green's theorem can be expressed

$$\oint_C M\,dx + N\,dy = \int_{C_1} M dx + N dy + \int_{C_2} M dx + N dy + \int_{C_3} M dx + N dy.$$

On the curve C_1, where $y = 0,\ dy = 0$, the first integral reduces to

$$\int_{C_1} M dx + N dy = \int_0^a x^2\, dx = \frac{a^3}{3}$$

On the curve C_2, where

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad 0 \le \theta \le \frac{\pi}{4}$$

the second integral reduces to

$$\begin{aligned}\int_{C_2} M dx + N dy &= \int_0^{\pi/4} -a^2 \cdot a\sin\theta\, d\theta + a^2 \sin\theta\cos\theta \cdot a\cos\theta\, d\theta \\ &= a^3 \cos\theta \Big|_0^{\pi/4} - \frac{a^3}{3}\cos^3\theta \Big|_0^{\pi/4} \\ &= a^3(\frac{\sqrt{2}}{2} - 1) - \frac{a^3}{3}\left(\frac{\sqrt{2}}{4} - 1\right) \\ &= a^3\left(\frac{5\sqrt{2}}{12} - \frac{2}{3}\right).\end{aligned}$$

On the curve C_3, where $y = x,\ 0 \le x \le \frac{\sqrt{2}}{2}a$, the third integral can be expressed as

$$\int_{C_3} M dx + N dy = \int_{\frac{\sqrt{2}}{2}a}^0 2x^2\, dx + x^2 dx = -\frac{\sqrt{2}}{4}a^3.$$

Adding the three integrals give us the line integral portion of Green's theorem which is

$$\oint_C M\, dx + N\, dy = \frac{1}{3}a^3 + a^3\left(\frac{5\sqrt{2}}{12} - \frac{2}{3}\right) - \frac{\sqrt{2}}{4}a^3 = \frac{a^3}{3}\left(\frac{\sqrt{2}}{2} - 1\right).$$

The area integral representing the right-hand side of Green's theorem is now evaluated. One finds

$$\frac{\partial N}{\partial x} = y, \qquad \frac{\partial M}{\partial y} = 2y$$

and

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint -y\, dy\, dx$$

The geometry of the problem suggests a transformation to polar coordinates in order to evaluate the integral. Changing to polar coordinates the above integral becomes

$$\int_0^{\pi/4}\int_0^a (r\sin\theta)(r\, dr d\theta) = -\frac{r^3}{3}\Big|_0^a \int_0^{\pi/4} \sin\theta\, d\theta = \frac{a^3}{3}\left(\frac{\sqrt{2}}{2} - 1\right).$$

■

Solution of Differential Equations by Line Integrals

The total differential of a function $\phi = \phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy. \tag{8.17}$$

When the right-hand side of this equation is set equal to zero, the resulting equation is called **an exact differential equation**, and $\phi = \phi(x, y) = Constant$ is called **a primitive or integral of this equation**. The set of curves $\phi(x, y) = C = constant$ represents a family of solution curves to the exact differential equation.

A differential equation of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{8.17}$$

is **an exact differential equation** if there exists a function $\phi = \phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y).$$

If such a function ϕ exists, then the mixed second partial derivatives must be equal and

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial M}{\partial y} = M_y = \frac{\partial^2 \phi}{\partial y\, \partial x} = \frac{\partial N}{\partial x} = N_x. \tag{8.19}$$

Hence **a necessary condition that the differential equation be exact** is that the **partial derivative of M with respect to y must equal the partial derivative of N with respect to x or $M_y = N_x$**. If the differential equation is exact, then Green's theorem tells us that the line integral of $M\, dx + N\, dy$ around a closed curve must equal zero, since

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = 0 \quad \text{because} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \tag{8.19}$$

For an arbitrary path of integration, such as the path illustrated in figure 8-9, the above integral can be expressed

$$\begin{aligned} \oint_C M\, dx + N\, dy = & \int_{x_0}^{x} M(x, y_1(x))\, dx + N(x, y_1(x))\, dy \\ & + \int_{x}^{x_0} M(x, y_2(x))\, dx + N(x, y_2(x))\, dy = 0 \end{aligned}$$

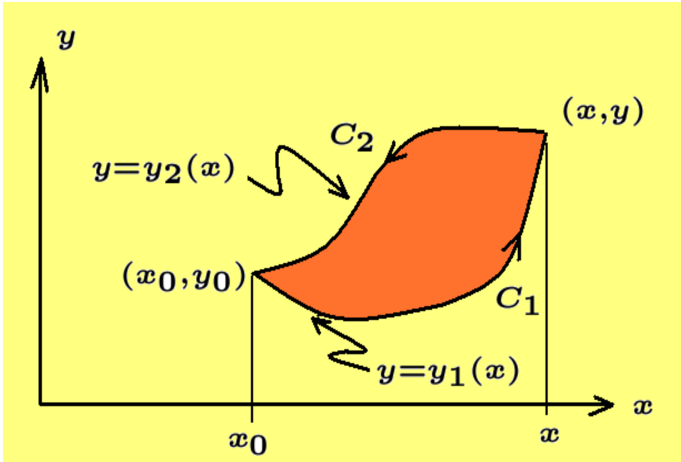


Figure 8-9. Arbitrary paths connecting points (x_0, y_0) and (x, y).

Consequently, one can write

$$\int_{x_0}^{x} M(x, y_1(x))\, dx + N(x, y_1(x))\, dy = \int_{x_0}^{x} M(x, y_2(x))\, dx + N(x, y_2(x))\, dy. \tag{8.20}$$

Equation (8.20) shows that the line integral of $M\,dx + N\,dy$ from (x_0, y_0) to (x, y) is **independent of the path joining these two points.**

It is now demonstrated that the line integral

$$\int_{(x_0,y_0)}^{(x,y)} M(x, y)\, dx + N(x, y)\, dy$$

is a function of x and y which is related to the solution of the exact differential equation $M\,dx + N\,dy = 0$. Observe that if $M\,dx + N\,dy$ is an exact differential, there exists a function $\phi = \phi(x, y)$ such that $\phi_x = M$ and $\phi_y = N$, and the above line integral reduces to

$$\int_{(x_0,y_0)}^{(x,y)} \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = \int_{(x_0,y_0)}^{(x,y)} d\phi = \phi \Big|_{(x_0,y_0)}^{(x,y)} = \phi(x, y) - \phi(x_0, y_0). \tag{8.21}$$

Thus the solution of the differential equation $M\,dx + N\,dy = 0$ can be represented as $\phi(x, y) = C = Constant$, where the function ϕ can be obtained from the integral

$$\phi(x, y) - \phi(x_0, y_0) = \int_{(x_0,y_0)}^{(x,y)} M(x, y)\, dx + N(x, y)\, dy. \tag{8.22}$$

Since the line integral is independent of the path of integration, it is possible to select **any convenient path of integration from** (x_0, y_0) **to** (x, y).

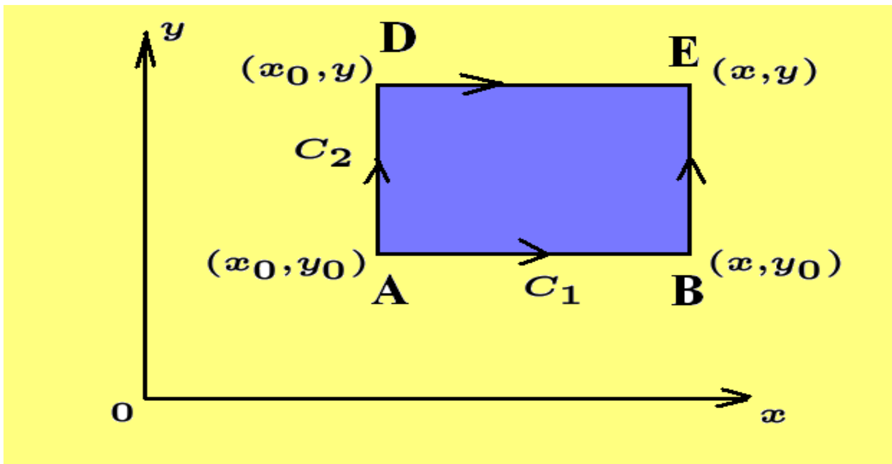


Figure 8-10. Path of integration for solution to exact differential equation.

Illustrated in figure 8-10 are two paths of integration consisting of straight-line segments. The point (x_0, y_0) may be chosen as any convenient point which guarantees that the functions M and N remain bounded and continuous along the line segments joining the point (x_0, y_0) to (x, y). If the path C_1 is selected, note that on the segment AB one finds $y = y_0$ is constant so $dy = 0$ and on the line segment BE one finds x is held constant so $dx = 0$. The line integral is then broken up into two parts and can be expressed in the form

$$\int_{(x_0,y_0)}^{(x,y)} d\phi = \phi(x, y) - \phi(x_0, y_0) = \int_{x_0}^{x} M(x, y_0)\, dx + \int_{y_0}^{y} N(x, y)\, dy, \tag{8.23}$$

where x is held constant in the second integral of equation (8.23). If the path C_2 is chosen as the path of integration note that on AD x is held constant so $dx = 0$ and on the segment DE y is held constant so that $dy = 0$. One should break up the line integral into two parts and express it in the form

$$\int_{(x_0,y_0)}^{(x,y)} d\phi = \phi(x, y) - \phi(x_0, y_0) = \int_{y_0}^{y} N(x_0, y)\, dy + \int_{x_0}^{x} M(x, y)\, dx, \tag{8.24}$$

where y is held constant in the second integral of equation (8.24).

Example 8-6. Find the solution of the exact differential equation

$$(2xy - y^2)\,dx + (x^2 - 2xy)\,dy = 0$$

Solution After verifying that $M_y = N_x$ one can state that the given differential equation is exact. Along the path C_1 illustrated in the figure 8-10 one finds

$$\begin{aligned}\phi(x,y) - \phi(x_0,y_0) &= \int_{x_0}^{x} (2xy_0 - y_0^2)\,dx + \int_{y_0}^{y} (x^2 - 2xy)\,dy \\ &= (x^2 y_0 - y_0^2 x)\Big|_{x_0}^{x} + (x^2 y - xy^2)\Big|_{y_0}^{y} \\ &= \left(x^2 y - xy^2\right) - (x_0^2 y_0 - x_0 y_0^2).\end{aligned}$$

Here $\phi(x,y) = x^2 y - xy^2 = Constant$ represents the solution family of the differential equation. It is left as an exercise to verify that this same result is obtained by performing the integration along the path C_2 illustrated in figure 8-10.

■

Area Inside a Simple Closed Curve.

A very interesting special case of Green's theorem concerns the area enclosed by a simple closed curve. Consider the simple closed curve such as the one illustrated in figure 8-11. Green's theorem in the plane allows one to find the area inside a simple closed curve if **one knows the values of x, y on the boundary of the curve.**

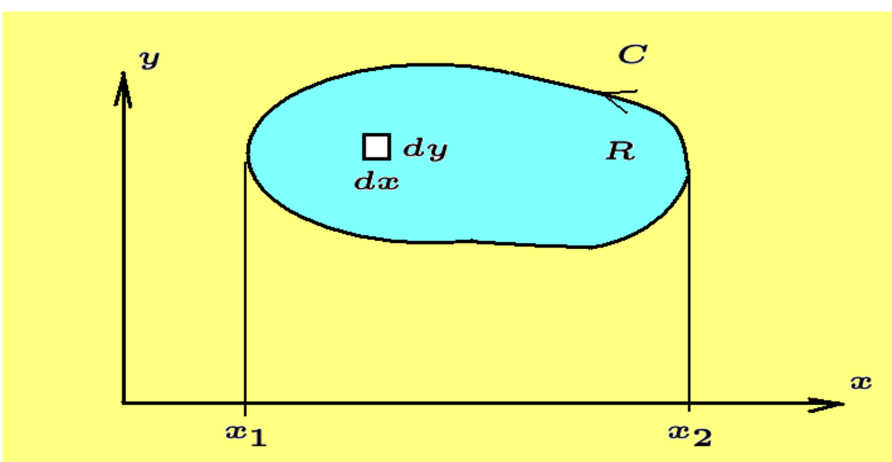


Figure 8-11. Area enclosed by a simple closed curve.

In Green's theorem the functions M and N are arbitrary. Therefore, in the special case $M=-y$ and $N=0$ one obtains

$$\oint_C -y\,dx = \iint_R dx\,dy = A = \text{Area enclosed by C.} \tag{8.25}$$

Similarly, in the special case $M=0$ and $N=x$, the Green's theorem becomes

$$\oint_C x\,dy = \iint_R dx\,dy = A = \text{Area enclosed by C.} \tag{8.26}$$

Adding the results from equations (8.25) and (8.26) produces

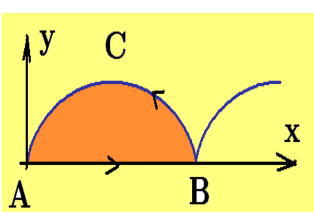
$$\begin{aligned} 2A =& \oint_C x\,dy - y\,dx \\ \text{or} \qquad A =& \frac{1}{2}\oint_C x\,dy - y\,dx = \iint_R dx\,dy = \text{Area enclosed by C.} \end{aligned} \tag{8.27}$$

Therefore the area enclosed by a simple closed curve C can be expressed as a line integral around the boundary of the region R enclosed by C. That is, by knowing the values of x and y on the boundary, one can calculate the area enclosed by the boundary. This is the concept of a device called **a planimeter**, which is a mechanical instrument used for measuring the area of a plane figure by moving a pointer around the surrounding boundary curve.

Example 8-7.

Find the area under the cycloid defined by $x=r(\phi-\sin\phi)$, $y=r(1-\cos\phi)$ for $0\le\phi\le 2\pi$ and $r>0$ is a constant. Find the area illustrated by using a line integral around the boundary of the area moving from A to B to C to A.

Solution



Let BCA and the line AB denote the bounding curves of the area under the cycloid between $\phi=0$ and $\phi=2\pi$. The area is given by the relation

$$A = \frac{1}{2}\int_{C_1}(x\,dy - y\,dx) + \frac{1}{2}\int_{C_2}(x\,dy - y\,dx),$$

where C_1 is the straight-line from A to B and C_2 is the curve

B to C to A. On the straight-line C_1 where $y=0$ and $dy=0$ the first line integral has the value of zero. On the cycloid one finds

$$x = r(\phi - \sin\phi) \qquad\qquad y = r(1-\cos\phi)$$
$$dx = r(1-\cos\phi)\,d\phi \qquad\qquad dy = r\sin\phi\,d\phi$$

and ϕ varies from 2π to zero. By substituting the known values of x and y on the boundary of the cycloid, the line integral along C_2 becomes

$$2A = \int_{2\pi}^{0} r(\phi-\sin\phi)\,r\sin\phi\,d\phi - r(1-\cos\phi)\,r(1-\cos\phi)\,d\phi$$

or
$$2A = r^2\int_0^{2\pi}\left[1-2\cos\phi+\cos^2\phi+\sin^2\phi-\phi\sin\phi\right]\,d\phi$$
$$2A = r^2\left[2\phi-2\sin\phi+\phi\cos\phi-\sin\phi\right]_0^{2\pi}$$
$$2A = 6\pi r^2$$

Hence, the area under the curve is given by $A = 3\pi r^2$.

■

Change of Variable in Green's Theorem

Often it is convenient to change variables in an integration in order to make the integrals more tractable. If x, y are variables which are related to another set of variables u, v by a set of transformation equations

$$x = x(u,v) \qquad y = y(u,v) \tag{8.28}$$

and if these equations are continuous and have partial derivatives, then one can calculate

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv \qquad dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv. \tag{8.29}$$

It is therefore possible to express the area integral (8.27) in the form

$$\begin{aligned}\iint_R dx\,dy =& \frac{1}{2}\oint_C x\,dy - y\,dx\\ \iint_R dx\,dy =& \frac{1}{2}\oint_C x(u,v)\left[\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right] - y(u,v)\left[\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right]\\ =& \frac{1}{2}\oint_C \left[x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\right]\,du + \left[x\frac{\partial y}{\partial v} - y\frac{\partial x}{\partial v}\right]\,dv.\end{aligned} \tag{8.30}$$

where R is a region of the x, y-plane where the area is to be calculated.

Let $M(u,v) = \frac{1}{2}\left(x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\right)$ and $N(u,v) = \frac{1}{2}\left(x\frac{\partial y}{\partial v} - y\frac{\partial x}{\partial v}\right)$ and apply Green's theorem to the integral (8.30). Using the results

$$\frac{\partial M}{\partial v} = \frac{1}{2}\left[x\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - y\frac{\partial^2 x}{\partial u\,\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}\right] \quad \text{and} \quad \frac{\partial N}{\partial u} = \frac{1}{2}\left[x\frac{\partial^2 y}{\partial v\,\partial u} + \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - y\frac{\partial^2 x}{\partial v\,\partial u} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right]$$

one finds

$$\left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = J\left(\frac{x,y}{u,v}\right), \tag{8.31}$$

where the determinant J is called **the Jacobian determinant** of the transformation from (x,y) to (u,v). The area integral can then be expressed in the form

$$A = \iint_R dx\,dy = \iint_{u,v} J\left(\frac{x,y}{u,v}\right) du\,dv, \tag{8.32}$$

where the limits of integration are over that range of the variables u,v which define the region R.

Example 8-8. In changing from rectangular coordinates (x,y) to polar coordinates (r,θ) the transformation equations are

$$x = x(r,\theta) = r\cos\theta \qquad y = y(r,\theta) = r\sin\theta,$$

and the Jacobian of this transformation is

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

and the area can be expressed as

$$\iint_{R_{xy}} dx\,dy = \iint_{R_{r\theta}} r\,dr\,d\theta \tag{8.33}$$

which is the familiar area integral from polar coordinates.

■

In general, an integral of the form $\iint_{R_{xy}} f(x,y)\,dx\,dy$ under a change of variables $x = x(u,v),\ y = y(u,v)$ becomes $\iint_{R_{uv}} f(x(u,v), y(u,v))\,J\left(\frac{x,y}{u,v}\right) du\,dv$, where the integrand is expressed in terms of u and v and the element of area $dx\,dy$ is replaced by the new element of area $J\left(\frac{x,y}{u,v}\right) du\,dv$.

The Curl of a Vector Field

Let $\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,x)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$ denote a continuous vector field possessing continuous derivatives, and let P_0 denote a point in this vector field having coordinates (x_0, y_0, z_0). Insert into this field an arbitrary surface S which contains the point P_0 and construct a unit normal $\hat{\mathbf{e}}_n$ to the surface at point P_0. On the surface construct a simple closed curve C which encircles the point P_0. The work done in moving around the closed curve is called **the circulation at point** P_0. The circulation is a scalar quantity and is expressed as

$$\oint_C \vec{F}\cdot d\vec{r} = \text{Circulation of } \vec{F} \text{ around } C \text{ on the surface } S,$$

where the integration is taken counterclockwise. If the circulation is divided by the area ΔS enclosed by the simple closed curve C, then the limit of the ratio $\frac{\text{Circulation}}{\text{Area}}$ as the area ΔS tends toward zero, is called **the component of the** curl **of** $\vec{F}$ **in the direction** $\hat{\mathbf{e}}_n$ and is written as

$$(\operatorname{curl}\vec{F})\cdot\hat{\mathbf{e}}_n = \lim_{\Delta S\to 0}\frac{\oint_C \vec{F}\cdot d\vec{r}}{\Delta S}. \tag{8.34}$$

To evaluate one component of the curl of a vector field $\vec{F}$ at the point $P_0(x_0, y_0, z_0)$, construct the plane $z = z_0$ which passes through P_0 and is parallel to the xy plane. This plane has the unit normal $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_3$ at all points on the plane. In this plane, consider the circulation at P_0 due to a circle of radius ϵ centered at P_0. The equation of this circle in parametric form is

$$x = x_0 + \epsilon\cos\theta, \qquad y = y_0 + \epsilon\sin\theta, \qquad z = z_0$$

and in vector form $\vec{r} = (x_0 + \epsilon\cos\theta)\,\hat{\mathbf{e}}_1 + (y_0 + \epsilon\sin\theta)\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$. The circulation can be expressed as

$$I = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(x_0 + \epsilon\cos\theta, y_0 + \epsilon\sin\theta, z_0)\,[-\epsilon\sin\theta\,\hat{\mathbf{e}}_1 + \epsilon\cos\theta\,\hat{\mathbf{e}}_2]\,d\theta.$$

By expanding $\vec{F} = \vec{F}(x_0 + \epsilon\cos\theta, y_0 + \epsilon\sin\theta, z_0)$ in a Taylor series about $\epsilon = 0$, one finds

$$\vec{F}(x_0 + \epsilon\cos\theta, y_0 + \epsilon\sin\theta, z_0) = \vec{F}(x_0, y_0, z_0) + \epsilon\frac{d\vec{F}}{d\epsilon} + \frac{\epsilon^2}{2!}\frac{d^2\vec{F}}{d\epsilon^2} + \cdots,$$

where all the derivatives are evaluated at $\epsilon = 0$. The circulation can be written as

$$I = \oint_C \vec{F}\cdot d\vec{r} = \epsilon\mu_0 + \epsilon^2\mu_1 + \epsilon^3\mu_2 + \cdots,$$

where

$$\mu_0 = \int_0^{2\pi} \vec{F}_0 \cdot d\vec{\xi}, \qquad F_0 = F(x_0, y_0, z_0)$$
$$\mu_1 = \int_0^{2\pi} \frac{d\vec{F}}{d\epsilon} \cdot d\vec{\xi}$$
$$\mu_2 = \int_0^{2\pi} \frac{1}{2!} \frac{d^2\vec{F}}{d\epsilon^2} \cdot d\vec{\xi}$$
$$\cdots$$

where all the derivatives are evaluated at $\epsilon = 0$ and $d\vec{\xi} = (-\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2)\,d\theta$. The vector $\vec{F}_0$ is a constant and the integral μ_0 is easily shown to be zero. The vector $\frac{d\vec{F}}{d\epsilon}$ evaluated at $\epsilon = 0$, when expanded is given by

$$\begin{aligned}\frac{d\vec{F}}{d\epsilon} = \frac{\partial \vec{F}}{\partial x}\cos\theta + \frac{\partial \vec{F}}{\partial y}\sin\theta = & \left(\frac{\partial F_1}{\partial x}\cos\theta + \frac{\partial F_1}{\partial y}\sin\theta\right)\hat{\mathbf{e}}_1 \\ & + \left(\frac{\partial F_2}{\partial x}\cos\theta + \frac{\partial F_2}{\partial y}\sin\theta\right)\hat{\mathbf{e}}_2 \\ & + \left(\frac{\partial F_3}{\partial x}\cos\theta + \frac{\partial F_3}{\partial y}\sin\theta\right)\hat{\mathbf{e}}_3,\end{aligned}$$

where the partial derivatives are all evaluated at $\epsilon = 0$. It is readily verified that the integral μ_1 reduces to

$$\mu_1 = \pi\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

The area of the circle surrounding P_0 is $\pi\epsilon^2$, and consequently the ratio of the circulation divided by the area in the limit as ϵ tends toward zero produces

$$(\text{ curl } \vec{F}\,) \cdot \hat{\mathbf{e}}_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}. \tag{8.35}$$

Similarly, by considering other planes through the point P_0 which are parallel to the xz and yz planes, arguments similar to those above produce the relations

$$(\text{curl } \vec{F}\,) \cdot \hat{\mathbf{e}}_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \quad \text{and} \quad (\text{ curl } \vec{F}\,) \cdot \hat{\mathbf{e}}_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}. \tag{8.36}$$

Adding these components gives the mathematical expression for curl $\vec{F}$. One finds the curl $\vec{F}$ can be written as

$$\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{e}}_3. \tag{8.37}$$

The curl $\vec{F}$ can be expressed by using the operator ∇ in the determinant form

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{\mathbf{e}}_1[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}] - \hat{\mathbf{e}}_2[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}] + \hat{\mathbf{e}}_3[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}] \quad (8.38)$$

Example 8-9. Find the curl of the vector field

$$\vec{F} = x^2 y\, \hat{\mathbf{e}}_1 + (x^2 + y^2 z)\, \hat{\mathbf{e}}_2 + 4xyz\, \hat{\mathbf{e}}_3$$

Solution From the relation (8.38) one finds

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & x^2 + y^2 z & 4xyz \end{vmatrix}$$

$$\text{curl } \vec{F} = (4xz - y^2)\, \hat{\mathbf{e}}_1 - 4yz\, \hat{\mathbf{e}}_2 + (2x - x^2)\, \hat{\mathbf{e}}_3.$$

■

Physical Interpretation of Curl

The curl of a vector field is itself a vector field. If curl $\vec{F} = \vec{0}$ at all points of a region R, where $\vec{F}$ is defined, then the vector field $\vec{F}$ is called an **irrotational vector field**, otherwise the vector field is called **rotational.**

The circulation $\oint_C \vec{F} \cdot d\vec{r}$ about a point P_0 can be written as

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \oint_C \vec{F} \cdot \hat{\mathbf{e}}_t\, ds$$

where C is a simple closed curve about the point P_0 enclosing an area. The quantity $\vec{F} \cdot \hat{\mathbf{e}}_t$, evaluated at a point on the curve C, represents the projection of the vector $\vec{F}(x, y, z)$, onto the unit tangent vector to the curve C. If the summation of these tangential components around the simple closed curve is positive or negative, then this indicates that there is a moment about the point P_0 which causes a rotation. The circulation is **a measure of the forces tending to produce a rotation about a given point** P_0. The curl is the limit of the circulation divided by the area surrounding P_0 as the area tends toward zero. The curl can also be thought of as **a measure of the circulation density of the field** or as **a measure of the angular velocity produced by the vector field.**

Consider the two-dimensional velocity field $\vec{V} = V_0\, \hat{\mathbf{e}}_1, \quad 0 \le y \le h$, where V_0 is constant, which is illustrated in figure 8-12(a). The velocity field $\vec{V} = V_0\, \hat{\mathbf{e}}_1$ is uniform, and to each point (x, y) there corresponds a constant velocity vector in the $\hat{\mathbf{e}}_1$ direction. The curl of this velocity field is zero since the derivative of a constant is zero. The given velocity field is an example of an irrotational vector field.

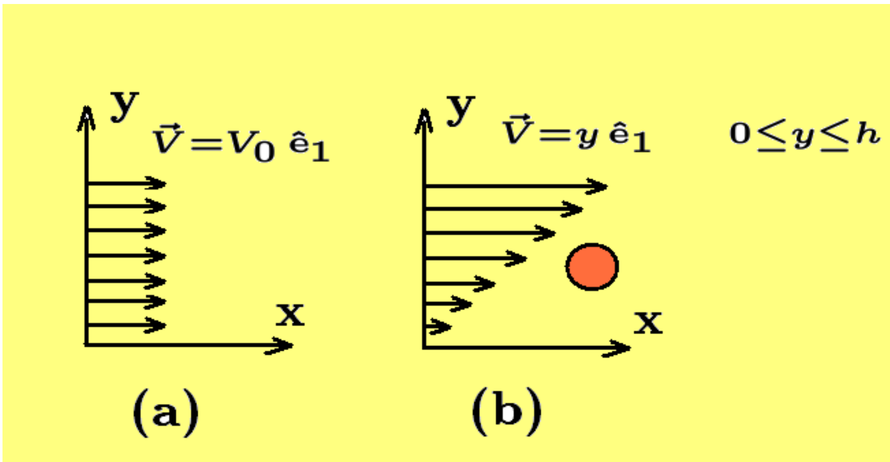


Figure 8-12. Comparison of two dimensional velocity fields.

In comparison, consider the two-dimensional velocity field $\vec{V} = y\,\hat{\mathbf{e}}_1, \quad 0 \le y \le h$, which is illustrated in figure 8-12(b). Here the velocity field may be thought of as representing the flow of fluid in a river. The curl of this velocity field is

$$\text{curl}\,\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\,\hat{\mathbf{e}}_3.$$

In this example, the velocity field is rotational. Consider a spherical ball dropped into this velocity field. The curl $\vec{V}$ tells us that the ball rotates in a clockwise direction about an axis normal to the xy plane. Observe the difference in velocities of the water particles acting upon the upper and lower surfaces of the sphere which cause the clockwise rotation.

Using the right-hand rule, let the fingers of the right hand move in the direction of the rotation. The thumb then points in the $-\,\hat{\mathbf{e}}_3$ direction.

The curl tells us the direction of rotation, but it does not tell us the angular velocity associated with a point as the following example illustrates. Consider a basin of water in which the water is rotating with a constant angular velocity $\vec{\omega} = \omega_0\,\hat{\mathbf{e}}_3$. The velocity of a particle of fluid at a position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ is given by

$$\vec{V} = \vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & \omega_0 \\ x & y & 0 \end{vmatrix} = -\omega_0 y\,\hat{\mathbf{e}}_1 + \omega_0 x\,\hat{\mathbf{e}}_2.$$

The curl of this velocity field is

$$\text{curl}\ \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega_0 y & -\omega_0 x & 0 \end{vmatrix} = 2\omega_0\, \hat{\mathbf{e}}_3.$$

The curl tells us the direction of the angular velocity but not its magnitude.

Stokes Theorem

Let $\vec{F} = \vec{F}(x, y, z)$ denote a vector field having continuous derivatives in a region of space. Let S denote an open two-sided surface in the region of the vector field. For any simple closed curve C lying on the surface S, the following integral relation holds

$$\iint_S (\text{curl}\ \vec{F}\,) \cdot d\vec{S} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{\mathbf{e}}_n\, dS = \oint_C \vec{F} \cdot d\vec{r}, \tag{8.39}$$

where the surface integrations are understood to be over the portion of the surface enclosed by the simple closed curve C lying on S and the line integral around C is in the positive sense with respect to the normal vector to the surface bounded by the simple closed curve C. The above integral relation is known as **Stokes theorem.**[1] In scalar form, the line and surface integrals in Stokes theorem can be expressed as

$$\begin{aligned} \iint_S (\text{ curl}\ \vec{F}\,) \cdot d\vec{S} &= \iint_S \text{curl}\ \vec{F} \cdot \hat{\mathbf{e}}_n\, dS \\ &= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\mathbf{e}}_3 \right] \cdot \hat{\mathbf{e}}_n\, dS \end{aligned} \tag{8.40}$$

and $$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_1\, dx + F_2\, dy + F_3\, dz,$$

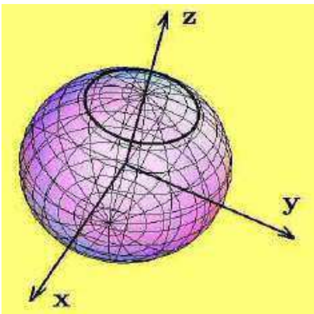
where $\hat{\mathbf{e}}_n$ is a unit normal to the surface S inside the closed curve C. In this case the path of integration C is counterclockwise with respect to this normal. By the right-hand rule if you place the thumb of your right hand in the direction of the normal, then your fingers indicate the direction of integration in the counterclockwise or positive sense.

[1] George Gabriel Stokes (1819-1903) An Irish mathematician who studied hydrodynamics.

Example 8-10. Illustrate Stokes theorem using the vector field

$$\vec{F} = yz\,\hat{\mathbf{e}}_1 + xz^2\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3,$$

where the surface S is a portion of a sphere of radius r inside a circle on the sphere.



The surface of the sphere can be described by the parametric equations.

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

for r constant, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. The position vector to a point on the sphere being represented

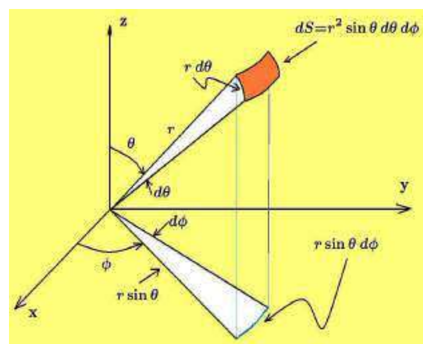
$$\vec{r} = \vec{r}(\theta,\phi) = r\sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + r\sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + r\cos\theta\,\hat{\mathbf{e}}_3 \qquad (r \text{ is constant})$$

From the previous chapter we found an element of surface area on the sphere can be represented

$$dS = \left|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\phi}\right| d\theta\,d\phi = \sqrt{EG-F^2}\,d\theta\,d\phi = r^2\sin\theta\,d\theta\,d\phi \qquad (r \text{ is constant}) \tag{8.41}$$

The physical interpretation of dS being that it is the area of the parallelogram having the sides $\frac{\partial\vec{r}}{\partial\theta}\,d\theta$ and $\frac{\partial\vec{r}}{\partial\phi}\,d\phi$ with diagonal vector

$$d\vec{r} = \frac{\partial\vec{r}}{\partial\theta}\,d\theta + \frac{\partial\vec{r}}{\partial\phi}\,d\phi$$



If one holds $\theta = \theta_0$ constant, one obtains a circle C on the sphere described by

$$\vec{r} = \vec{r}(\phi) = r\sin\theta_0\cos\phi\,\hat{\mathbf{e}}_1 + r\sin\theta_0\sin\phi\,\hat{\mathbf{e}}_2 + r\cos\theta_0\,\hat{\mathbf{e}}_3, \qquad 0 \le \phi \le 2\pi \tag{8.42}$$

A unit outward normal to the sphere and inside the circle C is given by

$$\hat{\mathbf{e}}_n = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3, \qquad \begin{matrix} 0 \le \phi \le 2\pi \\ 0 \le \theta \le \theta_0 \end{matrix} \tag{8.43}$$

The vector $\operatorname{curl}\vec{F}$ is calculated from the determinant

$$\begin{aligned}\operatorname{curl}\vec{F} =& \nabla\times\vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz^2 & xy \end{vmatrix} \\ =& \hat{\mathbf{e}}_1(x-2xz) + \hat{\mathbf{e}}_2(0) + \hat{\mathbf{e}}_3(z^2-z)\end{aligned}$$

The left-hand side of Stokes theorem can be expressed

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathbf{e}}_n \, dS = \iint_S \left[x(1-2z)\sin\theta\cos\phi + (z^2 - z)\cos\theta \right] \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\theta_0} \left[r\sin^2\theta\cos^2\phi(1-2r\cos\theta) + (r^2\cos^3\theta - r\cos^2\theta) \right] \, r^2\sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \frac{r^3}{12} \Big[4(-1+\cos^3\theta_0) - 3r(-1+\cos^4\theta_0) + 16(2+\cos\theta_0)\cos^2\phi\sin^4\left(\frac{\theta_0}{2}\right) - 6r\cos^2\phi\sin^4\theta_0 \Big] \, d\phi$$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \pi r^3 \cos\theta_0(-1 + r\cos\theta_0)\sin^2\theta_0$$

The right-hand side of Stokes theorem can be expressed

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C yz\,dx + xz^2\,dy + xy\,dz$$

Substitute the values of x, y, z on the curve C using

$$x = r\sin\theta_0\cos\phi, \qquad y = r\sin\theta_0\sin\phi, \qquad z = r\cos\theta_0$$

$$dx = -r\sin\theta_0\sin\phi\,d\phi, \qquad dy = r\sin\theta_0\cos\phi, \qquad dz = 0$$

The right-hand side of Stokes theorem then becomes

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \left[yz(-r\sin\theta_0\sin\phi) + xz^2(r\sin\theta_0\cos\phi) + xy(0) \right] \, d\phi$$

$$= \int_0^{2\pi} \left[r\sin\theta_0\sin\phi(r\cos\theta_0)(-r\sin\theta_0\sin\phi) + (r\sin\theta_0\cos\phi)(r^2\cos^2\theta_0)(r\sin\theta_0\cos\phi) \right] \, d\phi$$

$$= \pi r^3\cos\theta_0(-1 + r\cos\theta_0)\sin^2\theta_0$$

■

Proof of Stokes Theorem

To prove Stokes theorem one could verify each of the following integral relations

$$\begin{aligned}
\iint_S \left(\frac{\partial F_1}{\partial z} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n - \frac{\partial F_1}{\partial y} \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n \right) dS &= \oint_C F_1\,dx \\
\iint_S \left(\frac{\partial F_2}{\partial x} \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n - \frac{\partial F_2}{\partial z} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n \right) dS &= \oint_C F_2\,dy \\
\iint_S \left(\frac{\partial F_3}{\partial y} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n - \frac{\partial F_3}{\partial x} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n \right) dS &= \oint_C F_3\,dz.
\end{aligned} \tag{8.44}$$

Then an addition of these integrals would produce the Stokes theorem as given by equation (8.40). However, the arguments used in proving the above integrals are repetitious, and so only the first integral is verified.

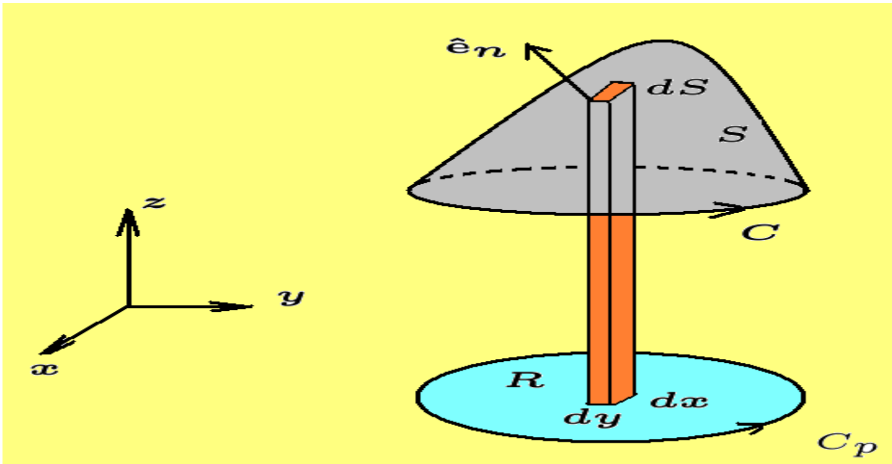


Figure 8-13. Surface S bounded by a simple closed curve C on the surface.

Let $z = z(x, y)$ define the surface S and consider the projections of the surface S and the curve C onto the plane $z = 0$ as illustrated in figure 8-13. Call these projections R and C_p. The unit normal to the surface has been shown to be of the form

$$\hat{\mathbf{e}}_n = \frac{-\dfrac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1 - \dfrac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}} \tag{8.45}$$

Consequently, one finds

$$\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2 = \frac{-\frac{\partial z}{\partial y}}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}}, \qquad \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}} \qquad (8.46))$$

The element of surface area can be expressed as

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

and consequently the integral on the left side of equation (8.44) can be simplified to the form

$$\iint\limits_S \left(\frac{\partial F_1}{\partial z}\,\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_n - \frac{\partial F_1}{\partial y}\,\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n\right) dS = \iint\limits_S \left(-\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial F_1}{\partial y}\right) dx\, dy \tag{8.47}$$

Now on the surface S defined by $z = z(x,y)$ one finds $F_1 = F_1(x,y,z) = F_1(x,y,z(x,y))$ is a function of x and y so that a differentiation of the composite function F_1 with respect to y produces

$$\frac{\partial F_1(x,y,z(x,y))}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} \tag{8.48}$$

which is the integrand in the integral (8.47) with the sign changed. Therefore, one can write

$$\iint_S \left(-\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial F_1}{\partial y}\right) dx\,dy = -\iint_S \frac{\partial F_1(x,y,z(x,y))}{\partial y}\,dx\,dy \tag{8.49}$$

Now by using Greens theorem with $M(x,y) = F_1(x,y,z(x,y))$ and $N(x,y) = 0$, the integral (8.49) can be expressed as

$$-\iint_S \frac{\partial F_1(x,y,z(x,y))}{\partial y}\,dx\,dy = \int_{C_p} F_1(x,y,z(x,y))\,dx = \int_C F_1(x,y,z)\,dx \tag{8.50}$$

which verifies the first integral of the equations (8.44). The remaining integrals in equations (8.44) may be verified in a similar manner.

Example 8-11. Verify Stokes theorem for the vector field

$$\vec{F} = 3x^2y\,\hat{\mathbf{e}}_1 + x^2y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3,$$

where S is the upper half of the sphere $x^2+y^2+z^2=1$.

Solution The given vector field has the curl vector

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^2y & z \end{vmatrix} = (2xy - 3x^2)\,\hat{\mathbf{e}}_3.$$

The unit normal to the sphere at a general point (x,y,z) on the sphere is given by

$$\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

and the element of surface area dS when projected upon the xy plane is

$$dS = \frac{dx\,dy}{\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3} = \frac{dx\,dy}{z}.$$

The surface integral portion of Stokes theorem can therefore by expressed as

$$\begin{aligned}
\iint_S \text{curl } \vec{F}\, d\vec{S} =& \iint_S (2xy - 3x^2)\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_n \, dS = \iint_S (2xy - 3x^2)\, dx\, dy \\
=& \int_{-1}^{1} \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} x(2y - 3x)\, dy\, dx = \int_{-1}^{1} x(y^2 - 3xy)\Big|_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} dx \\
=& \int_{-1}^{1} \left\{ x\left[(1-x^2) - 3x\sqrt{1-x^2}\right] - x\left[(1-x^2) + 3x\sqrt{1-x^2}\right]\right\} dx \\
=& \int_{-1}^{1} -6x^2\sqrt{1-x^2}\, dx = -\frac{3\pi}{4}.
\end{aligned}$$

For the line integral portion of Stokes theorem one should observe the boundary of the surface S is the circle

$$x = \cos\theta \qquad y = \sin\theta \qquad z = 0 \qquad 0 \le \theta \le 2\pi.$$

Consequently, there results

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} =& \oint_C 3x^2y\, dx + x^2y\, dy + z\, dz \\
=& \int_0^{2\pi} -3\cos^2\theta \sin^2\theta\, d\theta + \cos^3\theta \sin\theta\, d\theta = -\frac{3\pi}{4}.
\end{aligned}$$

Think of the unit circle $x^2 + y^2 = 1$ with a rubber sheet over it. The hemisphere in this example is assumed to be formed by stretching this rubber sheet. All two-sided surfaces that result by deforming the rubber sheet in a continuous manner are surfaces for which Stokes theorem is applicable.

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Related Integral Theorems

Let ϕ denote a scalar field and $\vec{F}$ a vector field. These fields are assumed to be continuous with continuous derivatives. For the volumes, surfaces, and simple closed curves of Stokes theorem and the divergence theorem, there exist the additional integral relationships

$$\iiint_V \text{curl } \vec{F}\, dV = \iiint_V \nabla \times \vec{F}\, dV = \iint_S \hat{\mathbf{e}}_n \times \vec{F} dS \tag{8.51}$$

$$\iiint_V \text{grad } \phi\, dV = \iiint_V \nabla\, \phi\, dV = \iint_S \phi\, \hat{\mathbf{e}}_n\, dS \tag{8.52}$$

$$\iint_S \hat{\mathbf{e}}_n \times \text{grad } \phi\, dS = \iint_S \hat{\mathbf{e}}_n \times \text{grad } \phi\, dS = -\iint_S \text{grad } \phi \times d\vec{S} = \oint_C \phi\, d\vec{r}. \tag{8.53}$$

The integral relation (8.51) follows from the divergence theorem. In the divergence theorem, substitute $\vec{F} = \vec{H} \times \vec{C}$, where $\vec{C}$ is an arbitrary constant vector. By using the vector relations

$$\begin{aligned}\nabla \cdot (\vec{H} \times \vec{C}) =&\vec{C} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{C})\\ \text{div } \vec{F} = \nabla \cdot \vec{F} = \nabla \cdot (\vec{H} \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{H}) \qquad &\text{and} \\ \vec{F} \cdot \hat{\mathbf{e}}_n = (\vec{H} \times \vec{C}) \cdot \hat{\mathbf{e}}_n = \vec{H} \cdot (\vec{C} \times \hat{\mathbf{e}}_n) = \vec{C} \cdot (\hat{\mathbf{e}}_n \times \vec{H}) \qquad &\text{(triple scalar product)},\end{aligned} \tag{8.54}$$

the divergence theorem can be written as

$$\iiint_V \text{div } (\vec{H} \times \vec{C})\, dV = \iiint_V \vec{C} \cdot (\nabla \times \vec{H})\, dV = \iint_S (\vec{H} \times \vec{C}) \cdot \hat{\mathbf{e}}_n \, dS = \iint_S \vec{C} \cdot (\hat{\mathbf{e}}_n \times \vec{H})\, dS \tag{8.55}$$

Since $\vec{C}$ is a constant vector one may write

$$\vec{C} \cdot \iint_V \nabla \times \vec{H} \, dV = \vec{C} \cdot \iint_S \hat{\mathbf{e}}_n \times \vec{H} \, dS. \tag{8.56}$$

For arbitrary $\vec{C}$ this relation implies

$$\iiint_V \nabla \times \vec{H} \, dV = \iint_S \hat{\mathbf{e}}_n \times \vec{H} \, dS. \tag{8.57}$$

In this integral replace $\vec{H}$ by $\vec{F}$ ($\vec{H}$ is arbitrary) to obtain the relation (8.51).

The integral (8.52) also is a special case of the divergence theorem. If in the divergence theorem one makes the substitution $\vec{F} = \phi \vec{C}$, where ϕ is a scalar function of position and $\vec{C}$ is an arbitrary constant vector, there results

$$\iint_V \text{div } \vec{F} \, dV = \iiint_V \nabla(\phi \vec{C})\, dV = \iiint_V \vec{C} \cdot \nabla\phi \, dV = \iint_S \vec{C} \phi \, d\vec{S}. \tag{8.58}$$

where the vector identity $\nabla(\phi \vec{C}) = (\nabla\phi) \cdot \vec{C} + \phi(\nabla \times \vec{C})$ has been employed. The relation given by equation (8.58), for an arbitrary constant vector $\vec{C}$, produces the integral relation (8.52).

The integral (8.53) is a special case of Stokes theorem. If in Stokes theorem one substitutes $\vec{F} = \phi \vec{C}$, where $\vec{C}$ is a constant vector, there results

$$\begin{aligned}\iint_S (\text{ curl } \vec{F}\,) \cdot d\vec{S} = \iint_S \nabla \times (\phi \vec{C}) \cdot \hat{\mathbf{e}}_n \, dS =& \iint_S (\nabla \phi \times \vec{C}) \cdot \hat{\mathbf{e}}_n \, dS \\ =& \iint_S (\hat{\mathbf{e}}_n \times \nabla \phi) \cdot \vec{C} \, dS = \oint_C \vec{C} \phi \, d\vec{r}.\end{aligned} \tag{8.59}$$

For arbitrary $\vec{C}$, this integral implies the relation (8.53). That is, one can factor out the constant vector $\vec{C}$ as long as this vector is different from zero. Under these conditions the integral relation (8.53) must hold.

Region of Integration

Green's, Gauss' and Stokes theorems are valid only if certain conditions are satisfied. In these theorems it has been assumed that the integrands are continuous inside the region and on the boundary where the integrations occur. Also assumed is that all necessary derivatives of these integrands exist and are continuous over the regions or boundaries of the integration. In the study of the various vector and scalar fields arising in engineering and physics, there are times when discontinuities occur at points inside the regions or on the boundaries of the integration. Under these circumstances the above theorems are still valid but one must modify the theorems slightly. Modification is done by using superposition of the integrals over each side of a discontinuity and under these circumstances there usually results some kind of a jump condition involving the value of the field on either side of the discontinuity.

If a region of space has the property that every simple closed curve within the region can be deformed or shrunk in a continuous manner to a single point within the region, without intersecting a boundary of the region, then the region is said to be **simply connected.** If in order to shrink or reduce a simply closed curve to a point the curve must leave the region under consideration, then the region is said to be **a multiply connected region.** An example of a multiply connected region is the surface of a torus. Here a circle which encloses the hole of this doughnut-shaped region cannot be shrunk to a single point without leaving the surface, and so the region is called a multiply-connected region.

If a region is multiply connected it usually can be **modified by introducing imaginary cuts or lines within the region and requiring that these lines cannot be crossed.** By introducing appropriate cuts, one can usually modify a multiply connected region into a simply connected region. **The theorems of Gauss, Green, and Stokes are applicable to simply connected regions or multiply connected regions which can be reduced to simply connected regions by introducing suitable cuts.**

Example 8-12. Consider the evaluation of a line integral around a curve in a multiply connected region. Let the multiply connected region be bounded by curves like $C_0, C_1, \ldots, C_n$ as illustrated in figure 8-14(a).

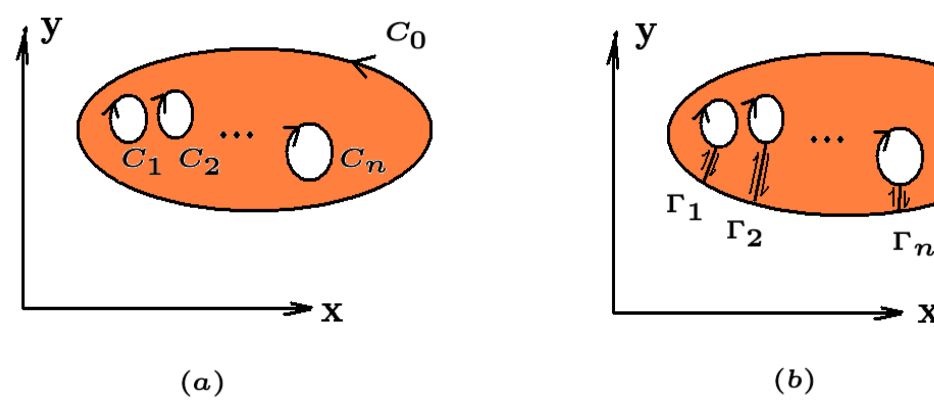


Figure 8-14. Multiply-connected region.

Such a region can be converted to a simply connected region by introducing cuts Γ_i, $i = 1, \ldots, n$. Observe that one can integrate along C_0 until one comes to a cut, say for example the cut Γ_1 in figure 8-14. Since it is not possible to cross a cut, one must integrate along Γ_1 to the curve C_1, then move about C_1 clockwise and then integrate along Γ_1 back to the curve C_0. Continue this process for each of the cuts one encounter as one moves around C_0. Note that the line integrals along the cuts add to zero in pairs (i.e. from C_0 to C_i and from C_i to C_0 for each $i = 1, 2, \ldots n$), then one is left with only the line integrals around the curves $C_0, C_1, \ldots, C_n$ in the sense illustrated in figure 8-14(b).

■

Green's First and Second Identities

Two special cases of the divergence theorem, known as **Green's first and second identities**, are generated as follows.

In the divergence theorem, make the substitution $\vec{F} = \psi \nabla \phi$ to obtain

$$\iiint_V \nabla \vec{F}\, dV = \iiint_V \nabla\left(\psi \nabla \phi\right)\, dV = \iint_S \psi \nabla \phi \cdot d\vec{S} = \iint_S \psi \frac{\partial \phi}{\partial n}\, dS \tag{8.60}$$

where $\dfrac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{\mathbf{e}}_n$ is known as a normal derivative at the boundary. Using the relation

$$\nabla(\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi$$

one can express equation (8.60) in the form

$$\iint\limits_V \left(\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi\right)\,dV = \iint\limits_S \psi\nabla\phi\cdot d\vec{S} = \iint\limits_S \psi\frac{\partial\phi}{\partial n}\,dS \tag{8.61}$$

This result is known as **Green's first identity.**

In Green's first identity interchange ψ and ϕ to obtain

$$\iint\limits_V \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)\,dV = \iint\limits_S \phi\nabla\psi\cdot d\vec{S} = \iint\limits_S \phi\frac{\partial\psi}{\partial n}\,dS \tag{8.62}$$

Subtracting equation (8.61) from equation (8.62) produces **Green's second identity**

$$\iiint\limits_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,dV = \iint\limits_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot d\vec{S} \tag{8.63}$$

or

$$\iiint\limits_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,dV = \iint\limits_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\,dS$$

Green's first and second identities have many uses in studying scalar and vector fields arising in science and engineering.

Additional Operators

The del operator in Cartesian coordinates

$$\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3 \tag{8.64}$$

has been used to express the gradient of a scalar field and the divergence and curl of a vector field. There are other operators involving the operator ∇. In the following list of operators let $\vec{A}$ denote a vector function of position which is both continuous and differentiable.

1. The operator $\vec{A}\cdot\nabla$ is defined as

$$\begin{aligned}\vec{A}\cdot\nabla =& \left(A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3\right)\cdot\left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right)\\ =& A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\end{aligned} \tag{8.65}$$

Note that $\vec{A}\cdot\nabla$ is an operator which can operate on vector or scalar quantities.

2. The operator $\vec{A}\times\nabla$ is defined as

$$\begin{aligned}\vec{A}\times\nabla =& \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ A_1 & A_2 & A_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}\\ =& \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{\mathbf{e}}_1 + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{\mathbf{e}}_2 + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{\mathbf{e}}_3\end{aligned} \tag{8.66}$$

This operator is a vector operator.

3. The Laplacian operator $\nabla^2 = \nabla \cdot \nabla$ in rectangular Cartesian coordinates is given by

$$\nabla^2 = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right) \cdot \left(\frac{\partial}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial}{\partial z}\,\hat{\mathbf{e}}_3\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2} \tag{8.67}$$

This operator can operate on vector or scalar quantities.

One must be careful in the use of operators because **in general, they are not commutative.** They operate only on the quantities to their immediate right.

Example 8-13. For the vector and scalar fields defined by

$$\vec{B} = xyz\,\hat{\mathbf{e}}_1 + (x+y)\,\hat{\mathbf{e}}_2 + (z-x)\,\hat{\mathbf{e}}_3 = B_1\,\hat{\mathbf{e}}_1 + B_2\,\hat{\mathbf{e}}_2 + B_3\,\hat{\mathbf{e}}_3$$

$$\vec{A} = x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3 = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$$

$$\text{and} \quad \phi = x^2y^2 + z^2yx$$

evaluate each of the following.

$$(a)\quad (\vec{A}\cdot\nabla)\vec{B} \qquad (b)\quad (\vec{A}\times\nabla)\cdot\vec{B}$$

$$(c)\quad \nabla^2\vec{A} \qquad (d)\quad (\vec{A}\times\nabla)\phi$$

Solution

(a)

$$\begin{aligned}
(\vec{A}\cdot\nabla)\vec{B} =& \; x^2\frac{\partial\vec{B}}{\partial x} + xy\frac{\partial\vec{B}}{\partial y} + y^2\frac{\partial\vec{B}}{\partial z}\\
=& \; x^2\left(\frac{\partial B_1}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial B_2}{\partial x}\,\hat{\mathbf{e}}_2 + \frac{\partial B_3}{\partial x}\,\hat{\mathbf{e}}_3\right)\\
&+ xy\left(\frac{\partial B_1}{\partial y}\,\hat{\mathbf{e}}_1 + \frac{\partial B_2}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial B_3}{\partial y}\,\hat{\mathbf{e}}_3\right)\\
&+ y^2\left(\frac{\partial B_1}{\partial z}\,\hat{\mathbf{e}}_1 + \frac{\partial B_2}{\partial z}\,\hat{\mathbf{e}}_2 + \frac{\partial B_3}{\partial z}\,\hat{\mathbf{e}}_3\right)\\
=& \; \left[x^2(yz) + xy(xz) + y^2(xy)\right]\,\hat{\mathbf{e}}_1 + \left[x^2(1) + xy(1)\right]\,\hat{\mathbf{e}}_2 + \left[x^2(-1) + y^2(1)\right]\,\hat{\mathbf{e}}_3
\end{aligned}$$

(b)

$$\begin{aligned}
(\vec{A}\times\nabla)\cdot\vec{B} =& \left[\left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{\mathbf{e}}_1 + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{\mathbf{e}}_2 + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{\mathbf{e}}_3\right]\cdot\vec{B}\\
=& \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)B_1 + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)B_2 + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)B_3\\
=& \left[xy\frac{\partial(xyz)}{\partial z} - y^2\frac{\partial(xyz)}{\partial y}\right] + \left[y^2\frac{\partial(x+y)}{\partial x} - x^2\frac{\partial(x+y)}{\partial z}\right] + \left[x^2\frac{\partial(z-x)}{\partial y} - xy\frac{\partial(z-x)}{\partial x}\right]\\
=& \; x^2y^2 - y^2xz + y^2 + xy
\end{aligned}$$

(c)

$$\nabla^2 \vec{A} = \frac{\partial^2}{\partial x^2}(x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3) + \frac{\partial^2}{\partial y^2}(x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3) + \frac{\partial^2}{\partial z^2}(x^2\,\hat{\mathbf{e}}_1 + xy\,\hat{\mathbf{e}}_2 + y^2\,\hat{\mathbf{e}}_3)$$
$$= 2\,\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_3$$

(d)

$$(\vec{A} \times \nabla)\phi = (xy\frac{\partial \phi}{\partial z} - y^2\frac{\partial \phi}{\partial y})\,\hat{\mathbf{e}}_1 + (y^2\frac{\partial \phi}{\partial x} - x^2\frac{\partial \phi}{\partial z})\,\hat{\mathbf{e}}_2 + (x^2\frac{\partial \phi}{\partial y} - xy\frac{\partial \phi}{\partial x})\,\hat{\mathbf{e}}_3,$$

where $\frac{\partial \phi}{\partial x} = 2xy^2 + yz^2, \qquad \frac{\partial \phi}{\partial y} = 2yx^2 + xz^2, \qquad \frac{\partial \phi}{\partial z} = 2xyz$

and
$$(\vec{A} \times \nabla)\phi = (2x^2y^2z - 2x^2y^3 - xy^2z^2)\,\hat{\mathbf{e}}_1 + (2xy^4 + y^3z^2 - 2x^3yz)\,\hat{\mathbf{e}}_2 + (2x^4y + x^3z^2 - 2x^2y^3 - xy^2z^2)\,\hat{\mathbf{e}}_3$$

■

Relations Involving the Del Operator

In summary, the following table illustrates a variety of relations involving the del operator. In these tables the functions f, g are assumed to be differentiable scalar functions of position and $\vec{A}$, $\vec{B}$ are vector functions of position, which are continuous and differentiable.

The ∇ operator and differentiation

1. $\nabla(f+g) = \nabla f + \nabla g$ or $\text{grad}\,(f+g) = \text{grad}\,f + \text{grad}\,g$
2. $\nabla \cdot (\vec{A}+\vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$ or $\text{div}\,(\vec{A}+\vec{B}) = \text{div}\,\vec{A} + \text{div}\,\vec{B}$
3. $\nabla \times (\vec{A}+\vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$ or $\text{curl}\,(\vec{A}+\vec{B}) = \text{curl}\,\vec{A} + \text{curl}\,\vec{B}$
4. $\nabla(f\vec{A}) = (\nabla f) \cdot \vec{A} + f(\nabla \cdot \vec{A})$
5. $\nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$
6. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B}(\nabla \times \vec{A}) - \vec{A}(\nabla \times \vec{B})$
7. $(\vec{A} \times \nabla)f = \vec{A} \times \nabla f$
8. For $f = f(u)$ and $u = u(x,y,z)$, then $\nabla f = \frac{df}{du}\nabla u$
9. For $f = f(u_1, u_2, \ldots, u_n)$ and $u_i = u_i(x,y,z)$ for $i = 1, 2, \ldots, n$, then
$$\nabla f = \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \cdots + \frac{\partial f}{\partial u_n}\nabla u_n$$
10. $\nabla \times (\vec{A} \times \vec{B}) = \nabla(\nabla \cdot \vec{A}) - \vec{B}(\nabla \cdot \vec{A})$
11. $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{B}\nabla)\vec{A} + (\vec{A}\nabla)\vec{B}$
12. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$
13. $\nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
14. $\nabla \times \nabla f) = \vec{0}$ The curl of a gradient is the zero vector.
15. $\nabla \cdot (\nabla \times \vec{A}) = 0$ The divergence of a curl is zero.

The ∇ operator and integration

16. $\iiint_V \nabla f \, dV = \iint_S f \, \hat{\mathbf{e}}_n \, dS$ Special case of divergence theorem

17. $\iiint_V \nabla \times \vec{A} \, dV = \iint_S \hat{\mathbf{e}}_n \times \vec{A} \, dS$ Special case of divergence theorem

18. $\oint_C d\vec{r} \times \vec{A} = \iint_S (\hat{\mathbf{e}}_n \times \nabla) \times \vec{A} \, dS$ Special case of Stokes theorem

19. $\oint_C f \, d\vec{r} = \iint_S d\vec{S} \times \nabla f$ Special case of Stokes theorem

Vector Operators in curvilinear coordinates

In this section the concept of curvilinear coordinates is introduced and the representation of scalars and vectors in these new coordinates are studied.

If associated with each point (x, y, z) of a rectangular coordinate system there is a set of variables (u, v, w) such that x, y, z can be expressed in terms of u, v, w by a set of functional relationships or transformations equations, then (u, v, w) are called the curvilinear coordinates[2] (x, y, z). Such transformation equations are expressible in the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w) \tag{8.68}$$

and the inverse transformation can be expressed as

$$u = u(x, y, z), \quad v = v(x, y, z) \quad w = w(x, y, z) \tag{8.69}$$

It is assumed that the transformation equations (8.68) and (8.69) are single valued and continuous functions with continuous derivatives. It is also assumed that the transformation equations (8.68) are such that the inverse transformation (8.69) exists, because this condition assures us that the correspondence between the variables (x, y, z) and (u, v, w) is a one-to-one correspondence.

The position vector

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \tag{8.70}$$

[2] Note how coordinates are defined and the order of their representation because there are no standard representation of angles or directions. Depending upon how variables are defined and represented, sometime left-hand coordinates are confused with right-handed coordinates.

of a general point (x, y, z) can be expressed in terms of the curvilinear coordinates (u, v, w) by utilizing the transformation equations (8.68). The position vector $\vec{r}$, when expressed in terms of the curvilinear coordinates, becomes

$$\vec{r} = \vec{r}(u, v, w) = x(u, v, w)\,\hat{\mathbf{e}}_1 + y(u, v, w)\,\hat{\mathbf{e}}_2 + z(u, v, w)\,\hat{\mathbf{e}}_3 \tag{8.71}$$

and an element of arc length squared is $ds^2 = d\vec{r} \cdot d\vec{r}$. In the curvilinear coordinates one finds $\vec{r} = \vec{r}(u, v, w)$ as a function of the curvilinear coordinates and consequently

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv + \frac{\partial \vec{r}}{\partial w}\,dw. \tag{8.72}$$

From the differential element $d\vec{r}$ one finds the element of arc length squared given by

$$\begin{aligned} d\vec{r} \cdot d\vec{r} = ds^2 = \; & \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}\,du\,du + \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}\,du\,dv + \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial w}\,du\,dw \\ & + \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial u}\,dvdu + \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}\,dvdv + \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial w}\,dv\,dw \\ & + \frac{\partial \vec{r}}{\partial w} \cdot \frac{\partial \vec{r}}{\partial u}\,dwdu + \frac{\partial \vec{r}}{\partial w} \cdot \frac{\partial \vec{r}}{\partial v}\,dwdv + \frac{\partial \vec{r}}{\partial w} \cdot \frac{\partial \vec{r}}{\partial w}\,dwdw. \end{aligned} \tag{8.73}$$

The quantities

$$\begin{aligned} g_{11} &= \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u} & g_{12} &= \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} & g_{13} &= \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial w} \\ g_{21} &= \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial u} & g_{22} &= \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v} & g_{23} &= \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial w} \\ g_{31} &= \frac{\partial \vec{r}}{\partial w} \cdot \frac{\partial \vec{r}}{\partial u} & g_{32} &= \frac{\partial \vec{r}}{\partial w} \cdot \frac{\partial \vec{r}}{\partial v} & g_{33} &= \frac{\partial \vec{r}}{\partial w} \cdot \frac{\partial \vec{r}}{\partial w} \end{aligned} \tag{8.74}$$

are called the metric components of the curvilinear coordinate system. The metric components may be thought of as the elements of a symmetric matrix, since $g_{ij} = g_{ji}$, $i, j = 1, 2, 3$. These metrices play an important role in the subject area of tensor calculus.

The vectors $\frac{\partial \vec{r}}{\partial u}$, $\frac{\partial \vec{r}}{\partial v}$, $\frac{\partial \vec{r}}{\partial w}$, used to calculate the metric components g_{ij} have the following physical interpretation. The vector $\vec{r} = \vec{r}(u, c_2, c_3)$, where u is a variable and $v = c_2$, $w = c_3$ are constants, traces out a curve in space called a coordinate curve. Families of these curves create a coordinate system. Coordinate curves can also be viewed as being generated by the intersection of the coordinate surfaces $v(x, y, z) = c_2$ and $w(x, y, z) = c_3$. The tangent vector to the coordinate curve is calculated with the partial derivative $\frac{\partial \vec{r}}{\partial u}$. Similarly, the curves $\vec{r} = \vec{r}(c_1, v, c_3)$ and $\vec{r} = \vec{r}(c_1, c_2, w)$ are coordinate curves and have the respective tangent vectors $\frac{\partial \vec{r}}{\partial v}$ and $\frac{\partial \vec{r}}{\partial w}$. One can calculate the magnitude of these tangent vectors by defining the scalar magnitudes as

$$h_1 = h_u = |\frac{\partial \vec{r}}{\partial u}|, \qquad h_2 = h_v = |\frac{\partial \vec{r}}{\partial v}|, \qquad h_3 = h_w = |\frac{\partial \vec{r}}{\partial w}|. \tag{8.75}$$

The unit tangent vectors to the coordinate curves are given by the relations

$$\hat{\mathbf{e}}_u = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u}, \qquad \hat{\mathbf{e}}_v = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial v}, \qquad \hat{\mathbf{e}}_w = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial w}. \tag{8.76}$$

The coordinate surfaces and coordinate curves may be formed from the equations (8.68) and are illustrated in figure 8-15

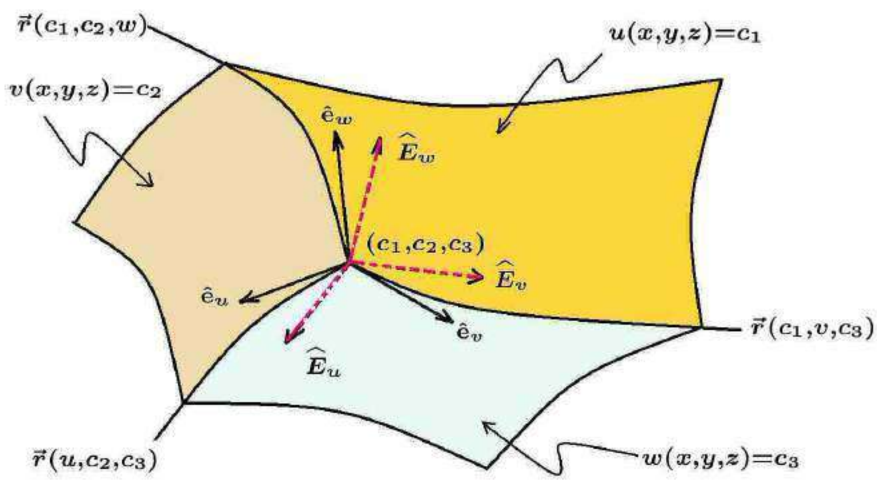


Figure 8-15. Coordinate curves and surfaces.

Consider the point $u = c_1,\ v = c_2,\ w = c_3$ in the curvilinear coordinate system. This point can be viewed as being created from the intersection of the three surfaces

$$u = u(x, y, z) = c_1$$
$$v = v(x, y, z) = c_2$$
$$w = w(x, y, z) = c_3$$

obtained from the inverse transformation equations (8.69).

For example, the figure 8-15 illustrates the surfaces $u = c_1$ and $v = c_2$ intersecting in the curve $\vec{r} = \vec{r}(c_1, c_2, w)$. The point where this curve intersects the surface $w = c_3$, is (c_1, c_2, c_3).

The vector $\text{grad}\, u(x, y, z)$ is a vector normal to the surface $u = c_1$. A unit normal to the $u = c_1$ surface has the form

$$\vec{E}_u = \frac{\text{grad } u}{|\text{grad } u|}.$$

Similarly, the vectors

$$\vec{E}_v = \frac{\text{grad } v}{|\text{grad } v|}, \qquad \text{and} \qquad \vec{E}_w = \frac{\text{grad } w}{|\text{grad } w|}$$

are unit normal vectors to the surfaces $v = c_2$ and $w = c_3$.

The unit tangent vectors $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$, $\hat{\mathbf{e}}_w$ and the unit normal vectors $\vec{E}_u$, $\vec{E}_v$, $\vec{E}_w$ are identical if and only if $g_{ij} = 0$ for $i \neq j$; for this case, the curvilinear coordinate system is called an orthogonal coordinate system.

Example 8-14. Consider the identity transformation between (x, y, z) and (u, v, w). We have $u = x$, $v = y$, and $w = z$. The position vector is

$$\vec{r}(x, y, z) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3,$$

and in this rectangular coordinate system, the element of arc length squared is given by $ds^2 = dx^2 + dy^2 + dz^2$. In this space the metric components are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the coordinate system is orthogonal.

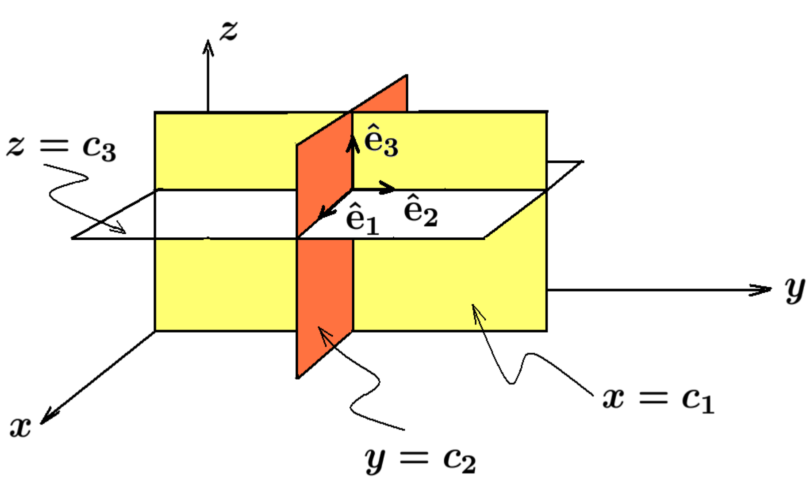


Figure 8-16. Cartesian coordinate system.

In rectangular coordinates consider the family of surfaces

$$x = c_1, \qquad y = c_2, \qquad z = c_3,$$

where c_1, c_2, c_3 take on the integer values $1, 2, 3, \ldots$. These surfaces intersect in lines which are the coordinate curves. The vectors

$$\text{grad } x = \hat{\mathbf{e}}_1, \qquad \text{grad } y = \hat{\mathbf{e}}_2, \quad \text{and} \quad \text{grad } z = \hat{\mathbf{e}}_3$$

are the unit vectors which are normal to the coordinate surfaces. The vectors

$$\frac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1, \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\mathbf{e}}_2, \qquad \frac{\partial \vec{r}}{\partial z} = \hat{\mathbf{e}}_3$$

can also be viewed as being tangent to the coordinate curves. The situation is illustrated in figure 8-16.

■

Example 8-15. In cylindrical coordinates (r, θ, z), the transformation equations (8.68) become

$$x = x(r, \theta, z) = r\cos\theta$$
$$y = y(r, \theta, z) = r\sin\theta$$
$$z = z(r, \theta, z) = z$$

and the inverse transformation (8.69) can be written

$$r = r(x, y, z) = \sqrt{x^2 + y^2}$$
$$\theta = \theta(x, y, z) = \arctan\frac{y}{x}$$
$$z = z(x, y, z) = z.$$

where the substitutions $u = r,\ v = \theta, w = z$ have been made. The position vector (8.70) is then

$$\vec{r} = \vec{r}(r, \theta, z) = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3.$$

The curve

$$\vec{r} = \vec{r}(c_1, \theta, c_3) = c_1\cos\theta\,\hat{\mathbf{e}}_1 + c_1\sin\theta\,\hat{\mathbf{e}}_2 + c_3\,\hat{\mathbf{e}}_3,$$

where c_1 and c_3 are constants, represents the circle $x^2 + y^2 = c_1^2$ in the plane $z = c_3$ and is illustrated in figure 8-17. The curve

$$\vec{r} = \vec{r}(c_1, c_2, z) = c_1\cos c_2\,\hat{\mathbf{e}}_1 + c_1\sin c_2\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

represents a straight line parallel to the z-axis which is normal to the xy plane at the point $r = c_1,\ \theta = c_2$. The curve

$$\vec{r} = \vec{r}(r, c_2, c_3) = r\cos c_2\,\hat{\mathbf{e}}_1 + r\sin c_2\,\hat{\mathbf{e}}_2 + c_3\,\hat{\mathbf{e}}_3$$

represents a straight line in the plane $z = c_3$, which extends in the direction $\theta = c_2$.

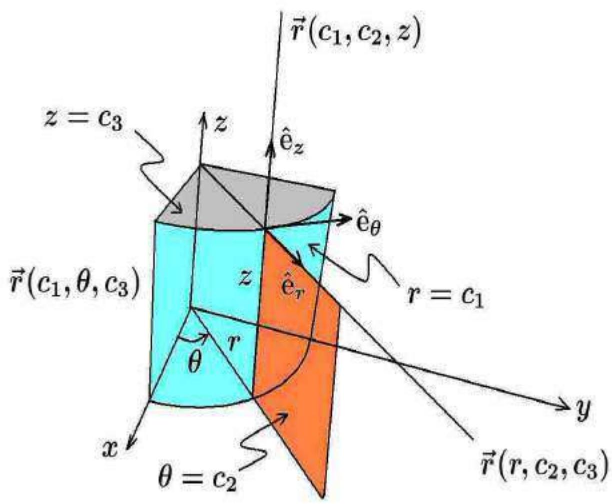


Figure 8-17. Cylindrical coordinates.

The tangent vectors to the coordinate curves are given by

$$\begin{aligned}\frac{\partial \vec{r}}{\partial r} &= \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2\\ \frac{\partial \vec{r}}{\partial \theta} &= -r\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\theta\,\hat{\mathbf{e}}_2\\ \frac{\partial \vec{r}}{\partial z} &= \hat{\mathbf{e}}_3\end{aligned}$$

and are illustrated in figure 8-17. The element of arc length squared is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

and the metric components of the space are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Observe that this is an orthogonal system where $g_{ij} = 0$ for $i \neq j$. The surface $r = c_1$ is a cylinder, whereas the surface $\theta = c_2$ is a plane perpendicular to the xy plane and passing through the z-axis. The surface $z = c_3$ is a plane parallel to the xy plane. The cylindrical coordinate system is an orthogonal system.

■

Example 8-16. The spherical coordinates (ρ, θ, ϕ) are related to the rectangular coordinates through the transformation equations

$$x = x(\rho, \theta, \phi) = \rho \sin\theta \cos\phi$$
$$y = y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi$$
$$z = z(\rho, \theta, \phi) = \rho \cos\theta$$

which can be obtained from the geometry of figure 8-18.

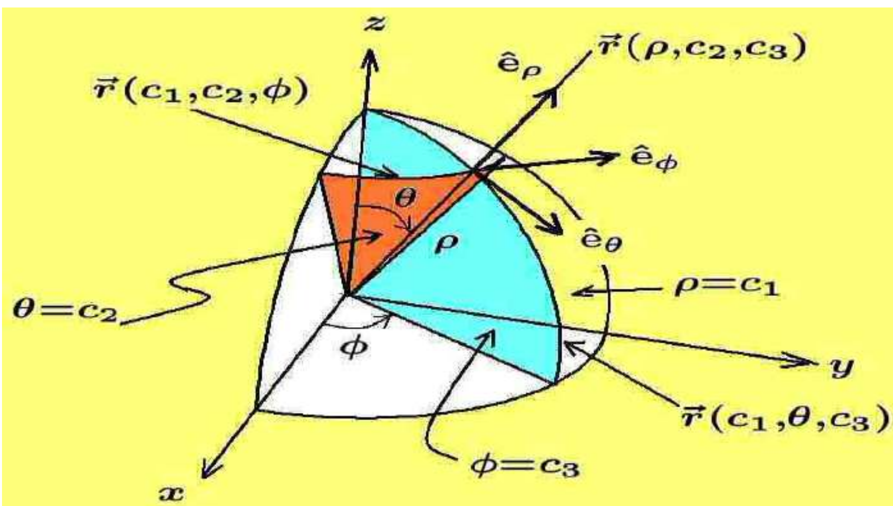


Figure 8-18. Spherical coordinate system.

The position vector (8.70) becomes

$$\vec{r} = \vec{r}(\rho, \theta, \phi) = \rho \sin\theta \cos\phi \, \hat{\mathbf{e}}_1 + \rho \sin\theta \sin\phi \, \hat{\mathbf{e}}_2 + \rho \cos\theta \, \hat{\mathbf{e}}_3,$$

and from this position vector one can generate the curves

$$\vec{r} = \vec{r}(c_1, c_2, \phi), \quad \vec{r} = \vec{r}(c_1, \theta, c_3), \quad \vec{r} = \vec{r}(\rho, c_2, c_3),$$

where c_1, c_2, c_3 are constants. These curves are, respectively, circles of radius $c_1 \sin c_2$, meridian lines on the surface of the sphere, and a line normal to the sphere. These curves are illustrated in figure 8-18. The surfaces $r = c_1$, $\theta = c_2$, and $\phi = c_3$ are, respectively, spheres, circular cones, and planes passing through the z-axis.

The unit tangent vectors to the coordinate curves and scale factors are given by

$$\begin{aligned}
\hat{\mathbf{e}}_\rho &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3, & h_1 &= h_\rho = 1\\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \sin\theta\,\hat{\mathbf{e}}_3, & h_2 &= h_\theta = \rho\\
\hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{e}}_1 + \cos\phi\,\hat{\mathbf{e}}_2, & h_3 &= h_\phi = \rho\sin\theta.
\end{aligned}$$

The element of arc length squared is

$$ds^2 = d\rho^2 + \rho^2\,d\theta^2 + \rho^2\sin^2\theta\,d\phi^2,$$

and the metric components of this space are given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0\\ 0 & \rho^2 & 0\\ 0 & 0 & \rho^2\sin^2\theta \end{pmatrix}.$$

Note that the spherical coordinate system is an orthogonal system.

■

Example 8-17.

An example of a curvilinear coordinate system which is not orthogonal is the oblique cylindrical coordinate system (r, θ, η) illustrated in figure 8-19

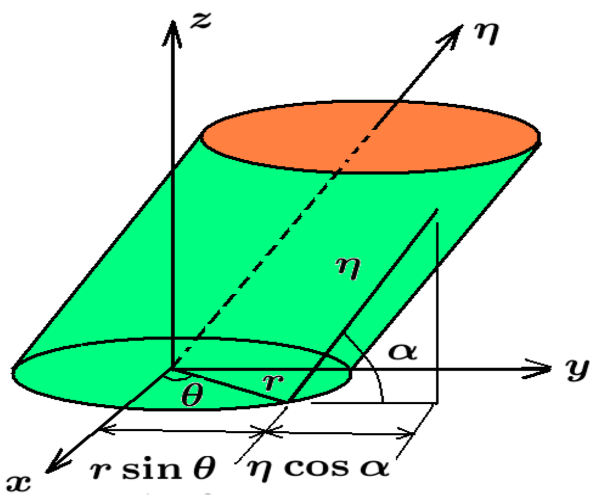


Figure 8-19. Oblique cylindrical coordinate system.

The transformation equations (8.68) are obtained from the geometry in figure 8-19. These equations are

$$x = r\cos\theta$$
$$y = r\sin\theta + \eta\cos\alpha$$
$$z = \eta\sin\alpha,$$

which for $\alpha = 90^\circ$ reduces to the transformation equations for cylindrical coordinates.

The unit tangent vectors are

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_\eta = \cos\alpha\,\hat{\mathbf{e}}_2 + \sin\alpha\,\hat{\mathbf{e}}_3,$$

and the metric components of this space are

$$g_{ij} = \begin{pmatrix} 1 & 0 & \sin\theta\cos\alpha \\ 0 & r^2 & r\cos\theta\cos\alpha \\ \sin\theta\cos\alpha & r\cos\theta\cos\alpha & 1 \end{pmatrix}.$$

■

Orthogonal Curvilinear Coordinates

The following is a list of some orthogonal curvilinear coordinates which have applications in many different scientific investigations.

Cylindrical coordinates (r, θ, z) :

$$\begin{aligned} x &= r\cos\theta \qquad 0 \le \theta \le 2\pi \\ y &= r\sin\theta \qquad r \ge 0 \\ z &= z \qquad -\infty < z < \infty \\ ds^2 &= h_r^2\,dr^2 + h_\theta^2\,d\theta^2 + h_z^2\,dz^2 \qquad (8.77) \\ h_r &= 1, \quad h_\theta = r, \quad h_z = 1 \\ g_{ij} &= \begin{pmatrix} h_r^2 & 0 & 0 \\ 0 & h_\theta^2 & 0 \\ 0 & 0 & h_z^2 \end{pmatrix} \end{aligned}$$

Spherical coordinates (ρ, θ, ϕ) :

$$\begin{aligned}
x &= \rho \sin\theta \cos\phi, \qquad \rho \geq 0 \\
y &= \rho \sin\theta \sin\phi, \qquad 0 \leq \phi \leq 2\pi \\
z &= \rho \cos\theta, \qquad 0 \leq \theta \leq \pi \\
ds^2 &= h_\rho^2\, d\rho^2 + h_\theta^2\, d\theta^2 + h_\phi^2\, d\phi^2 \\
h_\rho &= 1, \quad h_\theta = \rho, \quad h_\phi = \rho \sin\theta \\
g_{ij} &= \begin{pmatrix} h_\rho^2 & 0 & 0 \\ 0 & h_\theta^2 & 0 \\ 0 & 0 & h_\phi^2 \end{pmatrix}
\end{aligned} \tag{8.78}$$

Parabolic cylindrical coordinates (ξ, η, z) :

$$\begin{aligned}
x &= \xi\eta, \qquad -\infty < \xi < \infty \\
y &= \frac{1}{2}\left(\xi^2 - \eta^2\right), \qquad -\infty < z < \infty \\
z &= z, \qquad \eta \geq 0 \\
ds^2 &= h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_z^2\, dz^2 \\
h_\eta &= h_\xi = \sqrt{\eta^2 + \xi^2}, \qquad h_z = 1 \\
g_{ij} &= \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_z^2 \end{pmatrix}
\end{aligned} \tag{8.79}$$

Parabolic coordinates (ξ, η, ϕ) :

$$\begin{aligned}
x &= \xi\eta \cos\phi, \qquad \xi \geq 0, \quad \eta \geq 0 \\
y &= \xi\eta \sin\phi, \qquad 0 < \phi < 2\pi \\
z &= \frac{1}{2}(\xi^2 - \eta^2) \\
ds^2 &= h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_\phi^2\, d\phi^2 \\
h_\xi &= h_\eta = \sqrt{\eta^2 + \xi^2}, \qquad h_\phi = \xi\eta \\
g_{ij} &= \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_\phi^2 \end{pmatrix}
\end{aligned} \tag{8.80}$$

Elliptic cylindrical coordinates (ξ, η, z) :

$$
\begin{aligned}
x &= \cosh\xi\cos\eta, \qquad \xi \geq 0 \\
y &= \sinh\xi\sin\eta, \qquad 0 \leq \eta \leq 2\pi \\
z &= z, \qquad -\infty < z < \infty \\
ds^2 &= h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_z^2\, dz^2 \\
h_\xi &= h_\eta = \sqrt{\sinh^2\xi + \sin^2\eta}, \qquad h_z = 1 \\
g_{ij} &= \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_z^2 \end{pmatrix}
\end{aligned} \tag{8.81}
$$

Elliptic coordinates (ξ, η, ϕ) :

$$
\begin{aligned}
x &= \sqrt{(1-\eta^2)(\xi^2-1)}\cos\phi, \qquad -1 \leq \eta \leq 1 \\
y &= \sqrt{(1-\eta^2)(\xi^2-1)}\sin\phi, \qquad 1 \leq \xi < \infty \\
z &= \xi\eta, \qquad 0 \leq \phi \leq 2\pi \\
ds^2 &= h_\xi^2\, d\xi^2 + h_\eta^2\, d\eta^2 + h_\phi^2\, d\phi^2 \\
h_\xi &= \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}, \qquad h_\eta = \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}, \qquad h_\phi = \sqrt{(\xi^2-1)(1-\eta^2)} \\
g_{ij} &= \begin{pmatrix} h_\xi^2 & 0 & 0 \\ 0 & h_\eta^2 & 0 \\ 0 & 0 & h_\phi^2 \end{pmatrix}
\end{aligned} \tag{8.82}
$$

Transformation of Vectors

A vector field defined by

$$
\vec{A} = \vec{A}(x, y, z) = A_1(x, y, z)\,\hat{\mathbf{e}}_1 + A_2(x, y, z)\,\hat{\mathbf{e}}_2 + A_3(x, y, z)\,\hat{\mathbf{e}}_3
$$

represents a magnitude and direction associated to each point (x, y, z) in some region R or three dimensional cartesian coordinates. This vector field is to remain invariant under a coordinate transformation. However, the form used to represent the vector field will change. For example, under a transformation to cylindrical coordinates, where

$$
x = r\cos\theta \qquad y = r\sin\theta \qquad z = z, \tag{8.83}
$$

the above vector can be represented in terms of the unit orthogonal vectors $\hat{\mathbf{e}}_r,\ \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{e}}_z$ in the form

$$
\vec{A} = \vec{A}(r, \theta, z) = A_r(r, \theta, z)\,\hat{\mathbf{e}}_r + A_\theta(r, \theta, z)\,\hat{\mathbf{e}}_\theta + A_z(r, \theta, z)\,\hat{\mathbf{e}}_z. \tag{8.84}
$$

Here the quantities A_1, A_2, A_3 represent the components of the vector field $\vec{A}$ in rectangular coordinates, and A_r, A_θ, A_z represent the components of the same vector field $\vec{A}$ when referenced with respect to cylindrical coordinates. The unit vectors $\hat{\mathbf{e}}_r,\ \hat{\mathbf{e}}_\theta,\ \hat{\mathbf{e}}_z$ are orthogonal unit vectors and hence

$$\begin{aligned}
\vec{A}\cdot\hat{\mathbf{e}}_r &= A_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_r + A_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_r + A_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_r = A_r\\
&= \text{Component of } \vec{A} \text{ in the } \hat{\mathbf{e}}_r \text{ direction}\\
\vec{A}\cdot\hat{\mathbf{e}}_\theta &= A_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_\theta + A_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_\theta + A_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_\theta = A_\theta\\
&= \text{Component of } \vec{A} \text{ in the } \hat{\mathbf{e}}_\theta \text{ direction}\\
\vec{A}\cdot\hat{\mathbf{e}}_z &= A_1\,\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_z + A_2\,\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_z + A_3\,\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_z = A_z\\
&= \text{Component of } \vec{A} \text{ in the } \hat{\mathbf{e}}_z \text{ direction.}
\end{aligned}$$

These equations can be expressed in the matrix form as follows:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_r & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_z & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_z & \hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_z \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{8.85}$$

For example, it is known that the unit vectors in cylindrical coordinates are

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2\\
\hat{\mathbf{e}}_\theta &= -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2\\
\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_3,
\end{aligned}$$

and consequently the matrix (8.85) can be expressed as

$$\begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{8.86}$$

Equation (8.86) illustrates how to represent the vector field components A_r, A_θ, A_z of cylindrical coordinates, in terms of the components A_1, A_2, A_3 of rectangular coordinates. In using the above transformation equation, be sure to convert all x, y, z coordinates to r, θ, z cylindrical coordinates using the transformation equations (8.83). Note also that the coefficient matrix in equation (8.83) is an orthonormal matrix.

Example 8-18. Express the vector

$$\vec{A} = 2y\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2 + 2x\,\hat{\mathbf{e}}_3$$

in cylindrical coordinates.

Solution The rectangular components of $\vec{A}$ are $A_1 = 2y,\ A_2 = z,\ A_3 = 2x$, and from equation (8.86) the cylindrical components are

$$\begin{aligned} A_r &= 2y\cos\theta + z\sin\theta \\ A_\theta &= -2y\sin\theta + z\cos\theta \\ A_z &= 2x, \end{aligned}$$

where the variables x, y, z must be expressed in terms of the variables r, θ, z. From the transformation equations from rectangular to cylindrical coordinates one finds

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

so that

$$\begin{aligned} A_r &= 2r\sin\theta\cos\theta + z\sin\theta \\ A_\theta &= -2r\sin^2\theta + z\cos\theta \\ A_z &= 2r\cos\theta \end{aligned}$$

and the vector $\vec{A}$ in cylindrical coordinates can be represented as

$$\vec{A} = \vec{A}(r,\theta,z) = A_r\,\hat{\mathbf{e}}_r + A_\theta\,\hat{\mathbf{e}}_\theta + A_z\,\hat{\mathbf{e}}_z.$$

■

General Coordinate Transformations

In general, a vector in rectangular coordinates

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3$$

can be expressed in terms of the orthogonal unit vectors $\hat{\mathbf{e}}_u,\ \hat{\mathbf{e}}_v,\ \hat{\mathbf{e}}_w$ associated with a set of orthogonal curvilinear coordinates defined by the transformation equations given in equation (8.68). Let the representation of this vector in the orthogonal curvilinear coordinates system be denoted by

$$\vec{A} = \vec{A}(u,v,w) = A_u\,\hat{\mathbf{e}}_u + A_v\,\hat{\mathbf{e}}_v + A_w\,\hat{\mathbf{e}}_w,$$

where A_u, A_v, A_w denote the components of $\vec{A}$ in the new coordinate system and are functions of these coordinates. The transformation equations from rectangular coordinates to curvilinear coordinates is represented by the matrix equation

$$\begin{pmatrix} A_u \\ A_v \\ A_w \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_u & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_u & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_u \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_v & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_v & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_w \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_w & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_w & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_w \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{8.87}$$

which is derived by taking projections of the vector $\vec{A}$ onto the u,v and w directions.

Let us find the representation of the gradient, divergence and curl in a general orthogonal curvilinear coordinate system. Recall that the gradient, divergence, and curl in rectangular coordinates are given by

$$\begin{aligned} \text{grad } \phi &= \frac{\partial \phi}{\partial x}\, \hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\, \hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\, \hat{\mathbf{e}}_3 \\ \text{div}\, \vec{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ \text{curl } \vec{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \hat{\mathbf{e}}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \hat{\mathbf{e}}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \hat{\mathbf{e}}_3. \end{aligned}$$

Gradient in a General Orthogonal System of Coordinates

In an orthogonal curvilinear coordinate system, let the vector grad ϕ have the representation

$$\text{grad } \phi = A_u\, \hat{\mathbf{e}}_u + A_v\, \hat{\mathbf{e}}_v + A_w\, \hat{\mathbf{e}}_w.$$

By using the matrix equation (8.87), the component A_u in the curvilinear coordinates is

$$\text{grad } \phi \cdot \hat{\mathbf{e}}_u = A_u = \frac{\partial \phi}{\partial x}\, \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_u + \frac{\partial \phi}{\partial y}\, \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_u + \frac{\partial \phi}{\partial z}\, \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_u.$$

By employing equations (8.75) and (8.76), this result simplifies and

$$A_u = \frac{1}{h_1}\left[\frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}\right] = \frac{1}{h_1}\frac{\partial \phi}{\partial u}. \tag{8.88}$$

In a similar manner, it can be shown that the other components have the form

$$\text{grad } \phi \cdot \hat{\mathbf{e}}_v = A_v = \frac{1}{h_2}\frac{\partial \phi}{\partial v} \qquad \text{and} \qquad \text{grad } \phi \cdot \hat{\mathbf{e}}_w = A_w = \frac{1}{h_3}\frac{\partial \phi}{\partial w}.$$

Thus the gradient can be represented in the curvilinear coordinate system as

$$\nabla \phi = \text{grad } \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u}\, \hat{\mathbf{e}}_u + \frac{1}{h_2}\frac{\partial \phi}{\partial v}\, \hat{\mathbf{e}}_v + \frac{1}{h_3}\frac{\partial \phi}{\partial w}\, \hat{\mathbf{e}}_w. \tag{8.89}$$

Since

$$\frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{e}}_3 = \left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z}\right)\hat{\mathbf{e}}_3,$$

equation (8.89) can be expressed in the form

$$\nabla \phi = \text{grad}\ \phi = \nabla u \frac{\partial \phi}{\partial u} + \nabla v \frac{\partial \phi}{\partial v} + \nabla w \frac{\partial \phi}{\partial w}. \tag{8.92}$$

Equation (8.92) suggests how the operator ∇ can be expressed in a general curvilinear coordinate system. In a general curvilinear coordinate system (u, v, w) one finds the operator ∇ has the form

$$\nabla = \nabla u \frac{\partial}{\partial u} + \nabla v \frac{\partial}{\partial v} + \nabla w \frac{\partial}{\partial w}. \tag{8.91}$$

Divergence in a General Orthogonal System of Coordinates

To find the divergence in an orthogonal curvilinear system, the following relations are employed:

$$\nabla u = \frac{1}{h_1}\,\hat{\mathbf{e}}_u, \qquad \nabla v = \frac{1}{h_2}\,\hat{\mathbf{e}}_v, \qquad \nabla w = \frac{1}{h_3}\,\hat{\mathbf{e}}_w \tag{8.92}$$

which are special cases of the result in equation (8.89). Equations (8.92) imply

$$\begin{aligned} \hat{\mathbf{e}}_u &= \hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w = h_2 h_3 (\nabla v) \times (\nabla w) \\ \hat{\mathbf{e}}_v &= \hat{\mathbf{e}}_w \times \hat{\mathbf{e}}_u = h_1 h_3 (\nabla w) \times (\nabla u) \\ \hat{\mathbf{e}}_w &= \hat{\mathbf{e}}_u \times \hat{\mathbf{e}}_v = h_1 h_2 (\nabla u) \times (\nabla v) \end{aligned} \tag{8.93}$$

Example 8-19. Derive the divergence of a vector which is represented in the generalized orthogonal coordinates (u, v, w) in the form

$$\vec{F} = \vec{F}(u, v, w) = F_u\,\hat{\mathbf{e}}_u + F_v\,\hat{\mathbf{e}}_v + F_w\,\hat{\mathbf{e}}_w.$$

Solution: By using the properties of the del operator one finds

$$\nabla \cdot \vec{F} = \nabla(F_u\,\hat{\mathbf{e}}_u) + \nabla(F_v\,\hat{\mathbf{e}}_v) + \nabla(F_w\,\hat{\mathbf{e}}_w). \tag{8.94}$$

The first term in equation (8.94) can be expanded, and

$$\begin{aligned}\nabla(F_u\,\hat{\mathbf{e}}_u) &= \nabla(F_u)\cdot\,\hat{\mathbf{e}}_u + F_u\nabla(\,\hat{\mathbf{e}}_u)\\ &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + F_u\nabla[h_2h_3(\nabla v)\times(\nabla w)] \qquad \text{(See eqs. (8.89) and (8.93))}\\ &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + F_u\left\{\nabla(h_2h_3)\cdot[(\nabla v)\times(\nabla w)] + h_2h_3\nabla\cdot[(\nabla v)\times(\nabla w)]\right\},\end{aligned}$$

where properties of the del operator were used to obtain this result. With the result div (grad v × grad w) $= 0$, so that

$$\begin{aligned}\nabla(F_u\,\hat{\mathbf{e}}_u) &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + F_u\nabla(h_2h_3)\cdot\frac{\hat{\mathbf{e}}_u}{h_2h_3} \qquad \text{(See eq. (8.93)}\\ &= \frac{1}{h_1}\frac{\partial F_u}{\partial u} + \frac{F_u}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial u} = \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3F_u)}{\partial u} \qquad \text{(See eq. (8.89)).}\end{aligned}$$

Similarly it can be verified that the remaining terms in equation (8.94) can be expressed as

$$\nabla(F_v\,\hat{\mathbf{e}}_v) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2F_v)}{\partial v}$$

$$\text{and} \qquad \nabla(F_w\,\hat{\mathbf{e}}_w) = \frac{1}{h_1h_2h_3}\frac{\partial(h_1h_2F_w)}{\partial w}.$$

Hence, the divergence in generalized orthogonal curvilinear coordinates can be expressed as

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_u)}{\partial u} + \frac{\partial(h_1h_3F_v)}{\partial v} + \frac{\partial(h_1h_2F_w)}{\partial w}\right]. \tag{8.95}$$

■

Curl in a General Orthogonal System of Coordinates

Our problem is to derive an expression for the curl of a vector $\vec{F}$ which is represented in the generalized orthogonal coordinates (u, v, w) in the form

$$\vec{F} = \vec{F}(u,v,w) = F_u\,\hat{\mathbf{e}}_u + F_v\,\hat{\mathbf{e}}_v + F_w\,\hat{\mathbf{e}}_w$$

one can write

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \nabla\times(F_u\,\hat{\mathbf{e}}_u) + \nabla\times(F_v\,\hat{\mathbf{e}}_v) + \nabla\times(F_w\,\hat{\mathbf{e}}_w). \tag{8.96}$$

The first term in equation (8.96) can be expanded by using properties of the del operator and

$$\begin{aligned}\nabla\times(F_u\,\hat{\mathbf{e}}_u) &= \nabla\times(F_uh_1\nabla u) \quad \text{(See eq. (8.89))}\\ &= \nabla(F_uh_1)\times\nabla u + F_uh_1\nabla\times\nabla u.\end{aligned}$$

Since curl grad $u = 0$, the above simplifies to

$$\begin{aligned}
\nabla \times (F_u\,\hat{\mathbf{e}}_u) &= \nabla(F_u h_1) \times \frac{\hat{\mathbf{e}}_u}{h_1} \\
&= \left[\frac{1}{h_1}\frac{\partial(F_u h_1)}{\partial u}\,\hat{\mathbf{e}}_u + \frac{1}{h_2}\frac{\partial(F_u h_1)}{\partial v}\,\hat{\mathbf{e}}_v + \frac{1}{h_3}\frac{\partial(F_u h_1)}{\partial w}\,\hat{\mathbf{e}}_w\right] \times \frac{\hat{\mathbf{e}}_u}{h_1} \\
&= \frac{1}{h_1 h_3}\frac{\partial(F_u h_1)}{\partial w}\,\hat{\mathbf{e}}_v - \frac{1}{h_1 h_2}\frac{\partial(F_u h_1)}{\partial v}\,\hat{\mathbf{e}}_w
\end{aligned}$$

In a similar manner it may be verified that the remaining terms in equation (8.96) can be expressed as

$$\begin{aligned}
\nabla \times (F_v\,\hat{\mathbf{e}}_v) &= \frac{1}{h_1 h_2}\frac{\partial(F_v h_2)}{\partial u}\,\hat{\mathbf{e}}_w - \frac{1}{h_2 h_3}\frac{\partial(F_v h_2)}{\partial w}\,\hat{\mathbf{e}}_u \\
\text{and} \qquad \nabla \times (F_w\,\hat{\mathbf{e}}_w) &= \frac{1}{h_2 h_3}\frac{\partial(F_w h_3)}{\partial v}\,\hat{\mathbf{e}}_u - \frac{1}{h_1 h_3}\frac{\partial(F_w h_3)}{\partial u}\,\hat{\mathbf{e}}_v.
\end{aligned}$$

Hence, the curl of a vector in generalized curvilinear coordinates can be represented in the form

$$\begin{aligned}
\nabla \times \vec{F} = &\frac{1}{h_2 h_3}\left[\frac{\partial(F_w h_3)}{\partial v} - \frac{\partial(F_v h_2)}{\partial w}\right]\hat{\mathbf{e}}_u \\
&+ \frac{1}{h_1 h_3}\left[\frac{\partial(F_u h_1)}{\partial w} - \frac{\partial(F_w h_3)}{\partial u}\right]\hat{\mathbf{e}}_v \\
&+ \frac{1}{h_1 h_2}\left[\frac{\partial(F_v h_2)}{\partial u} - \frac{\partial(F_u h_1)}{\partial v}\right]\hat{\mathbf{e}}_w.
\end{aligned} \tag{8.97}$$

Equation (8.97) can also be represented in the determinant form as

$$\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\,\hat{\mathbf{e}}_u & h_2\,\hat{\mathbf{e}}_v & h_3\,\hat{\mathbf{e}}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ F_u h_1 & F_v h_2 & F_w h_3 \end{vmatrix}. \tag{8.98}$$

The Laplacian in Generalized Orthogonal Coordinates

Using the definition $\nabla^2\phi = \nabla\nabla\phi$ and the relation for the gradient given by equation (8.89) and show that

$$\nabla\nabla\phi = \nabla\left[\frac{1}{h_1}\frac{\partial\phi}{\partial u}\,\hat{\mathbf{e}}_u + \frac{1}{h_2}\frac{\partial\phi}{\partial v}\,\hat{\mathbf{e}}_v + \frac{1}{h_3}\frac{\partial\phi}{\partial w}\,\hat{\mathbf{e}}_w\right]. \tag{8.99}$$

The result of equation (8.95) simplifies equation (8.99) to the final form given as

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial w}\right)\right]. \tag{8.100}$$

The equation $\nabla^2 U = 0$ is known as Laplace's equation.

Example 8-20.

The Laplacian in rectangular coordinates (x, y, z) is given by

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \tag{8.101}$$

The Laplacian in cylindrical coordinates (r, θ, z) is given by

$$\begin{aligned} \nabla^2 U =& \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} \\ \nabla^2 U =& \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} \end{aligned} \tag{8.102}$$

The Laplacian in spherical coordinates (ρ, θ, ϕ) is given by

$$\begin{aligned} \nabla^2 U =& \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} \\ \nabla^2 U =& \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial U}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} \end{aligned} \tag{8.103}$$

Special cases of the Laplace equation $\nabla^2 U = 0$ are easy to solve.

1. If $y = z = 0$ in rectangular coordinates, the Laplace equation, with Laplacian (8.101), reduces to $\frac{d^2U}{dx^2} = 0$. One integration produces $\frac{dU}{dx} = C_1$, where C_1 is a constant of integration. Another integration gives $U = C_1 x + C_2$, where C_2 is another constant of integration.
2. If $\theta = z = 0$ in cylindrical coordinates, the Laplace equation, with Laplacian (8.102), reduces to $\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) = 0$. An integration of this equation gives $r\frac{dU}{dr} = C_1$, where C_1 is a constant of integration. Separate the variables in this equations and integrate again to show the solution of the special Laplace equation is given by $U = C_1 \ln r + C_2$, where C_2 is another constant of integration.
3. If $\theta = \phi = 0$ in spherical coordinates, the Laplace equation, with Laplacian (8.103), reduces to $\frac{1}{\rho^2}\left(\rho^2\frac{dU}{d\rho}\right) = 0$. An integration of this equation gives $\rho^2\frac{dU}{d\rho} = C_1$, where C_1 is a constant of integration. Separate the variables and perform another integration to show $U = \frac{-C_1}{\rho} + C_2$, where C_2 is another constant of integration.

■

Exercises

▶ **8-1.** Sketch some level curves $\phi = k$ for the given values of k and then find the gradient vector.

(i) $\phi = 4x - 2y, \quad k = -2, -1, 0, 1, 2$

(ii) $\phi = xy, \quad k = -2, -1, 0, 1, 2$

(iii) $\phi = x^2 + y^2, \quad k = 0, 1, 9, 25$

(iv) $\phi = 9x^2 + 4y^2, \quad k = 0, 36, 72$

▶ **8-2.** Find the gradient vector associated with the given functions and then evaluate the gradient at the points indicated.

(i) $\phi = 4x - 2y, \quad (4, 9), (0, 0), (-4, -9)$

(ii) $\phi = xy, \quad (0, 1), (-1, 0), (0, -1), (1, 0), (1, 1), (-1, 1), (-1, -1), (1, -1)$

(iii) $\phi = x^2 + y^2, \quad (1, 0), (3, 4), (0, 1), (-3, 4), (-1, 0), (-3, -4), (0, -1), (3, -4)$

(iv) $\phi = 9x^2 + 4y^2, \quad (2, 0), (0, 3), (-2, 0), (0, -3)$

▶ **8-3.** Find a normal vector to the given surfaces at the point indicated and describe the surface.

(i) $4x + 3y + 6z = 13 \qquad P(1, 1, 1)$

(ii) $x^2 + y^2 + z^2 = 9 \qquad P(1, 2, 2)$

(iii) $z - x^2 - y^2 = 0 \qquad P(3, 4, 25)$

(iv) $z = xy \qquad P(2, 3, 6)$

▶ **8-4.** Discuss the critical points associated with the function $z = z(x, y) = xy$. Graph the level curves $z = k$, where $k = -2, -1, 0, 1, 2$ and describe the surface.

▶ **8-5.** Find the unit tangent vector at the point $P(3, 2, 6)$ on the curve of intersection of the surfaces

$$x^2 + y^2 + z^2 = 49, \qquad \text{and} \qquad x + y + z = 11.$$

▶ **8-6.** Let r denote the magnitude of the position vector $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$.

(i) Show that $\nabla(r^n) = nr^{n-2}\vec{r}$

(ii) Show that $\nabla(\ln r) = \frac{\vec{r}}{r^2}$

(iii) Show that $\nabla\left(f(r)\right) = f'(r)\frac{\vec{r}}{r}$, where f is differentiable.

(iv) Does the result in part (iii) check with the solutions given in parts (i) and (ii)?

▶ **8-7.** Find the minimum distance between the lines defined by the parametric equations

$$L_1: \quad x = \tau - 1, \quad y = -\tau + 16, \quad z = 2\tau - 2$$

$$L_2: \quad x = -t, \quad y = 2t, \quad z = 3t$$

▶ **8-8.** Find the minimum distance from the origin to the plane $x + y + z = 1$

▶ **8-9.** The special symbol $\frac{d\phi}{dn}$ is used to denote the normal derivative of a function ϕ on the boundary of a region R. The normal derivative is defined

$$\frac{d\phi}{dn} = \text{grad } \phi \cdot \hat{\mathbf{e}}_n = \nabla\phi \cdot \hat{\mathbf{e}}_n,$$

where $\hat{\mathbf{e}}_n$ is the unit exterior normal vector to the boundary of the region. Find the normal derivative of $\phi = x^3y + xy^2$ on the boundary of the regions given.

(i) The unit circle $x^2 + y^2 = 1$

(ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iii) The square with vertices $(0,0), (1,0), (1,1), (0,1)$

▶ **8-10.** Find the critical points associated with the given functions and test for relative maxima and minima.

(i) $z = (x-2)^2 + (y-3)^2$

(ii) $z = (x-2)^2 - (y-3)^2$

(iii) $z = -(x-2)^2 - (y-3)^2$

▶ **8-11.** Let $u(x,y,z)$ denote a scalar field which is continuous and differentiable. Let $x = x(t), y = y(t)$ and $z = z(t)$ denote the position vector of a particle moving through the scalar field. Show that on the path of the particle one finds

$$\frac{du}{dt} = (\text{grad } u) \cdot \frac{d\vec{r}}{dt}.$$

▶ **8-12.** Let $f(x,y,z,t)$ denote a scalar field which is changing with time as well as position. Let $x = x(t), y = y(t)$ and $z = z(t)$ denote the position vector of a particle moving through the scalar field. Show that on the path of the particle

$$\frac{df}{dt} = (\text{grad } f) \cdot \frac{d\vec{r}}{dt} + \frac{\partial f}{\partial t}.$$

In hydrodynamics, where $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ represents the velocity of the particle, the above derivative $\frac{df}{dt}$ is called **a material derivative** and is represented using the notation $\frac{Df}{Dt}$. Note that the material derivative represents the change of f as one follows the motion of the fluid.

▶ **8-13.** A force field $\vec{F}$ is said to be conservative if it is derivable from a scalar potential function V such that

$$\vec{F} = \pm \text{grad } V.$$

One uses either a plus sign or a minus sign depending upon the particular application being represented.

Consider the motion of a spring-mass system which oscillates in the x-direction. Assume the force acting on the mass m is derivable from the potential function $V = \frac{1}{2}kx^2$, where k is the spring constant. Use Newton's second law (vector form) and derive the equation of motion of the spring-mass system.

▶ **8-14.** (**Divergence of a vector quantity**)

Let

$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3$$

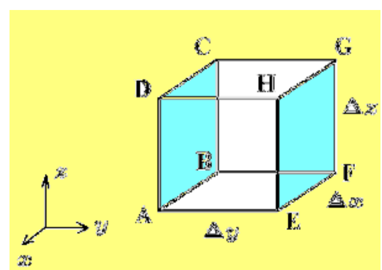
denote a vector field and consider a volume element $\Delta x \Delta y \Delta z$ located at the point (x,y,z) in this vector field.

(a) Use the first couple of terms of a Taylor series expansion to calculate the vector field at

(i) $\vec{F}(x+\Delta x, y, z)$

(ii) $\vec{F}(x, y+\Delta y, z)$

(iii) $\vec{F}(x, y, z+\Delta z)$



(b) Use the results in part (a) and calculate the flux over the surface of the cubic volume element $\Delta V = \Delta x\,\Delta y\,\Delta z$ and then divided by the volume of this element in the limit as the volume tends toward zero.

▶ **8-15.** Determine whether the given vector fields are solenoidal or irrotational

(i) $\vec{F} = (2xyz - z^2)\,\hat{\mathbf{e}}_1 + x^2 z\,\hat{\mathbf{e}}_2 + (x^2 y - 2xz)\,\hat{\mathbf{e}}_3$

(ii) $\vec{F} = \hat{\mathbf{e}}_1 + (x^2 y - y^2 z)\,\hat{\mathbf{e}}_2 + (yz^2 - x^2 z)\,\hat{\mathbf{e}}_3$

(iii) $\vec{F} = 2xy\,\hat{\mathbf{e}}_1 + (x^2 - 2yz)\,\hat{\mathbf{e}}_2 - y^2\,\hat{\mathbf{e}}_3$

(iv) $\vec{F} = 2x(z-y)\,\hat{\mathbf{e}}_1 + (y^2 - yx^2)\,\hat{\mathbf{e}}_2 + (zx^2 - z^2)\,\hat{\mathbf{e}}_3$

▶ **8-16.** Show that div (curl $\vec{F}$) $= 0$

▶ **8-17.** Show that curl (grad ϕ) $= \vec{0}$

▶ **8-18.** For $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and $r = |\vec{r}|$ show

(i) $\text{curl } r^n\vec{r} = \vec{0}$

(ii) $\text{div } \vec{r} = 3$

(iii) $\text{curl } \vec{r} = \vec{0}$

(iv) $\text{div } r^n\vec{r} = (n+3)r^n$

▶ **8-19.** Show that the vector field $\nabla\phi$ is both solenoidal and irrotational if ϕ is a scalar function of position which satisfies the Laplace equation $\nabla^2\phi = 0$.

▶ **8-20.** Show that the following functions are solutions of the Laplace equation in two dimensions.

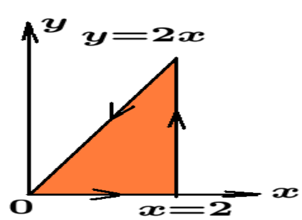
(i) $\phi = x^2 - y^2$

(ii) $\phi = 3x^2y - y^3$

(iii) $\phi = \ln(x^2 + y^2)$

▶ **8-21.** Verify the divergence theorem for $\vec{F} = xy\,\hat{\mathbf{e}}_1 + y^2\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ over the region bounded by the cylindrical surface $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 4$. Whenever possible, integrate by using cylindrical or polar coordinates. Find the sections of this surface which has a flux integral.

▶ **8-22.**



(i) Verify Green's theorem in the plane for

$$M(x,y) = x^2 + y^2 \quad \text{and} \quad N(x,y) = xy,$$

where C is the closed curve illustrated in the figure.

(ii) Use line integration and appropriate values for M and N in Green's theorem to determine the shaded area of the attached figure.

▶ **8-23.** Verify Stokes theorem for $\vec{F} = y\,\hat{\mathbf{e}}_3$ over that portion of the unit sphere in the first octant. Hint: Use spherical coordinates.

▶ **8-24.** Verify the given differential equations are exact and then use line integrals to find solutions.

(i) $(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$

(ii) $(3x^2y + 2xy^2)\,dx + (x^3 + 2yx^2 + 2)\,dy = 0$

► **8-25.** Use line integrals to find the area enclosed by the given curves.

(i) The ellipse, $x = a\cos t \quad y = b\sin t, \quad 0 \le t \le 2\pi.$

(ii) The circle, $x = \cos t \quad y = \sin t, \quad 0 < t < 2\pi.$

(iii) The unit square whose boundaries are $x = 0, x = 1, y = 0, y = 1.$

► **8-26.** Verify the divergence theorem in the case $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. Hint: Use spherical coordinates.

► **8-27.** Calculate the flux of the vector field $\vec{F} = z\,\hat{\mathbf{e}}_3$ entering and leaving the volume enclosed by the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Does the Gauss divergence theorem hold for this volume and surface?

► **8-28.** Calculate the flux of the vector field $\vec{F} = y\,\hat{\mathbf{e}}_2$ entering and leaving the volume enclosed by the two cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, bounded by the planes $z = 0$ and $z = 2$. Does the Gauss divergence theorem hold for this volume and surface?

► **8-29.**

Let S denote the surface of a rectangular parallelepiped with unit surface normals $\pm\,\hat{\mathbf{e}}_1, \pm\,\hat{\mathbf{e}}_2, \pm\,\hat{\mathbf{e}}_3$ and write the surface integral

$$I = \iint_S \vec{F}\cdot d\vec{S} = \int_{S_1} \vec{F}\cdot d\vec{S} + \int_{S_2} \vec{F}\cdot d\vec{S} + \int_{S_3} \vec{F}\cdot d\vec{S} + \int_{S_4} \vec{F}\cdot d\vec{S} + \int_{S_5} \vec{F}\cdot d\vec{S} + \int_{S_6} \vec{F}\cdot d\vec{S}$$

as a summation of the flux over the six faces of the parallelepiped. Calculate the above flux integral for $\vec{F} = y\,\hat{\mathbf{e}}_1 + z\,\hat{\mathbf{e}}_2 + x\,\hat{\mathbf{e}}_3$

(iii) Consider a unit cube with one vertex at the origin. Calculate the flux entering or leaving each face of the cube. Sum these fluxes and comment on your result.

► **8-30.** Let $\vec{F} = M(x,y)\,\hat{\mathbf{e}}_1 + N(x,y)\,\hat{\mathbf{e}}_2$ and use $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ to represent the position of the curve C and show Green's theorem in the plane can be represented in either of the forms

$$(a)\ \oint_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})\cdot\hat{\mathbf{e}}_3\,dxdy \quad \text{or} \quad (b)\ \oint_C (\vec{F}\times\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_n\,ds = \iint_R \nabla\cdot(\vec{F}\times\hat{\mathbf{e}}_3), dxdy$$

where $\hat{\mathbf{e}}_n$ is a unit outward normal to the boundary curve C.
Hint: Use triple scalar product.

▶ **8-31.** Let $\vec{r}$ denote a position vector to a general point on a closed surface S, which encloses a volume V. Evaluate the surface integral

$$\iint_S \vec{r}\cdot d\vec{S} = \iint_S \vec{r}\cdot \hat{\mathbf{e}}_n\, dS$$

▶ **8-32.** **The Gauss Theorem** Let $\vec{r}$ denote the position vector from the origin to a general point on a closed surface S. Show that

$$\iint_S \frac{\hat{\mathbf{e}}_n\cdot\vec{r}}{r^3}\, dS = \begin{cases} 0, & \text{if the origin is outside the closed surface } S \\ 4\pi, & \text{if the origin is inside the closed surface } S \end{cases}$$

Hint: Use the divergence theorem and when the origin is inside S, construct a small sphere of radius ϵ about the origin.

▶ **8-33.** For $\vec{F} = x^2z\,\hat{\mathbf{e}}_1 + xyz\,\hat{\mathbf{e}}_2 + yz\,\hat{\mathbf{e}}_3$ and $\phi = xyz^2$, calculate $\nabla(\phi\vec{F})$

▶ **8-34.** For $\vec{A},\vec{B}$ vector fields and f a scalar field, verify each of the following:

(i) $\text{curl }(f\vec{A}) = (\text{grad } f)\times\vec{A} + f\,\text{curl }\vec{A}$

(ii) $\text{curl }(\vec{A}\times\vec{B}) = \vec{A}(\text{div }\vec{B}) - \vec{B}(\text{div }\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

(iii) $\text{div }(f\vec{A}) = (\text{grad } f)\cdot\vec{A} + f\,\text{div }\vec{A}$

(iv) $\text{grad }(\vec{A}\times\vec{B}) = \vec{B}\cdot\text{curl }\vec{A} - \vec{A}\cdot\text{curl }\vec{B}$

(v) $\text{grad }(fg) = f\,\text{grad } g + g\,\text{grad } f$

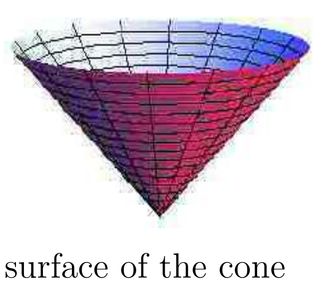
(vi) $\text{grad }(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times\text{curl }\vec{B} + \vec{B}\times\text{curl }\vec{A}$

▶ **8-35.** Evaluate the line integral

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

around the triangle having the vertices $(0,0)$, $(b,0)$ and (c,h) where b,c,h are positive constants. Evaluate this integral using Green's theorem in the plane.

▶ **8-36.** Evaluate the integral



$$I = \iint_S (\nabla\times\vec{F})\cdot d\vec{S},$$

where $\vec{F} = (y-2x)\,\hat{\mathbf{e}}_1 + (3x+2y)\,\hat{\mathbf{e}}_2$ and S is the surface of the cone $x = u\cos v,\ y = u\sin v,\ z = u$ for $0\le u\le 9$ and $0\le v\le 2\pi$.
Hint: If you use Stoke's theorem be sure to note direction of integration.

► **8-37.** Let $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and evaluate the surface integral

$$I = \iint_S \vec{F}\cdot d\vec{S},$$

where S is the surface enclosing the volume bounded by the planes $x=0,\ y=0,\ z=0$ and $2x+3y+4z=12$ Hint: The volume of a tetrahedron having sides $a,\ b$ and c is given by $V = \frac{1}{6}abc$.

► **8-38.** Use Stokes theorem to evaluate the integral

$$I = \oint_C \vec{F}\cdot d\vec{r}, \quad \text{where} \quad \vec{F} = y\,\hat{\mathbf{e}}_1 + 2z\,\hat{\mathbf{e}}_2 + (4y+2x)\,\hat{\mathbf{e}}_3$$

and C is the simple closed curve consisting of the line segments

$$\overline{P_1P_2} + \overline{P_2P_3} + \overline{P_3P_1}$$

connecting the points $P_1(0,0,0),\ P_2(1,1,0)$, and $P_3(0,0,2\sqrt{2})$.

► **8-39.** Let $\vec{v} = \vec{v}(x,y,z,t)$ denote the velocity of a fluid having density$\rho = \rho(x,y,z,t)$. Construct an imaginary volume of fluid V enclosed by a surface S lying within the fluid.

(a) Show the mass of the fluid inside V is given by $M = \iiint \rho(x,y,z,t)\,dV$

(b) Show the time rate of change of mass is $\dfrac{\partial M}{\partial t} = \iint \dfrac{\partial \rho}{\partial t}\,dV$

(c) Show the mass of fluid leaving V per unit of time is given by $\dfrac{\partial M}{\partial t} = -\iint_S \rho\vec{v}\cdot\vec{n}\,dS$

(d) Use the divergence theorem to show $\iiint \dfrac{\partial \rho}{\partial t}\,dV = -\iint \rho\vec{v}\cdot\vec{n}\,dS = -\iiint_V \nabla(\rho\vec{v})\,dV$

(e) Since V is an arbitrary volume show that $\nabla\vec{J} + \dfrac{\partial\rho}{\partial t} = 0$, where $\vec{J} = \rho\vec{v}$. This equation is known as the continuity equation of fluid dynamics.

► **8-40.** In parabolic cylindrical coordinates (ξ,η,z), find

(a) The unit vectors $\hat{\mathbf{e}}_\xi,\ \hat{\mathbf{e}}_\eta,\ \hat{\mathbf{e}}_z$

(b) The metric components g_{ij}

► **8-41.** In the paraboloidal coordinates (ξ,η,ϕ), find

(a) The unit vectors $\hat{\mathbf{e}}_\xi, \hat{\mathbf{e}}_\eta, \hat{\mathbf{e}}_\phi$

(b) The metric components g_{ij}

▶ **8-42.** In cylindrical coordinates (r, θ, z), show that

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

▶ **8-43.** In cylindrical coordinates (r, θ, z), show that

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial \theta}(F_\theta) + \frac{\partial}{\partial z}(rA_z)\right]$$

▶ **8-44.** In cylindrical coordinates (r, θ, z), show that

$$\text{grad } u = \nabla u = \frac{\partial u}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{\partial u}{\partial z}\,\hat{\mathbf{e}}_z$$

▶ **8-45.** In cylindrical coordinates (r, θ, z), show that

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

▶ **8-46.** In spherical coordinates (r, θ, ϕ), show that

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

▶ **8-47.** In spherical coordinates (r, θ, ϕ), show that

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta F_r\right) + \frac{\partial}{\partial \theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right]$$

▶ **8-48.** In spherical coordinates (r, θ, ϕ), show that

$$\text{grad } u = \frac{\partial u}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{1}{r\sin^2\theta}\frac{\partial u}{\partial \phi}\,\hat{\mathbf{e}}_\phi$$

▶ **8-49.** In spherical coordinates (r, θ, ϕ), show that

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

▶ **8-50.** Show that

(a) In cylindrical coordinates (r, θ, z), the element of volume is $dV = r\,dr\,d\theta dz.$

(b) In spherical coordinates (r, θ, ϕ), the element of volume is $dV = r^2\,\sin\theta\,dr\,d\phi\,d\theta.$

(c) In a general orthogonal curvilinear coordinate system (u, v, w), the element of volume can be expressed as $dV = h_u h_v h_w\,du\,dv\,dw.$

▶ **8-51.** Show in a general orthogonal coordinate system $\operatorname{div}(\operatorname{grad} v \times \operatorname{grad} w) = 0$

▶ **8-52.** For $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ independent orthogonal unit vectors (base vectors), one can express any vector $\vec{A}$ as

$$\vec{A} = A_1\,\hat{\mathbf{e}}_1 + A_2\,\hat{\mathbf{e}}_2 + A_3\,\hat{\mathbf{e}}_3,$$

where A_1, A_2, A_3 are the coordinates of $\vec{A}$ relative to the base vectors chosen.

(a) Show that these components are the projection of $\vec{A}$ onto the base vectors and

$$\vec{A} = (\vec{A}\cdot\hat{\mathbf{e}}_1)\,\hat{\mathbf{e}}_1 + (\vec{A}\cdot\hat{\mathbf{e}}_2)\,\hat{\mathbf{e}}_2 + (\vec{A}\cdot\hat{\mathbf{e}}_3)\,\hat{\mathbf{e}}_3.$$

(b) By selecting any three independent orthogonal vectors,$\vec{E}_1, \vec{E}_2, \vec{E}_3$, not necessarily of unit length, show that one can write

$$\vec{A} = \left(\frac{\vec{A}\cdot\vec{E}_1}{\vec{E}_1\cdot\vec{E}_1}\right)\vec{E}_1 + \left(\frac{\vec{A}\cdot\vec{E}_2}{\vec{E}_2\cdot\vec{E}_2}\right)\vec{E}_2 + \left(\frac{\vec{A}\cdot\vec{E}_3}{\vec{E}_3\cdot\vec{E}_3}\right)\vec{E}_3.$$

Consequently,

$$\frac{\vec{A}\cdot\vec{E}_i}{\vec{E}_i\cdot\vec{E}_i}, \qquad i = 1, 2, \text{or}\, 3$$

are the components of $\vec{A}$ relative to the chosen base vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$.

▶ **8-53.** Two bases $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}^1, \vec{E}^2, \vec{E}^3$ are said to be reciprocal if they satisfy the condition

$$\vec{E}_i\cdot\vec{E}^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(i.e., A vector from one basis is orthogonal to two of the vectors from the other basis). Show that if $\vec{E}_1, \vec{E}_2, \vec{E}_3$ is a given set of base vectors, then

$$\vec{E}^1 = \frac{1}{V}\vec{E}_2\times\vec{E}_3, \qquad \vec{E}^2 = \frac{1}{V}\vec{E}_3\times\vec{E}_1, \qquad \vec{E}^3 = \frac{1}{V}\vec{E}_1\times\vec{E}_2$$

is a reciprocal basis, where $V = \vec{E}_1\cdot(\vec{E}_2\times\vec{E}_3)$ is a triple scalar product and represents the volume of the parallelepiped having the basis vectors for its sides. Show also that $\vec{E}^1\cdot(\vec{E}^2\times\vec{E}^3) = \dfrac{1}{V}$

▸ **8-54.** Let $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}^1, \vec{E}^2, \vec{E}^3$ be a system of reciprocal basis. (See previous problem).

(a) If $\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3$ find the components A^1, A^2, A^3 of $\vec{A}$ relative to the base vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$.

(b) If $\vec{A} = A_1\vec{E}^1 + A_2\vec{E}^2 + A_3\vec{E}^3$ find the components A_1, A_2, A_3 relative to the basis $\vec{E}^1, \vec{E}^2, \vec{E}^3$. The numbers A^i are called the contravariant components of $\vec{A}$ and the numbers A_i are called the covariant components of $\vec{A}$.

(c) Using the notation

$$\vec{E}_i \cdot \vec{E}_j = g_{ij} = g_{ji}, \qquad \text{and} \qquad \vec{E}^i \cdot \vec{E}^j = g^{ij} = g^{ji},$$

where $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}^1, \vec{E}^2, \vec{E}^3$ is a reciprocal system of basis, show that

$$A_i = \sum_{k=1}^{3} g_{ik}A^k \qquad \text{and} \qquad A^i = \sum_{k=1}^{3} g_{ik}A_k,$$

where i is called the free index and k is a summation index. Here g^{ij} are called the conjugate metric components of the space and satisfy $\sum_{j=1}^{3} g_{ij}g^{jk} = \delta_i^k$ is the **Kronecker delta**.

(d) Show that

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \sum_{j=1}^{3} g_{ij}g^{jk} = \delta_i^k$$

▸ **8-55.** Show that in an orthogonal curvilinear coordinate system (u, v, w), the vectors

$$(\vec{E}_1, \vec{E}_2, \vec{E}_3) = \left(\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w}\right)$$

$$\text{and} \qquad (\vec{E}^1, \vec{E}^2, \vec{E}^3) = (\text{grad } u,\ \text{grad } v,\ \text{grad } w)$$

are a reciprocal system of basis.

Chapter 9

Applications of Vectors

The use of vectors in mathematics, physics, engineering and the sciences is extensive. The applications presented within these pages have been selected mainly from the study areas of physics and engineering.

Approximation of Vector Field

The Kriging[1] method is a numerical method to **approximate a quantity using a statistical weighting of known data values**. The Kriging method can be used to approximate many different kinds of quantities. The following illustrates an application for the **approximation of a vector field using interpolation**. The weighted average associated with a set of data values $\{Q_1, Q_2, Q_3, \ldots, Q_n\}$ is defined

$$Q = \frac{w_1 Q_1 + w_2 Q_2 + w_3 Q_3 + \cdots + w_n Q_n}{w_1 + w_2 + w_3 + \cdots w_n} \tag{9.1}$$

where $w_1, \ldots, w_n$ are the assigned weighting factors. Note that if all the weights equal unity, then equation (9.1) reduces down to a regular average of the given data values.

The following discussion illustrates how the Kriging method can be used to **approximate a vector field** in the neighborhood of **known points and known vectors associated with these points**. Note that the discussion presented can be generalized and made applicable to any quantity $Q = Q(x, y, z)$ which is a function of position that one wants to approximate.

Given a finite number of **known vectors**

$$\vec{F}_1 = \vec{F}(x_1, y_1, z_1), \vec{F}_2 = \vec{F}(x_2, y_2, z_2), \ \cdots, \vec{F}_n = \vec{F}(x_n, y_n, z_n)$$

which are associated with the **known points** $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$. It is assumed that these known vectors are associated with a vector field $\vec{F} = \vec{F}(x, y, z)$, but we don't know the form for $\vec{F}$. In order to approximate the representation of the vector field $\vec{F} = \vec{F}(x, y, z)$ in some neighborhood of the known points (x_i, y_i, z_i), $i = 1, \ldots, n$ and known vectors at these points one can proceed as follows. In order to use the known data values to estimate the value of $\vec{F} = \vec{F}(x, y, z)$ at a general point (x, y, z) one can define the distances

[1] Danie Gerhardus Krige(1919-) A South African geologist and mining engineer.

$$
\begin{aligned}
d_1 =& \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} \\
d_2 =& \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2} \\
&\vdots \\
d_n =& \sqrt{(x-x_n)^2+(y-y_n)^2+(z-z_n)^2}
\end{aligned}
\tag{9.2}
$$

of a general point (x, y, z) from each of the known data points. It is then possible to use these distances to construct the weights

$$
w_1 = d_2 d_3 d_4 \cdots d_n, \quad w_2 = d_1 d_3 d_4 \ldots d_n, \quad w_3 = d_1 d_2 d_4 \ldots d_n, \quad \ldots \quad w_n = d_1 d_2 \cdots d_{n-1} \tag{9.3}
$$

Note that to form the weight w_i, for some fixed value of i in the range $1 \le i \le n$, one can form a product of all the distances $\prod_{j=1}^{n} d_j = d_1 d_2 d_3 \cdots d_{i-1} d_i d_{i+1} \cdots d_n$ and then remove the term d_i from this product to form the weight $w_i = d_1 d_2 d_3 \cdots d_{i-1} d_{i+1} \cdots d_n$. A shorthand notation to represent the above weights is given by the product formula

$$
w_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} d_j \quad \text{where} \quad d_j = \sqrt{(x-x_j)^2+(y-y_j)^2+(z-z_j)^2} \tag{9.4}
$$

for $j = 1, 2, 3, \ldots, n$. The vector $\vec{F}$ at the interpolation position (x, y, z) is then approximated by the weighted average

$$
\vec{F} = \vec{F}(x, y, z) = \frac{w_1 \vec{F}_1 + w_2 \vec{F}_2 + \cdots + w_n \vec{F}_n}{w_1 + w_2 + \cdots + w_n} \tag{9.5}
$$

Observe that if $(x, y, z) = (x_i, y_i, z_i)$ for some fixed value of i in the range $1 \le i \le n$, then $d_i = 0$ and equation (9.5) reduces to the identity $\vec{F}(x_i, y_i, z_i) = \vec{F}_i$. The Kriging method examines the distances between the coordinates of the known quantities and the selected interpolation point (x, y, z). It then forms weights where points closest to the interpolation point have the highest weight. This can be seen by writing the coefficients of the vectors in equation (9.5) in the form

$$
Coefficient_i = \frac{\frac{1}{d_i}}{\sum_{j=1}^{n} \frac{1}{d_j}} \tag{9.6}
$$

so that the smaller the $d_i \neq 0$, the higher the weighting coefficient. If $d_i = 0$, then all the coefficients with index different from i are zero so that an identity with the known value at i occurs. This approximation method is **an interpolation method** associated with the given data values and is known as **a weighted prediction method.**

A modification of the above method is obtained as follows. The Kriging weights can be generalized by requiring the coefficients in equation (9.6) to have the form

$$Coefficient_i = \frac{\frac{1}{d_i^\beta}}{\sum_{j=1}^{n} \frac{1}{d_j^\beta}} \qquad \beta > 0 \text{ a constant} \tag{9.7}$$

and then adjusting the parameter β to achieve some kind of desired result.

Spherical Trigonometry

The figure 9-1 illustrates three points A, B, C on the surface of **a unit sphere** with a great circle passing through any two of the selected points. This forms a spherical triangle. Let α, β, γ denote the angles[2] at the points A, B, C and let a, b, c denote the length of the sides opposite these angles. On a circle of radius r the arc length s swept out by an angle θ is given by $s = r\theta$. The sphere being a unit sphere dictates that the arc lengths $a = \angle B0C$, $b = \angle A0C$, $c = \angle A0B$. **One of the basic problems in spherical trigonometry is to find a relation between the angles** α, β, γ **and the sides of arc lengths** a, b, c **of a spherical triangle.** The following illustrates how vectors can be employed to find such relationships.

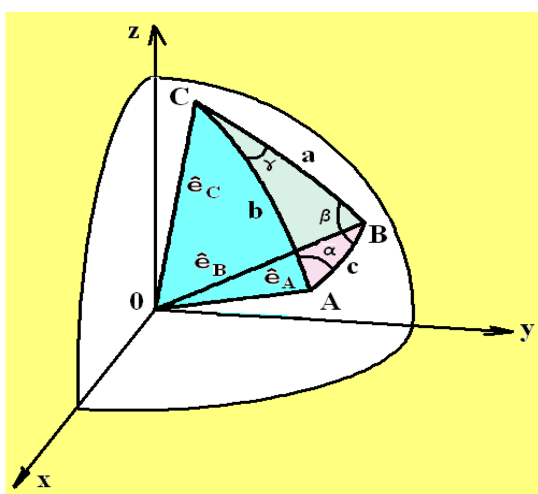


Figure 9-1. Spherical triangle with unit vectors $\hat{\mathbf{e}}_A$, $\hat{\mathbf{e}}_B$, $\hat{\mathbf{e}}_C$

[2] The angles are the same as the angles between the tangent lines to the great circles.

Define the **unit vectors** $\hat{\mathbf{e}}_A$, $\hat{\mathbf{e}}_B$, $\hat{\mathbf{e}}_C$ from **the center of the unit sphere** to the points A, B, C on the surface of the sphere and observe that by using the definition of a cross product and dot product one obtains

$$\begin{aligned} |\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C| &= \sin b, & |\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B| &= \sin c, & |\,\hat{\mathbf{e}}_C \times \hat{\mathbf{e}}_B| &= \sin a \\ \hat{\mathbf{e}}_A \cdot \hat{\mathbf{e}}_C &= \cos b, & \hat{\mathbf{e}}_A \cdot \hat{\mathbf{e}}_B &= \cos c, & \hat{\mathbf{e}}_C \cdot \hat{\mathbf{e}}_B &= \cos a \end{aligned} \tag{9.8}$$

Note that since **the sphere is a unit sphere** the angles a, b, c are given respectively by the arcs $\widehat{BC}$, $\widehat{AC}$ and $\widehat{AB}$ or arcs opposite the vertices A, B, C.

The angle between two intersecting planes is called a **dihedral angle**. The dihedral angle can be calculated from the unit normal vectors to the intersecting planes. In figure 9-1 , let

$$\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C = \sin a\, \hat{\mathbf{e}}_{0BC}, \qquad \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C = \sin b\, \hat{\mathbf{e}}_{0AC}, \qquad \hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B = \sin c\, \hat{\mathbf{e}}_{0AB} \tag{9.9}$$

define the unit vectors $\hat{\mathbf{e}}_{0BC}$, $\hat{\mathbf{e}}_{0AC}$, $\hat{\mathbf{e}}_{0AB}$ which are perpendicular to the planes defining the dihedral angles α, β, γ. The cross product relations given by the equation (9.8) together with the unit normal vectors can be used to calculate the cosines associated with the angle α, β, γ. One finds that

$$\hat{\mathbf{e}}_{0BC} \cdot \hat{\mathbf{e}}_{0AB} = \cos\beta, \qquad \hat{\mathbf{e}}_{0BC} \cdot \hat{\mathbf{e}}_{0AC} = \cos\gamma, \qquad \hat{\mathbf{e}}_{0AC} \cdot \hat{\mathbf{e}}_{0AB} = \cos\alpha \tag{9.10}$$

and with the aid of equations (9.9) one can write

$$\cos\gamma = \frac{|(\,\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C) \cdot (\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_B)|}{|\,\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C|\,|\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C|} \tag{9.11}$$

with similar expressions for the representation of $\cos\alpha$ and $\cos\beta$. The relation (9.11) can be simplified using the dot product relation (6.32) which is repeated here

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{9.12}$$

The numerator in equation (9.11) can then be expressed

$$\begin{aligned} (\,\hat{\mathbf{e}}_B \times \hat{\mathbf{e}}_C) \cdot (\,\hat{\mathbf{e}}_A \times \hat{\mathbf{e}}_C) =& (\,\hat{\mathbf{e}}_B \cdot \hat{\mathbf{e}}_A)(\,\hat{\mathbf{e}}_C \cdot \hat{\mathbf{e}}_C) - (\,\hat{\mathbf{e}}_B \cdot \hat{\mathbf{e}}_C)(\,\hat{\mathbf{e}}_C \cdot \hat{\mathbf{e}}_A) \\ =& \cos c - \cos a \cos b \end{aligned} \tag{9.13}$$

The results from equations (9.8) and (9.13) show that the equation (9.11) can be expressed in the form

$$\cos c = \cos a\, \cos b + \sin a\, \sin b\, \cos\gamma \tag{9.14}$$

Using similar arguments associated with the representation of $\cos\alpha$ and $\cos\beta$, one can show

$$\begin{aligned}\cos b =& \cos c\, \cos a + \sin c\, \sin a\, \cos\beta \\ \cos a =& \cos b\, \cos c + \sin b\, \sin c\, \cos\alpha\end{aligned} \tag{9.15}$$

The equations (9.14) and (9.15)) are known as the **law of cosines for the spherical triangle** ABC.

Replace the dot product in equation (9.11) by a cross product and show

$$\sin\gamma = \frac{|(\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C) \times (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C)|}{|\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C|\,|\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C|} \tag{9.16}$$

The cross product relation (6.30), repeated here as

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \vec{C}\left[\vec{D} \cdot (\vec{A} \times \vec{B})\right] - \vec{D}\left[\vec{C} \cdot (\vec{A} \times \vec{B})\right] \tag{9.17}$$

can be used to simplify the numerator of equation (9.16). One can use properties of the scalar triple product to write

$$\begin{aligned}(\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C) \times (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C) =& \hat{\mathbf{e}}_A\,[\,\hat{\mathbf{e}}_C \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] - \,\hat{\mathbf{e}}_C\,[\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] \\ =& \hat{\mathbf{e}}_A\,[\,\hat{\mathbf{e}}_B \cdot (\,\hat{\mathbf{e}}_C \times\, \hat{\mathbf{e}}_C)] - \,\hat{\mathbf{e}}_C\,[\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)] \\ =& -\,\hat{\mathbf{e}}_C\,[\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)]\end{aligned} \tag{9.18}$$

so that

$$|(\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C) \times (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_C)| = \,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)$$

The triple scalar product relation shows that

$$\begin{aligned}\sin\gamma\, \sin a\, \sin b =& |\,\hat{\mathbf{e}}_A \cdot (\,\hat{\mathbf{e}}_B \times\, \hat{\mathbf{e}}_C)| \\ \sin\alpha\, \sin b\, \sin c =& |\,\hat{\mathbf{e}}_B \cdot (\,\hat{\mathbf{e}}_C \times\, \hat{\mathbf{e}}_A)| \\ \sin\beta\, \sin c\, \sin a =& |\,\hat{\mathbf{e}}_C \cdot (\,\hat{\mathbf{e}}_A \times\, \hat{\mathbf{e}}_B)|\end{aligned} \tag{9.19}$$

and the scalar triple product relation implies that

$$\sin\alpha\, \sin b\, \sin c = \sin\beta\, \sin c\, \sin a = \sin\gamma\, \sin a\, \sin b \tag{9.20}$$

Divide each term in equation (9.20) by $\sin a\, \sin b\, \sin c$ to show

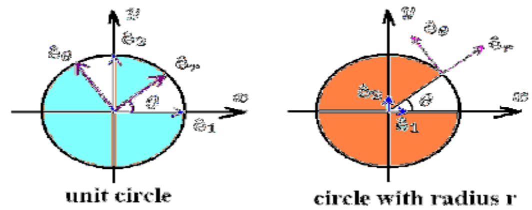
$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} \tag{9.21}$$

which is known as **the law of sines from spherical trigonometry**.

Velocity and Acceleration in Polar Coordinates

In polar coordinates (r, θ) one can employ the orthogonal unit vectors

$$\begin{aligned}\hat{\mathbf{e}}_r &= \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_\theta &= -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2\end{aligned} \tag{9.22}$$



to represent the position of a moving particle.

If in two-dimensional polar coordinates the position vector of a moving particle is represented in the form

$$\vec{r} = r\,\hat{\mathbf{e}}_r$$

where r, θ and consequently, $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ are changing with respect to time t, then the velocity of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{dr}{dt}\,\hat{\mathbf{e}}_r \tag{9.23}$$

Differentiate the vectors in equation (9.22) with respect to time t and show

$$\begin{aligned}\frac{d\,\hat{\mathbf{e}}_r}{dt} &= -\sin\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_1 + \cos\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_2 = \frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta \\ \frac{d\,\hat{\mathbf{e}}_\theta}{dt} &= -\cos\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_1 - \sin\theta\,\frac{d\theta}{dt}\,\hat{\mathbf{e}}_2 = -\frac{d\theta}{dt}\,\hat{\mathbf{e}}_r\end{aligned} \tag{9.24}$$

The first equation in (9.24) simplifies the equation (9.23) to the form

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\,\hat{\mathbf{e}}_r = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta \tag{9.25}$$

where $\dot{} = \frac{d}{dt}$. Here $v_r = \dot{r}$ is called **the radial component of the velocity** and the term $v_\theta = r\dot{\theta}$ is called **the transverse component of the velocity** or **tangential component of the velocity.** The **speed of the particle** is given by

$$v = |\vec{v}| = \sqrt{(r\dot{\theta})^2 + \dot{r}^2}$$

which represents the magnitude of the velocity.

The acceleration of the particle is obtained by differentiating the velocity. Differentiate the equation (9.25) and show

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} =& \dot{r}\frac{d\,\hat{\mathbf{e}}_r}{dt} + \ddot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\frac{d\,\hat{\mathbf{e}}_\theta}{dt} + \frac{d}{dt}(r\dot{\theta})\,\hat{\mathbf{e}}_\theta \\ =& \dot{r}(\dot{\theta}\,\hat{\mathbf{e}}_\theta) + \ddot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}(-\dot{\theta}\,\hat{\mathbf{e}}_r) + (r\ddot{\theta} + \dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta \\ \vec{a} =& (\ddot{r} - r(\dot{\theta})^2)\,\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta\end{aligned}$$

Here the **radial component of acceleration** is $(\ddot{r} - r(\dot{\theta})^2)$ and the **transverse component of acceleration** or **tangential component** is $(r\ddot{\theta}+2\dot{r}\dot{\theta})$. The **magnitude of the acceleration** is given by

$$a = |\vec{a}| = \sqrt{(\ddot{r} - r(\dot{\theta})^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

Velocity and Acceleration in Cylindrical Coordinates

In rectangular (x, y, z) coordinates the position vector, velocity vector and acceleration vector of a moving particle are given by

$$\begin{aligned} \vec{v} = \vec{r} =& x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \\ \vec{v} = \frac{d\vec{r}}{dt} =& \frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 \\ \vec{a} = \frac{d\vec{r}}{dt} = \frac{d^2\vec{r}}{dt^2} =& \frac{d^2x}{dt^2}\,\hat{\mathbf{e}}_1 + \frac{d^2y}{dt^2}\,\hat{\mathbf{e}}_2 + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_3 \end{aligned}$$

Upon changing to a cylindrical coordinates (r, θ, z) using the transformation equations

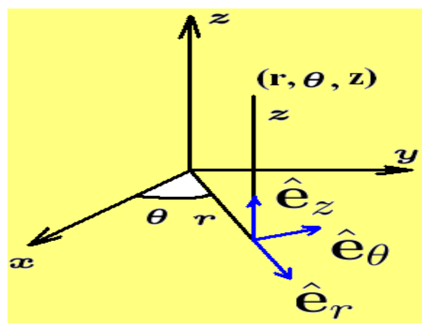
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

one can represent the position vector of the particle as

$$\vec{r} = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3 \tag{9.26}$$

Using the orthogonal unit vectors

$$\begin{aligned} \hat{\mathbf{e}}_r =& \frac{\partial\vec{r}}{\partial r} = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_\theta =& \frac{1}{r}\frac{\partial\vec{r}}{\partial\theta} = -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_z =& \frac{\partial\vec{r}}{\partial z} = \hat{\mathbf{e}}_3 \end{aligned} \tag{9.27}$$



obtained from equations (7.107), the position vector of a moving particle can be expressed in cylindrical coordinates as

$$\vec{r} = r\,\hat{\mathbf{e}}_r + z\,\hat{\mathbf{e}}_z \tag{9.28}$$

To obtain the velocity vector in cylindrical coordinates one must differentiate equation (9.28) with respect to time t to obtain

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\hat{\mathbf{e}}_r + r\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{dz}{dt}\,\hat{\mathbf{e}}_z \tag{9.29}$$

since $\hat{\mathbf{e}}_r$ changes with time, but $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_3$ remains constant. From equation (9.27) one can calculate the derivatives

$$\begin{aligned}\frac{d\,\hat{\mathbf{e}}_r}{dt} =& -\sin\theta\,\frac{d\theta}{dt}\;\hat{\mathbf{e}}_1 + \cos\theta\,\frac{d\theta}{dt}\;\hat{\mathbf{e}}_2 = \frac{d\theta}{dt}\;\hat{\mathbf{e}}_\theta\\ \frac{d\,\hat{\mathbf{e}}_\theta}{dt} =& -\cos\theta\frac{d\theta}{dt}\,\hat{\mathbf{e}}_1 - \sin\theta\frac{d\theta}{dt}\,\hat{\mathbf{e}}_2 = -\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta\\ \frac{d\,\hat{\mathbf{e}}_z}{dt} =& 0\end{aligned} \tag{9.30}$$

as these derivatives will be useful in simplifying any derivatives with respect to time of vectors in cylindrical coordinates. The equations (9.30) allows one to obtain the result

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\;\hat{\mathbf{e}}_\theta + \frac{dz}{dt}\;\hat{\mathbf{e}}_z \tag{9.31}$$

which can also be represented in the form

$$\vec{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta + \dot{z}\,\hat{\mathbf{e}}_z$$

where the dot notation is used to represent time differentiation. Here $v_r = \dot{r}$ is **the radial component of the velocity**, $v_\theta = r\dot{\theta}$ is **the azimuthal component of velocity** and $v_z = \dot{z}$ is **the vertical component of the velocity**.

The acceleration in cylindrical coordinates is obtained by differentiating the velocity. Differentiate the equation (9.31) with respect to time t and show

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} =& \frac{d}{dt}\left[\frac{dr}{dt}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\;\hat{\mathbf{e}}_\theta + \frac{dz}{dt}\;\hat{\mathbf{e}}_z\right]\\ =& \frac{dr}{dt}\frac{d\,\hat{\mathbf{e}}_r}{dt} + \frac{d^2r}{dt^2}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\frac{d\,\hat{\mathbf{e}}_\theta}{dt} + r\frac{d^2\theta}{dt^2}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_z\\ =& \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{d^2r}{dt^2}\,\hat{\mathbf{e}}_r - r\left(\frac{d\theta}{dt}\right)^2\;\hat{\mathbf{e}}_r + r\frac{d^2\theta}{dt^2}\,\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_z\\ =& \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\;\hat{\mathbf{e}}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\;\hat{\mathbf{e}}_\theta + \frac{d^2z}{dt^2}\,\hat{\mathbf{e}}_z\\ \vec{a} =& (\ddot{r} - r(\dot{\theta})^2)\,\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta + \ddot{z}\,\hat{\mathbf{e}}_z\end{aligned} \tag{9.32}$$

where $\dot{} = \frac{d}{dt}$ and $\ddot{} = \frac{d^2}{dt^2}$ are shorthand notations for the first and second derivatives with respect to time t. In calculating the derivatives in equation (9.32) make note that the results from equation (9.30) have been employed.

Velocity and Acceleration in Spherical Coordinates

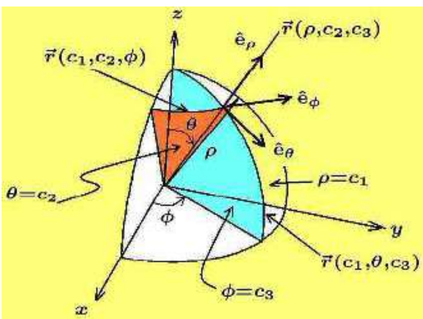
Upon changing to spherical[3] coordinates (ρ, θ, ϕ) the transformation equations are

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\theta$$

and consequently the position vector describing the position of a moving particle is given by

$$\vec{r} = \rho \sin\theta \cos\phi \,\hat{\mathbf{e}}_1 + \rho \sin\theta \sin\phi \,\hat{\mathbf{e}}_2 + \rho \cos\theta \,\hat{\mathbf{e}}_3 \tag{9.33}$$

Using the unit orthogonal vector $\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$ in spherical coordinates obtained from the equations (7.102) and having the representation



$$\begin{aligned}
\hat{\mathbf{e}}_\rho &= \frac{\partial \vec{r}}{\partial \rho} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_1 + \sin\theta\sin\phi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3 \\
\hat{\mathbf{e}}_\theta &= \frac{1}{\rho}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{\mathbf{e}}_1 + \cos\theta\sin\phi\,\hat{\mathbf{e}}_2 - \sin\theta\,\hat{\mathbf{e}}_3 \\
\hat{\mathbf{e}}_\phi &= \frac{1}{\rho\sin\theta} = -\sin\phi\,\hat{\mathbf{e}}_1 + \cos\phi\,\hat{\mathbf{e}}_2
\end{aligned} \tag{9.34}$$

the position vector $\vec{r}$ can be expressed in spherical coordinates by the equation

$$\vec{r} = \rho\,\hat{\mathbf{e}}_\rho \tag{9.35}$$

In order to obtain the first and second derivatives of equation (9.35) with respect to time t it is necessary that one first differentiate the equations (9.34) with respect to time t. As an exercise show that the derivatives of the equations (9.34) can be represented

$$\begin{aligned}
\frac{d\,\hat{\mathbf{e}}_\rho}{dt} &= \frac{\partial\,\hat{\mathbf{e}}_\rho}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\,\hat{\mathbf{e}}_\rho}{\partial\phi}\frac{d\phi}{dt} = \frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi \\
\frac{d\,\hat{\mathbf{e}}_\theta}{dt} &= \frac{\partial\,\hat{\mathbf{e}}_\theta}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\,\hat{\mathbf{e}}_\theta}{\partial\phi}\frac{d\phi}{dt} = -\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\rho + \cos\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi \\
\frac{d\,\hat{\mathbf{e}}_\phi}{dt} &= \frac{\partial\,\hat{\mathbf{e}}_\phi}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\,\hat{\mathbf{e}}_\phi}{\partial\phi}\frac{d\phi}{dt} = -\sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\rho - \cos\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\theta
\end{aligned} \tag{9.36}$$

One can then differentiate equation (9.35) and show the velocity in spherical coordinates has the form

[3] Note (ρ, θ, ϕ) gives a right-handed coordinate system, whereas the ordering (ρ, ϕ, θ) gives a left-handed coordinate system. Be aware that European textbooks, many times use left-handed coordinate systems.

$$\vec{v} = \frac{d\vec{r}}{dt} = \rho\frac{d\,\hat{\mathbf{e}}_\rho}{dt} + \frac{d\rho}{dt}\,\hat{\mathbf{e}}_\rho = \rho\left(\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi\right) + \frac{d\rho}{dt}\,\hat{\mathbf{e}}_\rho = \dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\theta}\,\hat{\mathbf{e}}_\theta + \rho\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\phi \tag{9.38}$$

Here $v_\rho = \dot{\rho}$ is **the radial component of the velocity**, $v_\theta = \rho\dot{\theta}$ is **the polar component of velocity** and $v_\phi = \rho\dot{\phi}\sin\theta$ is **the azimuthal component of velocity**.

Differentiating the velocity with respect to time gives the acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\theta}\,\hat{\mathbf{e}}_\theta + \rho\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\phi\right) = \dot{\rho}\frac{d\,\hat{\mathbf{e}}_\rho}{dt} + \ddot{\rho}\,\hat{\mathbf{e}}_\rho + (\rho\dot{\theta})\frac{d\,\hat{\mathbf{e}}_\theta}{dt} + \frac{d}{dt}(\rho\dot{\theta})\,\hat{\mathbf{e}}_\theta + (\rho\dot{\phi}\sin\theta)\frac{d\,\hat{\mathbf{e}}_\phi}{dt} + \frac{d}{dt}(\rho\dot{\phi}\sin\theta)\,\hat{\mathbf{e}}_\phi \tag{9.38}$$

Substitute the derivatives from equation (9.36) into the equation (9.38) and simplify the results to show the acceleration vector in spherical coordinates is represented

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = (\ddot{\rho} - \rho(\dot{\theta})^2 - \rho(\dot{\phi})^2\sin^2\theta)\,\hat{\mathbf{e}}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} - \rho(\dot{\phi})^2\sin\theta\cos\theta)\,\hat{\mathbf{e}}_\theta + (\rho\ddot{\phi}\sin\theta + 2\dot{\rho}\dot{\phi}\sin\theta + 2\rho\dot{\theta}\dot{\phi}\cos\theta)\,\hat{\mathbf{e}}_\phi \tag{9.39}$$

where $\dot{} = \frac{d}{dt}$ and $\ddot{} = \frac{d^2}{dt^2}$ is the dot notation for the first and second time derivatives.

In spherical coordinates an element of volume is given by $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

Introduction to Potential Theory

In this section some properties of irrotational and/or solenoidal vector fields are derived. Recall that a vector field $\vec{F} = \vec{F}(x,y,z)$ which is continuous and differentiable in a region R is called irrotational if $\text{curl}\,\vec{F} = \nabla\times\vec{F} = \vec{0}$ at all points of R and it is called solenoidal if $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = 0$ at all points of R.

Some properties of **irrotational vector fields** are now considered. If a vector field $\vec{F}$ is an irrotational vector field, then $\nabla\times\vec{F} = \vec{0}$ and under these conditions the vector field $\vec{F}$ is **derivable from a scalar field** $\phi = \phi(x,y,z)$ and can be calculated by the operation[4]

$$\vec{F} = \vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3 = \nabla\phi = \text{grad }\phi = \frac{\partial\phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{e}}_3$$

Note that it you have a choice to solve for three quantities

$$F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)$$

[4] Sometimes $\vec{F} = -\text{grad }\phi$. The selection of either a + or - sign in front of the gradient depends upon how the vector field is being used.

or to solve for one quantity $\phi = \phi(x, y, z)$, then it should be obvious that it would be easier to solve for the one quantity ϕ and then calculate the components F_1, F_2, F_3 by calculating the gradient $\text{grad}\,\phi$. The function ϕ, which defines the scalar field from which $\vec{F}$ is derivable is called the **potential function** associated with the irrotational vector field $\vec{F}$.

In a **simply-connected**[5] **region** R, let $\vec{F}$ define an irrotational vector field which is continuous with derivatives which are also continuous. The following statements are then equivalent.

1. $\nabla \times \vec{F} = \text{curl}\,\vec{F} = \vec{0}$ and the vector field $\vec{F}$ is irrotational.
2. $\vec{F} = \nabla\phi = \text{grad}\,\phi$ and $\vec{F}$ is derivable from a scalar potential function $\phi = \phi(x, y, z)$ by taking the gradient of this function.
3. The dot product $\vec{F} \cdot d\vec{r} = d\phi$, where $d\phi$ is an exact differential.
4. The line integral $W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is the work done in moving through the vector field $\vec{F}$ between two points P_1 and P_2, and this work done is independent of the curve selected for connecting the points P_1 and P_2.
5. The line integral $\oint_C \vec{F} \cdot d\vec{r} = 0$, which implies that the work done in moving around a simple closed path is zero.

If a vector field $\vec{F} = \vec{F}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{e}}_1 + F_2(x, y, z)\,\hat{\mathbf{e}}_2 + F_3(x, y, z)\,\hat{\mathbf{e}}_3$ is derivable from a scalar function $\phi = \phi(x, y, z)$ such that $\vec{F} = \text{grad}\,\phi = \nabla\phi$ (sometimes $\vec{F}$ is defined as the negative of the gradient due to a particular application that requires a negative sign), then $\vec{F}$ is called **a conservative vector field**, and ϕ is called **the potential function** from which the field is derivable. Set $\vec{F} = \text{grad}\,\phi$, and equate the like components of these vectors and obtain the scalar equations

$$F_1(x, y, z) = \frac{\partial\phi}{\partial x}, \qquad F_2(x, y, z) = \frac{\partial\phi}{\partial y}, \qquad F_3(x, y, z) = \frac{\partial\phi}{\partial z}.$$

These equations imply that

$$\vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = d\phi \tag{9.40}$$

is an exact differential. Consequently the statement 2 implies the statement 3.

If $\vec{F} = \text{grad}\,\phi$, then the line integral $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the path of integration joining the points P_1 and P_2. To show this, let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

---

[5] A region R where a closed curve can by continuously shrunk to a point, without the curve leaving the region, is called a simply-connected region.

denote two points in the simply connected region R of the vector field $\vec{F}$. The work done can be expressed by performing a line integral of the equation (9.40) to obtain

$$\begin{aligned} W &= \int_{P_1(x_1,y_1,z_1)}^{P_2(x_2,y_2,z_2)} \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_2} \nabla\phi\cdot d\vec{r} = \int_{P_1}^{P_2} d\phi = \phi\big|_{P_1}^{P_2} \\ \text{or} \qquad W &= \int_{P_1(x_1,y_1,z_1)}^{P_2(x_2,y_2,z_2)} \vec{F}\cdot d\vec{r} = \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1) \end{aligned} \tag{9.41}$$

which implies that the work done depends only on the end points P_1 and P_2 and is thus independent of the path which joins these two points. Thus statement 2 above implies statement 4. Note that this result does not necessarily hold for multiply connected regions.

The line integral given by equation (9.41) being independent of the path of integration which joins P_1 and P_2 can be expressed as

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}, \tag{9.42}$$

where the integral on the left is along a path C_1 and the integral on the right is along a path C_2, where both paths go from P_1 to P_2 as illustrated in figure 9-2.

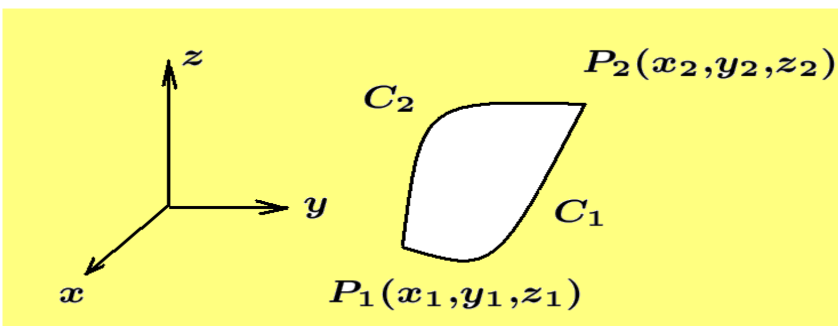


Figure 9-2. Paths of integration.

The integral (9.42) can be expressed in the form

$$\oint_C \vec{F}\cdot d\vec{r} = 0, \tag{9.43}$$

where the closed path C goes from P_1 to P_2 along the path C_1 and then from P_2 to P_1 along the path C_2. The curves C_1 and C_2 are arbitrary so that the work done in going around an arbitrary closed path is zero. Note that Stokes' theorem, with $\nabla \times \vec{F} = \vec{0}$, implies that the line integral around an arbitrary simple closed path is zero.

To show that statement 2 implies statement 1, let $\vec{F} = \text{grad}\,\phi = \nabla\phi$. In this case it is readily verified that

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \nabla \times \nabla\phi = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \vec{0}. \tag{9.44}$$

The relation (9.41) establishes that in a conservative vector field the line integral between any two points is independent of the path of integration. In this case one can write

$$\int_{P_0(x_0,y_0,z_0)}^{P(x,y,z)} \vec{F} \cdot d\vec{r} = \phi(x,y,z) - \phi(x_0,y_0,z_0), \tag{9.45}$$

and this line integral is independent of the path of integration which joins the two end points. The function ϕ can be evaluated from $\vec{F}$ by selecting the special path of integration which is the piecewise smooth curve constructed from straight line segments parallel to the coordinate axes. This special path of integration is illustrated in figure 9-3.

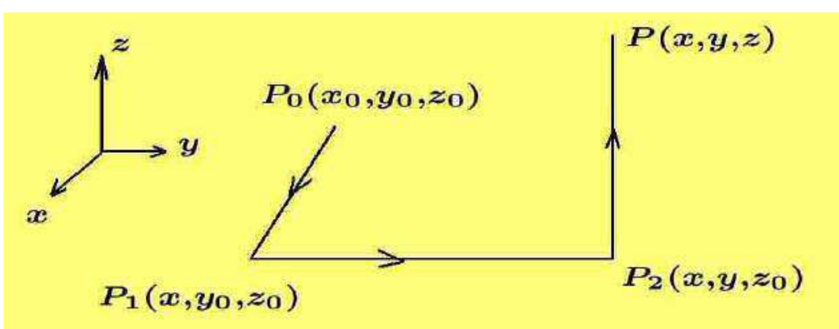


Figure 9-3. Straight line segments connecting end points of integration.

Along the sectionally continuous straight-line paths of integration illustrated in figure 9-3 the line integral (9.45) can be expressed in the component form as

$$\int_{P_0}^{P} \vec{F} \cdot d\vec{r} = \int_{P_0}^{P} F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz = \phi(x,y,z) - \phi(x_0,y_0,z_0).$$

The line integral (9.45) can be expressed as the sum of the line integrals along the straight line paths $\overline{P_0P_1}$, $\overline{P_1P_2}$, $\overline{P_2P}$ illustrated in figure 9-3, where

Along $\overline{P_0P_1}$, one finds $\quad dy = dz = 0,\ y = y_0,\ z = z_0$

Along $\overline{P_1P_2}$, there exists the conditions $\quad dx = dz = 0,\ z = z_0,\ x$ held constant

Along $\overline{P_2P}$, use $\quad dx = dy = 0,\ x$ and y both held constant.

This produces the integral

$$\begin{aligned}\int_{P_0}^{P} \vec{F}\cdot d\vec{r} =& \int_{x_0}^{x} F_1(x,y_0,z_0)\,dx + \int_{y_0}^{y} F_2(x,y,z_0)\,dy + \int_{z_0}^{z} F_3(x,y,z)\,dz \\ =& \phi(x,y,z) - \phi(x_0,y_0,z_0).\end{aligned} \tag{9.46}$$

If $\vec{F}$ is irrotational, then $\nabla\times\vec{F} = \vec{0}$ which implies that

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{9.47}$$

This hypothesis leads to the result $\vec{F} = \text{grad}\ \phi = \nabla\phi$ or its equivalence

$$\frac{\partial\phi}{\partial x} = F_1, \quad \frac{\partial\phi}{\partial y} = F_2, \quad \frac{\partial\phi}{\partial z} = F_3.$$

To demonstrate this take the partial derivatives of both sides of the equation (9.46) and show

$$\begin{aligned}\frac{\partial\phi}{\partial x} =& F_1(x,y_0,z_0) + \int_{y_0}^{y} \frac{\partial F_2(x,y,z_0)}{\partial x}\,dy + \int_{z_0}^{z} \frac{\partial F_3(x,y,z)}{\partial x}\,dz \\ \frac{\partial\phi}{\partial y} =& F_2(x,y,z_0) + \int_{z_0}^{z} \frac{\partial F_3(x,y,z)}{\partial y}\,dz \\ \frac{\partial\phi}{\partial z} =& F_3(x,y,z).\end{aligned} \tag{9.48}$$

Use the results from equation (9.47), to simplify the first set of integrals and find

$$\begin{aligned}\frac{\partial\phi}{\partial x} =& F_1(x,y_0,z_0) + \int_{y_0}^{y} \frac{\partial F_1(x,y,z_0)}{\partial y}\,dy + \int_{z_0}^{z} \frac{\partial F_1(x,y,z)}{\partial z}\,dz \\ =& F_1(x,y_0,z_0) + F_1(x,y,z_0)\Big|_{y_0}^{y} + F_1(x,y,z)\Big|_{z_0}^{z} \\ =& F_1(x,y_0,z_0) + F_1(x,y,z_0) - F_1(x,y_0,z_0) + F_1(x,y,z) - F_1(x,y,z_0) \\ =& F_1(x,y,z).\end{aligned}$$

Similarly, the relations of equations (9.47) can be used to simplify the second integral of equation (9.48) and one can show

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= F_2(x,y,z_0) + \int_{z_0}^{z} \frac{\partial F_2(x,y,z)}{\partial z}\, dz \\
&= F_2(x,y,z_0) + F_2(x,y,z)\Big|_{z_0}^{z} \\
&= F_2(x,y,z_0) + F_2(x,y,z) - F_2(x,y,z_0) \\
&= F_2(x,y,z).
\end{aligned}
$$

Thus, from the hypothesis that $\nabla \times \vec{F} = \vec{0}$, one finds that $\vec{F} = \text{grad}\, \phi = \nabla \phi$. Consequently, one can say that an irrotational vector field is derivable from a potential function ϕ.

Example 9-1. Show that

$$
\vec{F} = (y^2 + z)\,\hat{\mathbf{e}}_1 + (2xy + z^2)\,\hat{\mathbf{e}}_2 + (2yz + x)\,\hat{\mathbf{e}}_3
$$

is an irrotational vector field and find the corresponding potential function from which $\vec{F}$ is derivable.

Solution: It is readily verified that $\text{curl}\, \vec{F} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2+z) & (2xy+z^2) & (2yz+x) \end{vmatrix} = \vec{0}$ and hence $\vec{F}$ is irrotational. Two methods of finding the corresponding potential function are as follows.

Method 1 By line integral integration, where the path of integration consists of the straight-line segments illustrated in figure 9-3, one can show

$$
\begin{aligned}
\phi(x,y,z) - \phi(x_0,y_0,z_0) &= \int_{x_0}^{x} (y_0^2 + z_0)\, dx + \int_{y_0}^{y} (2xy + z_0^2)\, dy + \int_{z_0}^{z} (2yz + x)\, dz \\
&= (y_0^2 x + z_0 x)\Big|_{x_0}^{x} + (xy^2 + z_0^2 y)\Big|_{y_0}^{y} + (yz^2 + xz)\Big|_{z_0}^{z} \\
&= (xy^2 + yz^2 + xz) - (x_0 y_0^2 + y_0 z_0^2 + x_0 z_0)
\end{aligned}
$$

where in the second integral x is held constant and in the third integral both x and y are held constant. The resulting integral implies

$$
\phi(x,y,z) = xy^2 + yz^2 + xz.
$$

Method 2 The components of the relation $\vec{F} = \text{grad}\, \phi$ produce the scalar equations

$$
\frac{\partial \phi}{\partial x} = y^2 + z, \qquad \frac{\partial \phi}{\partial y} = 2xy + z^2, \qquad \frac{\partial \phi}{\partial z} = 2yz + x.
$$

Integrating the first equation with respect to x, the second equation with respect to y and the third equation with respect to z produces

$$
\phi = y^2 x + zx + f_1(y,z), \qquad \phi = y^2 x + z^2 y + f_2(x,z), \qquad \phi = xz + z^2 y + f_3(x,y)
$$

where f_1, f_2, f_3 are arbitrary functions which have been held constant during the partial differentiation process. Now add the first and second equations, add the first and third equation and add the second and third equations to obtain

$$\begin{aligned}2\phi =&2xy^2 + xz + yz^2 + f_1 + f_2\\2\phi =&xy^2 + 2xz + yz^2 + f_1 + f_3\\2\phi =&xy^2 + xz + 2yz^2 + f_2 + f_3\end{aligned}$$

In order that these three equations be the same, require that

$$f_1 + f_2 = xz + yz^2, \qquad f_1 + f_3 = xy^2 + yz^2, \qquad f_2 + f_3 = xy^2 + xz \tag{9.49}$$

Now solve the equations (9.49) for f_1, f_2 and f_3 to show that

$$f_1 = z^2y, \qquad f_2 = xz, \qquad f_3 = xy^2,$$

The potential function can then be expressed as

$$\phi = xy^2 + xz + yz^2.$$

Observe that any constant can be added to this potential function to obtain a more general result, since the derivative of a constant is zero.

■

Solenoidal Fields

A vector field which is **solenoidal** satisfies the property that **the divergence of the vector field is zero**. An alternate definition of a solenoidal vector field is obtained from Gauss' divergence theorem

$$\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}. \tag{9.50}$$

If $\nabla \cdot \vec{F} = 0$, then $\iint_S \vec{F} \cdot d\vec{S} = 0$ which implies that the total flux, through the simple closed surface surrounding the volume V, is zero.

It has been shown that an irrotational vector field is derivable from a potential function. An analogous result holds for solenoidal vector fields. That is, if $\vec{F}$ is **a solenoidal vector field which is continuous and differentiable**, then there exists **a vector potential $\vec{V}$ such that $\vec{F} = \nabla \times \vec{V}$** $= \text{curl}\, \vec{V}$. However, this vector potential **is not unique**, for if $\vec{V}$ is a vector satisfying $\vec{F} = \text{curl}\, \vec{V}$, then the vector potential $\vec{V}^* = \vec{V} + \nabla\psi$, where

ψ is any scalar function, is also a vector satisfying $\vec{F} = \text{curl}\, \vec{V}^*$. This result is verified by using the distributive property of the curl since

$$\vec{F} = \text{curl}\, \vec{V}^* = \text{curl}\, (\vec{V} + \nabla\psi) = \text{curl}\, \vec{V} + \text{curl}\, (\nabla\psi) = \text{curl}\, \vec{V}. \tag{9.51}$$

together with the fact that $\text{curl}\, (\nabla\psi) = \text{curl grad}\, \psi = \vec{0}$.

Since the vector potential $\vec{V}$ is not uniquely determined, it is only necessary to exhibit **one vector potential** of $\vec{F}$. Toward this end make note of the fact that $\vec{F}$ can be expressed in the form

$$\begin{aligned} \vec{F} &= F_1\, \hat{\mathbf{e}}_1 + F_2\, \hat{\mathbf{e}}_2 + F_3\, \hat{\mathbf{e}}_3 = \nabla \times \vec{V} \\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) \hat{\mathbf{e}}_1 + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) \hat{\mathbf{e}}_2 + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) \hat{\mathbf{e}}_3. \end{aligned} \tag{9.52}$$

Show that if the component $V_3 = 0$, then the components of $\vec{F}$ must satisfy the equations

$$F_1 = -\frac{\partial V_2}{\partial z}, \qquad F_2 = \frac{\partial V_1}{\partial z}, \qquad F_3 = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}. \tag{9.53}$$

An integration of the first two equations in (9.53) produces

$$\begin{aligned} V_1 &= \int_{z_0}^{z} F_2\, dz + f_1(x,y) \\ V_2 &= -\int_{z_0}^{z} F_1\, dz + f_2(x,y), \end{aligned} \tag{9.54}$$

where f_1, f_2 are arbitrary functions which are held constant during the partial differentiation processes used to calculate $\frac{\partial V_1}{\partial z}$ and $\frac{\partial V_2}{\partial z}$. The functions f_1 and f_2 must be selected in such a way that the last equation in (9.53) is also satisfied for all values of x, y, and z. Substitution of equations (9.54) into the last equation of (9.53) informs us that

$$\begin{aligned} \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} &= -\int_{z_0}^{z} \frac{\partial F_1}{\partial x}\, dz - \int_{z_0}^{z} \frac{\partial F_2}{\partial y}\, dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \\ &= -\int_{z_0}^{z} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}. \end{aligned} \tag{9.55}$$

Now by assumption, $\vec{F}$ is a solenoidal vector field and consequently

$$\text{div}\, \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0.$$

We therefore can write

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = -\frac{\partial F_3}{\partial z}$$

and thereby simplify the integral (9.55) to the form

$$\begin{aligned}\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} &= \int_{z_0}^{z} \frac{\partial F_3}{\partial z}\,dz + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \\ &= F_3(x,y,z) - F_3(x,y,z_0) + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.\end{aligned} \tag{9.56}$$

This equation tells us that if f_1, f_2 are selected to satisfy

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = F_3(x,y,z_0),$$

then the last equation of (9.53) is satisfied. One choice of f_1 and f_2 which satisfies the required condition is $f_1(x,y) = 0$ and

$$f_2(x,y) = \int_{x_0}^{x} F_3(x,y,z_0)\,dx.$$

For the special conditions assumed, the constructed vector potential $\vec{V}$ has the components

$$\begin{aligned}V_1 &= \int_{z_0}^{z} F_2(x,y,z)\,dz \\ V_2 &= -\int_{z_0}^{z} F_1(x,y,z)\,dz + \int_{x_0}^{x} F_3(x,y,z_0)\,dx \\ V_3 &= 0.\end{aligned} \tag{9.57}$$

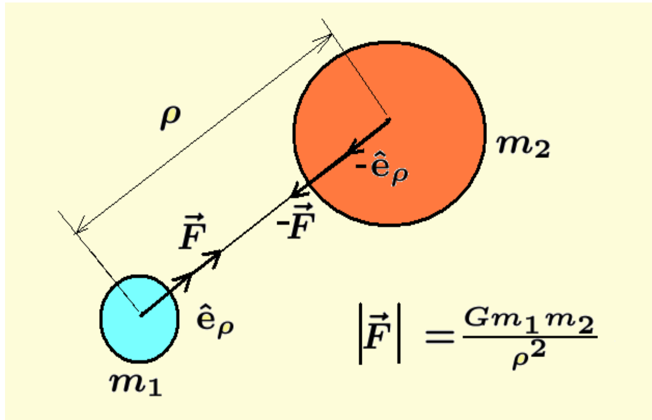
Other vector potential functions may be constructed by utilizing different assumptions on the components of $\vec{V}$ and performing similar integrations to those illustrated. Alternatively one could add the gradient of any arbitrary scalar function to the vector potential $\vec{V}$ and obtain other potential functions $\vec{V}^* = \vec{V} + \nabla\psi$.

Example 9-2.

By Newton's inverse square law, the force of attraction between two masses m_1 and m_2 is given by

$$\vec{F} = \frac{G m_1 m_2}{\rho^2}\,\hat{\mathbf{e}}_\rho \tag{9.58}$$

where $G = 6.6730\,(10)^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$ is called the gravitational constant, ρ is the distance between the center of mass of each body and $\hat{\mathbf{e}}_\rho$ is a unit vector pointing along the line connecting the center of mass of the two bodies. The force is an attractive force and so the direction of $\hat{\mathbf{e}}_\rho$ depends upon which center of mass is selected to sketch this force.



This force is derivable from the potential function

$$\phi = \frac{k}{\rho}, \qquad \text{where} \qquad \rho = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$$

is the distance between the two masses and $k = Gm_1m_2$ is a constant. In the above relation the coordinates of the center of mass of the bodies 1 and 2 are respectively $P_1(x_1, y_1, z_1)$ and $P_2(x, y, z)$. The quantity k is a constant, and $\hat{\mathbf{e}}_\rho$ is a unit vector with origin at P_1 and pointing toward P_2. The force of attraction of mass m_1 toward mass m_2 is calculated by the vector operation $\vec{F} = -\text{grad}\,\phi$. To calculate this force, first calculate the partial derivatives

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \frac{\partial\phi}{\partial\rho}\frac{\partial\rho}{\partial x} = \frac{-k}{\rho^2}\frac{(x-x_1)}{\rho} \\
\frac{\partial\phi}{\partial y} &= \frac{\partial\phi}{\partial\rho}\frac{\partial\rho}{\partial y} = \frac{-k}{\rho^2}\frac{(y-y_1)}{\rho} \\
\frac{\partial\phi}{\partial z} &= \frac{\partial\phi}{\partial\rho}\frac{\partial\rho}{\partial z} = \frac{-k}{\rho^2}\frac{(z-z_1)}{\rho}
\end{aligned}$$

and then the gradient is calculated and one obtains

$$\vec{F} = -\text{grad}\,\phi = \frac{k}{\rho^2}\left[\frac{(x-x_1)}{\rho}\,\hat{\mathbf{e}}_1 + \frac{(y-y_1)}{\rho}\,\hat{\mathbf{e}}_2 + \frac{(z-z_1)}{\rho}\,\hat{\mathbf{e}}_3\right] = \frac{k}{\rho^2}\,\hat{\mathbf{e}}_\rho$$

Here $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ is a position vector for the point P_2 and $\vec{r}_1 = x_1\,\hat{\mathbf{e}}_1 + y_1\,\hat{\mathbf{e}}_2 + z_1\,\hat{\mathbf{e}}_3$ is a position vector for the point P_1. The vector $\vec{r} - \vec{r}_1$ is a vector pointing from P_1 to P_2 and the vector $\dfrac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|} = \hat{\mathbf{e}}_\rho$ is a unit vector pointing from P_1 to P_2. Here **the vector field is called conservative since the force field is derivable from a potential function**. The potential function for Newton's law of gravitation is called **the gravitational potential**. By using the relation $\vec{F} = +\text{grad}\,\phi$ one obtains the force of attraction of mass m_2 toward mass m_1.

■

Example 9-3. Multiply both sides of Newton's second law $\vec{F} = m\vec{a} = m\dfrac{d^2\vec{r}}{dt^2}$ by $\dfrac{d\vec{r}}{dt}$ and then integrate from $P_0(x_0, y_0, z_0)$ to $P(x, y, z)$, to obtain

$$\begin{aligned}\int_{P_0}^{P} \vec{F}\cdot d\vec{r} =& \int_{P_0}^{P} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_{P_0}^{P} m\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}\,dt\\ =& \int_{P_0}^{P} \frac{m}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 dt = \int_{P_0}^{P}\frac{m}{2}\frac{d}{dt}\left(V^2\right)\,dt\\ =& \left.\frac{mV^2}{2}\right|_{P_0}^{P} = \left.\frac{mV^2}{2}\right|_{P} - \left.\frac{mV^2}{2}\right|_{P_0}\end{aligned}\tag{9.59}$$

which states that the work done in moving from P_0 to P_1 equals the change in kinetic energy. Now if $\vec{F}$ is derivable from a potential function ϕ such that $\vec{F} = -\nabla\phi$, then

$$\int_{P_0}^{P} \vec{F}\cdot d\vec{r} = \int_{P_0}^{P} -\nabla\phi\, d\vec{r} = \int_{P_0}^{P} -d\phi = -\phi\Big|_{P_0}^{P} = \phi(x_0, y_0, z_0) - \phi(x, y, z)\tag{9.60}$$

Equating the results from equations (9.59) and (9.60) and rearranging terms shows that

$$\phi(x_0, y_0, z_0) + \frac{m}{2}V^2(x_0, y_0, z_0) = \phi(x, y, z) + \frac{m}{2}V^2(x, y, z)\tag{9.61}$$

This equation states that **the sum of the kinetic energy and the potential energy has a constant value**. A result which is known as **the principal of conservation of energy**. As a result, any force fields which are derivable from potential functions are called **conservative force fields.**

■

Example 9-4. If $\vec{F}$ is a solenoidal vector field, then $\operatorname{div}\vec{F} = 0$ and one can write $\vec{F} = \operatorname{curl}\vec{V}$ for some vector potential $\vec{V}$. Consider an arbitrary region enclosed by a surface S and then select a simple closed curve C on this surface which divides the surface into two regions, call these regions S_1 and S_2. The flux through this volume is given by

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot d\vec{S}_1 + \iint_{S_2}\vec{F}\cdot d\vec{S}_2 = \iiint_V \operatorname{div}\vec{F}\,dV = 0$$

which implies that

$$\iint_{S_1}\vec{F}\cdot d\vec{S}_1 = -\iint_{S_2}\vec{F}\cdot d\vec{S}_2$$

By Stokes' theorem

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \iint_{S_1} (\nabla \times \vec{V}) \cdot d\vec{S}_1 = \oint_C \vec{V} \cdot d\vec{r}$$

and

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \iint_{S_2} (\nabla \times \vec{V}) \cdot d\vec{S}_2 = \oint_C -\vec{V} \cdot d\vec{r},$$

where the negative sign is due to the relative directions associated with the line integrals relative to the normals $\hat{\mathbf{e}}_{n_1}$ and $\hat{\mathbf{e}}_{n_2}$ to the respective surfaces S_1 and S_2. That is, Stokes theorem requires the line integral around the closed curve C be in the positive direction with respect to the normal on the surface. When the above integrals are added, the result is the net flux through an arbitrary closed surface is zero.

■

Two-dimensional Conservative Vector Fields

If corresponding to each point (x, y) in a region R of the plane $z = 0$, there corresponds a vector

$$\vec{F} = \vec{F}(x, y) = M(x, y)\,\hat{\mathbf{e}}_1 + N(x, y)\,\hat{\mathbf{e}}_2, \tag{9.62}$$

a vector field is said to exist in the region. Further, this field is said to be conservative if a scalar function of position $\phi(x, y)$ exists such that

$$\text{grad }\phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = M(x, y)\,\hat{\mathbf{e}}_1 + N(x, y)\,\hat{\mathbf{e}}_2 = \vec{F}. \tag{9.63}$$

The scalar function ϕ is called a potential function for the vector field $\vec{F}$. (Again, note that sometimes $\vec{F} = -\text{grad}\,\phi$ is more convenient to use.) The vector $\vec{F}$ is also referred to as an irrotational vector field and is derivable from the scalar potential function ϕ which satisfies

$$\frac{\partial \phi}{\partial x} = M \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N.$$

Differentiating these relations produces

$$\frac{\partial^2 \phi}{\partial x\,\partial y} = \frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial y\,\partial x} = \frac{\partial N}{\partial x} \tag{9.64}$$

so that a necessary condition that $\vec{F} = M\,\hat{\mathbf{e}}_1 + N\,\hat{\mathbf{e}}_2$ be a conservative field is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

An equivalent statement is that $\text{curl }\vec{F} = \vec{0}$.

Definition: (Equipotential curves) ***If $\vec{F} = \vec{F}(x, y)$ is a given conservative vector field with potential $\phi(x, y)$, then the family of curves $\phi(x, y) = c$ are called equipotential curves.***

By selecting a constant value c and graphing the equipotential curves

$$\phi = c, \quad \phi = c + 1, \quad \phi = c + 2, \ldots,$$

one can determine by the spacing of these curves **an estimate of the field intensity** in a given region.

The equipotential family of curves $\phi(x, y) = c$ satisfies the differential equation

$$\begin{aligned} d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0 \\ \text{or} \qquad M(x, y)\, dx + N(x, y)\, dy = 0. \end{aligned} \tag{9.65}$$

If this differential equation is exact, then it can be expressed in the form

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = M(x, y)\, dx + N(x, y)\, dy$$

and using the figure 8-10, its solution may be expressed as a line integral in either of the forms

$$\begin{aligned} \phi(x, y) &= \int_{x_0}^{x} M(x, y_0)\, dx + \int_{y_0}^{y} N(x, y)\, dy = c \\ \text{or} \qquad \phi(x, y) &= \int_{x_0}^{x} M(x, y)\, dx + \int_{y_0}^{y} N(x_0, y)\, dy = c \end{aligned} \tag{9.66}$$

depending upon the straight-line path of integration from P_0 to P. At any point (x, y), except singular points, where M and N are undefined, there is a tangent vector and a normal vector to the point (x, y) on the curve $\phi(x, y) = c$. The vector $\vec{F} = \vec{F}(x, y)$ lies in the direction of the normal to the curve since $\text{grad}\, \phi = \vec{F}$ is a vector normal to $\phi(x, y) = c$ at the point (x, y)

Field Lines and Orthogonal Trajectories

Field lines are lines or curves such that **at each point on these curves the direction of the tangent vector to the curve is the same as the direction of the vector field at that point**. An **orthogonal trajectory** of a family of plane curves is a curve which intersects every member of the family at right angles. The set of all curves which intersect every member of $\phi(x, y)$ orthogonally are called **the orthogonal trajectories** of the family. Let $\psi(x,y)=c^*$ denote the family of orthogonal trajectories to the family of equipotential curves $\phi(x,y)=c$. The family of curves $\psi(x,y)=c^*$ describes the field lines associated with the vector field $\vec{F}$. That is, every orthogonal trajectory of the family of equipotential curves $\phi(x,y)=c$ has a tangent vector which lies along the same direction as the vector $\vec{F}$ (same direction as the normal to $\phi(x,y)=c$.) If $\vec{r}=\vec{r}(x,y)$ defines a field line, then $d\vec{r}$ points in the direction of vector field so that the slope of the field lines are in direct proportion to the components of $\vec{F}$. If $d\vec{r}=k\vec{F}$, where k is some constant, then one can write $dx\,\hat{\mathbf{e}}_1+dy\,\hat{\mathbf{e}}_2=k[M(x,y)\,\hat{\mathbf{e}}_1+N(x,y)\,\hat{\mathbf{e}}_2]$ or after equating like components

$$\frac{dx}{M}=\frac{dy}{N}=k \qquad \text{or} \qquad -N\,dx+M\,dy=0. \tag{9.67}$$

This gives the differential equation which defines the field lines. An equivalent statement is that $d\vec{r}\times\vec{F}=\vec{0}$, where $\vec{r}$ is the position vector to a point on the field line curve $\psi(x,y)=c$.

Example 9-5. Show that the vector field

$$\vec{F}=M(x,y)\,\hat{\mathbf{e}}_1+N(x,y)\,\hat{\mathbf{e}}_2=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2$$

is conservative and sketch **the equipotential curves** and **field lines** associated with this vector field.

Solution: The vector field is conservative, since $\text{curl}\,\vec{F}=\vec{0}$. If $\phi(x,y)=c$ is a family of equipotential curves, then $d\phi=\text{grad}\,\phi\cdot d\vec{r}=\vec{F}\cdot d\vec{r}=0$ produces the differential equation of the equipotential curves and one can write

$$d\phi=M\,dx+N\,dy=0 \qquad \text{or} \qquad d\phi=x\,dx+y\,dy=0.$$

By integrating this equation, there results the equipotential curves

$$\phi(x,y)=\frac{x^2}{2}+\frac{y^2}{2}=c,$$

which are circles centered at the origin.

If $\vec{r}$ is the position vector to a point on a field line, then $d\vec{r}$ is in the direction of the tangent to the field line and must have the same direction as the vector field $\vec{F}$ so that one can write $d\vec{r}=k\vec{F}$, where k is a proportionality constant. Equating like components one then finds the differential equation describing the field lines as

$$d\vec{r}=dx\,\hat{\mathbf{e}}_1+dy\,\hat{\mathbf{e}}_2=kF_1\,\hat{\mathbf{e}}_1+kF_2\,\hat{\mathbf{e}}_2 \quad \text{or} \quad \frac{dx}{F_1}=\frac{dy}{F_2}=k \quad \text{or} \quad \frac{dx}{x}=\frac{dy}{y}=k.$$

This differential equation is derived by requiring the direction of the vector field at an arbitrary point (x, y) have the same direction as the tangent to the field line curve which passes through the same point (x, y). An integration of the differential equation defining the field lines produces

$$\ln x = \ln y + \ln c$$

or the family curves defining the field lines is given by

$$\psi(x, y) = \frac{y}{x} = k,$$

where k is an arbitrary constant.

The equipotential curves and field lines associated with the given vector field $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ are illustrated in figure 9-4. In two dimensions, the vector fields are best visualized by sketches of the equipotential curves and field lines. In sketching the vector fields **be sure to distinguish the field lines from the equipotential curves by placing arrows at various points on the field lines**. These arrows indicate the **direction of the vector field** at various points.

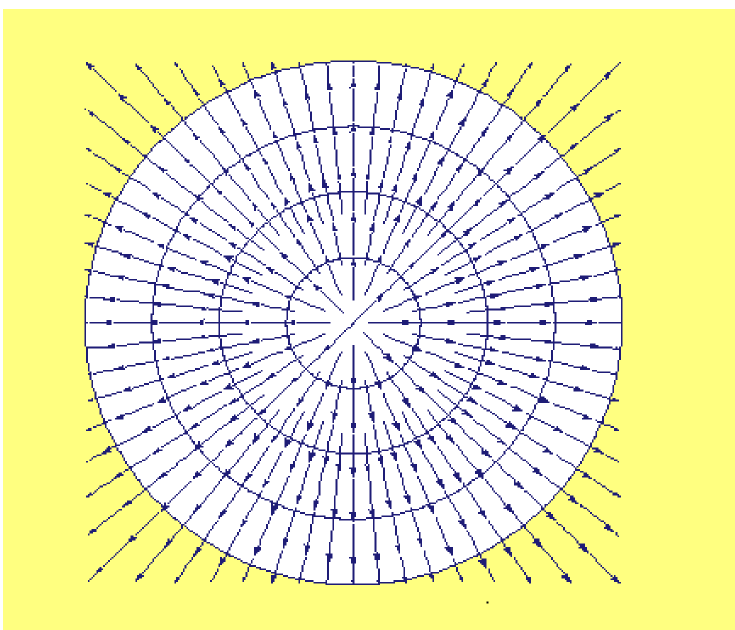


Figure 9-4. Equipotential curves and field lines for $\vec{F} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$

■

Vector Fields Irrotational and Solenoidal

If in addition to being conservative, the two-dimensional vector field given by $\vec{F} = M(x,y)\,\hat{\mathbf{e}}_1 + N(x,y)\,\hat{\mathbf{e}}_2$ is also solenoidal and

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0,$$

then

(i) The equipotential curves $\phi(x,y) = c$, c constant, are obtained from the exact differential equation

$$d\phi = \operatorname{grad}\phi \cdot d\vec{r} \quad \text{or} \quad d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0 \quad \text{or} \quad d\phi = M(x,y)\,dx + N(x,y)\,dy = 0$$

(ii) The family of field lines $\psi(x,y) = c^*$, c^* constant, are obtained from the differential equation

$$d\vec{r} = k\vec{F} \quad \text{or} \quad \frac{dx}{M} = \frac{dy}{N} = k \quad \text{or} \quad -\,N\,dx + M\,dy = 0$$

The solution of the differential equation defining the field lines is easily obtained since it also is an exact differential equation. The solution can be represented as a line integral in either of the forms

$$\begin{aligned} \psi(x,y) &= \int_{x_0}^{x} -N(x,y_0)\,dx + \int_{y_0}^{y} M(x,y)\,dy = c \\ \text{or} \quad \psi(x,y) &= \int_{x_0}^{x} -N(x,y)\,dx + \int_{y_0}^{y} M(x_0,y)\,dy = c, \end{aligned} \tag{9.68}$$

depending upon the choice of the path connecting the end points. These curves represent the field lines associated with the vector field $\vec{F}$, where

$$\frac{\partial \psi}{\partial x} = -N \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = M.$$

It follows that **if the vector field is both irrotational and solenoidal**, then the equipotential curves $\phi(x,y) = c$ and the field lines $\psi(x,y) = c^*$ are such that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{9.69}$$

These equations are called **the Cauchy–Riemann equations.** In vector form these equations may be expressed as

$$\text{grad}\ \phi = (\text{grad}\ \psi) \times\ \hat{\mathbf{e}}_3.$$

That is

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 = \frac{\partial \psi}{\partial y}\,\hat{\mathbf{e}}_1 - \frac{\partial \psi}{\partial x}\,\hat{\mathbf{e}}_2. \tag{9.70}$$

Differentiate the Cauchy–Riemann equations (9.69) and show

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial x\,\partial y}. \tag{9.71}$$

Addition of these equations produces **the Laplace equation**

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{9.72}$$

Similarly, it can be shown that

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y\,\partial x}$$

so that by addition

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{9.73}$$

Hence, both ϕ and ψ are solutions of Laplace's equation.

> **Definition: (Harmonic function)** ***Any function which is a solution of Laplace's equation $\nabla^2\omega = 0$ and which has continuous second-order derivatives is called a harmonic function.***

Orthogonality of Equipotential Curves and Field Lines

To show that the equipotential curves $\phi = c_1$ and the field lines $\psi = c^*$ are orthogonal, consider the dot product of the vectors normal to these curves at a common point of intersection. These normal vectors are

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{e}}_2 \quad \text{and} \quad \text{grad } \psi = \frac{\partial \psi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial \psi}{\partial y}\,\hat{\mathbf{e}}_2$$

and their dot product produces

$$\text{grad } \phi \cdot \text{grad } \psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y}.$$

With the use of the Cauchy-Riemann equations it can be shown that this dot product is zero. Thus the vector $\text{grad}\,\psi$ is perpendicular to the vector $\text{grad}\,\phi$ and the equipotential curves and field lines are orthogonal.

In various branches of science and engineering, the quantities ϕ and ψ have many different physical interpretations. For example, in fluid dynamics, the velocity field is derivable from a velocity potential ϕ, and the field lines are called streamlines. In the study of heat flow, the heat flow vector is derivable from a potential ϕ which represents temperature and the equipotential curves $\phi =$ Constant are called isothermal curves (curves of constant temperature.) The field lines associated with this vector field are termed heat flow lines. In the study of electric and magnetic fields the potential functions from which these fields are derivable are termed, respectively, the electric and magnetic field potentials. The field lines associated with these potentials are called lines of electric and lines of magnetic force. Usually the harmonic functions ϕ and ψ are expressed as the real and imaginary parts of a function of a complex variable.

Laplace's Equation

For $\vec{F} = \vec{F}(x,y,z)$, a vector field which is both irrotational and solenoidal, then $\vec{F}$ satisfies

$$\text{curl } \vec{F} = \vec{0} \qquad \text{and} \qquad \text{div } \vec{F} = 0. \tag{9.74}$$

It has been shown that for these circumstances $\vec{F}$ is derivable from a scalar potential function Φ. In particular, $\vec{F} = \text{grad } \Phi = \nabla\, \Phi$. Hence, Φ must be a solution of Laplace's equation $\nabla^2\Phi = 0$. That is,

$$\text{div } \vec{F} = \text{div}\,(\text{grad}\,\Phi) = \nabla \cdot (\nabla\, \Phi) = \nabla^2 \Phi = 0.$$

In expanded form the Laplace equation is expressed

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0$$

$$\text{or} \qquad \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

This partial differential equation has many physical applications associated with it and arises in many areas of science, physics and engineering. The Laplace equation can be expressed in different forms depending upon the coordinate system in which it is represented.

Three-dimensional Representations

In **a rectangular right-handed** (x,y,z) **system of coordinates**, the Laplace equation is expressed as

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad U = U(x,y,z) \tag{9.75}$$

In **a cylindrical** (r, θ, z) **coordinate system**, the Laplace equation takes the form

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad U = U(r, \theta, z) \tag{9.76}$$

and in **a spherical** (ρ, θ, ϕ) **coordinate system**, the Laplace equation is represented has the form

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial U}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2} = 0, \qquad U = U(\rho, \theta, \phi) \tag{9.80}$$

Two-dimensional Representations

In a two-dimensional (x, y) coordinate system the Laplace equation is represented

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \qquad U = U(x, y) \tag{9.78}$$

In a polar (r, θ) coordinate system the Laplace equation becomes

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0, \qquad U = U(r, \theta) \tag{9.79}$$

In spherical coordinates, where there is symmetry with respect to the variable ϕ, the Laplacian is represented

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial U}{\partial \theta} = 0, \qquad U = U(\rho, \theta) \tag{9.80}$$

One-dimensional Representations

In one-dimension, the Laplace equation becomes

$$\begin{aligned} \frac{d^2 U}{dx^2} &= 0, \qquad U = U(x) \quad \text{rectangular} \\ \frac{d^2 U}{dr^2} + \frac{1}{r}\frac{dU}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) &= 0, \qquad U = U(r) \quad \text{polar} \\ \frac{d^2 U}{d\rho^2} + \frac{2}{\rho}\frac{dU}{d\rho} = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dU}{d\rho}\right) &= 0, \qquad U = U(\rho) \quad \text{spherical} \end{aligned} \tag{9.162}$$

Three-dimensional Conservative Vector Fields

Analogous to what has been done in studying two-dimensional vector fields, one can state that if a three-dimensional vector field

$$\vec{F} = \vec{F}(x, y, z) = F_1(x, y, z)\,\hat{\mathbf{e}}_1 + F_2(x, y, z)\,\hat{\mathbf{e}}_2 + F_3(x, y, z)\,\hat{\mathbf{e}}_3$$

is derivable from a potential function $\phi(x, y, z)$ such that $\vec{F} = \text{grad}\ \phi,$, then the family of surfaces $\phi(x, y, z) = c$ are called equipotential surfaces. The differential equation satisfied by the equipotential surfaces is obtained by differentiating $\phi(x,y,z)=c$ to obtain the exact differential

$$\begin{aligned} d\phi = & \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z} = 0 \\ \text{or} \quad & F_1\,dx + F_2\,dy + F_3\,dz = \vec{F}\cdot d\vec{r} = 0 \end{aligned} \tag{9.82}$$

The solution of this differential equation may be obtained by line integration methods.

The field lines associated with the vector field $\vec{F}$ are those curves which are everywhere tangent to the field vectors. The direction of the field lines are in direct proportion to the components of $\vec{F}$ and thus the differential equation satisfied by the field lines is $d\vec{r} = k\vec{F}$, where k is a proportionality constant. Equating like components produces the equations

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)} = k \tag{9.83}$$

which is equivalent to the statement that $d\vec{r}\times\vec{F} = \vec{0}$ since $d\vec{r}$ has the same direction as $\vec{F}$. Another way of picturing this is to let $\vec{r}$ denote the position vector to a point (x,y,z) on a field line. The differential element $d\vec{r}$ will then be in the direction of the tangent to the field line which, by definition, is also in the same direction as $\vec{F}$ at the common point (x,y,z). Thus, $d\vec{r} = k\vec{F}$, where k is a proportionality constant. This equation can be written in the component form as

$$d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3 = k\left[F_1(x,y,z)\,\hat{\mathbf{e}}_1 + F_2(x,y,z)\,\hat{\mathbf{e}}_2 + F_3(x,y,z)\,\hat{\mathbf{e}}_3\right].$$

Equating like components produces the differential relation (9.83). Geometrically, the field lines defined by equation (9.83) are orthogonal to the equipotential surfaces defined by equation (9.82). That is, $\text{grad}\,\phi$ is perpendicular to the tangent element $d\vec{r}$.

A solution of the differential system (9.83) consists of two independent relations or integrals of the form

$$\mu_1(x,y,z) = c_1 \qquad \text{and} \qquad \mu_2(x,y,z) = c_2,$$

which represents two families of surfaces having c_1 and c_2 as parameters. The field lines are the curves of intersection of these two family of surfaces, and these curves (field lines) are called a two-parameter family of curves, where the constants c_1 and

c_2 are the two parameters. Two methods for obtaining independent integrals of equations (9.83) are now presented.

Theory of Proportions

From the theory of proportions one can make use of the following result:

For constants $\alpha,\ \beta,\ \gamma$, not all zero, one can write

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} = \frac{\alpha\, dx + \beta\, dy + \gamma\, dz}{\alpha F_1 + \beta F_2 + \gamma F_3}.$$

In many instances one can choose appropriate values for the constants $\alpha,\ \beta,\ \gamma$ to construct equations which can be easily integrated to produce a family of surfaces representing a solution of the differential equations. Using the method of proportions, by trial and error, one tries to construct two independent family of solutions. Consider one surface from each family. These surfaces intersect and the curve of intersection defines a field line. The two family of surfaces intersect in a family of field lines.

Example 9-6. Find the field lines associated with the vector field

$$\vec{F} = \vec{F}(x,y,z) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$$

Solution The field lines are obtained from the differential equations

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} \tag{9.84}$$

In this equation make note that an addition of the numerators and denominators of the first two fractions produces an exact differential and

$$\frac{dx+dy}{x+y} = \frac{dz}{z} = \frac{d(x+y)}{x+y}.$$

In this new equation the variables are separated and then an integration produces $\ln(x+y) = \ln z + \ln c_1$ where $\ln c_1$ is selected for the constant of integration in order to simplify the algebra. This result can be expressed as

$$\mu_1(x,y,z) = \frac{x+y}{z} = c_1$$

and represents one family of solution surfaces. Return to the equations (9.84) defining the field lines and observe that from the first two fractions one can write

$$\frac{dx}{y} = \frac{dy}{x}.$$

This is an equation where the variables can be separated and then an integration produces another independent family of surfaces

$$\mu_2(x,y,z) = \frac{x^2}{2} - \frac{y^2}{2} = c_2.$$

Hence the field lines are the intersection of the family of cylindrical surfaces, defined by hyperbola with rulings parallel to the z-axis, with the family of planes $x+y-c_1 z = 0$.

■

Method of Tangents.

Observe that if the field lines are defined as the intersection of two families of surfaces

$$\mu_1(x,y,z) = c_1 \qquad \text{and} \qquad \mu_2(x,y,z) = c_2,$$

then by differentiation one obtains

$$\frac{\partial \mu_1}{\partial x}\,dx + \frac{\partial \mu_1}{\partial y}\,dy + \frac{\partial \mu_1}{\partial z}\,dz = \text{grad}\ \mu_1 \cdot d\vec{r} = 0$$

In a similar fashion one can show

$$\frac{\partial \mu_2}{\partial x}\,dx + \frac{\partial \mu_2}{\partial y}\,dy + \frac{\partial \mu_2}{\partial z}\,dz = \text{grad}\ \mu_2 \cdot d\vec{r} = 0.$$

Note that at a point (x,y,z) on a curve of intersection of two surfaces $\mu_1 = c_1$ and $\mu_2 = c_2$, the tangential direction $d\vec{r} = dx\,\hat{\mathbf{e}}_1 + dy\,\hat{\mathbf{e}}_2 + dz\,\hat{\mathbf{e}}_3$ is the same as the direction of the field line at that point. Therefore $d\vec{r}$ must be proportional to $\vec{F}$. At the common point (x,y,z) on both surfaces the gradient vectors $\text{grad}\ \mu_1$ and $\text{grad}\ \mu_2$ are perpendicular to the surfaces $\mu_1 = c_1$ and $\mu_2 = c_2$ respectively. These vectors must therefore be perpendicular to the vector field $\vec{F}$ at this common point. Consequently, one can write $\text{grad}\ \mu_1 \cdot \vec{F} = 0$ or

$$\frac{\partial \mu_1}{\partial x} F_1 + \frac{\partial \mu_1}{\partial y} F_2 + \frac{\partial \mu_1}{\partial z} F_3 = 0$$

and similarly $\text{grad}\ \mu_2 \cdot \vec{F} = 0$ or

$$\frac{\partial \mu_2}{\partial x} F_1 + \frac{\partial \mu_2}{\partial y} F_2 + \frac{\partial \mu_2}{\partial z} F_3 = 0.$$

These equations are the basis for the method of tangents. One tries to find, by using a trial and error method, two vector functions $\vec{V} = \text{grad}\ \mu_1$ and $\vec{W} = \text{grad}\ \mu_2$ such that $\vec{V} \cdot \vec{F} = 0$ and $\vec{W} \cdot \vec{F} = 0$. Then the equations

$$\vec{V} \cdot d\vec{r} = \text{grad}\ \mu_1 \cdot d\vec{r} = 0 \quad \text{and} \quad \vec{W} \cdot d\vec{r} = \text{grad}\ \mu_2 \cdot d\vec{r} = 0$$

are exact differential equations which are easily integrated. From these integrations one finds two independent family of surfaces $\mu_1 = c_1$ and $\mu_2 = c_2$.

Example 9-7. The field lines of the vector field $\vec{F} = \vec{F}(x, y, z) = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ are determined from the differential system

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z}.$$

By trial and error one can construct the functions

$$V_1 = \frac{1}{z} \qquad V_2 = \frac{1}{z} \qquad V_3 = -\frac{(x+y)}{z^2}$$

so that $\vec{V} \cdot \vec{F} = 0$. One can then construct the exact differential equation

$$\vec{V} \cdot d\vec{r} = \frac{1}{z}\,dx + \frac{1}{z}\,dy - \frac{(x+y)}{z^2}\,dz = \text{grad}\;\mu_1 \cdot d\vec{r} = d\mu_1 = 0$$

from which to determine

$$\mu_1 = \frac{x+y}{z} = c_1$$

Similarly, by using trial and error, one can show that the functions

$$W_1 = x \qquad W_2 = -y \qquad W_3 = 0$$

are such that $\vec{W} \cdot \vec{F} = 0$. This produces the exact differential equation

$$\vec{W} \cdot d\vec{r} = x\,dx - y\,dy = \text{grad}\;\mu_2 \cdot d\vec{r} = d\mu_2 = 0$$

which is easily integrated. One finds that

$$\mu_2 = \frac{x^2}{2} - \frac{y^2}{2} = c_2.$$

Note also that the trial and error method might produce all kinds of results. For example, let

$$P_1 = \frac{1}{2}z \qquad P_2 = -\frac{1}{2}z \qquad P_3 = \frac{1}{2}(x-y),$$

then one can show $\vec{P} \cdot \vec{F} = 0$. Consequently,

$$\vec{P} \cdot d\vec{r} = \frac{1}{2}z\,dx - \frac{1}{2}z\,dy + \frac{1}{2}(x-y)\,dz = \text{grad}\;\mu_3 \cdot d\vec{r} = d\mu_3 = 0 \tag{9.85}$$

is an exact differential which can be integrated. The equation (9.85) implies that

$$\frac{\partial \mu_3}{\partial x} = \frac{1}{2}z, \qquad \frac{\partial \mu_3}{\partial y} = -\frac{1}{2}z, \qquad \frac{\partial \mu_3}{\partial z} = \frac{1}{2}(x-y)$$

and an integration of each of these functions produces

$$\mu_3 = \frac{1}{2}xz + f(y,z)$$
$$\mu_3 = -\frac{1}{2}yz + g(x,z)$$
$$\mu_3 = \frac{1}{2}(x-y)z + h(x,y),$$

where $f(y,z), g(x,z), h(x,y)$ are treated as constants of integration during the integration of partial derivatives. One finds that by selecting

$$f = -\frac{1}{2}yz, \qquad g = \frac{1}{2}xz, \qquad h = 0$$

there results the family of surfaces

$$\mu_3 = \frac{1}{2}(x-y)z = c_3$$

At first glance it appears that $\mu_3 = c_3$ is a solution family different from $\mu_1 = c_1$ and $\mu_2 = c_2$. However, from $\mu_1 = c_1$ there results

$$z = \frac{x+y}{c_1}$$

which can be substituted into μ_3 to produce

$$\mu_3 = \frac{1}{2c_1}(x^2 - y^2).$$

Hence the solution $\mu_3 =$ Constant reduces to the solution $\mu_2 =$ Constant. When one of the surfaces $\mu_i = c_i$, $(i = 1 \text{ or } 2)$ has been obtained, this known solution may be used to determine the second surface. The known solution can be used to eliminate one of the variables in the differential system and thereby reduce it to a two-dimensional equation which theoretically can be solved. Three-dimensional field lines are in general more difficult to obtain and illustrate than their two-dimensional counterparts. ■

Solid Angles

A cone is described as a family of intersecting lines. A right circular cone is an example which is easily recognized, however, this is only one special kind of a cone.

A general cone is described by a line having one point fixed in space which is free to rotate. The figure 9-5 illustrates two cones which differ from a right circular cone.

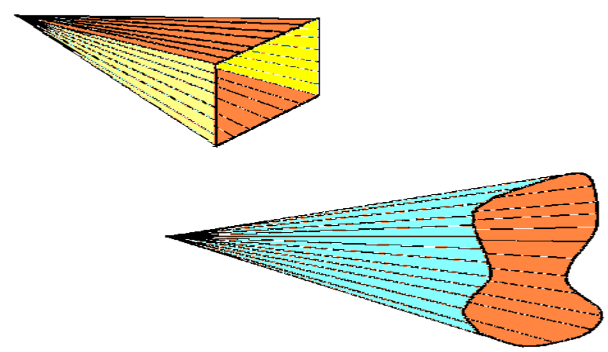
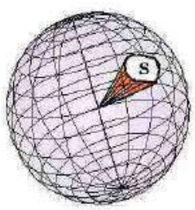


Figure 9-5. Cones generated by moving line about fixed point.



Consider a sphere of radius r and use the origin of the sphere to construct a cone which intersects the sphere and cuts out an area S on the surface as illustrated in the accompanying figure. The area S on the surface of the sphere of radius r will be proportional to r^2 since S is some fraction of the total surface area $4\pi r^2$. The ratio $\frac{S}{r^2}$ is therefore a dimensionless ratio and the quantity $\Omega = \frac{S}{r^2}$ is called **the solid angle subtended at the center of the sphere by the cone**. The **solid angle is a measure of how large an object appears to be when viewed from the origin of the sphere**. Solid angles are measured in units called steradians[6] (abbreviated sr) and by definition 1 steradian is the solid angle represented by the surface area of a sphere equal to the radius of the sphere squared. For example, if the area S in the accompanying figure equals r^2, then the solid angle subtended at the center of the sphere is said to be 1 steradian. The total solid angle about the center of the sphere being 4π steradians.

For a given oriented surface make the following constructions.

(i) A position vector $\vec{r}$ from the origin to the point on the oriented surface.
(ii) An element of surface area dS at the terminus of the vector $\vec{r}$.
(iii) A unit normal $\hat{\mathbf{e}}_n$ to the surface at the terminus of the vector $\vec{r}$.
(iv) A sphere of radius $r = |\vec{r}|$ centered at the origin 0.

[6] The solid angle is really dimensionless and sometimes the terminology of steradians is not used.

(v) A unit sphere about the origin.

(vi) Connect the points on the boundary of dS to the origin and form a cone which will intersect both the unit sphere and the sphere of radius r.

(vii) Use the dot product given by $\hat{\mathbf{e}}_n \cdot \vec{r} = r\cos\theta$ to find the angle θ between the unit normal to the surface and the position vector $\vec{r}$.

An example using the above constructions is illustrated in the figure 9-6. In the figure 9-6, the cone, constructed using the boundary of the element of surface area dS, intersects the sphere of radius r to produce an element of surface area $d\Omega$. The element of surface area $d\Omega$ can also be thought of as the projection of dS onto the sphere of radius r. This projection is given by $d\Omega = \cos\theta\, dS$ where θ is the angle between the normal to the surface and the position vector $\vec{r}$.

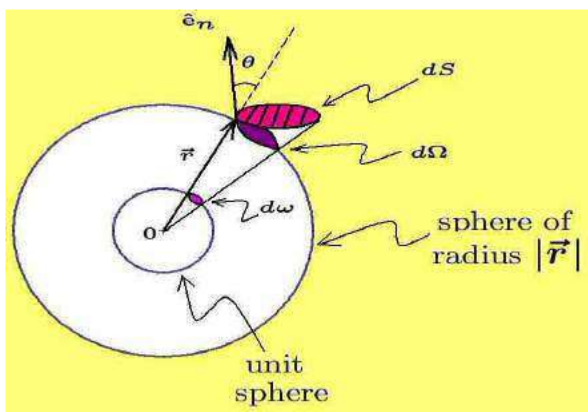


Figure 9-6. Solid angle as surface area on unit sphere.

The solid angle subtended at the origin does not depend upon the size of the sphere about the origin and so one can write

$$\frac{d\omega}{(1)^2} = \frac{d\Omega}{r^2} \qquad \Longrightarrow \qquad d\omega = \frac{d\Omega}{r^2}$$

Using the result $\hat{\mathbf{e}}_n \cdot \vec{r} = r\cos\theta$ or $\cos\theta = \dfrac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r}$ one obtains

$$d\Omega = \cos\theta\, dS = \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r}\, dS = \frac{\vec{r} \cdot d\vec{S}}{r}$$

where $d\vec{S} = \hat{\mathbf{e}}_n\, dS$ is a vector element of area.

Consider the special surface integral

$$\iint_S d\omega = \iint_S \frac{d\Omega}{r^2} = \iint_S \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r^3}\, dS = \iint_S \frac{\vec{r} \cdot d\vec{S}}{r^3}$$

where the surface S encloses a bounded, closed, simply connected region. Surface integrals of this type represent the total sum of the solid angles subtended by the element dS, summed over the surface S. For the solid angle summed about a point $0'$ outside the surface, the resulting sum of the solid angles is zero. This is because for each positive sum $+d\omega$ there is a corresponding negative sum $-d\omega$, and these add to zero in pairs. If the solid angle is summed about a point 0 inside the surface, the resulting sum is not zero. Here the sum of the areas $d\omega$ on the unit sphere, subtended by the elements dS, do not add together in pairs to produce zero but instead give the total surface area of the unit sphere which is 4π steradians. From these discussions one obtains the following

$$\iint_S d\omega = \iint_S \frac{\vec{r} \cdot d\vec{S}}{r^3} = \begin{cases} 0 & \text{if origin is outside closed surface} \\ 4\pi & \text{if origin is inside closed surface} \end{cases} \tag{9.86}$$

This result is **utilized in the study of inverse square law potentials and is known as Gauss' theorem.**

Example 9-8. Find the solid angle subtended by a right circular cone of radius r and height h.

Solution Let $\tan\theta_0 = \frac{r}{h}$ and construct a sphere of radius r which intersects the circular cone to form a spherical cap. On this spherical cap construct a ring-shaped element of area where the thickness of the ring is $ds = r\,d\theta$ and this element of thickness is rotated about the cone axis to form a ring as illustrated in the figure 9-7.

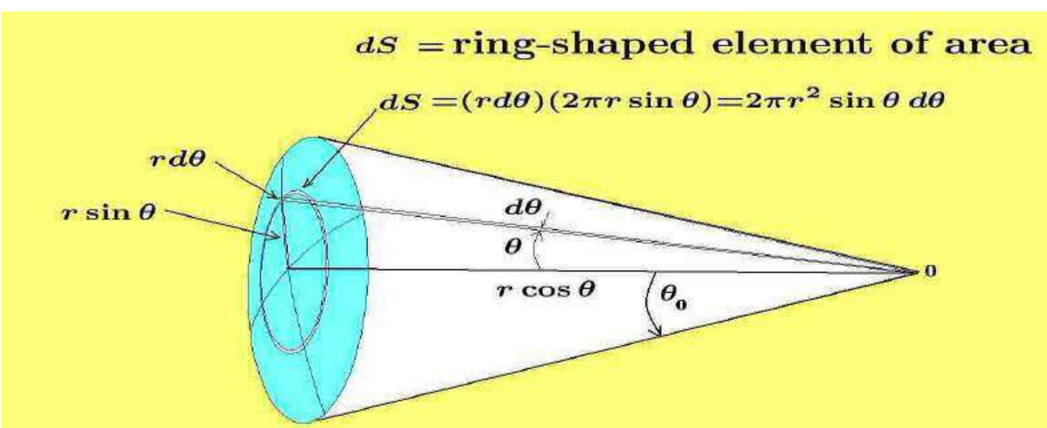


Figure 9-7. Area of spherical cap using ring-shaped element of area.

This produces an element of area

$$dS = (r d\theta)(2\pi r \sin\theta) = 2\pi r^2 \sin\theta \, d\theta \qquad \text{for} \qquad 0 \le \theta \le \theta_0$$

The total surface area of the spherical cap is obtained by a summation of the ring elements to produce the integral

$$S = \int_0^{\theta_0} 2\pi r^2 \sin\theta \, d\theta = 2\pi r^2 \left[-\cos\theta\right]_0^{\theta_0} = 2\pi r^2 (1 - \cos\theta_0)$$

The solid angle subtended by this right circular cone is therefore

$$\Omega = \frac{S}{r^2} = 2\pi(1 - \cos\theta_0)$$

■

Potential Theory

Potential theory is concerned with the solutions of Laplace's equation $\nabla^2 u = 0$, which satisfy **prescribed boundary conditions**. Two important problems of potential theory are **the Dirichlet problem** and **the Neumann problem**.

The Dirichlet problem deals with finding a solution U of Laplace's equation throughout a region R such that U takes on certain **pre assigned values on the boundary of the region** R.

The Neumann problem is concerned with obtaining a solution of Laplace's equation in a region R such that **on the boundary of** R**, the normal derivative**

$$\frac{\partial U}{\partial n} = \text{grad } U \cdot \hat{\mathbf{e}}_n$$

has prescribed values. Here $\hat{\mathbf{e}}_n$ is the unit outward normal to the boundary of the region R.

In obtaining a solution to a Dirichlet or Neumann problem in an infinite region there is the additional requirement that U satisfy certain conditions far from the origin.

Velocity Fields and Fluids

Let $\vec{V}$ denote the velocity field of a fluid in motion and let $\rho(x, y, z, t)$ denote the density of this fluid. Place within the fluid an arbitrary closed surface and consider an element of surface area dS on this surface. Let the mass of fluid flowing in a normal direction across this element of surface, in a time interval Δt, be denoted by

ΔM. It is assumed that the velocity is the same at all points over the tiny element of surface area. In a time interval Δt, the amount of fluid which crosses the element dS is given by $\Delta M = \rho \vec{V} \cdot \hat{\mathbf{e}}_n dS \Delta t$. The total mass of fluid flowing out of the volume $\mathcal{V}$ bounded by the surface S is given by

$$\Delta M = \Delta t \iint_S \rho \vec{V} \cdot d\vec{S} = \Delta t \iiint_{\mathcal{V}} \operatorname{div}(\rho \vec{V})\, dV.$$

Also the total mass of the fluid enclosed within the volume $\mathcal{V}$ bounded by S can be represented as the integral

$$M = \iiint_{\mathcal{V}} \rho dV. \tag{9.87}$$

The rate of change of the mass with time is

$$\frac{\partial M}{\partial t} = \iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, dV. \tag{9.88}$$

Hence, in a time interval Δt, the amount of fluid in the volume $\mathcal{V}$ diminishes by the amount

$$\Delta M = -\Delta t \iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, dV. \tag{9.89}$$

The amount of fluid flowing out of the arbitrary volume is equated to the amount of fluid decreasing within the volume to obtain

$$\begin{aligned} \Delta t \iiint_{\mathcal{V}} \operatorname{div}(\rho \vec{V})\, dV &= -\Delta t \iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, dV \\ \text{or} \qquad \iiint_{\mathcal{V}} \left[\operatorname{div}(\rho \vec{V}) + \frac{\partial \rho}{\partial t}\right] dV &= 0. \end{aligned} \tag{9.90}$$

For an arbitrary volume $\mathcal{V}$ within the fluid, the relation (2.42) must hold and consequently

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{V}) = 0. \tag{9.91}$$

This equation is called **the continuity equation of hydrodynamics** which can also be expressed in the form

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{V} + \rho \nabla \vec{V} = 0. \tag{9.92}$$

The first two terms on the left-hand side of this last equation represents the time rate of change of the density ρ, that is,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{V}. \tag{9.93}$$

If $\frac{d\rho}{dt} = 0$, then the fluid is an incompressible fluid, and the velocity field is solenoidal.

If the fluid flow is also irrotational, then $\vec{V}$ is derivable from a potential function Φ called the velocity potential of the fluid flow. The potential function must be a solution of Laplace's equation. The field lines associated with the velocity field $\vec{V}$ produce a family of curves which are termed streamlines.

Heat Conduction

In the basic equations describing heat conduction in materials, the following assumptions and terminology are employed:

1. Let $T = T(x,y,z,t)$ denote the temperature (°C) at a point (x,y,z) within the material at time t.
2. Heat flow within the material is denoted by the vector $\vec{q}$ having units $[\vec{q}] = \frac{\text{J}}{\text{cm}^2\cdot\text{sec}}$
3. Heat flows from regions of higher temperature to regions of lower temperature and the direction of heat flow is in the direction of the greatest rate of change of the temperature. Expressing this as a mathematics statement, we write

$$\vec{q} = -k \operatorname{grad} T, \tag{9.94}$$

where k is a proportionality constant having units of $\frac{\text{J}}{\text{cm}\cdot\text{sec}\cdot{}^\circ\text{C}}$ and is called the thermal conductivity of the material. Since the gradient of temperature points in the direction of increasing temperature, the negative sign in the relation (9.94) indicates that heat is flowing in the direction of decreasing temperature.
4. The symbol c is used to denote the specific heat of the material which is a measure of the heat capacity per unit mass of material. The specific heat c is measured in units $\frac{\text{J}}{\text{g}\,{}^\circ\text{C}}$.
5. The symbol ρ is used to denote the density of the material $[\frac{\text{g}}{\text{cm}^3}]$.
6. The total amount of heat in an arbitrary volume $\mathcal{V}$ bounded by a closed surface S is given by

$$H = \iiint_{\mathcal{V}} c\rho T\, dV, \tag{9.95}$$

where H is in joules.

If an imaginary closed surface S enclosing a volume $\mathcal{V}$ is placed within a body in which heat is flowing, then the heat flux across this surface is given by the integral

$$\iint_S \vec{q}\, d\vec{S} = \iint_S \vec{q}\cdot \hat{\mathbf{e}}_n\, dS \tag{9.96}$$

which by the divergence theorem can be expressed in the form

$$\iint_S \vec{q} \cdot d\vec{S} = \iiint_{\mathcal{V}} \text{div } \vec{q}\, dV. \tag{9.97}$$

Substituting the heat flow given by equation (2.46) into equation (2.49) produces the relation

$$\iint_S \vec{q} \cdot d\vec{S} = -\iiint_{\mathcal{V}} k\, \text{div (grad } T)\, dV, \tag{9.98}$$

which depicts the total amount of heat leaving the arbitrary volume $\mathcal{V}$ enclosed by S. From equation (2.47), one can calculate the rate of change of decreasing heat within the volume. Such a change is given by

$$-\frac{\partial H}{\partial t} = -\iint_{\mathcal{V}} c\rho \frac{\partial T}{\partial t}\, dV \tag{9.99}$$

and must equal the change given by the flux integral (2.50). Equating these quantities produces the relation

$$\iiint_{\mathcal{V}} \left[c\rho \frac{\partial T}{\partial t} - k\, \text{div (grad } T) \right] dV = 0 \tag{9.100}$$

which must hold for any arbitrary volume $\mathcal{V}$ within the material. Since the volume is arbitrary, it is required that the integrand be identically zero and write

$$c\rho \frac{\partial T}{\partial t} - k\, \text{div (grad } T) = 0 \implies \frac{c\rho}{k} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \implies \frac{c\rho}{k} \frac{\partial T}{\partial T} = \nabla^2 T \tag{9.101}$$

This result is known as the heat equation.

For steady-state temperature distributions, write $\frac{\partial T}{\partial t} = 0$, and consequently equation (9.101) reduces to Laplace's equation.

In the study of heat flow the level curves $T(x, y, z) = c$ are called isothermal surfaces, and the field lines associated with the heat flow $\vec{q}$ within the material are called heat flow lines.

Two-Body Problem

Newton's law of gravitation states that two masses m and M, are attracted toward each other with a force of magnitude $\frac{GmM}{r^2}$, where G is a constant and r is the distant between the masses. Let M represent the mass of the Sun and m represent the mass of a planet and assume that the motion of one mass with respect to the other mass takes place in a plane. Construct a set of x, y axes with origin located at the center of mass of M. Further, let $\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2$ denote a unit vector at the origin of our coordinate system and pointing in the direction of the mass m. One can then express the vector force of attraction of mass M on mass m by the equation

$$\vec{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{e}}_r \tag{9.102}$$

To find the equation of motion of mass m with respect to mass M, use Newton's second law. Let $\vec{r} = r\,\hat{\mathbf{e}}_r$ denote the position vector of mass m with respect to our origin. The equation of motion of mass m is determined from Newton's second law and is

$$\vec{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{e}}_r = m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{V}}{dt} \tag{9.103}$$

From this equation it can be shown that the motion of mass m can be described as a conic section. In order to accomplish this, let us review some facts about conic sections.

Recall that a conic section was defined as a locus of points $P(x, y)$ such that the distance of P from a fixed point (or points), called a focus, is proportional to the distance of P from a fixed line, called the directrix. The constant of proportionality is called the eccentricity and is denoted by the symbol ϵ. If $\epsilon = 1$, a parabola results; if $0 < \epsilon < 1$, an ellipse results; if $\epsilon > 1$, a hyperbola results; and if $\epsilon = 0$, the conic section is a circle.

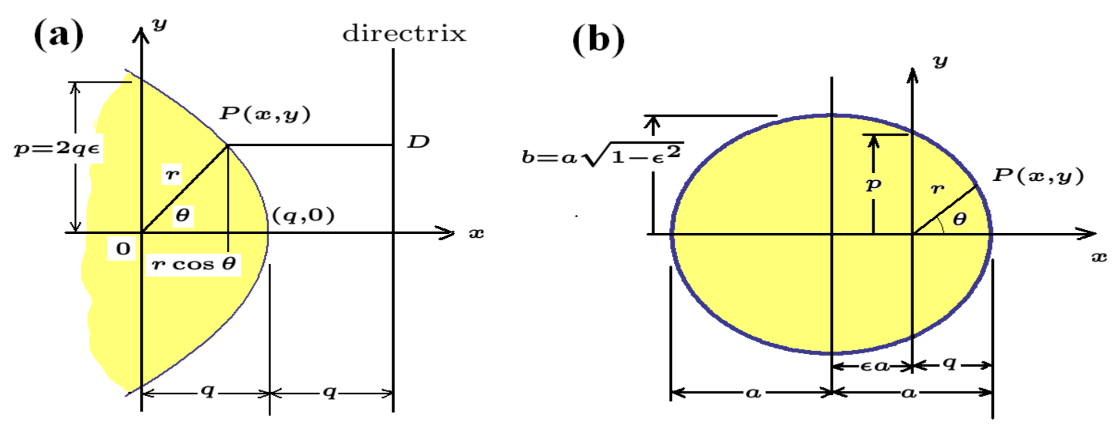


Figure 9-8. Parabolic and elliptic conic sections.

With reference to figure 9-8, a conic section is defined in terms of the ratio $\frac{\overline{OP}}{\overline{PD}} = \epsilon$, where $\overline{OP} = r$, and $\overline{PD} = 2q - r\cos\theta$. From this ratio, solve for the radius r and obtain the representation

$$r = \frac{p}{1 + \epsilon\cos\theta}, \tag{9.104}$$

where $p = 2q\epsilon$. Equation (9.104) is the equation of a conic section. The following terminology is applied to the variables and parameters in this equation:

1. The angle θ is called the true anomaly associated with the orbit.
2. The symbol a is introduced to denote the semi-major axis of an elliptical orbit. The symbol a can be shown to be related to r, p and ϵ.
3. The quantity p is called the semiparameter of the conic section and is illustrated in figure 9-8. Note that when θ has the value $\pi/2$, then $r = p$.

An important relation connecting the parameters p, a and ϵ is obtained from equation (9.104) by setting θ equal to zero. This gives

$$r = \frac{p}{1+\epsilon} = q = a(1-\epsilon) \qquad \text{which implies} \qquad p = a(1-\epsilon^2). \tag{9.105}$$

In order to demonstrate that the motion of mass m with respect to mass M is a conic section, show that the magnitude r of the position vector $\vec{r}$ satisfies an equation having the exact same form as equation (9.104).

Kepler's Laws

Johannes Kepler[7], an astronomer and mathematician, discovered three laws concerning the motion of the planets. **He discovered these laws from experimental data without the aid of calculus or vector analysis**. Newton, using calculus, verified these laws with the model for the inverse square law of attraction. These three laws are now derived.

To derive Kepler's three laws one can make use of the following vector identities:

$$\vec{r} \times \hat{\mathbf{e}}_r = r\,\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_r = \vec{0} \tag{9.106}$$

$$\frac{d}{dt}(\vec{r} \times \frac{d\vec{r}}{dt}) = \vec{r} \times \frac{d^2\vec{r}}{dt^2} \tag{9.107}$$

$$\hat{\mathbf{e}}_r \cdot \frac{d\,\hat{\mathbf{e}}_r}{dt} = 0 \tag{9.108}$$

$$\hat{\mathbf{e}}_r \times (\,\hat{\mathbf{e}}_r \times \frac{d\,\hat{\mathbf{e}}_r}{dt}) = -\frac{d\,\hat{\mathbf{e}}_r}{dt} \tag{9.109}$$

[7] Johannes Kepler (1571-1630), German astronomer and mathematician.

Note that the Newton law of gravitation implies that the derivative given by equation (9.107) is zero. That is, if

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} = \vec{F} = -\frac{GmM}{r^2}\,\hat{\mathbf{e}}_r$$

$$\text{then} \quad \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = -\frac{GM}{r^2}\vec{r}\times\hat{\mathbf{e}}_r = \vec{0}. \tag{9.110}$$

An integration of this equation produces the result

$$\vec{r}\times\frac{d\vec{r}}{dt} = \vec{h} = \text{Constant} \tag{9.111}$$

Recall that the vector $\vec{H} = \vec{r}\times m\vec{v}$ is defined as **the angular momentum.** The quantity $\vec{h} = \frac{1}{m}\vec{H} = \vec{r}\times\frac{d\vec{r}}{dt}$ appearing in equation (9.111) is called **the angular momentum per unit mass.** Equation (9.111) tells us that the angular momentum is a constant for the two-body system under consideration. Since $\vec{h}$ is a constant vector, it can be verified that

$$\begin{aligned}\frac{d}{dt}\left(\vec{v}\times\vec{h}\right) &= \frac{d\vec{v}}{dt}\times\vec{h} = -\frac{GM}{r^2}\,\hat{\mathbf{e}}_r\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\\ &= -\frac{GM}{r^2}\,\hat{\mathbf{e}}_r\times\left[r\,\hat{\mathbf{e}}_r\times\left(r\frac{d\,\hat{\mathbf{e}}_r}{dt}+\frac{dr}{dt}\,\hat{\mathbf{e}}_r\right)\right]\\ &= -GM\,\hat{\mathbf{e}}_r\times\left(\hat{\mathbf{e}}_r\times\frac{d\,\hat{\mathbf{e}}_r}{dt}\right)\\ &= GM\frac{d\,\hat{\mathbf{e}}_r}{dt}.\end{aligned} \tag{9.112}$$

Note that the result (9.112) was obtained by making use of the equations (9.106) and (9.109). An integration of the result (9.112) gives us the relation

$$\vec{v}\times\vec{h} = GM\,\hat{\mathbf{e}}_r + \vec{C}, \tag{9.113}$$

where $\vec{C}$ is a constant vector of integration. Using the triple scalar product formula it is readily verified that

$$\vec{r}\cdot\left(\vec{v}\times\vec{h}\right) = \vec{h}\cdot\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = h^2 = \vec{r}\cdot\left(\vec{v}\times\vec{h}\right) = GM\vec{r}\cdot\,\hat{\mathbf{e}}_r + \vec{r}\cdot\vec{C}$$

or

$$h^2 = GMr + Cr\cos\theta, \tag{9.114}$$

where θ is the angle between the vectors $\vec{C}$ and $\vec{r}$. In the equation (9.114) one can solve for r and find

$$r = \frac{p}{1+\epsilon\cos\theta}, \tag{9.115}$$

where $p = h^2/GM$ and $\epsilon = C/GM$. This result is known as **Kepler's first law** and implies that all the planets of the solar system describe elliptical paths with the sun at one focus.

Kepler's second law states that the position vector $\vec{r}$ sweeps out equal areas in equal time intervals. Consider the area swept out by the position vector of a planet during a time interval Δt. This element of area, in polar coordinates, is written as

$$dA = \frac{1}{2} r^2 d\theta$$

and therefore the rate of change of this area with respect to time is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

It has been demonstrated that the angular momentum per unit mass $\vec{h} = \vec{r} \times \vec{v}$ is a constant. For $\vec{r} = r\cos\theta\,\hat{\mathbf{e}}_1 + r\sin\theta\,\hat{\mathbf{e}}_2$, the angular momentum has components which can be calculated from the determinant

$$\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\dot{\theta} + \dot{r}\cos\theta & r\cos\theta\dot{\theta} + \dot{r}\sin\theta & 0 \end{vmatrix}$$

By expanding the above determinant and simplifying one can verify that

$$\vec{h} = r^2 \frac{d\theta}{dt}\,\hat{\mathbf{e}}_3 = h\,\hat{\mathbf{e}}_3 = \text{Constant} \tag{9.116}$$

which in turn implies

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{is a constant.} \tag{9.117}$$

This result is known as Kepler's second law. Analysis of this second law informs us that the position vector sweeps out equal areas during equal time intervals.

The time it takes for mass m to complete one orbit about mass M is called the period of the motion. Denote this period by the Greek letter τ. Note that equation (9.117) tells us that when r^2 is small $\frac{d\theta}{dt}$ becomes large and, conversely, when $\frac{d\theta}{dt}$ is small r^2 becomes large. The resulting motion is for planets to move faster when they are closer to the Sun and slower when they are farther away. Express equation (9.117) in the form $dA = \frac{1}{2}h\,dt$ and integrate the result from $t = 0$ to $t = \tau$, to show

$$A = \frac{h}{2}\tau, \tag{9.118}$$

where A is the area of the ellipse and τ is the period of one orbit. The area of an ellipse is given by the formula $A = \pi ab$, where a is the semi-major axis and $b = a\sqrt{1-\epsilon^2}$ is the semi-minor axis. Equation (9.118) can therefore be expressed in the form

$$A = \pi a^2 \sqrt{1-\epsilon^2} = \frac{h}{2}\tau$$

from which the period of the orbit is

$$\tau = \frac{2\pi a^2}{h}\sqrt{1-\epsilon^2}.$$

With the substitutions

$$1-\epsilon^2 = \frac{p}{a} \qquad \text{and} \qquad p = \frac{h^2}{GM},$$

the period of the orbit can be expressed

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{GM}} \qquad \text{or} \qquad \tau^2 = \frac{4\pi^2 a^3}{GM}. \tag{9.119}$$

This result is known as **Kepler's third law** and depicts the fact that the square of the period of one revolution is proportional to the cube of the semi-major axis of the elliptical orbit.

Planets, comets, and asteroids have either elliptic, parabolic or hyperbolic orbits about the sun.

Vector Differential Equations

A homogeneous vector differential equation, such as

$$\frac{d^2\vec{y}}{dt^2} + \alpha\frac{d\vec{y}}{dt} + \beta\vec{y} = \vec{0} \tag{9.135}$$

where α and β are scalar constants is solved by first solving the homogeneous scalar differential equation

$$\frac{d^2 y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = 0 \tag{9.137}$$

The solution of the homogeneous differential equation is called the complementary solution and is expressed using the notation y_c. By assuming an exponential solution $y = e^{\lambda t}$ and substituting it into the homogeneous equation one obtains the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0 \tag{9.136}$$

There are three cases to consider.

Case 1 The roots of characteristic equation (9.136) are real and unique. If r_1, r_2 are these roots, then the scalar homogeneous differential equation (9.137) has the fundamental set of solutions $\{e^{r_1 t}, e^{r_2 t}\}$ and the general of equation (9.137)is $y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where c_1 and c_2 are arbitrary scalar constants.

Case 2 The roots of the characteristic equation (9.136) are real and equal, say $r_1 = r_2$. In this case the fundamental set of solutions is given by $\{e^{r_1 t}, te^{r_1 t}\}$ and the general solution of equation (9.137) is $y_c = c_1 e^{r_1 t} + c_2\, te^{r_1 t}$ where c_1 and c_2 are arbitrary scalar constants.

Case 3 The roots of the characteristic equation (9.136) are complex roots, say $r_1 = a + ib$ and $r_2 = a - ib$. In this case the fundamental set of solutions can be represented in the form $\{e^{(a+ib)t},\, e^{(a-ib)t}\}$ or one can make use of Euler's equation $e^{ibt} = \cos bt + i \sin bt$ and take appropriate linear combination of solutions to write the fundamental set of solutions in the form $\{e^{at}\cos bt,\, e^{at}\sin bt\}$. The general solution to the scalar homogeneous equation can then be expressed in either of the forms

$$\begin{aligned} y =& c_1 e^{(a+ib)t} + c_2 e^{(a-ib)t} \\ \text{or} \qquad y_c =& e^{at}\left(c_1 \cos bt + c_2 \sin bt\right) \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

If $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions to the homogeneous scalar differential equation (9.137), then

$$\vec{y}_c = \vec{c}_1\, y_1(t) + \vec{c}_2\, y_2(t) \tag{9.138}$$

where $\vec{c}_1$ and $\vec{c}_2$ are arbitrary vector constants, is the representation of the general solution to the vector differential equation (9.135). Substitute equation (9.138) into equation (9.135) and show there results the vector equation

$$\vec{c}_1\left(\frac{d^2 y_1}{dt^2} + \alpha\frac{dy_1}{dt} + \beta y_1\right) + \vec{c}_2\left(\frac{d^2 y_2}{dt^2} + \alpha\frac{dy_2}{dt} + \beta y_2\right) = \vec{0} \tag{9.139}$$

Observe that if $\vec{c}_1$ and $\vec{c}_2$ are arbitrary independent vectors, then in order for equation (9.139) to be satisfied, the scalar components of the arbitrary vectors $\vec{c}_1$ and $\vec{c}_2$ must equal zero.

The solution of the nonhomogeneous vector differential equation

$$\frac{d^2\vec{y}}{dt^2} + \alpha\frac{d\vec{y}}{dt} + \beta\vec{y} = \vec{F}(t) \tag{9.125}$$

where α and β are scalar constants is obtained by first solving the homogeneous equation

$$\frac{d^2\vec{y}}{dt^2} + \alpha\frac{d\vec{y}}{dt} + \beta\vec{y} = \vec{0}$$

given by equation (9.138) and then finding any particular solution $\vec{y}_p = \vec{y}_p(t)$ which satisfies

$$\frac{d^2\vec{y}_p}{dt^2} + \alpha\frac{d\vec{y}_p}{dt} + \beta\vec{y}_p = \vec{F}(t)$$

The general solution to the vector differential equation (9.125) is then given by

$$\vec{y} = \vec{y}_c + \vec{y}_p = \vec{c}_1 y_1(t) + \vec{c}_2 y_2(t) + \vec{y}_p(t) \tag{9.126}$$

Example 9-9. Solve the vector differential equation

$$\frac{d^2\vec{y}}{dt^2} + 2\alpha\frac{d\vec{y}}{dt} + \beta^2\vec{y} = \sin 3t\ \hat{\mathbf{e}}_3 \tag{9.133}$$

Solution First solve the homogeneous vector differential equation

$$\vec{y}'' + 2\alpha\vec{y}' + \beta^2\vec{y} = \vec{0} \tag{9.128}$$

If $\vec{y} = \vec{c}_1 y_1(t) + \vec{c}_2 y_2(t)$ is the general solution of equation (9.128), then $y_1(t)$ and $y_2(t)$ must be independent solutions of the scalar differential equation

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \beta^2 y = 0 \tag{9.129}$$

This is an equation with constant coefficients. The general procedure to solve a differential equation with constant coefficients is to assume an exponential solution $y = e^{\lambda t}$. Substituting the assumed exponential solution into the differential equation (9.129) produce the characteristic equation

$$\lambda^2 + 2\alpha\lambda + \beta^2 = 0 \tag{9.130}$$

for determining values of λ to be substituted into the assumed solution. Solving equation (9.130) for λ gives the characteristic roots

$$\lambda = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\beta^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \beta^2} \tag{9.131}$$

Case 1 If $\alpha^2 - \beta^2 = \omega^2 > 0$, then a fundamental set of solutions is given by $\{e^{-(\alpha-\omega)t}, e^{-(\alpha+\omega)t}\}$ and the general solution to equation (9.128) is

$$\vec{y} = \vec{c}_1 e^{-(\alpha-\omega)t} + \vec{c}_2 e^{-(\alpha+\omega)t}$$

Case 2 If $\alpha^2 - \beta^2 = 0$, the characteristic equation has the repeated roots $\lambda = -\alpha$. The first root gives the first member of the fundamental set as $e^{-\alpha t}$ and using the rule for repeated roots, the second member of the fundamental set of solutions is $te^{-\alpha t}$. The general solution to equation (9.128) can then be expressed in the form

$$\vec{y} = \vec{c}_1 e^{-\alpha t} + \vec{c}_2\, t\, e^{-\alpha t}$$

Case 3 If $\alpha^2 - \beta^2 = -\omega^2 < 0$, then the fundamental set of solutions is $\{e^{-\alpha t}\cos\omega t, e^{-\alpha t}\sin\omega t\}$ and the general solution to equation (9.128) is given by

$$\vec{y} = \vec{c}_1 e^{-\alpha t}\cos\omega t + \vec{c}_2 e^{-\alpha t}\sin\omega t$$

The solution to the homogeneous vector equation is then given by

$$\vec{y}_c = \vec{c}_1 y_1(t) + \vec{c}_2 y_2(t)$$

where $\{y_1(t), y_2(t)\}$ are the functions from one of the cases previously examined.

To find a particular solution which gives the right-hand side $\sin 3t\,\hat{\mathbf{e}}_3$ examine this function and its first couple of derivatives $3\cos 3t\,\hat{\mathbf{e}}_3,\ -9\sin 3t\,\hat{\mathbf{e}}_3$. The basic terms in the set containing the function and its derivatives are the terms $\sin 3t$ and $\cos 3t$ multiplied by some constant. One can then assume a particular solution has the form

$$\vec{y}_p = \vec{y}_p(t) = A\sin 3t\ \hat{\mathbf{e}}_3 + B\cos 3t\ \hat{\mathbf{e}}_3 \tag{9.132}$$

where A, B are undetermined coefficients. Substitute the equation (9.132) into the nonhomogeneous equation (9.133) and show there results after simplification

$$[6\alpha A + (\beta^2 - 9)B]\cos 3t + [(\beta^2 - 9)A - 6\alpha B]\sin 3t = \sin 3t \tag{9.133}$$

Compare like terms in equation (9.133) and show A and B must be selected to satisfy the simultaneous equations

$$\begin{aligned} 6\alpha A + (\beta^2 - 9)B =& 0 \\ (\beta^2 - 9)A - 6\alpha B =& 1 \end{aligned}$$

One finds

$$A = \frac{\beta^2 - 9}{(\beta^2 - 9)^2 + 36\alpha^2} \qquad\qquad B = \frac{-6\alpha}{(\beta^2 - 9)^2 + 36\alpha^2}$$

and so the particular solution is given by

$$\vec{y}_p = \vec{y}_p(t) = \left[\frac{(\beta^2 - 9)}{(\beta^2 - 9)^2 + 36\alpha^2}\sin 3t - \frac{6\alpha}{(\beta^2 - 9)^2 + 36\alpha^2}\cos 3t\right]\ \hat{\mathbf{e}}_3$$

The general solution can then be represented

$$\vec{y} = \vec{y}(t) = \vec{y}_c + \vec{y}_p$$

■

Maxwell's Equations

James Clerk Maxwell (1831-1874), a Scottish mathematician, studied properties of electric and magnetic fields and came up with a set of 20 partial differential equations in 20 unknowns which described mathematically how electric and magnetic fields interact. Much later, an English electrical engineer by the name of Oliver Heaviside (1850-1925), greatly simplified Maxwell's equations to four equations in two unknowns. A modern day version of the **Maxwell equations** in SI units[8] are

$$\begin{aligned}
\nabla \cdot \vec{E} =& \frac{\rho}{\epsilon_0} \\
\nabla \times \vec{E} =& -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} =& 0 \\
\nabla \times \vec{B} =& \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned} \tag{9.134}$$

In the Maxwell equations (9.134) one finds the following quantities

$\vec{E} = \vec{E}(x,y,z,t)$ Electric field intensity (N/coul)

$\vec{J} = \vec{J}(x,y,z)$ Total current density (amp/m^2)

$\vec{B} = \vec{B}(x,y,z,t)$ Magnetic field intensity (N/amp $\cdot$ m)

$\rho = \rho(x,y,z)$ charge density (coul/m^3)

$\mu_0 = 4\pi \times 10^{-7} \left(\frac{\text{N}}{\text{amp}^2}\right)$ the permeability of free space

$\epsilon_0 = 8.85 \times 10^{-12} \left(\frac{\text{coul}^2}{\text{N} \cdot \text{m}^2}\right)$ the permittivity of free space

It is left as an exercise to show that the Maxwell equations are dimensionally homogeneous.

Note 1: Warning! The symbols $\vec{B}$ and $\vec{H}$ occur in the study of electromagnetism. The symbol $\vec{H}$ is used to denote magnetic fields within a material medium. It has no name, but some textbooks call it a magnetic induction– which is wrong. To make matters worse many textbooks interchange the roles of $\vec{B}$ and $\vec{H}$. My only suggestion is be aware of these conflicts and study any textbook carefully and **see how things are defined.**

[8] There are two popular sets of units used to represent the Maxwell equations. These two popular units are the **International System of Units** or Système international dúnit es, designated **SI** (mks) in all languages and the **Gaussian** (cgs) set of units. The main advantage of the Gaussian units is that they simplify many of the basic equations of electricity and magnetism more so than the SI units.

Note 2: The product $\mu_0\epsilon_0 = \frac{1}{c^2}$, where $c = 3 \times (10)^{10}$ cm/sec is the speed of light. It will be demonstrated later in this chapter that the vector fields describing $\vec{E}$ and $\vec{B}$ of Maxwell equations are solutions of the wave equation.

Electrostatics

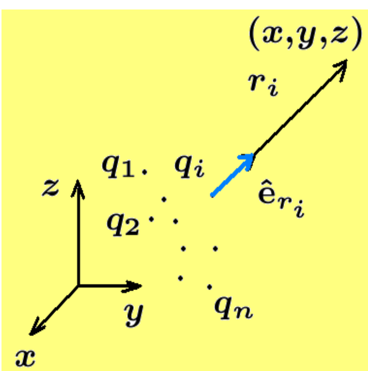
Coulomb's law[9] states that the force on a single test charge Q due to a single point charge q is given by

$$F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\,\hat{\mathbf{e}}_r \tag{9.135}$$

where $\epsilon_0 = 8.85 \times 10^{-12} \dfrac{\text{coul}^2}{\text{N}\cdot\text{m}^2}$ is the permittivity of free space, r is the distance between the charges and $\hat{\mathbf{e}}_r$ is a unit vector along the line connecting the charges. If q and Q have the same sign, the force is a repulsive force and if q and Q have opposite signs, then the force is attractive.

If there are many charges $q_1, q_2, \ldots, q_n$ at distances $r_1, r_2, \ldots, r_n$ from the test charge Q at the point (x, y, z), then one can use superposition to calculate the total force acting on the test charge. One finds

$$\vec{F} = \vec{F}(x,y,z) = \sum_{i=1}^{n} F_i = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{r_1^2}\,\hat{\mathbf{e}}_{r_1} + \frac{q_2 Q}{r_2^2}\,\hat{\mathbf{e}}_{r_2} + \cdots + \frac{q_n Q}{r_n^2}\,\hat{\mathbf{e}}_{r_n}\right) = Q\vec{E} \tag{9.136}$$



where $\hat{\mathbf{e}}_{r_i}$, for $i = 1, \ldots, n$, are unit vectors pointing from charge q_i to the point (x, y, z) of the test charge Q. The quantity

$$\vec{E} = \vec{E}(x,y,z) = \frac{1}{Q}\vec{F}(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\,\hat{\mathbf{e}}_{r_i} \tag{9.137}$$

is called the electric field produced by the n-charges.

---

[9] Charles Augustin de Coulomb (1736-1806) A French engineer who studied electricity and magnetism.

Example 9-10.

Consider the special case of a single point charge q located at the origin. The electric field due to this point charge is

$$\vec{E} = \vec{E}(x, y, z) = \frac{1}{Q}\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{e}}_r \tag{9.138}$$

where $\hat{\mathbf{e}}_r$ is a unit vector in spherical coordinates and r is the distance of a test charge Q from the origin to the position (x, y, z). This vector field produces field lines and the strength of the vector field is proportional to the flux across some surface placed within the electric field. In the case where a sphere of radius r and centered at the origin is placed within the electric field, then the flux is calculated from the surface integral

$$\text{Flux} = \iint\limits_S \vec{E}\cdot d\vec{S} = \iint\limits_S \vec{E}\cdot\hat{\mathbf{e}}_n\,dS \tag{9.139}$$

where $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_r$ in spherical coordinates. Also the element of area dS in spherical coordinates is given by $d\vec{S} = r^2\sin\theta\,d\theta d\phi\;\hat{\mathbf{e}}_r$ so that equation (9.139) can be expressed as

$$\text{Flux} = \iint\limits_S \vec{E}\cdot d\vec{S} = \int_0^{2\pi}\left[\int_0^{\pi}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,r^2\sin\theta\,d\theta\right]d\phi = \frac{q}{\epsilon_0} \tag{9.140}$$

This result states that the flux is a constant no matter what size sphere is placed about the point charge. If the sphere were made of rubber and could be deformed into some other simple closed surface, the number of field lines passing through the new surface would also be the same constant as above. This is because the dot product $\vec{E}\cdot\hat{\mathbf{e}}_n$ selects an element of area perpendicular to the field lines and the flux is proportional to the number of these lines. Note that if the point charge were outside the closed surface, then the flux would be zero, since field lines entering the surface at one point must exist at some other point and then the sum of the flux would be zero.

One can say that if there were n-charges $q_1, q_2, \ldots, q_n$ inside a simple closed surface and $\vec{E}_i$ was the electric field associated with the ith charge, then $\vec{E} = \sum_{i=1}^{n}\vec{E}_i$ would represent the total electric field and the flux across any simple closed surface due to this total electric field would be

$$\iint\limits_S \vec{E}\cdot d\vec{S} = \sum_{i=1}^{n}\left(\iint\limits_S \vec{E}_i\cdot d\vec{S}\right) = \sum_{i=1}^{n}\frac{q_i}{\epsilon_0} \tag{9.141}$$

If the discrete number of n-charges $q_1, \ldots, q_n$ were replaced by a continuous distribution of charges inside the surface, then the right-hand side of equation (9.141) would be replaced by $\iiint_V \frac{\rho}{\epsilon_0}\, dV$ where dV is an element of volume $\left(\text{meter}^3\right)$ and ρ is a charge density $\left(\frac{\text{coulomb}}{\text{meter}^3}\right)$ and the equation (9.141) would then be written

$$\iint_S \vec{E} \cdot d\vec{S} = \iiint_V \frac{\rho}{\epsilon_0}\, dV \tag{9.142}$$

Using the divergence theorem of Gauss, the equation (9.142) can also be expressed as

$$\iiint_V \left(\nabla \cdot \vec{E} - \frac{\rho}{\epsilon_0} \right) dV = 0 \tag{9.143}$$

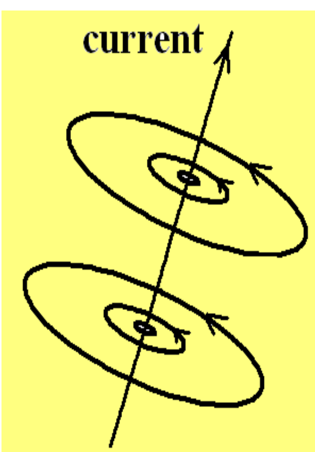
If the equation (9.143) is to hold for all arbitrary simple closed surfaces, then one must require that

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{9.144}$$

This is the first of Maxwell's equations (9.134) and is called the Gauss law of electrostatics.

■

Magnetostatics



A moving charge produces a current and a moving current produces a magnetic field. Consider a current moving along a wire considered as a line. The magnetic field created is described by circles around the wire. The strength of the magnetic field falls off as the perpendicular distance from the line increases. One can use the right-hand rule of letting the thumb point in the direction of the current flow, then the fingers of the right-hand point in the direction of the magnetic field lines.

The magnetic force on a charge Q moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by

$$\vec{F}_m = Q\,(\vec{v} \times \vec{B}) \qquad (\text{coul})\left(\frac{\text{m}}{\text{s}}\right)\left(\frac{\text{N}}{\text{amp} \cdot \text{m}}\right) \tag{9.145}$$

while the electric force acting on Q is

$$\vec{F}_e = Q\vec{E} \qquad (\text{coul}\,)\left(\frac{\text{N}}{\text{coul}}\right) \tag{9.146}$$

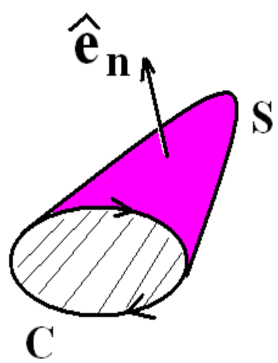
The total force acting on the moving charge Q is

$$\vec{F} = \vec{F}_m + \vec{F}_e = Q\,(\vec{E} + \vec{v} \times \vec{B}) \tag{9.147}$$

The magnetic forces can be summed over lines, surfaces or volumes. This gives rise to a one-dimensional, two-dimensional and three-dimensional representation for the magnetic force. In one-dimension the element of force acting on an element of wire is $d\vec{F}_m = (\vec{v} \times \vec{B})\,\rho_\ell\,ds$ where ρ_ℓ is the charge density per unit length (coul/m) and ds is an element of arc length. In two-dimensions the element of force acting on a surface is $d\vec{F}_m = (\vec{v} \times \vec{B})\,\rho_S\,d\sigma$ where ρ_S is the charge density per unit area (coul/m^2) and $d\sigma$ is an element of area. In three-dimensions the element of force acting within a volume is $d\vec{F}_m = (\vec{v} \times \vec{B})\,\rho_V\,dV$ where ρ_V is the charge density per unit volume (coul/m^3) and dV is an element of volume. An integration gives the total magnetic force as

$$\begin{aligned}
\vec{F}_m &= \int (\vec{v} \times \vec{B})\,\rho_\ell\,ds \quad \text{one-dimension}\\
\vec{F}_m &= \iint (\vec{v} \times \vec{B})\,\rho_S\,d\sigma \quad \text{two-dimension}\\
\vec{F}_m &= \iiint (\vec{v} \times \vec{B})\,\rho_V\,dV \quad \text{three-dimension}
\end{aligned} \tag{9.148}$$

Example 9-11. The Maxwell-Faraday Equation



Faraday's law[10] of induction investigates the magnetic flux $\iint_S \vec{B}\cdot d\vec{S}$ across a surface[11] S determined by a simple closed curve C. Think of a simple closed curve in space drawn on a sheet of rubber and then hold the simple closed curve fixed but deform the rubber surface into any kind of continuous surface S having C for its boundary. The direction of the unit normal $\hat{\mathbf{e}}_n$ to the surface S is determined by the right-hand rule of moving the fingers of the right

[10] Michael Faraday (1791-1867) English physicist who studied electricity and magnetism.
[11] Think of a rubber sheet across C and then deform the sheet to form the surface S.

hand in the direction around C so that the thumb points in the direction of the normal. Faraday's law, **obtained experimentally**, states that the line integral of the electric field around the closed curve C equals the negative of the time rate of change of the magnetic flux. This law can be written

$$\oint_C \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{S} \tag{9.149}$$

Here $\vec{E}$ is the electric field, $\vec{r}$ is the position vector defining the closed curve C, $\vec{B}$ is the magnetic field and $d\vec{S} = \hat{\mathbf{e}}_n\, d\sigma$ is a vector element of area on the surface S. The Faraday law, given by equation (9.149), assumes the curve C and surface S are fixed and do not change with time. The left-hand side of equation (9.149) is the work done in moving around the curve C within the electric field $\vec{E}$. The right-hand side of equation (9.149) is the negative time rate of change of the magnetic flux across the surface S. One can employ Stokes theorem and express equation (9.149) in the form

$$\iint_S \nabla\times\vec{E}\cdot d\vec{S} = -\iint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \qquad \text{or} \qquad \iint_S \left[\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right]\cdot d\vec{S} = 0 \tag{9.150}$$

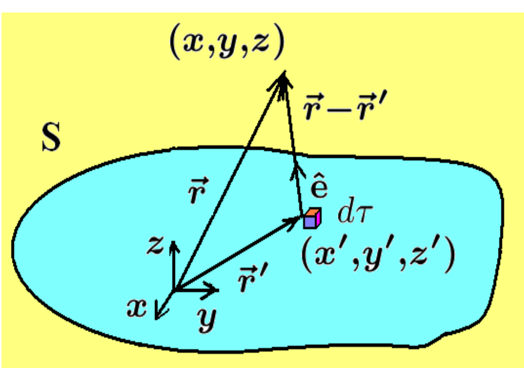
The equation (9.150) holds for all arbitrary surfaces S and consequently the integrand must equal zero giving the Maxwell-Faraday equation

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{9.152}$$

which is the second Maxwell equation.

■

Example 9-12. The Biot-Savart law



Consider a volume V enclosed by a surface S as illustrated and let $\vec{J} = \vec{J}(x',y',z')$ denote the current density within this volume. Let (x',y',z') denote a point inside V where an element of volume $dV = dx'dy'dz'$ is constructed. In addition, construct the vectors $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ to a general point (x,y,z) outside of the volume V and

$\vec{r}' = x'\,\hat{\mathbf{e}}_1 + y'\,\hat{\mathbf{e}}_2 + z'\,\hat{\mathbf{e}}_3$ to the point (x', y', z') inside the volume V. The vector $\vec{r} - \vec{r}'$ then points from the point (x', y', z') to the point (x, y, z). The vector $\hat{\mathbf{e}} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$ is a unit vector in this direction as illustrated in the accompanying figure.

The magnetic field $\vec{B} = \vec{B}(x, y, z)$ at the point (x, y, z) due to a current density $\vec{J} = \vec{J}(x', y', z')$ inside V is given by the Biot[12]-Savart[13] law

$$\vec{B} = \vec{B}(x, y, z) = \frac{\mu_0}{4\pi} \iiint_V \nabla \cdot \left[\frac{\vec{J}(x', y', z') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] dx'dy'dz' \tag{9.152}$$

The divergence of this magnetic field is determined by calculating

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \iiint_V \nabla \cdot \left[\frac{\vec{J} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] dV \tag{9.153}$$

In order to evaluate the divergence as given by equation (9.153) one can employ the vector identities

$$\begin{aligned} \nabla \times (f\vec{A}) &= (\nabla f) \times \vec{A} + f(\nabla \times \vec{A}) \\ \nabla \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \end{aligned} \tag{9.154}$$

Let $\vec{B} = \dfrac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$ and $\vec{A} = \vec{J}$ along with the second of the equations (9.154) to show

$$\nabla \cdot (\vec{J} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{J}) - \vec{J} \cdot (\nabla \times \vec{B}) = -\vec{J} \cdot (\nabla \times \vec{B}) \tag{9.155}$$

This holds because $\vec{J}$ is a function of the primed coordinates and ∇ involves differentiation with respect to the unprimed coordinates so that $\nabla \times \vec{J}$ is zero. Using the first equation in (9.154) with $f = \dfrac{1}{|\vec{r} - \vec{r}'|^3}$ one can write

$$\nabla \times \vec{B} = \nabla \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{1}{|\vec{r} - \vec{r}'|^3} \nabla \times (\vec{r} - \vec{r}') - (\vec{r} - \vec{r}') \times \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|^3} \right) \tag{9.156}$$

One can verify that

$$\nabla \times (\vec{r} - \vec{r}') = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - x') & (y - y') & (z - z') \end{vmatrix} = \vec{0}$$

and if $f = |\vec{r} - \vec{r}'|^{-3}$, then $\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial z}\,\hat{\mathbf{e}}_3$ where

$$\frac{\partial f}{\partial x} = -3|\vec{r} - \vec{r}'|^{-4} \frac{\partial}{\partial x} \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \frac{-3(x - x')}{|\vec{r} - \vec{r}'|^5}$$

[12] Jean Baptiste Biot (1774-1862) A French mathematician.

[13] F elix Savart (1791-1841) A French physician who studied physics.

In a similar fashion one can verify that

$$\frac{\partial f}{\partial y} = \frac{-3(y-y')}{|\vec{r}-\vec{r}'|^5} \qquad \text{and} \qquad \frac{\partial f}{\partial z} = \frac{-3(z-z')}{|\vec{r}-\vec{r}'|^5}$$

Using the above results verify that

$$\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|^3}\right) = \frac{-3}{|\vec{r}-\vec{r}'|^5}(\vec{r}-\vec{r}') \tag{9.157}$$

and show the right-hand side of equation (9.156) is zero because $(\vec{r}-\vec{r}')\times(\vec{r}-\vec{r}') = \vec{0}$. It then follows that

$$\nabla\cdot\vec{B} = 0$$

which is the third equation in the Maxwell's equations (9.134).

■

Example 9-13. If the charge density ρ (coul/m^3) moves with velocity $\vec{v}$ (m/s), then the current density is given by $\vec{J} = \rho\vec{v}$ (amp/m^2). Surround the current density field with a simple closed surface S which encloses a volume V. The flux across the surface S is then given by

$$\iint_S \vec{J}\cdot d\vec{S} = \iint_S \vec{J}\cdot \hat{\mathbf{e}}_n\, dS = \iint_V \nabla\cdot\vec{J}\, dV$$

where the divergence theorem of Gauss has been employed to express the flux surface integral with a volume integral. The charge must be conserved so that the flux out of the volume must be accounted for by the time rate of change of the charge density within the volume so that one can write

$$\iint_S \vec{J}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{J}\, dV = -\frac{d}{dt}\iiint_V \rho\, dV = -\iiint_V \frac{\partial\rho}{\partial t}\, dV$$

or

$$\iiint_V \left[\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t}\right] dV = 0 \tag{9.158}$$

The equation (9.158) must hold for all volumes V and consequently one must require

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{9.159}$$

The equation (9.159) is know as **the continuity equation** for the charge density.

■

Example 9-14. The last Maxwell equation is hard to derive. Historically, Ampere[14] showed that for straight line currents the curl of the magnetic field was proportional to the volume current density $\vec{J}$ so that one could write

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Maxwell realized that this equation did not hold in general because it did not satisfy the property that the divergence of the curl must be zero. Based upon theoretical reasoning Maxwell came up with the modified equation

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{9.160}$$

where the term $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is known as Maxwell's term for Ampere's law. Taking the divergence of equation (9.160) one can show

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{B}) =& \mu_0 \nabla \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial \nabla \cdot \vec{E}}{\partial t} \quad \text{Use the first Maxwell equation and show} \\
\nabla \cdot (\nabla \times \vec{B}) =& \mu_0 \nabla \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial (\rho / \epsilon_0)}{\partial t} \\
\nabla \cdot (\nabla \times \vec{B}) =& \mu_0 \left[\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right] = 0
\end{aligned}$$

where the continuity equation from the previous example has been employed to show the divergence of the curl is zero. The equation (9.160) is the last of the Maxwell equations from (9.134).

■

Example 9-15. If there are no charges or currents in space, then the Maxwell equations (9.134) simplify to the form

$$\begin{aligned}
\nabla \cdot \vec{E} =& 0 \\
\nabla \times \vec{E} =& -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} =& 0 \\
\nabla \times \vec{B} =& \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned} \tag{9.161}$$

Use the property of the del operator that

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

[14] André Marie Ampère (1775-1836) A French physicist, chemist and mathematician.

and take the curl of the second and fourth of the Maxwell's equations to obtain

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right) = -\mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \nabla \times \left(\mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\nabla \times \vec{E}\right) = -\mu_0\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

The first and third of the Maxwell equations require that $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$ so that the vector fields $\vec{E}$ and $\vec{B}$ must satisfy the wave equations

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{and} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

Here the product $\mu_0\epsilon_0 = \frac{1}{c^2}$, where $c = 3 \times (10)^{10}\,\text{cm/sec}$ is the speed of light.

■

Exercises

▶ **9-1.** Solve each of the one-dimensional Laplace equations

$$\begin{aligned} \frac{d^2U}{dx^2} &= 0, \qquad U = U(x) \quad \text{rectangular} \\ \frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) &= 0, \qquad U = U(r) \quad \text{polar} \\ \frac{d^2U}{d\rho^2} + \frac{2}{\rho}\frac{dU}{d\rho} = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dU}{d\rho}\right) &= 0, \qquad U = U(\rho) \quad \text{spherical} \end{aligned} \tag{9.162}$$

▶ **9-2.** Verify that the velocity field $\vec{V} = V_0 \cos\alpha\, \hat{\mathbf{e}}_1 - V_0 \sin\alpha\, \hat{\mathbf{e}}_2$, V_0, α are constants is both irrotational and solenoidal. Find and sketch the velocity field, streamlines. Find the velocity potential. Note that for $\alpha = 0$ the flow is a parallel flow and for $\alpha = \frac{\pi}{2}$ the flow is a vertical flow.

▶ **9-3.** Verify that the velocity field $\vec{V} = 2x\, \hat{\mathbf{e}}_1 - 2y\, \hat{\mathbf{e}}_2$ is both irrotational and solenoidal. Find and sketch the vector field and the streamlines for $0 < x < 2$, $0 < y < 2$. Also find the velocity potential. The velocity field for this type of fluid motion can be used to describe the flow in the vicinity of a corner.

▶ **9-4.** For the velocity field $\vec{V} = 2y\, \hat{\mathbf{e}}_1 + 2x\, \hat{\mathbf{e}}_2$ find and sketch the vector field and streamlines. Find the velocity potential.

▶ **9-5.** Consider the vector field $\vec{E} = \frac{1}{r^2}\,\hat{\mathbf{e}}_r$ in polar coordinates. (a) Show this vector field is irrotational. (b) Find a potential function $\phi = \phi(r)$ satisfying $\phi(r_0) = 0$, where $r_0 > 0$.

▶ **9-6.** True or false, if both $\vec{A}$ and $\vec{B}$ are irrotational, then the vector $\vec{F} = \vec{A} \times \vec{B}$ is solenoidal.

▶ **9-7.** Show that if $\phi = \phi(x, y, z)$ is a solution of the Laplace equation $\nabla^2\phi = 0$, then (a) Show the vector $\vec{V} = \nabla\phi$ is irrotational. (b) Show the vector $\vec{V} = \nabla\phi$ is solenoidal.

▶ **9-8.**

(a) Show the velocity field $\vec{V} = \dfrac{kx}{x^2+y^2}\,\hat{\mathbf{e}}_1 + \dfrac{ky}{x^2+y^2}\,\hat{\mathbf{e}}_2$ is both irrotational and solenoidal and has the potential function $\Phi = \dfrac{k}{2}\ln(x^2+y^2)$ and the stream function $\Psi = k\tan^{-1}\dfrac{y}{x}$

(b) Show that $$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \quad\text{and}\quad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}$$

(c) Express the potential function and stream function in polar coordinates and sketch the equipotential curves and streamlines. This type of velocity field is said to correspond to a source at the origin if $k > 0$ or a sink at the origin if $k < 0$.

▶ **9-9.** Verify that the velocity field $\vec{V} = \dfrac{-ky}{x^2+y^2}\,\hat{\mathbf{e}}_1 + \dfrac{kx}{x^2+y^2}\,\hat{\mathbf{e}}_2$ is both irrotational and solenoidal. Find the potential and streamlines for this velocity field. This type of flow is termed a circulation about the origin of strength k.

▶ **9-10.** Sketch the field lines and analyze the vector fields defined by:

$$\begin{array}{llll} (a) & \vec{F} = y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 & (d) & \vec{F} = 2xy\,\hat{\mathbf{e}}_1 + (x^2 - y^2)\,\hat{\mathbf{e}}_2 \\ (b) & \vec{F} = y\,\hat{\mathbf{e}}_1 - x\,\hat{\mathbf{e}}_2 & (e) & \vec{F} = (x^2 + y^2)\,\hat{\mathbf{e}}_1 + 2xy\,\hat{\mathbf{e}}_2 \\ (c) & \vec{F} = a\,\hat{\mathbf{e}}_1 + b\,\hat{\mathbf{e}}_2 & (f) & \vec{F} = a\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2 \end{array}$$

▶ **9-11.** Show in polar coordinates that the Cauchy-Riemann equations can be expressed as

$$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad \frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

► **9-12.** At all points (x,y) between the circles $x^2+y^2=1$ and $x^2+y^2=9$, the vector function $\vec{F} = \dfrac{-y\,\hat{\mathbf{e}}_1 + x\,\hat{\mathbf{e}}_2}{x^2+y^2}$ is continuous and equals the gradient of the scalar function

$$\Phi(x,y) = \tan^{-1}\frac{y}{x}.$$

Show that $\displaystyle\int_{(-2,0)}^{(2,0)} \vec{F}\cdot d\vec{r}$ is not independent of the path of integration by computing this line integral along the upper half and then the lower half of the circle $x^2+y^2=4$. Is the region of integration a simply-connected region?

► **9-13.** Find a vector potential for

$$(a)\quad \vec{F} = 2y\,\hat{\mathbf{e}}_1 + 2x\,\hat{\mathbf{e}}_2 \qquad\qquad (b)\quad \vec{F} = (x-y)\,\hat{\mathbf{e}}_1 - z\,\hat{\mathbf{e}}_3$$

► **9-14.** For the gravity field $\vec{F} = -mg\,\hat{\mathbf{e}}_3$

((a) Show that this vector field is irrotational.

(b) Find the potential function from which this field is derivable.

(c) Show that the work done in moving from a height h_1 to a height h_2 is the change in potential energy.

► **9-15.** (**Conservation of Energy**)

(a) If $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2}$ show that $\vec{F}\cdot\dfrac{d\vec{r}}{dt} = \dfrac{1}{2}m\dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right)^2$

(b) Show $\displaystyle\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F}\cdot d\vec{r} = \frac{1}{2}mv^2\Big|_{(x_0,y_0,z_0)}^{(x,y,z)} = \frac{1}{2}mv^2\Big|_{(x,y,z)} - \frac{1}{2}mv^2\Big|_{(x_0,y_0,z_0)}$

(c) If $\vec{F}$ is a conservative vector field such that $\vec{F} = -\nabla\phi$, show that

$$\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F}\cdot d\vec{r} = -\int_{(x_0,y_0,z_0)}^{(x,y,z)} \nabla\phi\cdot d\vec{r} = -\int_{(x_0,y_0,z_0)}^{(x,y,z)} d\phi = -\phi(x,y,z) + \phi(x_0,y_0,z_0)$$

(d) Show that $\phi(x_0,y_0,z_0) + \dfrac{1}{2}mv^2\Big|_{(x_0,y_0,z_0)} = \phi(x,y,z) + \dfrac{1}{2}mv^2\Big|_{(x,y,z)}$ which states that for a conservative vector field the sum of the potential energy and kinetic energy at point (x_0,y_0,z_0) is the same as the sum of the potential energy and kinetic energy at the point (x,y,z).

► **9-16.** A conservative vector field has the family of equipotential curves

$$x^2 - y^2 = c.$$

Find the field lines and vector field associated with this potential.

▶ **9-17.** **(Research project on orbital motion)**

Assume a mass m located at a position $\vec{r} = r\,\hat{\mathbf{e}}_r$ experiences a central force $\vec{F} = mf(r)\,\hat{\mathbf{e}}_r$

(a) Show the equation of motion is given by $m\dfrac{d^2\vec{r}}{dt^2} = \vec{F}$ which can be written in the form $\dfrac{d\vec{v}}{dt} = f(r)\,\hat{\mathbf{e}}_r$

(b) Show that $\vec{r}\times\vec{v} = \vec{h}$ is a constant.

(c) Show the motion of m is in a plane and that the mass m sweeps out an area at a constant rate. (Kepler's law of areas).

(d) Show that in the special case $mf(r) = -\dfrac{GmM}{r^2}$ the mass m is attracted toward mass M, assumed to be at the origin, and $\dfrac{d\vec{v}}{dt} = -\dfrac{k}{r^2}\,\hat{\mathbf{e}}_r$ where $k = GM$ is a constant.

(e) Show that $\vec{h} = r^2\,\hat{\mathbf{e}}_r\times\dfrac{d\,\hat{\mathbf{e}}_r}{dt}$, $\dfrac{d\vec{v}}{dt}\times\vec{h} = k\dfrac{d\,\hat{\mathbf{e}}_r}{dt}$ and $\vec{v}\times\vec{h} = k(\,\hat{\mathbf{e}}_r + \vec{\epsilon})$ where $\vec{\epsilon}$ is a constant vector.

(f) Use the results from part (e) and show

$$\vec{r}\times\vec{v}\cdot\vec{h} = h^2 \quad \text{and} \quad \vec{r}\cdot\vec{v}\times\vec{h} = kr(1+\epsilon\,\cos\theta)$$

where θ is the angle between $\vec{\epsilon}$ and $\vec{r}$, and consequently

$$r = \frac{\alpha}{1+\epsilon\cos\theta}, \quad \text{where} \quad \alpha = \frac{h^2}{k}$$

Note r describes a conic section having eccentricity ϵ.

(f) When $\epsilon < 1$ show an ellipse results with m having an orbital period

$$T = \frac{\text{area of ellipse}}{h/2} = \frac{2\pi}{\sqrt{k}}a^{3/2}, \quad \text{where} \quad \frac{T^2}{a^3} = \frac{4\pi^2}{k}$$

This is known as Kepler's third law.

▶ **9-18.** (a) Find the potential associated with the conservative vector field

$$\vec{F} = (y^2\cos x + z^3)\;\hat{\mathbf{e}}_1 + (2y\sin x - 4)\;\hat{\mathbf{e}}_2 + 3xz^2\;\hat{\mathbf{e}}_3$$

(b) Find the differential equation which describes the field lines.

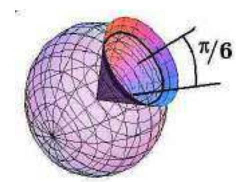
▶ **9-19.** Show that the vector field

$$\vec{F} = (2xyz + y)\;\hat{\mathbf{e}}_1 + (x^2z + x)\;\hat{\mathbf{e}}_2 + x^2y\;\hat{\mathbf{e}}_3$$

is conservative and find its scalar potential.

▶ **9-20.**

A right circular cone intersects a sphere of radius r as illustrated. Find the solid angle subtended by this cone.



Right circular cone intersecting sphere.

▶ **9-21.** Evaluate $\iint_S \vec{r} \cdot d\vec{S}$, where S is a closed surface having a volume V.

▶ **9-22.** In the divergence theorem $\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{\mathbf{e}}_n\, dS$ let $\vec{F} = \phi(x,y,z)\,\vec{C}$ where $\vec{C}$ is a nonzero constant vector and show $\iiint_V \nabla\phi\, dV = \iint_S \phi\; \hat{\mathbf{e}}_n\, dS$

▶ **9-23.** Assume $\vec{F}$ is both solenoidal and irrotational so that $\vec{F}$ is the gradient of a scalar function Φ (a) Show Φ is a solution of Laplace's equation and (b) Show the integral of the normal derivative of Φ over any closed surface must equal zero.

▶ **9-24.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and show $\nabla|\vec{r}|^\nu = \nu\,|\vec{r}|^{\nu-2}\vec{r} = \nu\,|\vec{r}|^{\nu-1}\,\hat{\mathbf{e}}_{\vec{r}}$ where $\hat{\mathbf{e}}_{\vec{r}} = \dfrac{\vec{r}}{|\vec{r}|}$ is a unit vector in the direction $\vec{r}$.

▶ **9-25.** For $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ and $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_1 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ show that

$$\frac{\partial|\vec{r}-\vec{r}_0|}{\partial x} = \frac{x-x_0}{|\vec{r}-\vec{r}_0|}$$
$$\frac{\partial|\vec{r}-\vec{r}_0|}{\partial y} = \frac{y-y_0}{|\vec{r}-\vec{r}_0|}$$
$$\frac{\partial|\vec{r}-\vec{r}_0|}{\partial z} = \frac{z-z_0}{|\vec{r}-\vec{r}_0|}$$

▶ **9-26.** Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3$ denote the position vector to the variable point (x,y,z) and let $\vec{r}_0 = x_0\,\hat{\mathbf{e}}_2 + y_0\,\hat{\mathbf{e}}_2 + z_0\,\hat{\mathbf{e}}_3$ denote the position vector to the fixed point (x_0, y_0, z_0).

(a) Show $\nabla|\vec{r}-\vec{r}_0|^\nu = \nu\,|\vec{r}-\vec{r}_0|^{\nu-1}\,\hat{\mathbf{e}}_{\vec{r}-\vec{r}_0}$ where $\hat{\mathbf{e}}_{\vec{r}-\vec{r}_0}$ is a unit vector in the direction $\vec{r}-\vec{r}_0$.

(b) Show $\nabla^2|\vec{r}-\vec{r}_0|^\nu = \nu(\nu+1)|\vec{r}-\vec{r}_0|^{\nu-2}$

(c) Write out the results from part (b) in the special cases $\nu = -1$ and $\nu = 2$.

▶ **9-27.** In thermodynamics the internal energy U of a gas is a function of pressure P and volume V denoted by $U = U(P, V)$. If a gas is involved in a process where the pressure and volume change with time, then this process can be described by a curve called a P-V diagram of the process. Let $Q = Q(t)$ denote the amount of heat obtained by the gas during the process. From the first law of thermodynamics which states that $dQ = dU + P dV$, show that $dQ = \frac{\partial U}{\partial P}\, dP + \left[\frac{\partial U}{\partial V} + P\right] dV$ and determine whether the line integral $\int_{t_0}^{t_1} dQ$, which represents the heat received during a time interval Δt, is independent of the path of integration or dependent upon the path of integration.

▶ **9-28.**

(a) If x and y are **independent** variables and you are given an equation of the form $F(x) = G(y)$ for all values of x and y what can you conclude if (i) x varies and y is constant and (ii) y varies but x is constant.

(b) Assume a solution to Laplace's equation $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ in Cartesian coordinates of the form $\phi = X(x)Y(y)$, where the variables are separated. If the variables x and y are **independent** show that there results two linear differential equations

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda,$$

where λ is termed a separation constant.

▶ **9-29.** Evaluate the line integral $I = \int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = yz\,\hat{\mathbf{e}}_1 + xz\,\hat{\mathbf{e}}_2 + xy\,\hat{\mathbf{e}}_3$ and C is the curve $\vec{r} = \vec{r}(t) = \cos t\ \hat{\mathbf{e}}_1 + (\frac{t}{\pi} + \sin t)\,\hat{\mathbf{e}}_2 + \frac{3t}{\pi}\,\hat{\mathbf{e}}_3$ between the points $(1, 0, 0)$ and $(-1, 1, 3)$.

▶ **9-30.** A particle moves along the x-axis subject to a restoring force $-Kx$. Find the potential energy and law of conservation of energy for this type of motion.

▶ **9-31.** Evaluate the line integral

$$I = \int_K (2x + y)\, dx + x\, dy, \qquad \text{where } K \text{ consists of straight line segments}$$

$\overline{OA} + \overline{AB} + \overline{BC}$ connecting the points $O(0, 0)$, $A(3, 3)$, $B(5, -1)$ and $C(7, 5)$.

▶ **9-32.** The problems below are concerned with obtaining a solution of Laplace's equation for temperature T. Chose an appropriate coordinate system and make necessary assumptions about the solution in order to reduce the problem to a one-dimensional Laplace equation.

(a) Find the steady-state temperature distribution along a bar of length L assuming that the sides of the bar are insulated and the ends are kept at temperatures T_0 and T_1. This corresponds to solving $\frac{d^2T}{dx^2} = 0$, $T(0) = T_0$ and $T(L) = T_1$.

(b) Find the steady-state temperature distribution in a circular pipe where the inside of the pipe has radius r_1 and temperature T_1, and the outside of the pipe has a radius r_2 and is maintained at a temperature T_2. This corresponds to solving $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$ such that $T(r_1) = T_1$ and $T(r_2) = T_2$

(c) Find the steady-state temperature distribution between two concentric spheres of radii ρ_1 and ρ_2, if the surface of the inner sphere is maintained at a temperature T_1, whereas the outer sphere is maintained at a temperature T_2. This corresponds to solving $\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dT}{d\rho}\right) = 0$ such that $T(\rho_1) = T_1$ and $T(\rho_2) = T_2$.

(d) Find the steady-state temperature distribution between two infinite and parallel plates $z = z_1$ and $z = z_2$ maintained, respectively, at temperatures of T_1 and T_2.

▶ **9-33.** Find the potential function associated with the conservative vector field

$$\vec{F} = 6xz\,\hat{\mathbf{e}}_1 + 8y\,\hat{\mathbf{e}}_2 + 3x^2\,\hat{\mathbf{e}}_3.$$

▶ **9-34.** Newton's law of attraction states that two particles of masses m_1 and m_2 attract each other with a force which acts in the direction of the line joining the two masses and whose magnitude is given by $F = Gm_1m_2/r^2$, where r is the distance between the masses and G is a universal constant.

(a) If mass m_1 is at the origin and mass m_2 is at a point (x, y, z), find the vector force of attraction of mass m_1 on mass m_2.

(b) If mass m_1 is at a fixed point $P_1(x_1, y_1, z_1)$ and mass m_2 is at the point (x, y, z), find the vector force of attraction of mass m_1 on mass m_2.

▶ **9-35.** Show that $u = u(x,t) = f(x-ct)+g(x+ct)$, f, g arbitrary functions, is a solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ Here f and g are wave shapes moving to the left and right.

▶ **9-36.** Express the Maxwell equations (9.161) as a system of partial differential equations.

▶ **9-37.** Assume solutions to the Maxwell equations (9.161) are waves moving in the x-direction only. This is accomplished by assuming exponential type solutions having the form $e^{i(kx-\omega t)}$ where i is an imaginary unit satisfying $i^2=-1$.

(a) Show that $\vec{E}=\vec{E}(x,t)=\vec{E}_0 e^{i(kx-\omega t)}$ and $\vec{B}=\vec{B}(x,t)=\vec{B}_0 e^{i(kx-\omega t)}$ are solutions of Maxwell's equations in this special case.

(b) Show that $\vec{B}_0=\dfrac{k}{\omega}(\hat{\mathbf{e}}_1\times\vec{E}_0)$

(c) Show that the waves for $\vec{E}$ and $\vec{B}$ are mutually perpendicular.

▶ **9-38.** Consider the following vector fields:

$\vec{B}$ a magnetic field intensity with units of amp/m

$\vec{E}$ an electrostatic intensity vector with units of volts/m

$\vec{Q}$ a heat flow vector with units of joules/cm$^2\cdot$sec

$\vec{V}$ a velocity vector with units of cm/sec

(a) Assign units of measurement to the following integrals and interpret the meanings of these integrals:

$$(a)\quad \iint_S \vec{E}\cdot d\vec{S} \qquad (b)\quad \iint_S \vec{Q}\cdot d\vec{S} \qquad (c)\quad \iint_S \vec{V}\cdot d\vec{S} \qquad (d)\quad \int_C \vec{B}\cdot d\vec{r}$$

(b) Assign units of measurements to the quantities:

$$(a)\quad \text{curl}\,\vec{H} \quad (b)\quad \text{div}\,\vec{E} \quad (c)\quad div\,\vec{Q} \quad (d)\quad \text{div}\,\vec{V}$$

▶ **9-39.** Solve each the following vector differential equations

$$(a)\quad \frac{d\vec{y}}{dt}=\hat{\mathbf{e}}_1 t+\hat{\mathbf{e}}_3\sin t \quad (b)\quad \frac{d^2\vec{y}}{dt^2}=\hat{\mathbf{e}}_1\sin t+\hat{\mathbf{e}}_2\cos t \quad (c)\quad \frac{d\vec{y}}{dt}=3\vec{y}+6\,\hat{\mathbf{e}}_3$$

▶ **9-40.** Solve the simultaneous vector differential equations $\dfrac{d\vec{y}_1}{dt}=\vec{y}_2,\qquad \dfrac{d\vec{y}_2}{dt}=-\vec{y}_1$

▶ **9-41.** A particle moves along the spiral $r=r(\theta)=r_0e^{\theta\cot\alpha}$, where r_0 and α are constants. If $\theta=\theta(t)$ is such that $\dfrac{d\theta}{dt}=\omega=constant$, find the components of velocity in the direction $\vec{r}$ and in the direction perpendicular to $\vec{r}$.

▶ **9-42.** Couloumb's law states that the force between two charges q_1 and q_2 acts along the line joining the two charges and the magnitude of the force varies directly to the product of charges and inversely as the square of the distance r between the charges. Symbolically the magnitude of this force is $F = q_1 q_2 / r^2$ in the appropriate system of units[15]. A charge Q is called a test charge if it is located at a variable point (x, y, z) and experiences a force from another charge q_1, located at a fixed point. The ratio of the force experienced by the test charge to the magnitude of the test charge is called the electrostatic intensity $\vec{E}$ at the point (x, y, z) and is given by $\vec{E} = \frac{\vec{F}}{Q}$. An equivalent statement is that a unit charge has been placed at the point P and the electrostatic intensity is the total force which acts on this test charge.

(a) Show that a charge q located at the origin produces an electrostatic intensity $\vec{E}$ at a point (x, y, z) given by

$$\vec{E} = \frac{q\vec{r}}{r^3}, \text{ where } r = \mid \vec{r} \mid \text{ and } \quad \vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3.$$

(b) Show that (i) $\vec{E} = -\nabla \frac{q}{r}$ (ii) $\vec{E}$ has the scalar potential $\phi = \frac{q}{r}$
(iii) $\operatorname{curl} \vec{E} = 0$ and (iv) $\operatorname{div} \vec{E} = -q\nabla^2 \frac{1}{r} = 0$

c) Let q_1 denote a fixed charge located at the point $P_1(x_1, y_1, z_1)$ and let q_2 denote another fixed charge located at the point $P_2(x_2, y_2, z_2)$. Show the electrostatic intensity on a test charge at (x, y, z) is given by

$$\vec{E} = -\frac{q_1 \vec{r}_1}{r_1^2} - \frac{q_2 \vec{r}_2}{r_2^3},$$

where $r_i = \mid r_i \mid$ for $i = 1, 2$, and $\vec{r}_i = (x_i - x)\,\hat{\mathbf{e}}_1 + (y_i - y)\,\hat{\mathbf{e}}_2 + (z_i - z)\,\hat{\mathbf{e}}_3$.

(d) Show for n charges q_i, $i = 1, 2, \ldots, n$, located respectively at the points $P_i(x_i, y_i, z_i)$, for $i = 1, 2, \ldots, n$, the electrostatic intensity at a general point (x, y, z) is given by

$$\vec{E} = -\sum_{i=1}^{n} \frac{q_i}{r_i^3} \vec{r}_i,$$

where $\vec{r}_i = (x_i - x)\,\hat{\mathbf{e}}_1 + (y_i - y)\,\hat{\mathbf{e}}_2 + (z_i - z)\,\hat{\mathbf{e}}_3$ and $r_i = \mid r_i \mid$ for $i = 1, 2, \ldots, n$.

[15] If charges are measured in units of statcoulombs and distance is measured in centimeters, then the force has units of dynes.

Chapter 10
Matrix and Difference Calculus

The matrix calculus is used in the study of linear systems and systems of differential equations and occurs in engineering mathematics, physics, statistics, biology, chemistry and many other scientific applications. The difference calculus is used to study discrete events.

The Matrix Calculus

A matrix is **a rectangular array of numbers or functions** and can be expressed in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \tag{10.1}$$

where the quantities a_{ij} for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ are called the elements of the matrix. Here the double subscript notation a_{ij} is used to denote the element in the ith row and jth column. A matrix with m rows and n columns is called a m by n matrix and expressed in the form "A is a $m \times n$ matrix". Matrices are usually denoted using capital letters and whenever it is necessary to emphasize the elements and size of the matrix it is sometimes expressed in the form $A = (a_{ij})_{m\times n}$. The rows of the matrix A are called row vectors and the columns of the matrix A are called column vectors.

For a and b positive integers, then matrices of the form $R = (r_{a1}\ r_{a2} \dots r_{aj} \dots r_{an})$ are called **n-dimensional row vectors** and matrices of the form

$$C = \begin{pmatrix} c_{1b} \\ c_{2b} \\ \vdots \\ c_{ib} \\ \vdots \\ c_{mb} \end{pmatrix} = \text{col}\,(c_{1b}, c_{2b}, \dots, c_{ib}, \dots, c_{mb}) \tag{10.2}$$

are called **m-dimensional column vectors**. The column notation $\text{col}(c_{1b}, \dots, c_{mb})$ is used to conserve space in typesetting the m-dimensional column vector.

Properties of Matrices

1. Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ having the same dimension are equal if $a_{ij} = b_{ij}$ for all values of i and j. Equality is expressed $A = B$.
2. Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ of the same size can be added or subtracted and the resulting matrices are denoted
$$\begin{aligned} C =& A + B \quad \text{where} \quad C = (c_{ij})_{m\times n} \text{ with } c_{ij} = a_{ij} + b_{ij} \\ D =& A - B \quad \text{where} \quad D = (d_{ij})_{m\times n} \text{ with } d_{ij} = a_{ij} - b_{ij} \end{aligned}$$
Here like elements are added or subtracted.
3. If the matrix $A = (a_{ij})_{m\times n}$ is multiplied by a scalar β, the resulting matrix is
$$\beta\, A = (\beta\, a_{ij})_{m\times n}$$
That is, each component a_{ij} of A is multiplied by the scalar β.
4. Matrices of the same size obey the following laws.
$$\begin{aligned} A + B =& B + A \qquad \text{commutative law} \\ A + (B + C) =& (A + B) + C \qquad \text{associative law} \end{aligned}$$
For α and β scalar quantities one can write the
$$\text{scalar distributive laws} \qquad \begin{cases} \alpha(A+B) = & \alpha A + \alpha B \\ (\alpha+\beta)A = & \alpha A + \beta A \\ \alpha\,(\beta A) = & (\alpha\beta)A \end{cases}$$
5. The zero matrix has all zeros for elements and can be expressed in one of the forms $[0]_{m\times n}$ or $[0]$ or $\underline{0}$ or $\tilde{0}$.
6. If the elements of the matrix A are functions of a single variable, say t, one can write $a_{ij} = a_{ij}(t)$ or $A = A(t) = (a_{ij}(t))$ to emphasize this fact, then the derivative of the matrix A is given by
$$\frac{dA}{dt} = \left(\frac{da_{ij}}{dt}\right) \tag{10.3}$$
and the integral of the matrix A is
$$\int A(t)\, dt = \left(\int a_{ij}(t)\, dt\right) + C \tag{10.4}$$
where C is a constant matrix of appropriate size. Here the derivative of a matrix is obtained by differentiating each element of the matrix and the integral of the matrix is obtained by integrating each element within the matrix and the constants of integration are collected into a constant matrix.

Example 10-1. Find the derivative and integral of the matrix $A = \begin{bmatrix} 1 & x \\ \sin x & e^{-2x} \end{bmatrix}$

Solution Taking the derivative of each element one finds

$$\frac{dA}{dx} = \begin{bmatrix} 0 & 1 \\ \cos x & -2e^{-2x} \end{bmatrix}$$

Taking the integral of each element one finds

$$\int A\,dt = \begin{bmatrix} x & \frac{1}{2}x^2 \\ -\cos x & -\frac{1}{2}e^{-2x} \end{bmatrix} + C$$

where $C = (c_{ij})_{2\times 2}$ is an arbitrary constant matrix.

∎

The Dot or Inner Product

The **dot or inner product** of a n-dimensional row vector R and n-dimensional column vector C, where

$$R = (r_1, r_2, r_3, \ldots, r_n) \qquad \text{and} \qquad C = \text{col}(c_1, c_2, c_3, \ldots, c_n)$$

is **a single number** written as the matrix product

$$RC = (r_1, r_2, r_3, \ldots, r_n) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = r_1c_1 + r_2c_2 + r_3c_3 + \cdots + r_nc_n = \sum_{m=1}^{n} r_m c_m \tag{10.5}$$

representing **the summation of the products of the mth row vector element with the mth column vector element, as m varies from 1 to n.** In order to calculate an inner product the row vector and column vector **must have the same number of elements**.

Matrix Multiplication

Let $A = (a_{ij})_{m\times n}$ denote an $m \times n$ matrix and let $B = (b_{ij})_{p\times q}$ denote a $p \times q$ matrix. If the dimensions n, p have the proper size, then the matrix A can be right-multiplied by the matrix B to produce a new matrix C. This matrix product is written $C = AB$ and this matrix product **can only occur when the matrices A and B have the proper dimensions**. For the matrix product $AB = A_{m\times n}B_{p\times q}$ to exist it is required that **the dimension p of B** must equal **the dimension n of A** and whenever this condition is satisfied, then the matrices are said to satisfy the **compatibility condition for matrix multiplication to occur**. If the column dimension of A **does not**

equal the row dimension of B, then the matrices A and B **cannot be multiplied**. The matrix product of two matrices A and B, having the proper dimensions, is written $C = AB$ and then one can say either

$$(a) \quad A \text{ premultiplies } B$$
$$(b) \quad \text{or } B \text{ postmultiplies } A$$

If the **row dimension of** B **equals the column dimension of** A one can write $p = n$, then the two matrices A and B can be multiplied and the resulting matrix product C has dimension $m \times q$. This is sometimes expressed in the form

$$C_{m\times q} = A_{m\times \underbrace{n\, B_{n}}}{}_{\times q}$$

where attention is drawn to the fact that the matrices satisfy the compatibility condition for matrix multiplication. Expressing the matrices A, B and C in expanded form one can write

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2q} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{iq} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mq} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nq} \end{pmatrix}$$

The element c_{ij} belong to the matrix product C is calculated using **the elements from the** i**th row vector of** A **and the elements from the** j**th column vector of** B to represent c_{ij} as a dot or inner product. **The** $\boldsymbol{i}$**th row vector of** $\boldsymbol{A}$ **is dotted with the** $\boldsymbol{j}$ **column vector from** $\boldsymbol{B}$ **and the resulting single number is called** $\boldsymbol{c_{ij}}$**.** This inner or dot product is defined as above, but now a double subscript notation is in use so that one obtains

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Performing all possible inner products of the i**th row vector with the** j**th column vector as** i **varies from** 1 **to** m **and** j **varies from** 1 **to** q produces the product matrix $C = (c_{ij})_{m\times q}$.

Example 10-2. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \end{pmatrix}_{3\times 4}$ the matrices A and B satisfy the compatibility condition for matrix multiplication and the matrix product $C = AB$ will be a matrix having the dimension of 2 rows and 4 columns. One can write

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix} = AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \end{pmatrix}$$

where c_{11} is the inner product of row 1 with column 1 giving

$$c_{11} = 1(7) + 2(11) + 3(15) = 74$$

In a similar fashion one finds

c_{12} is the inner product of row 1 with column 2 giving
$$c_{12} = 1(8) + 2(12) + 3(16) = 80$$
c_{13} is the inner product of row 1 with column 3 giving
$$c_{13} = 1(9) + 2(13) + 3(17) = 86$$
c_{14} is the inner product of row 1 with column 4 giving
$$c_{14} = 1(10) + 2(14) + 3(18) = 92$$
c_{21} is the inner product of row 2 with column 1 giving
$$c_{21} = 4(7) + 5(11) + 6(15) = 173$$
c_{22} is the inner product of row 2 with column 2 giving
$$c_{22} = 4(8) + 5(12) + 6(16) = 188$$
c_{23} is the inner product of row 2 with column 3 giving
$$c_{23} = 4(9) + 5(13) + 6(17) = 203$$
c_{24} is the inner product of row 2 with column 4 giving
$$c_{24} = 4(10) + 5(14) + 6(18) = 218$$

This gives the matrix product

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \end{pmatrix} = C = \begin{pmatrix} 74 & 80 & 86 & 92 \\ 173 & 188 & 303 & 218 \end{pmatrix}$$

■

Matrices with the proper dimensions satisfy the properties

$$A(B+C) = AB + AC \qquad \text{left distributive law}$$
$$(B+C)A = BA + CA \qquad \text{right distributive law}$$
$$A(BC) = (AB)C \qquad \text{associative law}$$

Example 10-3. Note that only in **special cases** is matrix multiplication commutative. One can say in general $AB \neq BA$. Consider the matrix product of the 2×2 matrices given by $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$. One finds

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix}$$

which shows that in general matrix multiplication is not commutative.

In addition, if the matrix product of A and B produces $AB = [0]$, this **does not mean** $A = [0]$ or $B = [0]$. For example, if $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, then one can show

$$AB = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

■

Special Square Matrices

There are many special matrices which have interesting properties. The following are some definitions of **special square matrices** which arise in applied mathematics, engineering, physics and the sciences.

The identity matrix

The $n \times n$ **identity matrix** can be expressed $I = (\delta_{ij})_{n \times n}$ where $\boldsymbol{\delta_{ij}}$ **is the Kronecker delta** and defined

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This matrix is characterized by having all 1's along the main diagonal and zero's everywhere else. An example of a 3×3 identity matrix is given by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Secondary diagonal

Main diagonal

The identity matrix has the property

$$AI = IA = A \tag{10.6}$$

for all square matrices A where A and I **have the same dimensions**.

The transpose matrix

The **transpose of a matrix** $A = (a_{ij})_{m\times n}$ is obtained by **interchanging the rows and columns of the matrix** A. The transpose matrix is denote $A^T = (a_{ji})_{n\times m}$. That is, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n} \qquad \text{then} \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}_{n\times m}$$

Note that $(A^T)^T = A$. If $A^T = A$, then the matrix A is called **a symmetric matrix**. If $A^T = -A$, then A is called **a skew-symmetric matrix**. The matrix transpose of a product satisfies

$$(AB)^T = B^T A^T, \qquad (ABC)^T = C^T B^T A^T$$

so that **the transpose of a product is the product of the transposed matrices in reverse order**.

Lower triangular matrices

Matrices which satisfy

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = 0 \quad \text{for} \quad i < j,$$

are called **lower triangular matrices**. Such matrices have zero for elements everywhere above the main diagonal. Any example of a lower triangular matrix is given in the figure 10-1.

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 1 & -1 & -2 & 4 \end{pmatrix}$$

Figure 10-1. A 4×4 lower triangular matrix.

Upper triangular matrices

If a square matrix A satisfies

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = 0 \quad \text{for} \quad i > j,$$

it is called an **upper triangular matrix**. Such matrices have zero for elements everywhere below the main diagonal. An example upper triangular matrix is illustrated in the figure 10-2.

$$A = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Figure 10-2. A 4×4 upper triangular matrix.

Diagonal matrices

A matrix which has zeros for all elements above and below the main diagonal is called a **diagonal matrix**. Such a matrix can be described by

$$D = (d_{ij}), \quad \text{where} \quad d_{ij} = 0 \quad \text{for} \quad i \neq j.$$

Diagonal matrices are sometimes written $D = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The identity matrix is an example of a diagonal matrix. Another example of a diagonal matrix is given in the figure 10-3.

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 10-3. Example of a 4×4 diagonal matrix.

Tridiagonal matrices

A matrix A satisfying

$$A = (a_{ij}), \quad \text{where} \quad a_{ij} = \begin{cases} 0, & i > j+1 \\ 0, & i < j-1 \end{cases}$$

is called a **tridiagonal matrix**. Such a matrix is recognized as having elements along the main diagonal and the immediate diagonals above and below the main diagonal. All other elements within the matrix are zero. An example tridiagonal matrix is given in the figure 10-4.

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Figure 10-4. A 5×5 tridiagonal matrix.

The trace of a matrix

The **trace of a** $\boldsymbol{n\times n}$ **square matrix** $\boldsymbol{A}$ is denoted $\mathrm{Tr}(A)$ and represents a summation of the diagonal elements of the matrix A. One can write

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

If matrices A and B are conformable matrices, then the trace satisfies the properties

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B), \qquad \mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

The Inverse Matrix

If A and E are square matrices such that their matrix product produces the identity matrix, that is, if $AE = EA = I$, then E is called **the inverse of** $\boldsymbol{A}$ and the matrix E is written

$$E = A^{-1},$$

which is read **"$\boldsymbol{E}$ equals $\boldsymbol{A}$ inverse".** Thus, the inverse matrix has the property that

$$AA^{-1} = A^{-1}A = I.$$

The inverse matrix, if it exists, is unique. This statement can be proven by first assuming that the inverse is not unique and then showing that this assumption is wrong. This type of proof is known as the method of **reductio ad absurdum**[1] to verify something is true.

For example, if A_1 and A_2 are both inverses of the matrix A, then by hypothesis both of the statements

$$AA_1 = A_1A = I \qquad \text{and} \qquad AA_2 = A_2A = I$$

must be true. Consequently, one can write

$$A_2 = A_2I = A_2(AA_1) = (A_2A)A_1 = IA_1 = A_1.$$

Hence, $A_2 = A_1 = A^{-1}$ and the initial assumption is wrong and so the inverse matrix must be unique.

[1] The method of reductio ad absurdum is used to prove a statement in mathematics by assuming initially that the statement is true (or false) and then performing an analysis of this assumption (the reduction of the proposition) to arrive at a conclusion which is obviously absurd and contradicts the initial assumption. The method of reductio ad absurdum was used by the early Greek mathematicians as a method for proving many theorems.

Example 10-4. For A an $n \times n$ square matrix, show that $(A^{-1})^{-1} = A$. That is, show the inverse of an inverse matrix is again the original matrix A.

Solution Let $B = A^{-1}$ so that $B^{-1} = (A^{-1})^{-1}$, then by definition of an inverse matrix one can write

$$AB = AA^{-1} = I.$$

Right-multiply this equation on both sides by B^{-1} to obtain

$$ABB^{-1} = IB^{-1} = B^{-1}.$$

Using the result that $BB^{-1} = I$ and that $AI = A$, this last equation simplifies to

$$AI = A = B^{-1} = (A^{-1})^{-1}$$

which establishes the result.

■

Example 10-5. Show that $(AB)^{-1} = B^{-1}A^{-1}$. That is, show **the inverse of a product of two matrices is the product of the inverses in the reverse order**.

Solution By definition

$$(AB)^{-1}(AB) = I.$$

so that if one postmultiples both sides of this equation by B^{-1} and simplifies the results, one finds

$$\begin{aligned}
(AB)^{-1}(AB)B^{-1} &= IB^{-1} \\
(AB)^{-1}A(BB^{-1}) &= B^{-1}, \quad \text{associative law} \\
(AB)^{-1}AI &= B^{-1} \\
(AB)^{-1}A &= B^{-1}
\end{aligned}$$

Now postmultiply both sides of this last equation by A^{-1} to obtain

$$\begin{aligned}
(AB)^{-1}AA^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1}I &= B^{-1}A^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1}
\end{aligned}$$

which establishes the result.

■

Methods for calculating the inverse of a square matrix, if the inverse exists, are developed in a later section. In this section, the emphasis is on definitions, terminology and certain operational properties associated with square matrices.

Matrices with Special Properties

The following is some terminology associated with square matrices A and B.

(1) If $AB = -BA$, then A and B are called **anticommutative**.

(2) If $AB = BA$, then A and B are called **commutative**.

(3) If $AB \neq BA$, then A and B are called **noncommutative**.

(4) If $A^p = \overbrace{AA\cdots A}^{p\ times} = \widetilde{0}$ for some positive integer p, then A is called **nilpotent of order** p.

(5) If $A^2 = A$, then A is called **idempotent**.

(6) If $A^2 = I$, then A is called **involutory**.

(7) If $A^{p+1} = A$, then A is called **periodic with period** p. The smallest integer p for which $A^{p+1} = A$ is called the least period p.

(8) If $A^T = A$, then A is called a **symmetric matrix**.

(9) If $A^T = -A$, then A is called a **skew-symmetric matrix**.

(10) If A^{-1} **exists**, then A is called a **nonsingular matrix**.

(11) If A^{-1} **does not exist**, then A is called **a singular matrix.**

(12) If $A^T A = AA^T = I$, then A is called an **orthogonal matrix** and $A^T = A^{-1}$.

Example 10-6.

The matrix $A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ is periodic with least period 2 because

$A^2 = AA = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $A^3 = A^2 A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = A$

■

Example 10-7. The matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ is nilpotent of index 2 because

$$A^2 = AA = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \widetilde{0}$$

■

Example 10-8. The matrix

$$B = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

is idempotent because

$$B^2 = BB = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = B$$

■

An orthogonal matrix

If A is an $n \times n$ square matrix satisfying $A^T A = AA^T = I$, then A is called an **orthogonal matrix**, and $A^{-1} = A^T$. An example of an orthogonal matrix is given by

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad AA^T = I$$

Example 10-9. Some examples of special matrices are:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad \text{is lower triangular}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{bmatrix} \qquad \text{is upper triangular}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{is an identity matrix which is also diagonal}$$

$$T = \begin{bmatrix} \beta & \gamma & 0 & 0 & 0 \\ \alpha & \beta & \gamma & 0 & 0 \\ 0 & \alpha & \beta & \gamma & 0 \\ 0 & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \alpha & \beta \end{bmatrix} \qquad \text{is a tridiagonal matrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{is an orthogonal matrix satisfying} \qquad AA^T = I$$

If $f = f(\bar{x}) = f(x_1, x_2, \ldots, x_n)$ is a function of n-variables, then the Hessian matrix associated with f is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1\, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\, \partial x_n} \\ \frac{\partial^2 f}{\partial x_2\, \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2\, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\, \partial x_1} & \frac{\partial^2 f}{\partial x_n\, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$$

■

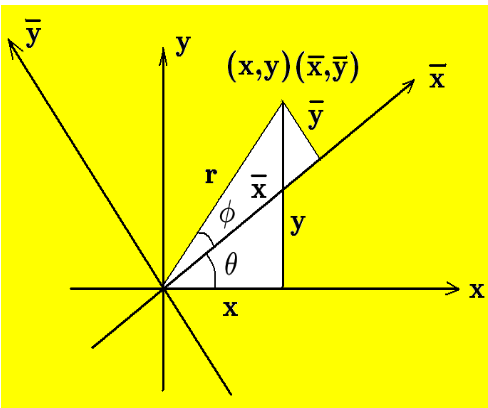
Example 10-10. Consider the fixed set of axes x, y and a set of barred axes $\bar{x}, \bar{y}$ where the barred set of axes is rotated about the origin through an angle θ as illustrated below. Consider a general point P having coordinates (x, y) with respect the unbarred axes. This same point P has coordinates $(\bar{x}, \bar{y})$ with respect to the barred set of axes. Let $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\bar{\hat{\mathbf{e}}}_1$, $\bar{\hat{\mathbf{e}}}_2$ denote unit vectors in the directions of the x, y and $\bar{x}, \bar{y}$-axes respectively. The position vector $\vec{r}$ of the point P can be expressed in either of the forms

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 \qquad \text{or} \qquad \vec{r} = \bar{x}\,\bar{\hat{\mathbf{e}}}_1 + \bar{y}\,\bar{\hat{\mathbf{e}}}_2$$

The transformation equations between the coordinates can be obtained by taking the dot product of $\vec{r}$ with the unit vectors $\bar{\hat{\mathbf{e}}}_1$ and $\bar{\hat{\mathbf{e}}}_2$ to obtain

$$\vec{r}\cdot\bar{\hat{\mathbf{e}}}_1 = \bar{x} = x\,\hat{\mathbf{e}}_1\cdot\bar{\hat{\mathbf{e}}}_1 + y\,\hat{\mathbf{e}}_2\cdot\bar{\hat{\mathbf{e}}}_1 = x\cos\theta + y\sin\theta$$

$$\vec{r}\cdot\bar{\hat{\mathbf{e}}}_2 = \bar{y} = x\,\hat{\mathbf{e}}_1\cdot\bar{\hat{\mathbf{e}}}_2 = y\,\hat{\mathbf{e}}_2\cdot\bar{\hat{\mathbf{e}}}_2 = x(-\sin\theta) + y\cos\theta$$



The above transformation equations between the $(\bar{x}, \bar{y})$ axes which have been rotated through an angle θ with respect to a fixed set of (x, y) axes can be represented by the matrix equation

$$\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \qquad \text{or} \qquad \bar{X} = AX \qquad (10.7)$$

where $\bar{X} = \text{col}(\bar{x}, \bar{y})$ and $X = \text{col}(x, y)$ are column vectors. Here the coefficient matrix of the above transformation is $A = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ and its transpose matrix is $A^T = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. If one calculates the matrix product of A times it transpose A^T, one finds $AA^T = I$, the identity matrix. Matrices with this property are called **orthogonal matrices.** Left-multiplication of equation (10.7) by $A^{-1} = A^T$ gives the inverse transformation $A^T\bar{X} = A^TAX = IX = X$ which can be expressed in expanded form

$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}.$$

The row and column vectors which make up the rows and columns of the matrix A are called orthogonal vectors.

■

Example 10-11. Represent the given system of differential equations in matrix form.

$$\frac{dy_1}{dt} = y_1 + y_2 - y_3 + \sin t, \qquad \frac{dy_2}{dt} = 2y_2 + y_3 + \cos t, \qquad \frac{dy_3}{dt} = 3y_3 + \sin 2t$$

Solution The above system of differential equations can be represented in the form

$$\frac{d\overline{y}}{dt} = A\overline{y} + \overline{f}(t) \tag{10.8}$$

where $\overline{y} = \overline{y}(t) = \text{col}(y_1, y_2, y_3)$ denotes a column vector, $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is a coefficient matrix and $\overline{f} = \overline{f}(t) = \text{col}(\sin t, \cos t,\ \sin 2t)$ represents a variable right-hand side to the differential system. Matrix differential equations of the form given by equation (10.8) subject to the initial condition $\overline{y}(0) = \bar{c}$, where $\bar{c}$ is a constant, are called **initial-value problems.**

■

Example 10-12. The nth order linear differential equation

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + a_2(t)\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_{n-2}(t)\frac{d^2y}{dt^2} + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = 0$$

is converted to matrix form by defining

$$\bar{y} = \text{col}(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \ldots, \frac{d^{n-2}y}{dt^{n-2}}, \frac{d^{n-1}y}{dt^{n-1}})$$

and

$$A = A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2} & -a_{n-3}(t) & \cdots & -a_2(t) & -a_1(t) \end{pmatrix}$$

The given scalar equation can then be represented by the matrix equation

$$\frac{d\bar{y}}{dt} = A(t)\bar{y}$$

The matrix $A = A(t)$ is called **the companion matrix**.

■

Example 10-13. Represent the differential equation $\frac{d^2y}{dt^2}+\omega^2 y = \sin 2t$ in matrix form.

Solution Let $y_1 = y$ and $y_2 = \frac{dy_1}{dt} = \frac{dy}{dt}$, then

$$\frac{dy_2}{dt} = \frac{d^2y_1}{dt^2} = \frac{d^2y}{dt^2} = -\omega^2 y + \sin 2t = -\omega^2 y_1 + \sin 2t$$

The given scalar differential equation can then be represented in the matrix form

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin 2t \end{pmatrix}$$

or

$$\frac{d\overline{y}}{dt} = A\overline{y} + \overline{f}(t) \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \overline{f}(t) = \begin{pmatrix} 0 \\ \sin 2t \end{pmatrix} \quad \text{and} \quad \overline{y} = \text{col}(y_1, y_2)$$

■

Example 10-14. Associated with the initial value problem

$$\frac{d\bar{y}}{dt} = A(t)\bar{y} + \bar{f}(t), \qquad \bar{y}(0) = \bar{c} \tag{10.9}$$

is the **matrix differential equation**

$$\frac{dX}{dt} = A(t)X, \qquad X(0) = I \tag{10.10}$$

where $X = (x_{ij})_{n\times n}$ and I is the $n \times n$ identity matrix. Associated with the matrix differential equation (10.10) is the **adjoint differential equation**

$$\frac{dZ}{dt} = -ZA(t), \qquad Z(0) = I, \quad Z = (z_{ij})_{n\times n} \tag{10.11}$$

The relationship between the three differential equations given by equations (10.9), (10.10), and (10.11), is as follows. Left-multiply equation (10.10) by Z and right-multiply equation (10.11) by X to obtain

$$Z\frac{dX}{dt} = ZA(t)X \tag{10.12}$$

$$\frac{dZ}{dt}X = -\, ZA(t)X \tag{10.13}$$

and then add the equation (10.12) and (10.13) to obtain

$$Z\frac{dX}{dt} + \frac{dZ}{dt}X = \frac{d}{dt}(ZX) = [0] \tag{10.14}$$

This implies that the matrix product $ZX = C$ is a constant. If the matrix X is nonsingular, then X^{-1} exists, so that one can solve for Z as $Z = X^{-1}C$. At time $t = 0$, it is required that $Z(0) = I$ and $X(0) = I$ so that $C = I$ and therefore $Z = X^{-1}$. Consequently, the adjoint equation can be expressed in the form

$$\frac{dX^{-1}}{dt} = -X^{-1}A(t), \qquad X^{-1}(0) = I \tag{10.15}$$

Now multiply equation (10.9) on the left by X^{-1} to obtain

$$X^{-1}\frac{d\bar{y}}{dt} = X^{-1}A\bar{y} + X^{-1}\bar{f}(t) \tag{10.16}$$

and then multiply equation (10.15) on the right by $\bar{y}$ to obtain

$$\frac{dX^{-1}}{dt}\bar{y} = -X^{-1}A\bar{y} \tag{10.17}$$

Sum the equations (10.16) and (10.17) to obtain

$$X^{-1}\frac{d\bar{y}}{dt} + \frac{dX^{-1}}{dt}\bar{y} = \frac{d}{dt}\left(X^{-1}\bar{y}\right) = X^{-1}\bar{f}(t) \tag{10.18}$$

Integrate equation (10.18) from 0 to t and show

$$\int_0^t \frac{d}{dt}\left(X^{-1}(t)\bar{y}(t)\right)\,dt = \int_0^t X^{-1}(t)\bar{f}(t)\,dt$$

which produces the result

$$X^{-1}(t)\bar{y}(t)\Big|_0^t = X^{-1}(t)\bar{y}(t) - X^{-1}(0)\bar{y}(0) = \int_0^t X^{-1}(t)\bar{f}(t)\,dt$$

which indicates that the solution to the matrix equation (10.9) can be represented in the form

$$\bar{y}(t) = X(t)\bar{c} + X(t)\int_0^t X^{-1}(\xi)\bar{f}(\xi)\,d\xi \tag{10.19}$$

■

The Determinant of a Square Matrix

A **fundamental principle from probability and statistics** is that if something can be done in n different ways and after it has been done in one of these ways, a second something can be done in m different ways, then the two somethings can be done in the order stated in $n \cdot m$ different ways. If a third something can be done in p different ways, then the three somethings can be done in $n \cdot m \cdot p$ different ways. This

principle of multiplication by the number of distinct ways a thing can be done can be extended to more than just two or three somethings.

A **permutation of a set of objects** represents some arrangement of the objects. The number of different permutations of n-objects is $n!$ (read n-factorial). This is because there are n choices for the first position of the arrangement, $(n-1)$ choices for the second position of the arrangement, $(n-3)$ choices for the third position of the arrangement, etc, and these quantities are being multiplied.

A **transposition** is **an interchanging of the positions of two objects within an arrangement** of the set of objects. In examining all possible permutations of the integers $(1, 2, 3, \dots, n)$ one finds these permutations can be divided into a group representing an even number of transpositions and another group representing the odd number of transpositions. For example, in going from $(1234\dots)$ to $(2134\dots)$ represents 1 transposition and going from $(1234,\dots)$ to $(2314\dots)$ would be two transpositions, etc.

The determinant of a $n \times n$ square matrix $A = (a_{ij})$ is denoted by either of the symbols $\det A$ or $\mid A \mid$. The determinant is **a single number** given by either of the summations

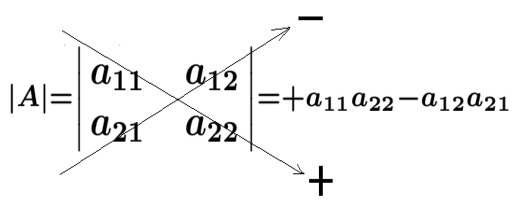
$$\det A = \mid A \mid = \sum (-1)^m a_{i1} a_{j2} a_{k3} \dots a_{\ell n} \qquad \text{column expansion}$$
$$\det A = \mid A \mid = \sum (-1)^m a_{1i} a_{2j} a_{3k} \cdots a_{n\ell} \qquad \text{row expansion}$$

The single number $\det A = \mid A \mid$ is **the sum of all possible products** in which there appears one and only one element from each row (or column) multiplied by the appropriate plus or minus sign. The sigma sign Σ denotes a sum over all $n!$ permutations of the numbers $(1, 2, 3, \dots, n)$ and the integers $(i, j, k, \dots, \ell)$ represent distinct permutations of the numbers from the set $(1, 2, 3, \dots, n)$. The appropriate plus or minus sign is assigned to each product within the sum and is based upon whether the permutation $(i, j, k, \dots, \ell)$ is either even (+1) or odd (-1). That is, $m = +1$ if $(i, j, k, \dots, \ell)$ represents an even number of transpositions associated with the set $(1, 2, 3, \dots, n)$ and $m = -1$ if $(i, j, k, \dots, \ell)$ represents an odd number of transpositions associated with the set $(1, 2, 3, \dots, n)$.

Example 10-15. The matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has the determinant

$$\mid A \mid = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

Here $(1,2)$ is an even permutation of $(1,2)$ and $(2,1)$ represents an odd permutation of $(1,2)$ A mnemonic device to remember this 2×2 determinant is illustrated by the following figure

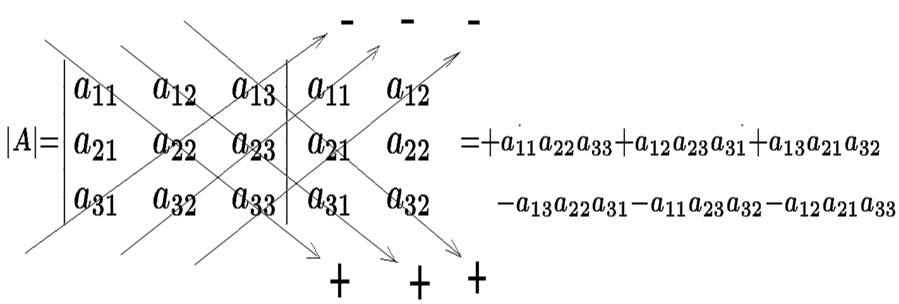


■

Example 10-16. The 3×3 matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has the determinant

$$| A |=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}$$

A mnemonic device to remember this 3×3 determinant is to append the first two columns to the end of the matrix and draw diagonal lines through the elements to create the following figure, where the elements on each diagonal are multiplied.



Note that the determinant of a $n\times n$ matrix has **n-factorial terms** and consequently if n is large, then mnemonic devices like those above **are not employed because the calculations become cumbersome** and sometimes extremely lengthy. Instead it has been found that by using **row reduction methods**[2] the given matrix can be converted to an equivalent upper triangular or lower triangular matrix having all zeros either below or above the main diagonal. The determinant of these special triangular matrices is then just **a product of the diagonal elements**.

■

Example 10-17. Find the derivative of the determinant

$$y=\det A=| A |=\begin{vmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{vmatrix}$$

---

[2] Row reduction methods are considered later in this chapter.

Solution

The definition of a determinant gives the relation $y = y(t) = \sum(-1)^m a_{i1}(t)a_{j2}(t)$ where the summation is over all permutations of the integers $(1,2)$. Differentiating this relation gives

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sum(-1)^m a_{i1}(t)a_{j2}(t)\right) = \sum(-1)^m\left[\frac{da_{i1}(t)}{dt}a_{j2}(t) + a_{i1}(t)\frac{da_{j2}(t)}{dt}\right]$$

which has the expanded form

$$\frac{dy}{dt} = \begin{vmatrix} \frac{da_{11}(t)}{dt} & \frac{da_{12}(t)}{dt} \\ a_{21}(t) & a_{22}(t) \end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) \\ \frac{da_{21}(t)}{dt} & \frac{da_{22}(t)}{dt} \end{vmatrix}$$

■

Minors and Cofactors

Associated with each element a_{pq} of a $n \times n$ square matrix A are the quantities m_{pq} and c_{pq} called **the minor and cofactor of the element** a_{pq}**.** The minor m_{pq} of an element a_{pq} is the determinant of the $(n-1)\times(n-1)$ matrix formed by **deleting the row and column of** A **which contains the element** a_{pq}. The cofactor of a_{pq} is then defined as $c_{pq} = (-1)^{p+q}m_{pq}$. That is, the cofactor is **the minor with the appropriate plus or minus sign** $(-1)^{p+q}$ **which is determined by the row number** p **and column number** q **of the element** a_{pq}. The matrix containing the cofactor elements c_{pq} of a_{pq} is written $C = (c_{pq})_{n\times n}$ and is called **the cofactor matrix** associated with A. The cofactor matrix has the property that $AC^T = \text{diag}\,[|A|, |A|, \ldots, |A|] = |A|I$, where $|A|$ is the determinant of A.

Example 10-18. (**(Minors and Cofactors)** For the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ calculate the cofactor matrix $C = (c_{ij})_{3\times 3}$ and then calculate AC^T.

Solution The minor of element a_{ij} is obtained by crossing out the row and column containing a_{ij} and then taking the determinant of the remaining elements. One finds

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$m_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad m_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad m_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$m_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad m_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad m_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{aligned} c_{11} &= m_{11}, & c_{12} &= -m_{12}, & c_{13} &= m_{13} \\ c_{21} &= -m_{21}, & c_{22} &= m_{22}, & c_{23} &= -m_{23} \\ c_{31} &= m_{31}, & c_{32} &= -m_{32}, & c_{33} &= m_{33} \end{aligned}$$

$$AC^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} c_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

where

$$|A| = \det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

In general for $A_{n\times n}$ one can write

$$|A| = \det A = \sum_{j=1}^{n} a_{ij}c_{ij} \quad \text{or} \quad |A| = \det A = \sum_{j=1}^{n} a_{ij}c_{ij}$$

for the column and row expansion of a determinant. If $n > 3$ these methods for calculating a determinant are ill advised as the method is very time consuming.

■

Properties of Determinants

Many of the properties of determinants are associated with performing elementary row (or column) operations upon the elements of the determinant. The **three basic elementary row operations being performed on determinants** are

(i) The interchange of any two rows.

(ii) The multiplication of a row by a nonzero scalar α

(iii) The replacement of the ith row by the sum of the ith row and α times the jth row, where $i \neq j$ and α is any nonzero scalar quantity.

The following are some properties of determinants stated without proof.

1. If two rows (or columns) of a determinant are equal or one row is a constant multiple of another row, then the determinant is equal to zero.
2. The interchange of any two rows (or two columns) of a determinant changes the numerical sign of the determinant.
3. If the elements of any row (or column) are all zero, then the value of the determinant is zero.
4. If the elements of any row (or column) of a determinant are multiplied by a scalar m and the resulting row vector (or column vector) is added to any other row (or column), then the value of the determinant is unchanged. As an example, take a 3×3 determinant and multiply row 3 by a nonzero constant m and add the result to row 2 to obtain

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ (d+mg) & (e+mh) & (f+mi) \\ g & h & i \end{vmatrix}.$$

5. If all the elements in a row (or column) are multiplied by the same scalar q, then the determinant is multiplied by q. This produces

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ qa_{i1} & qa_{i2} & \cdots & qa_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = q \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

6. The determinant of the product of two matrices is the product of the determinants and $|AB| = |A||B|$.
7. If each element of a row (or column) is expressible as the sum of two (or more) terms, then the determinant may also be expressed as the sum of two (or more) determinants. For example,

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}+b_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

8. Let c_{ij} denote the cofactor of a_{ij} in the determinant of A. The value of the determinant $|A|$ is the sum of the products obtained by multiplying each element of a row (or column) of A by its corresponding cofactor and

$$\begin{aligned} |A| =& a_{i1}c_{i1} + \cdots + a_{in}c_{in} = \sum_{k=1}^{n} a_{ik}c_{ik} \quad \text{row expansion} \\ \text{or} \quad |A| =& a_{1j}c_{1j} + \cdots + a_{nj}c_{nj} = \sum_{k=1}^{n} a_{kj}c_{kj} \quad \text{column expansion} \end{aligned}$$

If the elements of a row (or column) are multiplied by the cofactor elements from a different row (or column), then zero is obtained. These results can be used to write $AC^T = |A|I$

Example 10-19.

Show the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$ has the cofactor matrix $C = \begin{bmatrix} -5 & 5 & -5 \\ 2 & -4 & -2 \\ -1 & -3 & 1 \end{bmatrix}$.
If the elements from any row (or column) of A are multiplied by their respective cofactors, then the sum of these products gives us the determinant $|A|$. For example, using row expansions one can verify

$$|A| = (1)(-5) + (0)(5) + (1)(-5) = -10$$
$$|A| = (-1)(2) + (1)(-4) + (2)(-2) = -10$$
$$|A| = (3)(-1) + (2)(-3) + (-1)(1) = -10$$

and using a column expansion there results

$$|A| = (1)(-5) + (-1)(2) + (3)(-1) = -10$$
$$|A| = (0)(5) + (1)(-4) + (2)(-3) = -10$$
$$|A| = (1)(-5) + (2)(-2) + (-1)(1) = -10.$$

Observe also that if the elements from any row (or column) are multiplied by the cofactors from a different row (or column), then the sum of these elements is zero. For example, row 1 multiplied by the cofactors from row 2 gives

$$(1)(2) + (0)(-4) + (1)(-2) = 0.$$

Another example is row 2 multiplied by the cofactors from row 3

$$(-1)(-1) + (1)(-3) + (2)(1) = 0.$$

These results may be further illustrated by calculating the matrix product

$$AC^T = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \operatorname{diag}(1,1,1) = |A|I \tag{10.20}$$

■

Example 10-20.

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 6 \\ 1 & 1 & 0 & 3 & 9 \\ -2 & 0 & 1 & -3 & -9 \\ 0 & -1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 4 & 12 \end{bmatrix}.$$

Solution: Utilizing property 4, one can multiply any row by a constant and add the result to any other row **without changing the value of the determinant**. Perform the following operations on the above determinant: (a) subtract row 1 from row 5 (b) multiply row 1 by two and add the result to row 3, and (c) subtract row 1 from row 2. Performing these calculations produces

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 6 \end{vmatrix}.$$

Now perform the operations: (a) add row 2 to row 4 and (b) subtract row 3 from row 5. The determinant now has the form

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}.$$

Observe that the row operations performed have produced zeros both above and below the main diagonal. Next perform the operations of (a) subtracting twice row 5 from row 1, (b) subtracting row 5 from row 2, (c) subtracting row 5 from row 3, and (d) subtracting row 5 from row 4. These operations produce

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}.$$

By expanding $|A|$ using cofactors of the first rows and associated subdeterminants, there results

$$|A| = (1)(1)(1) \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5.$$

A much more general procedure for calculating the determinant of a matrix A is to use **row operations and reduce $|A| = \det(A)$ to a triangular form having all zeros below the main diagonal.** For example, reduce A to the form:

$$|A| = \det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix}.$$

The determinant of A is then obtained by **multiplying all the elements on the main diagonal** and

$$|A| = \det(A) = a_{11}a_{22}\dots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

■

Rank of a Matrix

The **rank of a** $m \times n$ **matrix** A is denoted using the notation rank(A). The **rank of the matrix** A is a real number defined as the size of the **largest nonzero determinant that can be formed using the elements of** A. If $A = (a_{ij})_{m\times n}$, one can show that the maximum possible rank(A) is the smaller of the numbers m and n.

Calculation of the Inverse Matrix

The following illustrates some methods for calculating the inverse of a square matrix **if such an inverse exists**. Previously it has been shown that if C is the cofactor matrix of A, then

$$AC^T = |A|I. \tag{10.21}$$

By multiplying this equation on the left by A^{-1} and dividing by $|A|$, one can verify the result

$$A^{-1} = \frac{1}{|A|}C^T. \tag{10.22}$$

as a formula for calculating the inverse matrix. Define the **transpose of the cofactor matrix** C^T to be the **adjoint of** A. The notation Adj A is used to denote the adjoint matrix. Using this definition, the above results can be expressed in the form

$$(\text{Adj}\,A)A = A(\text{Adj}\,A) = |A|I \qquad \text{or} \qquad A^{-1} = \frac{1}{|A|}\text{Adj}\,A \tag{10.23}$$

If A is an $n \times n$ square matrix and the determinant satisfies $\det A = |A| = 0$, then A is called **a singular matrix**. If A **is singular**, then the inverse matrix **does not exist**. If $\det A = |A| \neq 0$, then A is called **a nonsingular matrix**, and the inverse matrix A^{-1} exists under these conditions as can be discerned by examining the equation (10.23).

Example 10-21.

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

Solution: The cofactor matrix associated with A is given by $C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ and

$$\text{Adj}\,A = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}.$$

This gives $|A| = 10$ so that A is nonsingular and the inverse is given by

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \quad \text{As a check, verify that} \quad AA^{-1} = I$$

■

Elementary Row Operations

A very useful matrix operation is **an elementary row operation** performed on a matrix. These elementary row operations can be used to obtain a wide variety of results.

An **elementary row matrix** E is any matrix **formed from the identity matrix** $I = (\delta_{ij})$ **by performing any of the following elementary row operations upon the identity matrix.**

(a) Interchange any two rows of I

(b) Multiplication of a row of I by any nonzero scalar m

(c) Replacement of the ith row of I by the sum of the ith row and m times the jth row, where $i \neq j$ and m is any scalar.

An **elementary column matrix** E is obtained if column operations are used instead of row operations. An **elementary transformation of a matrix** A is **the multiplication of** A **by an elementary row matrix.**

Example 10-22. Consider the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and the elementary matrices

E_1 where row 1 and 2 of the identity matrix are interchanged.

E_2 where row 1 is interchanged with row 3 and then rows 1 and 2 are interchanged.

E_3 where row 1 of the identity matrix is multiplied by 3.

E_4 where row 2 of the identity matrix is multiplied by 3 and the result added to row 1.

These elementary matrices can be represented

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Observe that multiplication of the matrix A by an elementary matrix E produces the following elementary transformations of the matrix A.

$$E_1A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

where the first two rows of A are interchanged,

$$E_2A = \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$$

where simultaneously row 2 is moved to row 1, row 3 is moved to row 2 and row 1 is moved to row 3,

$$E_3A = \begin{bmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{bmatrix}$$

where row 1 is multiplied by the scalar 3, and

$$E_4A = \begin{bmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{bmatrix}$$

where row 2 of A is multiplied by 3 and the result is added to row 1. Observe that the elementary matrices E_1, E_2, E_3 and E_4 are obtained by performing elementary row operations on the identity matrix. When one of these elementary matrices multiplies the matrix A it has the same effect as performing the corresponding elementary row operation on the matrix A.

■

Let the product of successive elementary row transformations be denoted by

$$E_kE_{k-1}\cdots E_3E_2E_1 = P.$$

Similarly, one can define the product of successive elementary column transformations by

$$E_1E_2E_3\cdots E_m = Q.$$

The **equivalence of two matrices A and B** is defined as follows. Let P and Q denote, respectively, the product of successive elementary row and column transformations as defined above. If $B = PA$, then B is said to be **row equivalent to** A. If $B = AQ$, then

B is said to be **column equivalent to** A. If $B = PAQ$, then B is said to be **equivalent to to the matrix** A.

All elementary matrices have inverses and are therefore nonsingular matrices. If A is nonsingular, then A^{-1} exists. For A nonsingular, one can perform a sequence of elementary row transformations on the matrix A and reduce A to an identity matrix. These operations are denoted by

$$E_k E_{k-1} \cdots E_3 E_2 E_1 A = I \quad \text{or} \quad PA = I \tag{10.24}$$

Right-multiplication of equation (10.24) by A^{-1} gives

$$P = E_k E_{k-1} \cdots E_3 E_2 E_1 = A^{-1}. \tag{10.25}$$

This equation suggests how one might build a "machine" for finding the inverse matrix of a nonsingular matrix A. Write down the matrix $A_{n\times n}$ and append to the right of it the identity matrix $I_{n\times n}$. By doing this the identity matrix can then be used like a "recording device," to record all elementary row operations that are performed on A. That is, whatever a row operation is performed upon A you must also perform **the same row operation on the appended identity matrix**. The matrix A with the identity matrix appended to its right-hand side is called **an augmented matrix.** After writing down

$$A \mid I, \tag{10.26}$$

observe that if an elementary row transformation is applied to the matrix A, then it is possible to "record" this transformation on the right-hand side of the equation (10.26). For example, if E_1 is an elementary row transformation applied to the augmented matrix, one obtains

$$E_1 A \mid E_1 I.$$

By performing a sequence of elementary row transformations upon the augmented matrix, given by equation (10.26), one can change the augmented matrix to the form

$$E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1 I, \tag{10.27}$$

where the sequence of elementary transformations has been "recorded" on the right-hand side of the augmented matrix. If one can choose the elementary matrices E_i, $i = 1, \ldots, k$, in such a way that the left-hand side of the augmented matrix (10.27) becomes the identity matrix, there would result the equation (10.24) on the left-hand side of the transformed augmented matrix. Consequently, the right-hand side of equation (10.27) becomes an equation, which gives the inverse matrix. The following example illustrates this "machine."

Example 10-23. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$.

Solution Append to the matrix A the identity matrix I to obtain the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right]. \tag{10.28}$$

Now try to select a sequence of elementary row operations with the goal of reducing the left-hand side of the augmented matrix (10.28) to the identity matrix. Each time an elementary row operation is applied to to the left-hand side of the augmented matrix (10.28) be sure to "record" the operation on the right-hand side. To illustrate, consider the following row operations applied to the augmented matrix (10.28).

- Replace row 2 by adding row 1 to row 2
- Multiply row 1 by (-2) and add the result to row 3. This produces the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 0 & -1 & -4 & -2 & 0 & 1 \end{array}\right].$$

Next perform the elementary row operation of replacing row 3 by adding row 2 to row 3 to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Finally, perform the following row operations:

- Multiply row 3 by (-5) and add the result to row 2.
- Multiply row 3 by (-3) and add the result to row 1. The above row operations produce the desired result of producing the identity matrix on the left-hand side of the augmented matrix. The final form for the augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & 6 & -4 & -5 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

and an examination of the right-hand side of the augmented matrix gives the inverse matrix $A^{-1} = \begin{bmatrix} 4 & -3 & -3 \\ 6 & -4 & -5 \\ -1 & 1 & 1 \end{bmatrix}$. One can readily verify that $AA^{-1} = I$.

■

Eigenvalues and Eigenvectors

Consider the operator box illustrated in the figure 10-5 where the input to the operator box is the $n \times 1$ **nonzero column vector** $x = \text{col}(x_1, x_2, \ldots, x_n)$ and the output from the operator box is the $n \times 1$ column vector $y = Ax$ were A is a $n \times n$ nonzero constant matrix. The operator box is said to transform the **nonzero column vector** x to the column vector y by matrix multiplication. For example, if $n = 3$ one would have the situation illustrated.

Figure 10-5.
Transformation of vector x to vector y.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

If there are special **nonzero column vectors** x such that **the output** y **is proportional to the input** x, then these special vectors are called **eigenvectors** and the proportionality constants are called **eigenvalues**. If the output y is proportional to the nonzero input x, then the equation $y = Ax = \lambda x$ must be satisfied, where λ is the scalar proportionality constant. If the equation $Ax = \lambda x$ has **nonzero solutions**, then one can write

$$\begin{aligned} Ax =& \lambda x = \lambda I x \\ (A - \lambda I)\, x =& [0]_{n\times 1} \end{aligned}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{10.29}$$

Cramer's[3] rule states that in order for this last equation to have **a nonzero solution** it is required that the determinant of the unknowns $x_1, x_2, \ldots, x_n$ be zero. This requires that

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{13} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0 \tag{10.30}$$

[3] Gabriel Cramer (1704-1752) A Swiss mathematician who studied determinants.

Solving this equation for the values of λ gives the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ associated with the matrix A. Substituting an eigenvalue λ into the equation (10.29) enables one to solve for the corresponding eigenvector.

Example 10-24.

Find the eigenvalues and eigenvectors associated with the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Solution The eigenvalues and eigenvectors of the matrix A are determined by solving the matrix equation $Ax = \lambda x$ or

$$(A - \lambda I)\, x = \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10.31}$$

In order for this system to have a nonzero solution for the column vector x, Cramer's rule requires that

$$\det(A - \lambda I) = 0$$

or

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

Solving this equation for λ gives

$$(\lambda + 1)(\lambda - 5) = 0 \quad \text{with roots} \quad \lambda = -1 \quad \text{and} \quad \lambda = 5$$

which are called the eigenvalues of the matrix A. The eigenvector corresponding to the eigenvalue $\lambda = -1$ is found be substituting $\lambda = -1$ into the equation (10.31) to obtain

$$\begin{pmatrix} 1-(-1) & 4 \\ 2 & 3-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the equation $2x_1 + 4x_2 = 0$ or $x_1 = -2x_2$. This result specifies how the first component of the eigenvector is related to the second component of the eigenvector and gives $x = \text{col}(x_1, x_2) = \text{col}(-2x_2, x_2)$ for the eigenvector. Note that x_2 must be **some nonzero constant in order that the eigenvector be nonzero.** For convenience select the value $x_2 = 1$ to obtain the eigenvector $x = \text{col}(-2, 1)$. Note that any nonzero constant times an eigenvector is also an eigenvector. To summarize what has just been done, one can say the solution of the matrix equation (10.31), using the value $\lambda = -1$, tells us that $\text{col}(-2, 1)$ is an eigenvector of the matrix A and any constant times the eigenvector is also an eigenvector. In a similar fashion, substitute the value $\lambda = 5$ into the equation (10.31) to obtain

$$\begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies $x_1 = x_2$. This gives the eigenvector $x = \text{col}(x_1, x_2) = \text{col}(x_2, x_2)$ where the component x_2 **must be some nonzero constant.** Selecting the value $x_2 = 1$ gives the eigenvector $x = \text{col}(1, 1)$. This shows that corresponding to the eigenvalue $\lambda = 5$ there is the eigenvector $x = \text{col}(1, 1)$. Note also that any nonzero constant times $\text{col}(1, 1)$ is also and eigenvector.

■

Example 10-25.

Find the eigenvalues and eigenvectors associated with the matrix

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & -3 & 7 \end{pmatrix}$$

Solution The eigenvalues and eigenvectors of the matrix A are determined by solving the matrix equation

$$(A - \lambda I)\, x = \begin{pmatrix} 2-\lambda & 2 & 2 \\ 0 & -\lambda & 4 \\ 0 & -3 & 7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{10.32}$$

In order for this system to have a nonzero solution for the column vector x, Cramer's rule requires that

$$\det(A - \lambda I) = 0$$

or

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & -\lambda & 4 \\ 0 & -3 & 7-\lambda \end{vmatrix} = \lambda^3 + 9\lambda^2 - 26\lambda + 24 = 0$$

Solving this equation for λ one finds the factored form

$$(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0 \quad \text{with roots} \quad \lambda = 2,\ \lambda = 3,\ \text{and } \lambda = 4$$

which are called the eigenvalues of the matrix A. The eigenvector corresponding to the eigenvalue $\lambda = 2$ is found be substituting the value $\lambda = 2$ into the equation (10.32) to obtain

$$\begin{pmatrix} 2-2 & 2 & 2 \\ 0 & -2 & 4 \\ 0 & -3 & 7-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & -2 & 4 \\ 0 & -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives the equations $2x_2 + 2x_3 = 0$, $-2x_2 + 4x_3 = 0$ and $-3x_2 + 5x_3 = 0$. These equations imply $x_2 = x_3 = 0$ giving the eigenvector $\text{col}(x_1, 0, 0)$. Selecting the value of $x_1 = 1$ for convenience, the eigenvector corresponding to the eigenvalue $\lambda = 2$ is given by col$(1, 0, 0)$. Note that any nonzero constant times this vector is also an eigenvector. Substituting the eigenvalue $\lambda = 3$ into the equation (10.32) gives the equations $-x_1 + 2x_2 + 2x_3 = 0$, $-3x_2 + 4x_3 = 0$ and $-3x_2 + 4x_3 = 0$. These equations imply that $x_2 = \frac{2}{7}x_1$ and $x_3 = \frac{3}{14}x_1$. This gives the eigenvector col$(x_1, \frac{2}{7}x_1, \frac{3}{14}x_1)$. Selecting the value $x_1 = 14$ for convenience, one finds the eigenvector col$(14, 4, 3)$ corresponding to the eigenvalue $\lambda = 3$. Note that any nonzero constant times this eigenvector is also an eigenvector. Substituting the eigenvalue $\lambda = 4$ into the equation (10.32) gives the equations $-2x_1 + 2x_2 + 2x_3 = 0$, $-4x_2 + 4x_3 = 0$ and $-3x_2 + 3x_3 = 0$. These equations imply that $x_3 = \frac{1}{2}x_1$ and $x_2 = \frac{1}{2}x_1$ and so the eigenvector can be expressed col$(x_1, \frac{1}{2}x_1, \frac{1}{2}x_1)$. Selecting the value $x_1 = 2$ for convenience gives the eigenvector col$(2, 1, 1)$ corresponding to the eigenvalue $\lambda = 4$.

■

Properties of Eigenvalues and Eigenvectors

The following are some important properties concerning the eigenvalues and eigenvectors associated with an $n \times n$ square matrix A.

Property 1: If X is an eigenvector of A, then kX is also an eigenvector of A for any nonzero scalar k.

Assume that the vector X is an eigenvector of A, so that it must satisfy the equation $AX = \lambda X$. If this equation is multiplied by a nonzero constant k there results $kAX = k\lambda X$ which can be written $A(kX) = \lambda(kX)$ and given the interpretation kX is an eigenvector of A.

Property 2: An eigenvector of a square matrix cannot correspond to two different eigenvalues.

Let λ_1, λ_2 with $\lambda_1 \neq \lambda_2$ be two different eigenvalues of A. Assume X_1 is an eigenvector of A corresponding to both λ_1 and λ_2. Our assumption implies that the equations

$$AX_1 = \lambda_1 X_1 \qquad \text{and} \qquad AX_1 = \lambda_2 X_1$$

must be satisfied simultaneously. Subtracting these equations shows us that $(\lambda_1 - \lambda_2)X_1 = [0]$. But, if $\lambda_1 - \lambda_2 \neq 0$, then this equation would imply that $X_1 = [0]$, which contradicts the fact that X_1 must be a nonzero eigenvector. Hence, the original assumption must be false.

Property 3: If a matrix A has one of its eigenvalues as zero and $\lambda = 0$, then A is a singular matrix.

The eigenvalues of A are determined by the characteristic equation

$$C(\lambda) = \det(A - \lambda I) = |A - \lambda I| = (-1)^n \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0$$

If $\lambda = 0$ is a root of the characteristic equation, then $C(\lambda) = \det A = 0$ and consequently the matrix A is singular.

Property 4: Two matrices A and B are said **to be similar** if there exists a nonsingular matrix Q such that $B = Q^{-1}AQ$. If the matrices A and B are similar, then they have the same characteristic equation.

The above property is established if it can be shown that the characteristic equation of B equals the characteristic equation of A. For Q nonsingular and $B = Q^{-1}AQ$, one can write

$$\begin{aligned} B - \lambda I &= Q^{-1}AQ - \lambda I \\ &= Q^{-1}AQ - \lambda Q^{-1} I Q \\ &= Q^{-1}(A - \lambda I)Q. \end{aligned}$$

The determinant of this equation gives

$$C(\lambda) = |B - \lambda I| = \left|Q^{-1}(A - \lambda I)Q\right| = \left|Q^{-1}\right| |A - \lambda I| \, |Q| \, .$$

But $QQ^{-1} = I$ and $|Q| \left|Q^{-1}\right| = 1$ hence $C(\lambda) = |B - \lambda I| = |A - \lambda I|$ and the above property is established.

Additional Properties Involving Eigenvalues and Eigenvectors

The following are some additional properties and definitions relating to eigenvalues and eigenvectors of an $n \times n$ square matrix A. The properties are given without proof.

1. If the n eigenvalues $\lambda_1, \ \lambda_2, \ldots, \lambda_n$ of A are all distinct, then there exists n-linearly independent eigenvectors.
2. If an eigenvalue repeats itself, then the characteristic equation is said to have **a multiple root**. In such cases there may or may not exit n linearly independent eigenvectors. If the characteristic equation can be written

$$C(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k} = 0$$

where $\sum_{i=1}^{k} n_i = n$, then n_i is called the multiplicity of the eigenvalue λ_i.

3. If A is a symmetric matrix and λ_i is an eigenvalue of multiplicity r_i, then there are r_i linearly independent eigenvectors.
4. An $n \times n$ square matrix is **similar to a diagonal matrix** if it has n-independent eigenvectors.
5. The set of all eigenvalues of A is called the **spectrum of the matrix A.**
6. The largest (in absolute value) eigenvalue of the matrix A is called the **spectral radius of A.**
7. If A is a real symmetric matrix, then all eigenvalues are real.
8. If A is a real skew symmetric matrix, then all eigenvalues are imaginary.
9. If $a = \max |a_{ij}|$ and λ is an eigenvalue of A, then $|\lambda| \le na$.
10. An eigenvalue of A must lie within one of n circular disks whose centers are a_{ii}, $i = 1, 2, \dots, n$, and whose radii are

$$r_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|;$$

that is, the centers of these disks are determined by the elements along the main diagonal of A, and the radius of the disk with center at a_{ii} is obtained by deleting a_{ii} from the ith row and then summing the absolute value of the remaining elements in the ith row.

11. The eigenvalues of a real matrix A satisfy:

$$(a) \quad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \text{Trace}\,(A)$$

$$(b) \quad \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n = \det\,(A)$$

$$(c) \quad \sum_{i=1}^{n} \lambda_i^2 \le \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$$

Example 10-26. For the matrix $A = \begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix}$ find matrices Q and Q^{-1} such that $Q^{-1}AQ$ is a diagonal matrix.

Solution: Calculate the eigenvalues $\lambda_1,\ \lambda_2$ and eigenvectors $X_1,\ X_2$ of the given matrix A and show

$$\lambda_1 = 1, \quad X_1 = \text{col}[\sqrt{3}, 1] \qquad \text{and} \qquad \lambda_2 = 2, \quad X_2 = \text{col}[1, -\sqrt{3}].$$

Let $Q = [X_1, X_2] = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ denote the matrix containing the eigenvectors of A for its column vectors. By definition the eigenvalues and eigenvectors satisfy the equations

$$AX_1 = \lambda_1 X_1 \qquad \text{and} \qquad AX_2 = \lambda_2 X_2,$$

and these equations can be expressed using the above notation as

$$\begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} \lambda_1 x_{11} \\ \lambda_1 x_{21} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} \lambda_2 x_{12} \\ \lambda_2 x_{22} \end{bmatrix}.$$

These two sets of linear equations can be represented by the single matrix equation

$$\begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{10.33}$$

The matrix whose column vectors are n-linearly independent eigenvectors of A is called the **modal matrix** associated with A. Here Q is the modal matrix of A. Denote the diagonal matrix having the eigenvalues of A for the elements on the diagonal as $D = \text{diag}(\lambda_1, \lambda_2)$, then the equation (10.33) can be written as

$$AQ = QD \tag{10.34}$$

Left multiplication by Q^{-1} gives $Q^{-1}AQ = D$, where

$$Q = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \qquad Q^{-1} = \frac{1}{4} \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \qquad D = \text{diag}(1, 2)$$

This example illustrates that the modal matrix can be used to reduce a given matrix to a diagonal form.

Changes of variables of the form $D = Q^{-1}AQ$, for the proper choice of the matrix Q, is a transformation often used to produce diagonal matrices in a variety of applications.

■

Example 10-27. Find the eigenvalues and eigenvectors associated with the matrix

$$A = \begin{bmatrix} -1 & 4 & 6 & -6 \\ 1 & 4 & 0 & 2 \\ -4 & 0 & 7 & -8 \\ -1 & -2 & 0 & 0 \end{bmatrix}$$

Solution: The characteristic equation of A can be calculated by evaluating the determinant

$$C(\lambda) = |A - \lambda I| = \begin{vmatrix} -1-\lambda & 4 & 6 & -6 \\ 1 & 4-\lambda & 0 & 2 \\ -4 & 0 & 7-\lambda & -8 \\ -1 & -2 & 0 & -\lambda \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) = 0.$$

The eigenvalues are $\lambda = 1,\ 2,\ 3,\ 4$ and each eigenvector associated with A must be a nonzero solution vector which satisfies the equation

$$AX = \lambda X$$

or

$$\begin{bmatrix} -1-\lambda & 4 & 6 & -6 \\ 1 & 4-\lambda & 0 & 2 \\ -4 & 0 & 7-\lambda & -8 \\ -1 & -2 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Substitute successively the values $\lambda = 1,\ 2,\ 3,\ 4$ into this equation and each time solve for X to obtain the eigenvectors

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

The **modal matrix** Q, is the matrix having these eigenvectors for its column vectors. The modal matrix Q can be used to produce a diagonal matrix containing the eigenvalues of A such that

$$Q^{-1}AQ = D = \text{diag}(1, 2, 3, 4).$$

The proof of this statement is left as an exercise. It can also be verified that

$$\det(A) = 24 \quad \text{and} \quad (\text{rank}\, A) = 4.$$

As a final note, it should be pointed out that when one or more of the eigenvalues of a matrix A are repeated roots, then a set of n linearly independent eigenvectors may or may not exist. The number N of linearly independent eigenvectors associated with an eigenvalue λ_i is given by the formula

$$N = n - (\text{rank}\,[A - \lambda_i I]),$$

where n is the rank of A.

■

Infinite Series of Square Matrices

In the following discussions it is to be understood that the matrix A is an $n \times n$ constant square matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ which are distinct. Consider the infinite series

$$S(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{10.35}$$

and the corresponding matrix infinite series

$$S(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_k A^k + \cdots = \sum_{k=0}^{\infty} c_k A^k. \tag{10.36}$$

where the $n \times n$ matrix A has replaced the value x in equation (10.35) and the identity matrix has replaced the coefficient of the c_0 constant term. Convergence of the matrix infinite series can be defined in a manner analogous to that of the scalar infinite series. Examine the sequence of partial sums

$$S_N = \sum_{k=0}^{N} c_k A^k$$

and if $\lim_{N\to\infty} S_N$ exists, then the matrix series is said to converge, otherwise it is said to diverge. It can be shown that if the series in equation (10.35) is convergent for $x = \lambda_i$ $(i = 1, 2, \ldots, n)$, where λ_i is an eigenvalue of A, then the matrix series in equation (10.36) is convergent.

Some specific examples of series associated with a $n \times n$ constant square matrix A are the following.

1. The Exponential Series Corresponding to the scalar exponential series

$$e^{x\,t} = 1 + x\,t + x^2 \frac{t^2}{2!} + \cdots + x^k \frac{t^k}{k!} + \cdots$$

there is **the exponential matrix** $e^{A\,t}$ defined by the series

$$e^{A\,t} = I + A\,t + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!} + \cdots \tag{10.37}$$

Here A is constant so that one can differentiate equation (10.37) with respect to t to obtain

$$\begin{aligned}\frac{d}{dt}\left(e^{At}\right) =& A + A^2 t + A^3\frac{t^2}{2!} + A^4\frac{t^3}{3!} + \cdots + A^k\frac{t^{k-1}}{(k-1)!} + \cdots \\ \frac{d}{dt}\left(e^{At}\right) =& A\left(I + At + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots\right) \\ \frac{d}{dt}\left(e^{At}\right) =& Ae^{At} = e^{At}A\end{aligned} \tag{10.38}$$

The exponential matrix e^X is an important matrix used for solving systems of linear differential equations. The exponential matrix has the following properties which are stated without proofs.

1. If the matrices X and Y commute, so that $XY = YX$, then one can write

$$e^X e^Y = e^Y e^X = e^{X+Y} \tag{10.39}$$

   However, if the matrices X and Y **do not commute** so that $XY \neq YX$, then the equation (10.39) is not true.
2. $\left(e^X\right)^{-1} = e^{-X}$
3. $e^{[0]} = I$
4. $e^X e^{-X} = I$
5. $e^{\alpha X} e^{\beta X} = e^{(\alpha+\beta)X}$
6. $e^{X^T} = \left(e^X\right)^T$

2. The Sine Series Corresponding to the scalar sine series

$$\sin(xt) = xt - \frac{x^3 t^3}{3!} + \frac{x^5 t^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1} t^{2n+1}}{(2n+1)!} + \cdots$$

there is the matrix sine series

$$\sin(At) = At - \frac{A^3 t^3}{3!} + \frac{A^5 t^5}{5!} + \cdots + (-1)^n \frac{A^{2n+1} t^{2n+1}}{(2n+1)!} + \cdots \tag{10.40}$$

3. The Cosine Series Corresponding to the scalar cosine series

$$\cos(xt) = 1 - \frac{x^2 t^2}{2!} + \frac{x^4 t^4}{4!} - \cdots + (-1)^n \frac{x^{2n} t^{2n}}{(2n)!} + \cdots$$

there is the matrix cosine series

$$\cos(At) = I - \frac{A^2 t^2}{2!} + \frac{A^4 t^4}{4!} + \cdots + (-1)^n \frac{A^{2n} t^{2n}}{(2n)!} + \cdots \tag{10.41}$$

Differentiate equation (10.40) with respect to t and show

$$\begin{aligned}
\frac{d}{dt}\sin(At) =& A - A^3\frac{t^2}{2!} + A^5\frac{t^4}{4!} + \cdots + (-1)^n A^{2n+1}\frac{t^{2n}}{(2n)!} + \cdots \\
\frac{d}{dt}\sin(At) =& A\left(I - \frac{A^2\,t^2}{2!} + \frac{A^4\,t^4}{4!} + \cdots + (-1)^n\frac{A^{2n}\,t^{2n}}{(2n)!} + \cdots\right) \\
\frac{d}{dt}\sin(At) =& A\cos(At) = \cos(At)\,A
\end{aligned} \tag{10.42}$$

Differentiate equation (10.41) with respect to t and show

$$\begin{aligned}
\frac{d}{dt}\cos(At) =& -A^2\,t + A^4\frac{t^3}{3!} + \cdots + (-1)^n A^{2n}\frac{t^{2n-1}}{(2n-1)!} + \cdots \\
\frac{d}{dt}\cos(At) =& -A\left(At - \frac{A^3\,t^3}{3!} + \frac{A^5\,t^5}{5!} + \cdots + (-1)^n\frac{A^{2n+1}\,t^{2n+1}}{(2n+1)!} + \cdots\right) \\
\frac{d}{dt}\cos(At) =& -A\sin(At) = -\sin(At)\,A
\end{aligned} \tag{10.43}$$

Example 10-28. Show that if A is a constant matrix, then $X(t) = e^{A(t-t_0)}$ is a matrix solution to the initial-value problem to solve

$$\frac{dX}{dt} = AX, \qquad X(t_0) = I$$

Solution Differentiate X and show $\dfrac{dX}{dt} = Ae^{A(t-t_0)} = AX$ is satisfied. At the initial time $t = t_0$, one finds $X(t_0) = e^{A(0)} = I$.

■

The Hamilton-Cayley Theorem

The Hamilton[4]-Cayley[5] theorem states that **every $n \times n$ constant square matrix A satisfies its own characteristic equation.** That is, if $C(\lambda) = 0$ is the characteristic equation association with a $n \times n$ square matrix A, then the equation $C(A) = [0]$ must be satisfied. This result is known as the Hamilton-Cayley theorem.

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[4] William Rowan Hamilton (1806–1865),Irish mathematician and physicist.

[5] Arthur Cayley (1821–1895),English mathematician.

Example 10-29. The following is an example illustrating the Hamilton-Cayley theorem. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix},$$

then the characteristic equation associated with the matrix A is

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1 = 0.$$

Replacing the scalar λ by the matrix A one obtains $C(A) = A^2 - 4A + I$, where I is the 2×2 identity matrix. The given matrix A when squared gives

$$A^2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix}.$$

Substituting I,A and A^2 into $C(A)$ gives

$$C(A) = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} - 4 \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [0]$$

and, hence, A satisfies its own characteristic equation.

■

In order to prove the Hamilton-Cayley theorem, assume the $n \times n$ constant square matrix A is given and it has associated with it the characteristic polynomial of the form

$$C(\lambda) = |A - \lambda I| = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-2}\lambda^2 + \alpha_{n-1}\lambda + \alpha_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are appropriate scalar constants. Replace the scalar λ by the matrix A and replace the constant term α_n by $\alpha_n I$, to obtain the Hamilton-Cayley matrix equation

$$C(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-2}A^2 + \alpha_{n-1}A + \alpha_n I$$

To prove the Hamilton-Cayley theorem it must be demonstrated that $C(A) = [0]$. Toward this purpose replace the matrix A in equation (10.23) by the matrix $A - \lambda I$ to obtain

$$(A - \lambda I)\text{Adj}(A - \lambda I) = |A - \lambda I| I,$$

where the various elements of the matrix $\text{Adj}(A - \lambda I)$ are formed from $A - \lambda I$ by deleting a certain row and column and then taking the determinant of the $(n-1) \times (n-1)$ system that remains. This implies λ^{n-1} is the highest power of λ that can be in any element of the matrix $\text{Adj}(A-\lambda I)$. Observe that the equation $\text{Adj}(A-\lambda I)$ can be written in the form

$$\text{Adj}(A-\lambda I) = B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_{n-2}\lambda^2 + B_{n-1}\lambda + B_n,$$

where $B_1,\ B_2,\ldots,B_n$ are $n \times n$ matrices not containing λ and in general depend upon the elements of A. Using the relation

$$(A-\lambda I)\text{Adj}(A-\lambda I) = |A-\lambda I|I = C(\lambda)I$$

Write the right-hand side as

$$C(\lambda)I = \lambda^n I + \alpha_1\lambda^{n-1}I + \cdots + \alpha_{n-2}\lambda^2 I + \alpha_{n-1}\lambda I + \alpha_n I,$$

and write the left-hand side as

$$\begin{aligned}(A-\lambda I)(B_1\lambda^{n-1} &+ B_2\lambda^{n-2} + \cdots + B_{n-2}\lambda^2 + B_{n-1}\lambda + B_n)\\ &= -B_1\lambda^n - B_2\lambda^{n-1} - \ldots - B_{n-2}\lambda^3 - B_{n-1}\lambda^2 - B_n\lambda\\ &\quad + AB_1\lambda^{n-1} + \cdots + AB_{n-3}\lambda^3 + AB_{n-2}\lambda^2 + AB_{n-1}\lambda + AB_n\end{aligned}$$

By comparing the left and right-hand sides of this equation one can equate the coefficients of like powers of λ and obtain the following equations.

$$\begin{aligned}
-B_1 &= I && : A^n\\
AB_1 - B_2 &= \alpha_1 I && : A^{n-1}\\
AB_2 - B_3 &= \alpha_2 I && : A^{n-2}\\
&\vdots && \vdots\\
AB_{n-3} - B_{n-2} &= \alpha_{n-3} I && : A^3\\
AB_{n-2} - B_{n-1} &= \alpha_{n-2} I && : A^2\\
AB_{n-1} - B_n &= \alpha_{n-1} I && : A\\
AB_n &= \alpha_n I && : I
\end{aligned}$$

Now multiply the first equation by A^n, the second equation by A^{n-1}, the third equation by A^{n-2},..., the second to last equation by A and the last equation by I. The multiplication factors are illustrated to the right-hand side of the equations listed above. After multiplication, the equations are summed. Note that on the right-hand side there results the matrix equation $C(A)$ and on the left-hand side the summation produces the zero matrix. This establishes the Hamilton-Cayley theorem.

Evaluation of Functions

Let $f(x)$ denote a scalar function of x, where all derivatives with respect to x are defined at $x=0$. Functions which satisfy this condition can then be represented as a power series expansion about the point $x=0$. This power series expansion has the form

$$f(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where c_k are the coefficients of the power series. Let A denote an $n \times n$ matrix with characteristic equation $C(\lambda)=0$ which has the roots (eigenvalues) λ_i, $(i=1,2,\ldots,n)$.

The infinite power series for $f(x)$ can be represented in the alternative form

$$f(x) = C(x)\sum_{k=0}^{\infty} c_k^* x^k + R(x), \tag{10.44}$$

where c_k^* are new coefficients to be determined, $C(x)$ is the characteristic polynomial associated with the matrix A, and $R(x)$ is a remainder polynomial of degree less than or equal to $(n-1)$ which can be expressed in the form

$$R(x) = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_{n-1}x + \beta_n$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are constants. By the Hamiliton–Cayley theorem $C(A) = [0]$, thus, the matrix function $f(A)$ becomes

$$f(A) = R(A), \tag{10.45}$$

which implies the matrix function $f(A)$ can be expressed as some linear combination of the matrices $\{I, A, A^2, \ldots, A^{n-1}\}$ and consequently must have the form

$$f(A) = R(A) = \beta_1 A^{n-1} + \beta_2 A^{n-2} + \cdots + \beta_{n-1}A + \beta_n I$$

where $\beta_1, \ldots, \beta_n$ are constants to be determined. This result is not unexpected since it has been previously shown how one can use the Hamilton–Cayley theorem to express all powers of A, greater than or equal to the dimension n of A, in terms of linear combinations of the integer powers of A less than or equal to $n-1$. Also, from equation (10.44), one can write the n special relations

$$f(\lambda_i) = R(\lambda_i), \quad i = 1, 2, \ldots, n, \tag{10.46}$$

which must exist between the functions $f(x)$ and $R(x)$. These equations are n independent relations one can use to solve for the unknown coefficients β_i in the polynomial representation for $R(x)$. If λ_i is a repeated root of $C(\lambda)=0$, then the equations (10.46) do not form a set of n-linearly independent equations. However, if the eigenvalue λ_i is a repeated root of $C(\lambda)=0$, then the derivative relation

$$\frac{dC}{d\lambda}\bigg|_{\lambda=\lambda_i}=0$$

must also be true. By differentiating equation (10.44) and evaluating the result at $\lambda=\lambda_i$, the equations

$$\frac{df(x)}{dx}\bigg|_{x=\lambda_i}=\frac{dR}{dx}\bigg|_{x=\lambda_i}$$

can be used to determine the constants in the representation of $R(A)$. That is, if m_i is the multiplicity of the characteristic root λ_i, then the equations

$$f(\lambda_i)=R(\lambda_i),\quad \frac{df}{dx}\bigg|_{x=\lambda_i}=\frac{dR}{dx}\bigg|_{x=\lambda_i},\ldots,\frac{d^{m_i-1}f}{dx^{m_i-1}}\bigg|_{x=\lambda_i}=\frac{d^{m_i-1}R}{dx^{m_i-1}}\bigg|_{x=\lambda_i}, \tag{10.47}$$

form a set of m_i linear equations. These equations can be used to determine the coefficients in the remainder polynomial $R(x)$ and consequently the matrix $R(A)$ representing $f(A)$ can be determined.

Example 10-30. Given the matrix

$$A=\begin{bmatrix}2 & -1\\ -3 & 4\end{bmatrix}$$

find the matrix function $f(A)=A^k$, where k is a positive integer.

Solution: The characteristic equation of A is

$$C(\lambda)=|A-\lambda I|=\begin{vmatrix}2-\lambda & -1\\ -3 & 4-\lambda\end{vmatrix}=\lambda^2-6\lambda+5=(\lambda-5)(\lambda-1)=0.$$

Examination of the above equation one can see that $\lambda_1=1$ and $\lambda_2=5$ are the characteristic roots or eigenvalues of the given matrix A. The Hamilton–Cayley theorem requires that $C(A)=[0]$, which implies

$$A^2=6A-5I$$

Successive multiplications by the matrix A gives

$$\begin{aligned}A^3&=6A^2-5A=6(6A-5I)-5A=31A-30I\\ A^4&=31A^2-30A=31(6A-5I)-30A=156A-155I,\end{aligned}$$

and continuing in this manner, one finds the general form

$$R(A) = f(A) = A^k = \beta_1 A + \beta_2 I$$

for some constants β_1 and β_2 (i.e., $f(A)$ is some linear combination of $\{I, A\}$). For this example,

$$R(x) = \beta_1 x + \beta_2 \quad \text{and} \quad f(x) = x^k$$

and consequently

$$\begin{aligned} R(\lambda_1) &= R(1) = \beta_1 + \beta_2 = (1)^k = 1 = f(1) \\ R(\lambda_2) &= R(5) = 5\beta_1 + \beta_2 = (5)^k = f(5). \end{aligned} \tag{7.11}$$

From these equations the unknown constants β_1 and β_2 can be determined. As an exercise show that

$$\beta_1 = \frac{1}{4}(5^k - 1) \qquad \text{and} \qquad \beta_2 = \frac{1}{4}(5 - 5^k).$$

The matrix relation

$$R(A) = f(A) = A^k = \left(\frac{5^k - 1}{4}\right) A + \left(\frac{5 - 5^k}{4}\right) I$$

is a general formula for expressing the powers of the matrix A as a linear combination of the matrices $\{A, I\}$. Checking this result with the previous calculations obtained by use of the Hamilton–Cayley theorem, one finds

$$\begin{aligned} &\text{for } k = 0, \qquad A^0 = I \\ &\text{for } k = 1, \qquad A^1 = A \\ &\text{for } k = 2, \qquad A^2 = 6A - 5I \\ &\text{for } k = 3, \qquad A^3 = 31A - 30I \\ &\text{for } k = 4, \qquad A^4 = 156A - 155I \end{aligned}$$

which agrees with the previous results.

In general, the Hamilton-Cayley theorem implies that if A is a $n \times n$ square matrix, then powers of A, say A^m, for integer values of m, can be represented in the form

$$A^m = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$$

where $c_0, \ldots, c_{n-1}$ are constants to be determined.

■

Example 10-31. Given the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

find the matrix function $f(A) = e^{At}$.

Solution: Here $f(x) = e^{xt}$, and from the previous example, the eigenvalues of A have been determined as $\lambda_1 = 1$ and $\lambda_2 = 5$. If the matrix equation is to be represented in the form

$$f(A) = R(A) = \gamma_1 A + \gamma_2 I$$

which is a linear combination of $\{I, A\}$ then one can use the equation (10.46) and write

$$\begin{aligned} f(\lambda_1) &= e^{\lambda_1 t} = e^t = \gamma_1(1) + \gamma_2 \\ f(\lambda_2) &= e^{\lambda_2 t} = e^{5t} = \gamma_1(5) + \gamma_2 \end{aligned} \tag{10.48}$$

From these equations it is possible to solve for γ_1 and γ_2 and show

$$\gamma_1 = \frac{e^{5t} - e^t}{4}, \qquad \text{and} \qquad \gamma_2 = \frac{5e^t - e^{5t}}{4}.$$

The matrix function for $f(A)$ can then be represented as

$$f(A) = e^{At} = \left(\frac{e^{5t} - e^t}{4}\right) A + \left(\frac{5e^t - e^{5t}}{4}\right) I$$

which has the equivalent matrix form

$$e^{At} = \frac{1}{4} \begin{bmatrix} (e^{5t} + 3e^t) & (e^t - e^{5t}) \\ (3e^t - 3e^{5t}) & (3e^{5t} + e^t) \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

■

Example 10-32. Find the matrix function

$$f(A) = \sin At \quad \text{for} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation of A is

$$C(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (\lambda^2 - 1)(1 - \lambda) = 0.$$

The eigenvalues of A are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = -1.$$

Express $f(A)$ as a linear combination of the matrices $\{I, A, A^2\}$ and write

$$f(A) = R(A) = \sin At = c_0 I + c_1 A + c_2 A^2,$$

where c_0, c_1 and c_2 are functions of t to be determined. Since there is a repeated root, use differentiation and write the equations

$$\begin{aligned} R(x) &= c_0 + c_1 x + c_2 x^2 \\ R'(x) &= c_1 + 2c_2 x \end{aligned}$$

to obtain the system of equations

$$\begin{aligned} R(\lambda_1) = f(\lambda_1) &= \sin t = c_0 + c_1 + c_2 \\ R'(\lambda_1) = f'(\lambda_1) &= \cos t = c_1 + 2c_2 \\ R(\lambda_3) = f(\lambda_3) &= -\sin t = c_0 - c_1 + c_2 \end{aligned} \tag{10.49}$$

These are three independent equations which can be used to solve for the coefficients c_0, c_1 and c_2. Solving these equations for c_0, c_1 and c_2 one finds

$$c_0 = \frac{1}{2}(\sin t - \cos t), \qquad c_1 = \sin t, \qquad c_2 = \frac{1}{2}(\cos t - \sin t)$$

and consequently the matrix function $\sin At$ has the representation

$$\sin At = \left(\frac{\sin t - \cos t}{2}\right) I + A \sin t + \frac{1}{2}\left(\cos t - \sin t\right) A^2.$$

An alternate form for this result is the matrix form

$$\sin At = \frac{1}{2}\begin{bmatrix} 0 & 2\sin t & \cos t - \sin t \\ \cos t + \sin t & \cos t - \sin t & \cos t + \sin t \\ 2\cos t & 2\cos t & \cos t + \sin t \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

■

Four-terminal Networks

Consider the electrical networks illustrated in figure 10-6. No matter how complicated the circuit inside the boxes, there are only two input and two output terminals. Such devices are called four terminal networks and are represented by a box like those illustrated in figure 10-6, where the quantities Z, Z_1, and Z_2 are called impedances. Impedances Z are used in alternating current (a.c.) circuits and are analogous to the resistance R use in direct current (d.c.) circuits.

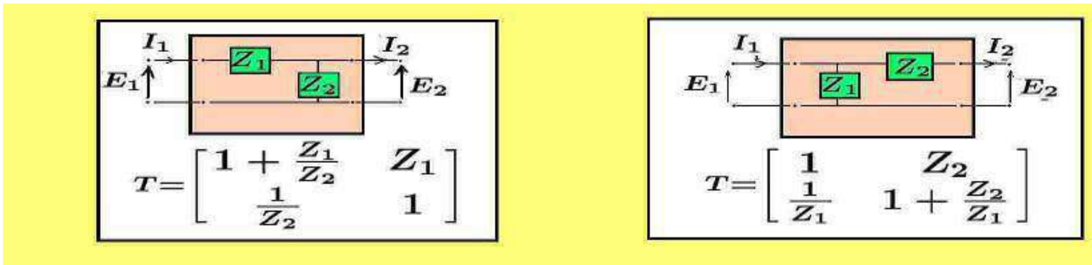


Figure 10-6. Four terminal networks

In figure 10-6 the quantities $I_1,\ E_1$ and $I_2,\ E_2$ are the input and output current and voltages. The column vectors $S_1 = \text{col}[E_1, I_1]$ and $S_2 = \text{col}[E_2, I_2]$ are called the input state vector and output state vector of the network. The networks are assumed to be linear so that the general relation between the input and output states can be expressed as the matrix equation

$$S_1 = TS_2, \quad \text{where} \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is called the transmission matrix . The element $T_{11}, T_{12}, T_{21}, T_{22}$ are in general complex numbers which satisfy the property $\det T = T_{11}T_{22} - T_{21}T_{12} = 1$. Solving for S_2 in terms of S_1 gives

$$S_2 = T^{-1}S_1 = PS_1$$

where the matrix P is called the transfer matrix of the network and is given by

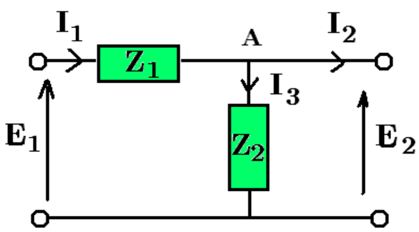
$$P = \begin{bmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{bmatrix}$$

If the input output current and voltages are linearly related, then it is easy to solve for the currents in terms of the voltages or the voltages in terms of the currents to obtain

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{T_{22}}{T_{12}} & -\frac{1}{T_{12}} \\ \frac{1}{T_{12}} & -\frac{T_{11}}{T_{12}} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \qquad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{T_{11}}{T_{21}} & -\frac{1}{T_{21}} \\ \frac{1}{T_{21}} & -\frac{T_{22}}{T_{21}} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Applying Kirchoff's laws to the short-circuit condition $E_2 = 0$ and the open-circuit condition $I_2 = 0$ allows for the determination of the transmission matrices given in the figure 10-6.

Example 10-33.



For the four terminal network illustrated one must have

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix} \implies \begin{aligned} E_1 =& T_{11}E_2 + T_{12}I_2 \\ I_1 =& T_{21}E_2 + T_{22}I_2 \end{aligned}$$

At the junction labeled A one must have $I_1 = I_2 + I_3$

(i) Examine the open-circuit condition $I_2 = 0$ and show $E_1 = T_{11}E_2$, $I_1 = T_{21}E_2$ and $I_1 = I_3$. This gives the relations

$$T_{11} = \frac{E_1}{E_2} = \frac{I_1Z_1 + I_1Z_2}{I_1Z_2} = 1 + \frac{Z_1}{Z_2} \qquad \text{and} \qquad I_1 = T_{21}(I_1Z_2) \quad \text{or} \quad T_{21} = \frac{1}{Z_2}$$

(ii) Examine the short-circuit condition $E_2 = 0$ and show $I_3 = 0$ so that $I_1 = I_2$, then under these conditions

$$E_1 = T_{12}I_2 \quad \text{or} \quad T_{12} = \frac{E_1}{I_2} = \frac{I_1Z_1}{I_1} = Z_1 \quad \text{and} \quad I_1 = T_{22}I_1 \qquad \implies T_{22} = 1$$

Also note the determinant of the transmission matrix is unity.

■

Calculus of Finite Differences

There are many concepts in science and engineering that can be approached from either a **discrete or a continuous** viewpoint. For example, consider how you might record the temperature outside at some specific place as a function of time. One technique would be to purchase a chart recorder capable of measuring and plotting the temperature as a function of time. This would give a continuous record of the temperature over some interval of time. Another way to record the temperature would be to measure the temperature, at the specified place, at discrete time intervals. The contrast between these two methods is that one method measures temperature continuously while the other method measures the temperature in a discrete fashion.

In any laboratory experiment, one must make a decision as to how data from the experiment is to be collected. Whether discrete measurements or continuous measurements are recorded depends upon many factors as well as the type of experiment being considered. The techniques used to analyze the data collected depends upon whether the data is continuous or discrete.

The investment of money at compound interest is an example of a physical problem which requires analysis of discrete values. Say, $1,000.00 is to be invested at R percent interest compounded quarterly. How does one determine the discrete values representing the amount of money available at the end of each compound period? To solve this problem, let P_0 denote the amount of money initially invested, R the percent interest yearly with $\frac{1}{4}\frac{R}{100} = i$ the quarterly interest and let P_n denote the principal due at the end of the nth compound period. The equations for the determination of P_n can be found by examining the discrete values produced. For P_0 the initial amount invested, one finds

$$
\begin{aligned}
P_1 &= P_0 + P_0 i = P_0(1+i)\\
P_2 &= P_1 + P_1 i = P_1(1+i) = P_0(1+i)^2\\
P_3 &= P_2 + P_2 i = P_2(1+i) = P_0(1+i)^3\\
&\;\;\vdots\\
P_n &= P_{n-1} + P_{n-1} i = P_{n-1}(1+i) = P_0(1+i)^n
\end{aligned}
$$

For $i = \frac{1}{4}\frac{R}{100}$ and $P_0 = 1,000.00$, figure 10-7 illustrates a graph of P_n vs time, for a 30 year period, where one year represents four payment periods. In this figure values of R for 4%, 5.5%, 7%, 8.5% and 10% were used in the above calculations.

Let us investigate some techniques that can be used in the analysis of discrete phenomena like the compound interest problem just considered.

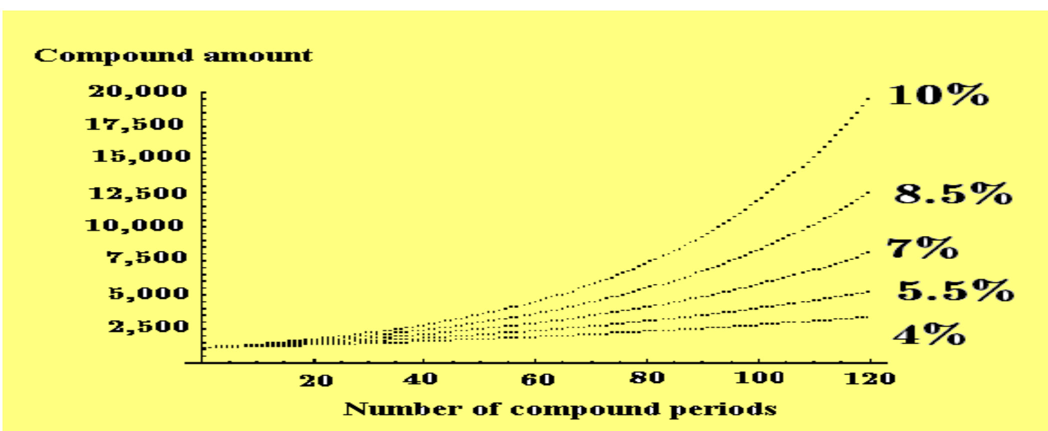


Figure 10-7.

Return from $1,000 investment compounded quarterly over 30 year period.

The study of calculus has demonstrated that derivatives are the mathematical quantities that represent **continuous change**. If derivatives (continuous change) are replaced by **differences (discrete change)**, then linear ordinary differential equations become **linear difference equations**. Let us begin our study of discrete phenomena by investigating difference equations and determining ways to construct solutions to such equations.

In the following discussions, note that the various techniques developed for analyzing discrete systems are very similar to many of the methods used for studying continuous systems.

Differences and Difference Equations

Consider the function $y = f(x)$ illustrated in the figure 10-8 which is evaluated at the equally spaced $x-$values of $x_0, x_1, x_2, \ldots, x_i, \ldots, x_n, x_{n+1}, \ldots$ where $x_{i+1} = x_i + h$ for $i = 0, 1, 2, \ldots, n$ where h is the distance between two consecutive points.

Let $y_n = f(x_n)$ and consider **the approximation of the derivative** $\dfrac{dy}{dx}$ **at the discrete value** x_n**.** Use the definition of a derivative and write the approximation as

$$\frac{dy}{dx}\Big|_{x=x_n} \approx \frac{y_{n+1} - y_n}{h}.$$

This is called a **forward difference approximation.** By letting $h = 1$ in the above equation one can define the first forward difference of y_n as

$$\Delta y_n = y_{n+1} - y_n. \tag{10.50}$$

There is no loss in generality in letting $h = 1$, as one can always rescale the x-axis by defining the new variable X defined by the transformation equation $x = x_0 + Xh$, then when $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh, \ldots$ the scaled variable X takes on the values $X = 0,\ 1,\ 2, \ldots, n, \ldots$.

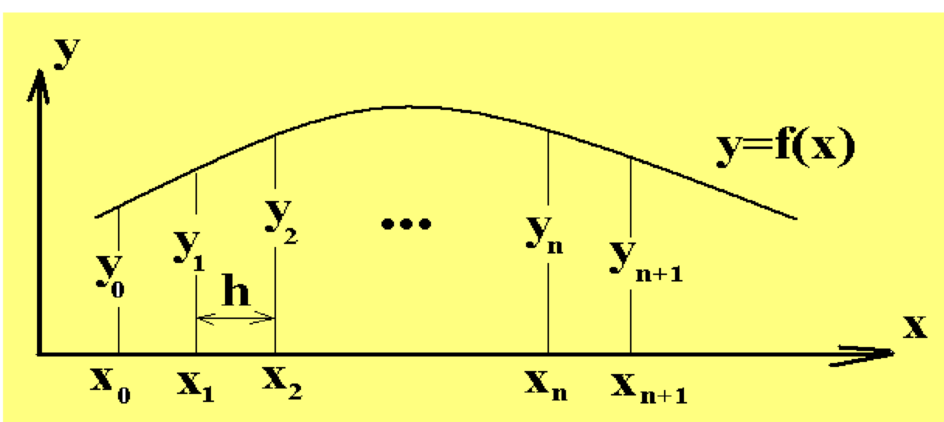


Figure 10-8. Discrete values of $y = f(x)$.

Define the **second forward difference as a difference of the first forward difference.** A second difference is denoted by the notation $\Delta^2 y_n$ and

$$\begin{aligned} &\Delta^2 y_n = \Delta(\Delta y_n) = \Delta y_{n+1} - \Delta y_n = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) \\ \text{or} \quad &\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n. \end{aligned} \tag{10.51}$$

Higher ordered difference are defined in a similar manner. A n**th order forward difference is defined as the difference of the** $(n-1)$**st forward difference, for** $n = 2, 3, \ldots$.

Analogous to the **differential operator** $D = \frac{d}{dx}$, **there is a stepping operator** E defined as follows:

$$\begin{aligned} Ey_n &= y_{n+1} \\ E^2 y_n &= y_{n+2} \\ &\cdots \\ E^m y_n &= y_{n+m}. \end{aligned} \tag{10.52}$$

From the definition given by equation (10.50) one can write the first ordered difference

$$\Delta y_n = y_{n+1} - y_n = Ey_n - y_n = (E-1)y_n$$

which illustrates that the difference operator Δ can be expressed in terms of the stepping operator E and

$$\Delta = E - 1. \tag{10.53}$$

This operator identity, enables us to express the second-order difference of y_n as

$$\begin{aligned} \Delta^2 y_n &= (E-1)^2 y_n \\ &= (E^2 - 2E + 1)y_n \\ &= E^2 y_n - 2Ey_n + y_n \\ &= y_{n+2} - 2y_{n+1} + y_n. \end{aligned}$$

Higher order differences such as $\Delta^3 y_n = (E-1)^3 y_n$, $\Delta^4 y_n = (E-1)^4 y_n, \ldots$ and higher ordered differences are quickly calculated by applying **the binomial expansion** to the operators operating on y_n.

Difference equations are equations which involve differences. For example, the equation

$$L_2(y_n) = \Delta^2 y_n = 0$$

is an example of a second-order difference equation, and

$$L_1(y_n) = \Delta y_n - 3y_n = 0$$

is an example of a first-order difference equation. The symbols $L_1(),\ L_2()$ are operator symbols representing linear operators. Using the operator E, the above equations can be written as

$$L_2(y_n) = \Delta^2 y_n = (E-1)^2 y_n = y_{n+2} - 2y_{n+1} + y_n = 0 \quad \text{and}$$
$$L_1(y_n) = \Delta y_n - 3y_n = (E-1)y_n - 3y_n = y_{n+1} - 4y_n = 0,$$

respectively.

There are many instances where variable quantities are assigned values at uniformly spaced time intervals. Let us study these discrete variable quantities by using differences and difference equations. **An equation which relates values of a function y and one or more of its differences is called a difference equation.** In dealing with difference equations one assumes that the function y and its differences $\Delta y_n,\ \Delta^2 y_n, \ldots$, evaluated at x_n, are all defined for every number x in some set of values $\{x_0,\ x_0+h,\ x_0+2h, \ldots, x_0+nh, \ldots\}$. A difference equation is called linear and of order m if it can be written in the form

$$L(y_n) = a_0(n)y_{n+m} + a_1(n)y_{n+m-1} + \cdots + a_{m-1}(n)y_{n+1} + a_m(n)y_n = g(n), \tag{10.54}$$

where the coefficients $a_i(n),\ i = 0,\ 1,\ 2, \ldots, m$, and the right-hand side $g(n)$ are known functions of n. If $g(n) \neq 0$, the difference equation is said to be **nonhomogeneous** and if $g(n) = 0$, the difference equation is called **homogeneous.**

The difference equation (10.54) can be written in the operator form

$$L(y_n) = [a_0(n)E^m + a_1(n)E^{m-1} + \cdots + a_{m-1}(n)E + a_m(n)]y_n = g(n),$$

where E is the stepping operator.

A mth-order **linear initial value problem** associated with a mth-order linear difference equation consists of a linear difference equation of the form given in the equation (10.54) together with a set of m **initial values** of the type

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad \ldots, \quad y_{m-1} = \alpha_{m-1},$$

where $\alpha_0,\ \alpha_1, \ldots, \alpha_{m-1}$ are specified constants.

Example 10-34.

Show $\Delta a^k = (a-1)a^k$, for a constant and k an integer.

Solution: Let $y_k = a^k$, then by definition

$$\Delta y_k = y_{k+1} - y_k = a^{k+1} - a^k = (a-1)a^k.$$

■

Example 10-35.

The function

$$k^{\underline{N}} = k(k-1)(k-2)\cdots[k-(N-2)][k-(N-1)], \qquad k^{\underline{0}} \equiv 1$$

is called **a factorial falling function** which is a polynomial function. Here $k^{\underline{N}}$ is a product of N terms. Show $\Delta k^{\underline{N}} = N\, k^{\underline{N-1}}$ for N a positive integer and fixed.

Solution: Observe that the factorial polynomials are

$$k^{\underline{0}} = 1, \quad k^{\underline{1}} = k, \quad k^{\underline{2}} = k(k-1), \quad k^{\underline{3}} = k(k-1)(k-2), \quad \cdots$$

Use $y_k = k^{\underline{N}}$ and calculate the forward difference

$$\begin{aligned} \Delta y_k = & \; y_{k+1} - y_k = (k+1)^{\underline{N}} - k^{\underline{N}} \\ = & \; (k+1)\underbrace{(k)(k-1)\cdots[k+1-(N-1)]}_{k^{\underline{N-1}}} - \underbrace{k(k-1)(k-2)\cdots[k-(N-2)]}_{k^{\underline{N-1}}}[k-(N-1)] \end{aligned}$$

which simplifies to

$$\Delta y_k = \Delta k^{\underline{N}} = \{(k+1) - [k-(N-1)]\}\, k^{\underline{N-1}} = N k^{\underline{N-1}}.$$

The function $k^{\overline{N}} = k(k+1)(k+2)\cdots[k+(N-2)][k+(N-1)]$ is the **factorial rising function**. As an exercise show $\Delta \dfrac{1}{k^{\overline{N}}} = \dfrac{-N}{k^{\overline{N+1}}}$

■

Example 10-36.

Verify the forward difference relation

$$\Delta(U_k V_k) = U_k \Delta V_k + V_{k+1} \Delta U_k$$

Solution: Let $y_k = U_k V_k$, then write

$$\begin{aligned} \Delta y_k &= y_{k+1} - y_k \\ &= U_{k+1}V_{k+1} - U_k V_k + [U_k V_{k+1} - U_k V_{k+1}] \\ &= U_k[V_{k+1} - V_k] + V_{k+1}[U_{k+1} - U_k] \\ &= U_k \Delta V_k + V_{k+1} \Delta U_k. \end{aligned}$$

■

Special Differences

The table 10.1 contains a list of some well known forward differences which are useful in many applications. The verification of these differences is left as an exercise.

Table 10.1 Some common forward differences		
1.	$\Delta a^k = (a-1)a^k$	
2.	$\Delta k^{\underline{N}} = N\, k^{\underline{N-1}}$ $\quad N$ fixed	$k^{\underline{N}}$ is factorial falling
3.	$\Delta \sin(\alpha+\beta k) = 2\sin(\beta/2)\cos(\alpha+\beta/2+\beta k)$	α, β constants
4.	$\Delta \cos(\alpha+\beta k) = -2\sin(\beta/2)\sin(\alpha+\beta/2+\beta k)$	α, β constants
5.	$\Delta \binom{k}{N} = \binom{k}{N-1}$ $\quad N$ fixed	$\binom{k}{N}$ are binomial coefficients
6.	$\Delta(k!) = k(k!)$	
7.	$\Delta(U_k V_k) = U_k \Delta V_k + V_{k+1}\Delta U_k$	
8.	$\Delta\left(\frac{1}{k^{\overline{N}}}\right) = \frac{-N}{k^{\overline{N+1}}}$, $\quad N$ fixed	$k^{\overline{N}}$ is factorial rising
9.	$\Delta k^2 = 2k+1$	
10.	$\Delta \log k = \log(1+1/k)$	

Finite Integrals

Associated with **finite differences** are **finite integrals**. If $\Delta y_k = f_k$, then the function y_k, whose difference is f_k, is called **the finite integral of** f_k. The inverse of the difference operation Δ is denoted Δ^{-1} and one can write $y_k = \Delta^{-1} f_k$, if $\Delta y_k = f_k$. For example, consider the difference of the factorial falling function $k^{\underline{N}}$. If $\Delta k^{\underline{N}} = N k^{\underline{N-1}}$, then $\Delta^{-1} N k^{\underline{N-1}} = k^{\underline{N}}$. Associated with the difference table 10.1 is the finite integral table 10.2. The derivation of the entries is left as an exercise.

Table 10.2 Some selected finite integrals		
1.	$\Delta^{-1}a^k = \dfrac{a^k}{a-1} \quad a \neq 1$	
2.	$\Delta^{-1}k^{\underline{n}} = \dfrac{k^{\underline{n+1}}}{n+1}$	$k^{\underline{n}}$ is factorial falling
3.	$\Delta^{-1}\sin(\alpha+\beta k) = \dfrac{-1}{2\sin(\beta/2)}\cos(\alpha-\beta/2+\beta k)$	α, β constants
4.	$\Delta^{-1}\cos(\alpha+\beta k) = \dfrac{1}{2\sin(\beta/2)}\sin(\alpha-\beta/2+\beta k)$	α, β constants
5.	$\Delta^{-1}\binom{k}{n} = \binom{k}{n+1}$ n fixed	$\binom{k}{n}$ are binomial coefficients
6.	$\Delta^{-1}(a+bk)^{\underline{n}} = \dfrac{(a+bk)^{\underline{n+1}}}{b(n+1)}$	a, b constants.

Summation of Series

Let $\Delta y_k = y_{k+1} - y_k = f_k$, then one can substitute $k = 0, 1, 2, \ldots$ to obtain

$$\begin{aligned} y_1 - y_0 =& f_0 \\ y_2 - y_1 =& f_1 \\ y_3 - y_2 =& f_2 \\ &\vdots \\ y_n - y_{n-1} =& f_{n-1} \\ y_{n+1} - y_n =& f_n \end{aligned} \tag{10.55}$$

Adding these equations one obtains

$$\sum_{i=0}^{n} f_i = y_{n+1} - y_0 = \Delta^{-1} f_i \big]_{i=0}^{n+1} = y_i \big]_{i=0}^{n+1} \qquad \text{where} \quad \Delta y_k = f_k.$$

One can verify that by adding the equations (10.55) from some point $i = m$ to n, one obtains the more general result

$$\sum_{i=m}^{n} f_i = y_{n+1} - y_m = \Delta^{-1} f_i \big]_{i=m}^{n+1} = y_i \big]_{i=m}^{n+1}. \tag{10.56}$$

Example 10-37.

Evaluate the sum

$$S = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$$

Solution: Let $f_k = k(k+1) = k^2 + k$ and show one can write f_k as the factorial falling function $f_k = (k+1)^{\underline{2}}$. Therefore,

$$S = \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} (i+1)^{\underline{2}} = \Delta^{-1} f_i \Big]_{i=1}^{n+1} = \frac{(i+1)^{\underline{3}}}{3} \Bigg]_{i=1}^{n+1} = \frac{(n+2)^{\underline{3}}}{3} - \frac{2^{\underline{3}}}{3}$$

which simplifies to $S = \dfrac{(n+2)(n+1)n}{3} - \dfrac{2 \cdot 1 \cdot 0}{3} = \dfrac{1}{3} n(n+1)(n+2)$.

■

Difference Equations with Constant Coefficients

Difference equations arise in a variety of situations. The following are some examples of where difference equations arise in applications. In assuming a power series solution to differential equations, the coefficients must satisfy certain recurrence formula which are nothing more than difference equations. In the study of stability of numerical methods there occurs difference equations which must be analyzed. In the computer simulation of various types of real-world processes, difference equations frequently occur. Difference equations also are studied in the areas of probability, statistics, economics, physics, and biology. We begin our investigation of difference equations by studying those with **constant coefficients** as these are the easiest to solve.

Example 10-38.

Given the difference equation

$$y_{n+1} - y_n - 2y_{n-1} = 0$$

with the initial conditions $y_0 = 1, \; y_1 = 0$. Find values for y_2 through y_{10}.

Solution: In the given difference equation, replace n by $n+1$ in all terms, to obtain

$$y_{n+2} = y_{n+1} + 2y_n,$$

then one can verify

$$\begin{aligned} n=0, &\qquad y_2 = y_1 + 2y_0 = 2 \\ n=1, &\qquad y_3 = y_2 + 2y_1 = 2 \\ n=2, &\qquad y_4 = y_3 + 2y_2 = 6 \\ n=3, &\qquad y_5 = y_4 + 2y_3 = 10 \\ n=4, &\qquad y_6 = y_5 + 2y_4 = 22 \\ n=5, &\qquad y_7 = y_6 + 2y_5 = 42 \\ n=6, &\qquad y_8 = y_7 + 2y_6 = 86 \\ n=7, &\qquad y_9 = 7_8 + 2y_7 = 170 \\ n=8, &\qquad y_{10} = y_9 + 2y_8 = 342. \end{aligned}$$

■

The study of difference equations with constant coefficients closely parallels the development of ordinary differential equations. Our goal is to determine functions $y_n = y(n)$, defined over a set of values of n, which reduce the given difference equation to an identity. Such functions are called solutions of the difference equation. For example, the function $y_n = 3^n$ is a solution of the difference equation $y_{n+1} - 3y_n = 0$ because $3^{n+1} - 3\cdot 3^n = 0$ for all $n = 0,\ 1,\ 2, \ldots$. Recall that for linear differential equations with constant coefficients one can assume a solution of the form $y(x) = \exp(\omega x)$. This assumption leads to producing the characteristic equation and consequently the characteristic roots associated with the differential equation. In the special case $x = n$, there results $y(n) = y_n = \exp(\omega n) = \lambda^n$, where $\lambda = \exp(\omega)$ is a constant. This suggests in our study of difference equations with constant coefficients that one should assume a solution of the form $y_n = \lambda^n$, where λ is a constant. Analogous to ordinary linear differential equations with constant coefficients, a linear, nth-order, homogeneous difference equation with constant coefficients has associated with it a characteristic equation with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. The characteristic equation is found by assuming a solution $y_n = \lambda^n$, where λ is a constant. The various cases that can arise are illustrated by the following examples.

Example 10-39. **(Characteristic equation with real roots)**

Solve the second-order difference equation

$$y_{k+2} - 3y_{k+1} + 2y_k = 0.$$

Solution: Assume solutions of the form $y_k = \lambda^k$, where λ is a constant. This assumed solution produces $y_{k+1} = \lambda^{k+1}$ and $y_{k+2} = \lambda^{k+2}$. Substituting these values into the difference equation produces the equation

$$(\lambda^2 - 3\lambda + 2)\lambda^k = 0,$$

which tells us the required values for λ in order that $y_k = \lambda^k$ satisfy the difference equation. For a nontrivial solution it is required that $\lambda \neq 0$. This produces the characteristic equation

$$\lambda^2 - 3\lambda + 2 = 0.$$

A short cut for writing down the characteristic equation is to observe the form of the given difference equation, when written in an operator form involving the stepping operator E. One can quickly obtain the characteristic equation from this operator form. For example, the given difference equation can be expressed in the form $(E^2 - 3E + 2)y_k = 0$, where the operator $E^2 - 3E + 2$ shows us the general form of the characteristic equation when E is replaced by λ. The characteristic equation has the roots $\lambda_1 = 2$ and $\lambda_2 = 1$, and hence two linearly independent solutions are

$$y_1(k) = 2^k \quad \text{and} \quad y_2(k) = 1^k = 1$$

which is called a **fundamental set of solutions.** The general solution can be written as a linear combination of this fundamental set and so one can write

$$y(k) = y_k = c_1(2)^k + c_2,$$

where c_1 and c_2 are arbitrary constants. Given a set of initial conditions of the form $y_0 = A$ and $y_1 = B$, where A and B are given constants, one can form the equations

$$y_0 = A = c_1 + c_2$$

$$y_1 = B = 2c_1 + c_2,$$

from which the constants c_1 and c_2 can be determined. Solving for these constants produces the solution of the initial value problem which satisfies the given initial conditions. The desired solution is unique and found to be

$$y_k = (B - A)(2)^k + (2A - B).$$

■

Example 10-40. **(Characteristic equation with repeated roots)**

Find the general solution to the difference equation

$$y_{n+2} - 4y_{n+1} + 4y_n = 0.$$

Solution: Write the difference equation in operator form $(E^2 - 4E + 4)y_n = 0$ and assume a solution of the form $y_n = \lambda^n$. By substituting the assumed solution into the difference equation one obtains the characteristic equation $\lambda^2 - 4\lambda + 4 = 0$ which has the repeated roots $\lambda = 2,\ 2$. As with ordinary differential equations, one solution is $y_1(n) = 2^n$ and the second independent solution can be obtained by a multiplication of the first solution by the independent variable n. This is analogous to the case of repeated roots for ordinary differential equations with constant coefficients. A second independent solution is therefore $y_2(n) = n2^n$, and the general solution can be expressed as

$$y(n) = y_n = c_0 2^n + c_1 n 2^n,$$

where c_0 and c_1 are arbitrary constants. To verify that $n2^n$ is a second independent solution, the method of variation of parameters is used. Assume that a second solution has the form $y_n = U_n 2^n$, where U_n is an unknown function of n to be determined. Substituting this assumed solution into the difference equation produces the equation

$$2^{n+2}(U_{n+2} - 2U_{n+1} + U_n) = 0$$

which can be written as

$$(E^2 - 2E + 1)U_n = (E-1)^2 U_n = \Delta^2 U_n = 0. \tag{10.57}$$

It is left as an exercise to verify that the general solution of $\Delta^k U_n = 0$ is given by

$$U_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + \cdots + c_{k-1} n^{k-1},$$

and therefore, equation (10.57) has the solution $U_n = c_0 + c_1 n$, which when substituted into the assumed solution gives the result $y(n) = c_0(2)^n + c_1 n(2)^n$ as the general solution.

■

In general, if the characteristic equation associated with a linear difference equation with constant coefficients has a characteristic root $\lambda = a$ of multiplicity k, then

$$y_1(n) = a^n,\ y_2(n) = na^n,\ y_3(n) = n^2 a^n, \ldots, y_k(n) = n^{k-1} a^n$$

are k linearly independent solutions of the difference equation. To show this, solve the difference equation

$$(E-a)^k y_n = 0, \tag{10.58}$$

as was done previously. Here the characteristic equation is $(\lambda - a)^k = 0$ with $\lambda = a$ as a root of multiplicity k. The method of **variation of parameters** starts by assuming a solution to equation (10.58) of the form $y_n = a^n U_n$. Observe that

$$\begin{aligned} Ey_n &= a^{n+1}U_{n+1} \quad \text{and} \quad (E-a)y_n = a^{n+1}\Delta U_n \\ E(E-a)y_n &= a^{n+2}\Delta U_{n+1} \quad \text{and} \quad (E-a)^2 y_n = a^{n+2}\Delta^2 U_n \\ E(E-a)^2 y_n &= a^{n+3}\Delta^2 U_{n+1} \quad \text{and} \quad (E-a)^3 y_n = a^{n+3}\Delta^3 U_n. \end{aligned}$$

Continuing in this manner, show that U_n must satisfy

$$(E-a)^k y_n = a^{n+k}\Delta^k U_n = 0.$$

For a nonzero solution it is required that a^{n+k} be different from zero, and so U_n must be chosen such that $\Delta^k U_n = 0$. The general solution of this equation is

$$U_n = c_0 + c_1 n + c_2 n^2 + \cdots + c_{k-1} n^{k-1},$$

and hence the general solution of equation (10.58) is $y_n = a^n U_n$.

One can compare the case of repeated roots for difference and differential equations and readily discern the analogies that exist.

Example 10-41. (Characteristic equation with complex or imaginary roots)

Solve the difference equation

$$y_{n+2} - 10y_{n+1} + 74y_n = (E^2 - 10E + 74)y_n = 0.$$

Solution: Assume a solution of the form $y_n = \lambda^n$ and obtain the characteristic equation $\lambda^2 - 10\lambda + 74 = 0$ which has the complex roots $\lambda_1 = 5 + 7i$ and $\lambda_2 = 6 - 7i$. Two independent solutions are therefore

$$y_1(n) = (5+7i)^n \quad \text{and} \quad y_2(n) = (6-7i)^n.$$

To obtain solutions in the form of real quantities, represent the roots $\lambda_1,\ \lambda_2$ in the polar form $re^{i\theta}$, as illustrated in figure 10-9.

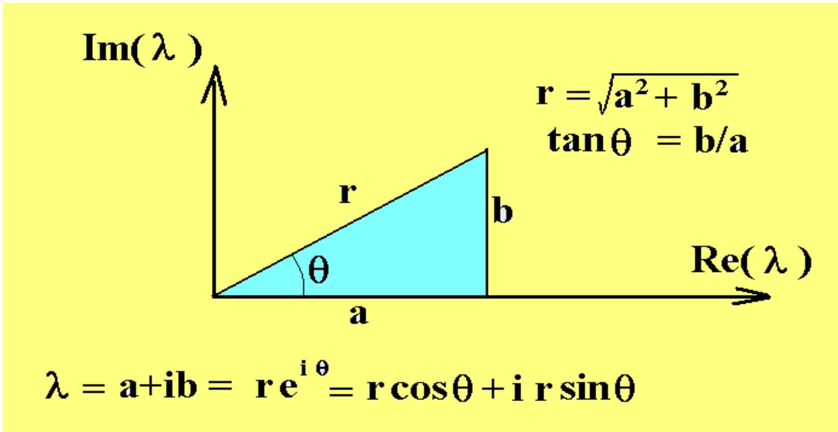


Figure 10-9. Polar form of complex numbers.

In this figure r is called the modulus or length of the complex number λ and θ is called an argument of the complex number λ. Values of 2π can be added to obtain other arguments of λ. A value of θ satisfying $-\pi < \theta \leq \pi$ is called the principal value of the argument of λ. The complex root λ_1 has a modulus and argument of

$$r = \sqrt{5^2 + 7^2} \qquad \text{and} \qquad \theta = \arctan(7/5). \tag{10.59}$$

The polar form of the characteristic roots produce solutions to the difference equations that can then be expressed in the form

$$y_1(n) = r^n e^{in\theta} \qquad \text{and} \qquad y_2(n) = r^n e^{-in\theta}.$$

The Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, is used to write these solutions in the form

$$y_1(n) = r^n(\cos n\theta + i\sin n\theta)$$
$$y_2(n) = r^n(\cos n\theta + i\sin n\theta).$$

The solutions $y_1(n)$ and $y_2(n)$ are independent solutions of the given difference equation and hence any linear combination of these solutions is also a solution. Form the linear combinations

$$y_3(n) = \frac{1}{2}[y_1(n) + y_2(n)] \quad \text{and}$$
$$y_4(n) = \frac{1}{2i}[y_1(n) - y_2(n)]$$

to obtain the real solutions

$$y_3(n) = r^n \cos n\theta$$

$$y_4(n) = r^n \sin n\theta.$$

The general solution is any linear combination of these functions and can be expressed

$$y(n) = r^n[c_1 \cos n\theta + c_2 \sin n\theta],$$

where r and θ are defined by equation (10.59) and c_1 and c_2 are arbitrary constants. Therefore, when complex roots arise, these roots are expressed in polar form in order to obtain a real solution and imaginary solution to the given difference equation. If real solutions are desired, then one can take linear combinations of the real solution and imaginary solution in order to construct a general solution.

Nonhomogeneous Difference Equations

Nonhomogeneous difference equations can be solved in a manner analogous to the solution of nonhomogeneous differential equations and one may use the **method of undetermined coefficients** or the **method of variation of parameters** to obtain particular solutions.

Example 10-42. (Undetermined coefficients)

Solve the first order difference equation

$$L(y_n) = y_{n+1} + 2y_n = 3n.$$

Solution: First solve the homogeneous equation

$$L(y_n) = y_{n+1} + 2y_n = 0$$

Assume a solution $y_n = \lambda^n$ and obtain the characteristic equation $\lambda + 2 = 0$ with characteristic root $\lambda = -2$. The complementary solution is then $y_c(n) = c_1(-2)^n$, where c_1 is an arbitrary constant. Next find any particular solution $y_p(n)$ which produces the right-hand side. Analogous to what has been done with differential equations, examine the differences of the right-hand side of the given equation. Let $r(n) = 3n$, then the first difference is a constant since $\Delta r(n) = r(n+1) - r(n) = 3$. The basic terms occurring in the right-hand side and the difference of the right-hand side are listed as members of the set $S = \{1, n\}$. If any member of S occurs in the complementary solution, then the set S is modified by multiplying each term of the set S by n. If any member of the new set S also occurs in the complementary solution, then members of the set S are modified again. This is analogous to what one does in the study of ordinary differential equations. Here one can assume a particular solution of the given difference equation which is some linear combination of the functions in S. This requires that an assumed particular solution have the form

$$y_p(n) = An + B$$

where A and B are undetermined coefficients. Substitute this assumed particular solution into the difference equation and obtain

$$A(n+1) + B + 2An + 2B = 3n$$

which simplifies to

$$(A + 3B) + 3An = 3n.$$

Comparing like terms produces the equations $3A = 3$ and $A + 3B = 0$. Solving for A and B produces $A = 1$ and $B = -1/3$. Hence, the particular solution becomes

$$y_p(n) = n - \frac{1}{3}.$$

The general solution can be written as the sum of the complementary and particular solutions.

$$y_n = y_c(n) + y_p(n) = c_1(-2)^n + n - \frac{1}{3}.$$

■

Example 10-43. **(Variation of parameters)**

Determine a particular solution to the difference equation

$$y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n = f_n, \tag{10.60}$$

where $a_1(n),\ a_2(n),\ f_n$ are given functions of n.

Solution: Assume that two independent solutions to the linear homogeneous equation

$$L(y_n) = y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n = 0$$

are known. Denote these solutions by u_n and v_n so that by hypothesis $L(u_n) = 0$ and $L(v_n) = 0$. Assume a particular solution to the nonhomogeneous equation (10.60) of the form

$$y_n = \alpha_n u_n + \beta_n v_n, \tag{10.61}$$

where α_n, β_n are to be determined. There are two unknowns and consequently two conditions are needed to determine these quantities. As with ordinary differential equations, assume for our first condition the relation

$$\Delta\alpha_n u_{n+1} + \Delta\beta_n v_{n+1} = 0. \tag{10.62}$$

The second condition is obtained by substituting the assumed solution, given by equation (10.61), into the given difference equation. Starting with the assumed solution given by equation (10.61) show

$$y_{n+1} = y_n + \Delta y_n = \alpha_n u_n + \beta_n v_n + \Delta(\alpha_n u_n + \beta_n v_n)$$
$$y_{n+1} = \alpha_n u_n + \beta_n v_n + \alpha_n \Delta u_n + \beta_n \Delta v_n + [(\Delta\alpha_n)u_{n+1} + (\Delta\beta_n)v_{n+1}].$$

This equation simplifies since by assumption equation (10.62) must hold. One can then show that y_{n+1} reduces to

$$y_{n+1} = \alpha_n u_{n+1} + \beta_n v_{n+1}. \tag{10.63}$$

In equation (10.63) replace n by $n+1$ everywhere and establish the result

$$y_{n+2} = \alpha_{n+1} u_{n+2} + \beta_{n+1} v_{n+2}. \tag{10.64}$$

By substituting equations (10.61), (10.63) and (10.64) into the equation (10.60), a second condition for determining the unknown constants is found. This second condition is that α_n and β_n must satisfy the equation

$$\alpha_{n+1} u_{n+2} + \beta_{n+1} v_{n+2} + a_1(n)(\alpha_n u_{n+1} + \beta_n v_{n+1}) + a_2(n)(\alpha_n u_n + \beta_n v_n) = f_n.$$

Rearrange terms in this equation, and show it can be written in the form

$$(\alpha_{n+1} - \alpha_n)u_{n+2} + (\beta_{n+1} - \beta_n)v_{n+2} + \alpha_n L(u_n) + \beta_n L(v_n) = f_n. \tag{10.65}$$

By hypothesis $L(u_n) = 0$ and $L(v_n) = 0$, thus simplifying the equation (10.65). The equations (10.62) and (10.65) are produce the two conditions

$$\Delta\alpha_n u_{n+1} + \Delta\beta_n v_{n+1} = 0$$
$$\Delta\alpha_n u_{n+2} + \Delta\beta_n v_{n+2} = f_n$$

for determining the constants α_n and β_n. This system of equations can be solve by Cramer's rule and written

$$\Delta\alpha_n = \alpha_{n+1} - \alpha_n = -\frac{f_n v_{n+1}}{C_{n+1}}, \qquad \Delta\beta_n = \beta_{n+1} - \beta_n = \frac{f_n u_{n+1}}{C_{n+1}}, \tag{10.66}$$

where $C_{n+1} = u_{n+1}v_{n+2} - u_{n+2}v_{n+1}$ is called the Casoratian (the analog of the Wronskian for continuous systems). It can be shown that C_{n+1} is never zero if u_n, v_n are linearly independent solutions of equation (10.60). The first order difference equations are a special case of problem 21 of the exercises at the end of this chapter, where it is demonstrated that the solutions can be written in the form

$$\begin{aligned} \alpha_n &= \alpha_0 - \sum_{i=0}^{n-1} \frac{f_i v_{i+1}}{C_{i+1}} \\ \beta_n &= \beta_0 + \sum_{i=0}^{n-1} \frac{f_i u_{i+1}}{C_{i+1}} \end{aligned} \tag{10.67}$$

and the general solution to equation (10.60) can be expressed as

$$y_n = \alpha_0 u_n + \beta_0 v_n - u_n \sum_{i=0}^{n-1} \frac{f_i v_{i+1}}{C_{i+1}} + v_n \sum_{i=0}^{n-1} \frac{f_i u_{i+1}}{C_{i+1}}. \tag{10.68}$$

■

Analogous to the nth-order linear differential equation with constant coefficients

$$L(D) = (D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)y = r(x) \tag{10.69}$$

with $D = \frac{d}{dx}$ a differential operator, there is the nth-order difference equation

$$L(E) = (E^n + a_1 E^{n-1} + \cdots + a_{n-1}E + a_n)y_k = r(k), \tag{10.70}$$

where E is the stepping operator satisfying $Ey_k = y_{k+1}$. Most theorems and techniques which can be applied to the ordinary differential equation (10.69) have analogous results applicable to the difference equation (10.70).

Exercises

In the following exercises if the size of the matrix is not specified, then assume that the given matrices are square matrices.

► **10-1.** Given the matrices:

$$A = \begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & 7 \\ 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 5 & -8 \\ -3 & 5 \end{bmatrix}$$

(a) Verify that $AA^{-1} = A^{-1}A = I$
(b) Verify that $BB^{-1} = B^{-1}B = I$
(c) Calculate AB
(d) Calculate $B^{-1}A^{-1}$
(e) Find $(AB)^{-1}$ and check your answer.

► **10-2.** Start with the definition $AA^{-1} = I$ and take the transpose of both sides of this equation. Note that $I^T = I$ and show that $(A^{-1})^T = (A^T)^{-1}$

► **10-3.** Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
Hint: $ABC = (AB)C$

► **10-4.** If A and B are nonsingular matrices which commute, then show that
(a) A^{-1} and B commute
(b) B^{-1} and A commute
(c) A^{-1} and B^{-1} commute Hint: If $AB = BA$, then $A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$

► **10-5.** If A is nonsingular and symmetric, show that A^{-1} is also symmetric.
Hint: If $AA^{-1} = I$, then $(AA^{-1})^T = (A^{-1})^T A^T = I$

► **10-6.** If A is nonsingular and $AB = AC$, show $B = C$

► **10-7.** Show that if $AB = A$ and $BA = B$, then A and B are idempotent.
Hint: Examine the products ABA and BAB

► **10-8.**
(a) Show $(A^2)^{-1} = (A^{-1})^2$
(b) Show for m a nonzero scalar that $(mA)^{-1} = \frac{1}{m}A^{-1}$

▶ **10-9.** Assume that A is a square matrix show that

(a) AA^T is symmetric (b) $A+A^T$ is symmetric (c) $A-A^T$ is skew-symmetric

(d) Show A can be written as the sum of a symmetric and skew symmetric matrix.

▶ **10-10.** If A and B are symmetric square matrices
(a) Show that AB is symmetric if $AB=BA$
(b) Show that $AB=BA$ if AB is symmetric.

▶ **10-11.** Show $AA^{-1}=A^{-1}A=I$ when

$$A=\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 0\\ 1 & 3 & -1\end{bmatrix} \qquad A^{-1}=\begin{bmatrix}-2 & 5 & -2\\ 1 & -2 & 1\\ 1 & -1 & 0\end{bmatrix}.$$

▶ **10-12.** For

$$A=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \qquad \text{and} \qquad B=\begin{bmatrix}a & b\\ c & d\end{bmatrix},$$

how should the constants a,b,c and d be chosen in order that A and B commute?

▶ **10-13.** For

$$A=\begin{bmatrix}a & 1-a\\ a & 1-a\end{bmatrix} \qquad \text{and} \qquad B=\begin{bmatrix}(\sqrt{2}-1) & 2\\ (\sqrt{2}-1) & -(\sqrt{2}-1)\end{bmatrix},$$

find A^2, A^3, B^2 and B^3 and identify the special matrices A and B.

▶ **10-14.** For

$$A=\begin{bmatrix}1 & -1 & -\frac{1}{2}\\ 0 & -1 & -1\\ 0 & 0 & 1\end{bmatrix},$$

find A^2, A^3, A^4, A^5, A^6 and A^7 and identify the matrix A.

▶ **10-15.** Assume that $A^2B=I$ and that $A^5=A$ (A is periodic with period 4). Solve for the square matrix B in terms of A.

▶ **10-16.** Let $AX=B$, where A is an $n\times n$ square matrix, and X and B are $n\times 1$ column vectors. Solve for the column vector X and state what conditions are required for the solution to exist.

▶ **10-17.** *Background material*

Definition: (Congruence) *Two integers I and J are said to be congruent modulo L, (written $I \equiv J(\bmod L)$) if $I - J = nL$ for some integer n*

This definition implies that two integers are congruent modulo L if and only if they have the same remainder when divided by L. Some examples are:

$$\begin{array}{lll} 13 \equiv 1(\bmod 12) & 7 \equiv 1(\bmod 3) & 31 \equiv 2(\bmod 29) \\ 26 \equiv 2(\bmod 12) & 34 \equiv 1(\bmod 3) & 77 \equiv 19(\bmod 29) \\ 54 \equiv 6(\bmod 12) & 305 \equiv 2(\bmod 3) & 46 \equiv 17(\bmod 29) \end{array}$$

CRYPTOGRAMS (A writing in cipher.) The message, "HOW ARE YOU?" could be written as a matrix of dimension 3×3 in the form

$$A = \begin{bmatrix} H & O & W \\ A & R & E \\ Y & O & U \end{bmatrix}.$$

Associate with each letter of the alphabet an integer as in the following scheme:

$$\begin{array}{llllll} A=1 & F=6 & K=11 & P=16 & U=21 & Z=26 \\ B=2 & G=7 & L=12 & Q=17 & V=22 & ?=27 \\ C=3 & H=8 & M=13 & R=18 & W=23 & \text{Blank}=28 \\ D=4 & I=9 & N=14 & S=19 & X=24 & !=29 \\ E=5 & J=10 & O=15 & T=20 & Y=25 & \end{array}$$

Here 29 symbols are used as it is desirable to do modulo arithmetic in the modulo 29 system (29 being a prime number) and the blank stands for a blank character. By replacing the letters in the above matrix A, by their number equivalents, there results

$$A = \begin{bmatrix} 8 & 15 & 23 \\ 1 & 18 & 5 \\ 25 & 15 & 21 \end{bmatrix}.$$

To disguise this message, the matrix A is multiplied by another matrix C, to form the matrix $B = AC$. In this multiplication, modulo 29 arithmetic is used as it is desired that only numbers between 1 and 29 are needed for our result. For example, using the matrix

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 4 \end{bmatrix} \quad \text{with} \quad C^{-1} = \begin{bmatrix} 2 & -4 & -1 \\ -1 & 4 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

there results

$$B = AC = \begin{bmatrix} 8 & 15 & 23 \\ 1 & 18 & 5 \\ 25 & 15 & 21 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 19 \\ 6 & 9 & 2 \\ 17 & 27 & 11 \end{bmatrix} = \begin{bmatrix} B & F & S \\ F & I & B \\ Q & ? & K \end{bmatrix}$$

Here modulo 29 arithmetic has been used. For example,

$$\begin{aligned} 8(1) + 15(0) + 23(1) = 31 &\equiv 2(\bmod 29) \\ 8(1) + 15(1) + 23(-2) = -23 &\equiv 6(\bmod 29) \\ 8(0) + 15(-1) + 23(4) = 77 &\equiv 19(\bmod 29). \end{aligned}$$

with similar results using inner products involving the second and third row vectors of A. Upon receiving the coded message, where you know that the matrix C was used to make up the code, then you can decipher the message by multiplying by $C^{-1}(\bmod 29)$, since $B = AC$ implies that $A = BC^{-1}$. For example,

$$A = BC^{-1} = \begin{bmatrix} 2 & 6 & 19 \\ 6 & 9 & 2 \\ 17 & 27 & 11 \end{bmatrix} \begin{bmatrix} 2 & -4 & -1 \\ -1 & 4 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 15 & 23 \\ 1 & 18 & 5 \\ 25 & 15 & 21 \end{bmatrix} = \begin{bmatrix} H & O & W \\ A & R & E \\ Y & O & U \end{bmatrix}.$$

Here are some coded messages which were coded modulo 29 using the matrix C above.

$$\begin{bmatrix} C & T & C & I & ? \\ ! & ! & D & ! & C \\ P & F & E & N & N \end{bmatrix}$$

$$\begin{bmatrix} Y & R & L \\ S & Y & ? \\ T & V & M \\ G & Q & T \\ ! & T & R \\ P & X & M \\ U & N & P \\ Y & J & X \\ W & V & U \\ W & Y & L \\ R & J & Q \\ & & O \\ S & G & M \\ N & E & H \end{bmatrix}$$

▶ **10-18.** Evaluate the following determinants:

$$(a)\quad \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} \qquad (c)\quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

▶ **10-19.** Evaluate the following determinants:

$$(a)\quad \begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \\ 2 & 0 & 3 \end{vmatrix} \qquad (c)\quad \begin{vmatrix} 2 & 0 & -1 \\ -1 & -1 & -1 \\ 2 & 4 & 3 \end{vmatrix}$$

▶ **10-20.** Given

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

(a) Find all minors M_{ij} (b) Find all cofactors C_{ij} (c) Calculate AC^T

▶ **10-21.** Evaluate the determinant

$$\begin{vmatrix} 2x & 3x & 4x \\ y & -y & 0 \\ z & 3z & 2z \end{vmatrix}$$

▶ **10-22.** Evaluate the following determinants:

$$(a)\quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 0 & 3 \end{vmatrix} \qquad (b)\quad \begin{vmatrix} 2 & 0 & 3 \\ -1 & 0 & \pi \\ e & 0 & .32 \end{vmatrix}$$

$$(c)\quad \begin{vmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ -1 & 0 & -2 & 5 \end{vmatrix} \qquad (d)\quad \begin{vmatrix} a_1 & \ell_1 & \ell_2 & \ell_3 \\ 0 & a_2 & \ell_4 & \ell_5 \\ 0 & 0 & a_3 & \ell_6 \\ 0 & 0 & 0 & a_4 \end{vmatrix}$$

$$(e)\quad \begin{vmatrix} a_1 & 0 & 0 & 0 \\ \ell_1 & a_2 & 0 & 0 \\ \ell_2 & \ell_3 & a_3 & 0 \\ \ell_4 & \ell_5 & \ell_6 & a_4 \end{vmatrix} \qquad (f)\quad \begin{vmatrix} 25 & 0 & -25 & 75 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 6 & 8 \\ -5 & 0 & -10 & 25 \end{vmatrix}$$

(g) How is the determinant in (f) related to the determinant in (c)?

▶ **10-23.**

(a) Find the inverse of

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

(b) What condition must be satisfied for the inverse to exist?

► **10-24.** Let

$$A = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}$$

(a) Calculate $C = AB$ (b) Find $|A|$, $|B|$, $|C|$ (c) Verify that $|C| = |A||B|$

► **10-25.** Show that the equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2) can be represented by the determinant

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

► **10-26.** Find A^{-1} and verify that $AA^{-1} = I$.

$$(a) \quad A = \begin{bmatrix} 4 & 1 \\ 11 & 3 \end{bmatrix} \qquad (b) \quad A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} \qquad (c) \quad A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 3 & -3 \\ 2 & 3 & -5 \end{bmatrix}$$

► **10-27.** Verify that the given matrices are orthogonal

$$(a) \quad A = \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad (b) \quad U = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$(c) \qquad V = \begin{bmatrix} \cos\theta & \sin\theta\cos\phi & \sin\theta\sin\phi \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

► **10-28.** Find the inverse of the following matrices:

(a) A lower triangular matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$$

(b) An upper triangular matrix

$$B = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) A symmetric matrix

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.0 \\ 0.0 & 0.1 & 0.2 & 0.1 \\ 0.0 & 0.0 & 0.1 & 0.2 \end{bmatrix}$$

(d) A diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \qquad \lambda_i \neq 0 \text{ for all } i$$

► **10-29.** Find values of α_1, α_2 and α_3 such that the given matrix is orthogonal

$$A = \begin{bmatrix} \frac{1}{2} & \alpha_2 & 0 \\ \alpha_1 & \frac{1}{2} & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

▶ **10-30.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{ij} = a_{ij}(t)$, $i, j = 1, 2, 3$ are differentiable functions of t.

(a) Show

$$\frac{d}{dt}(\det\, A) = \begin{vmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} \end{vmatrix}$$

(b) Evaluate $\frac{d}{dt}(\det\, A)$ when

$$A = \begin{bmatrix} 2 & t \\ t^2 & t^3 \end{bmatrix}$$

▶ **10-31.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{ij} = a_{ij}(t)$, $i, j = 1, 2, 3$ are differentiable functions of t.

(a) Show

$$\frac{d}{dt}(\det\, A) = \begin{vmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} & \frac{da_{13}}{dt} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} & \frac{da_{23}}{dt} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \frac{da_{31}}{dt} & \frac{da_{32}}{dt} & \frac{da_{33}}{dt} \end{vmatrix}$$

(b) Evaluate $\frac{d}{dt}(\det\, A)$ when

$$A = \begin{bmatrix} 1 & t & t+1 \\ 0 & t^2 & 2t \\ t-1 & t & 0 \end{bmatrix}$$

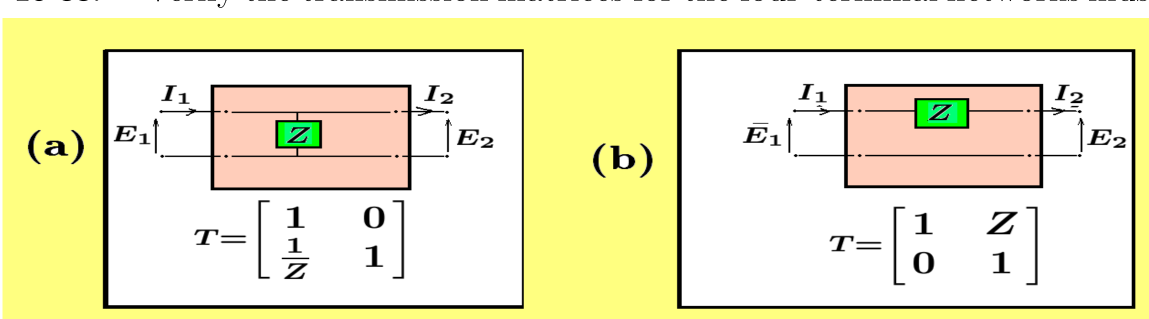
▶ **10-32.** Is the statement

$$\det(A+B) = \det(A) + \det(B)$$

true for arbitrary $n \times n$ square matrices A and B? Test your answer by using the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

▶ **10-33.** Verify the transmission matrices for the four-terminal networks illustrated.



▶ **10-34.** Let $A = \begin{bmatrix} t^3+t & \cos 3t \\ e^{4t} & \tanh 2t \end{bmatrix}$ and find

$$(a) \quad \frac{dA}{dt} \qquad\qquad (b) \quad \int A\,dt$$

▶ **10-35.** Let $C(\lambda) = |A - \lambda I| = \det(A - \lambda I) = 0$ denote the characteristic equation associated with the matrix A having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$.

(a) Show that $C(\lambda) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots c_{n-1}\lambda + c_n = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$

(b) Show that $c_n = \lambda_1\lambda_2\cdots\lambda_n = |A| = \det A$

(c) Show A is singular if any eigenvalue is zero.

(c) Use the fact that the matrix A satisfies its own characteristic equation and show

$$A^{-1} = \frac{-1}{c_n}\left[(-1)^n A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I\right]$$

▶ **10-36.**

(a) Show that $A\int_0^t e^{At}\,dt + I = e^{At}$

(b) Show that $\int_0^t e^{At}\,dt = A^{-1}\left[e^{At} - I\right] = \left[e^{At} - I\right]A^{-1}$

▶ **10-37.** Verify the following matrix relations for the $n \times n$ matrix A.

(a) $\sin A = \dfrac{e^{iA} - e^{-iA}}{2i}$ where $i^2 = -1$

(b) $\cos A = \dfrac{e^{iA} + e^{-iA}}{2}$

(c) $\sinh A = \dfrac{e^{A} - e^{-A}}{2}$

(d) $\cosh A = \dfrac{e^{A} + e^{-A}}{2}$

(e) $\sin^2 A + \cos^2 A = I$

(f) $\cosh^2 A - \sinh^2 A = I$

▶ **10-38.** Consider the initial-value matrix differential equation

$$\frac{dX(t)}{dt} = AX(t) + F(t), \qquad X(t_0) = C$$

where A is a constant matrix.

(a) Left-multiply the given matrix differential equation by the matrix function $e^{-A(t-t_0)}$ and show

$$\frac{d}{dt}\left(e^{-A(t-t_0)}X\right) = e^{-A(t-t_0)}F(t)$$

(b) Integrate both sides of the result from part (a) from $t = t_0$ to t and show

$$X = X(t) = e^{A(t-t_0)}C + e^{At}\int_{t_0}^t e^{-A\xi}F(\xi)\,d\xi$$

(c) Show in the special case $F = [0]$, the solution reduces to $X = X(t) = e^{A(t-t_0)}C$

▶ **10-39.** Show the relation between the vector differential equation

$$\frac{d\bar{y}}{dt} = A(t)\bar{y} + \bar{f}(t), \qquad \bar{y}(0) = \bar{c}$$

and the matrix differential equation

$$\frac{dY}{dt} = A(t)Y, \qquad Y(0) = I$$

is given by

$$\bar{y} = Y(t)\bar{c} + Y(t)\int_0^t Y^{-1}(\xi)\bar{f}(\xi)\,d\xi$$

▶ **10-40.** Verify the forward differences in table 10.1.

▶ **10-41.** Verify the finite integrals in table 10.2.

▶ **10-42.** Solve the given difference equations.

(a) $y_{n+2} - 5y_{n+1} + 6y_n = 0$ $\qquad$ (d) $y_{n+2} + 4y_{n+1} + 3y_n = 0$

(b) $y_{n+2} - 6y_{n+1} + 9y_n = 0$ $\qquad$ (e) $y_{n+2} + 2y_{n+1} + y_n = 0$

(c) $y_{n+2} - 6y_{n+1} + 13y_n = 0$ $\qquad$ (f) $y_{n+2} + 2y_{n+1} + 10y_n = 0$

▶ **10-43.** Find the finite integrals

$$(a) \quad \Delta^{-1}x^2 \qquad (b) \quad \Delta^{-1}\frac{1}{(3+2x)^{\overline{n}}} \qquad (c) \quad \Delta^{-1}\frac{1}{x^{\overline{n}}}$$

▶ **10-44.**

(a) For $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, verify the matrix function $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ e^{3t} - e^{2t} & e^{3t} \end{bmatrix}$

(b) For $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, verify the matrix function $e^{Bt} = \begin{bmatrix} e^{2t} & e^{2t} - e^{t} \\ 0 & e^{t} \end{bmatrix}$

▶ **10-45.**

(a) Verify the matrix differential equation

$$\frac{dX(t)}{dt} = AX(t) + F(t), \qquad X(0) = C$$

has the matrix solution

$$X = X(t) = e^{At}C + e^{At}\int_0^t e^{-A\xi}F(\xi)\,d\xi$$

(b) For $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $F(t) = \begin{bmatrix} e^{t} \\ 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ solve the matrix differential equation

$$\frac{dX(t)}{dt} = AX + F(t), \qquad X(0) = C$$

Chapter 11
Introduction to Probability and Statistics

The collecting of some type of data, organizing the data, determining how some characteristic of the data is to be presented as well conducting some type of analysis of the data, all comes under the category of **probability and statistics.**

Random Sampling

To determine some characteristic associated with a very large group of objects, called the **population**, it is impractical to examine every member of the group in order to perform an analysis of the population. Instead **a random selection of data associated with objects from the group** is examined. This is called **a random sample** from the population. Populations can be finite or infinite and by selecting a sample from the population one expects that some characteristics of the population can be inferred from an analysis of the sample data.

Analysis of the sample data, without trying to infer conclusions about the population from which the sample data comes, is called **descriptive or deductive statistics.** An analysis of sample data which tries to predict some characteristic of the population is called **inductive statistics or statistical inference**.

Simulations

Consider the figure 11-1 where some complicated system is described by n-input variables, j-parameter values, k-output variables and m-neglected or unknown variables. One replaces the complicated system with a model that in some way mimics or approximates the behavior of the real system. Those quantities that effect the model but whose behavior the model is not designed to study are called **exogenous variables**. These are usually the independent variables such as the input variables $x_1, \dots, x_n$ and parameters $p_1, \dots, p_j$ effecting system behavior. The behavior of those quantities from the complicated system that the model is designed to study are called **endogenous variables**. These are usually the dependent variables such as the outputs $y_1, \dots, y_k$ produced by the system.

Simulation is the process of designing **a mechanical or mathematical model of a real system** and then conducting experiments with this model for various purposes such as (i) obtaining a better understanding of the system (ii) to help construct theories for observed behavior (iii) aid in predicting future behavior (iv) to study how changes in inputs and parameters values effect the behavior of the system

(v) Finding ways to improve the system by experimenting with how inputs and parameter values produce changes in the system. (vi) to aid in making inferences and speculations on future system behavior.

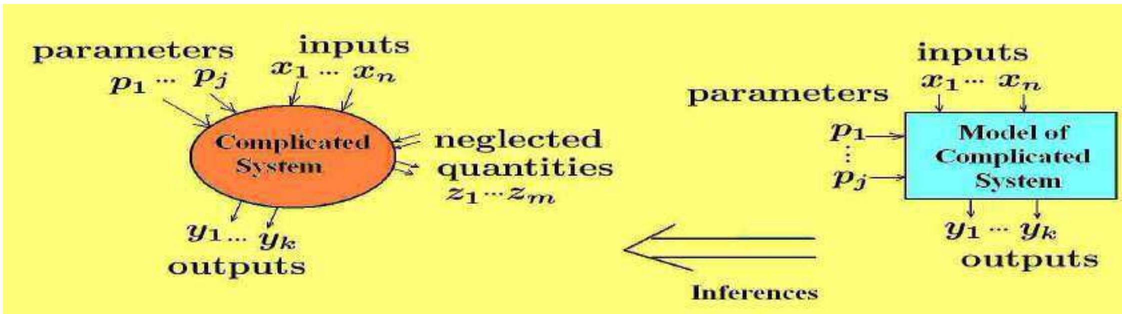


Figure 11-1.

Replacing complicated system with approximating mathematical model.

The model constructed can be continuous or discrete. By constructing a mathematical model one can use a computer to generate thousands of data values representing various outputs under a variety of input scenarios and then one can use statistics on the data values generated to make inferences concerning the behavior of the real system. For example, **Monte Carlo simulations** are discrete models where random numbers are used in specific ways to help simulate the behavior of a system. Monte Carlo methods can be used to study a wide variety of things. A small sampling of disciplines where Monte Carlo techniques are employed are the study areas of aerodynamics, fluid dynamics, atomic physics, radiation analysis, material research, oil exploration, and to verify theoretical predictions.

An example of a Monte Carlo method is the calculation of the value of π using random numbers. It can be shown that generating enough random numbers and using them in the proper way, one can calculate π as accurately as you desire.

The Representation of Data

The data from a population can be either **discrete or continuous**. If Y is a variable representing the characteristic being sampled and Y can take on any value between two given values, then Y is called a continuous variable. If Y is not a continuous variable, then it is called a discrete variable.

Some examples of discrete data is presented in the table 11.1. These numbers can be plotted as vertical or horizontal bar graphs, either stacked or grouped or as a line graph. The figure 11-2 illustrates these type of graphs.

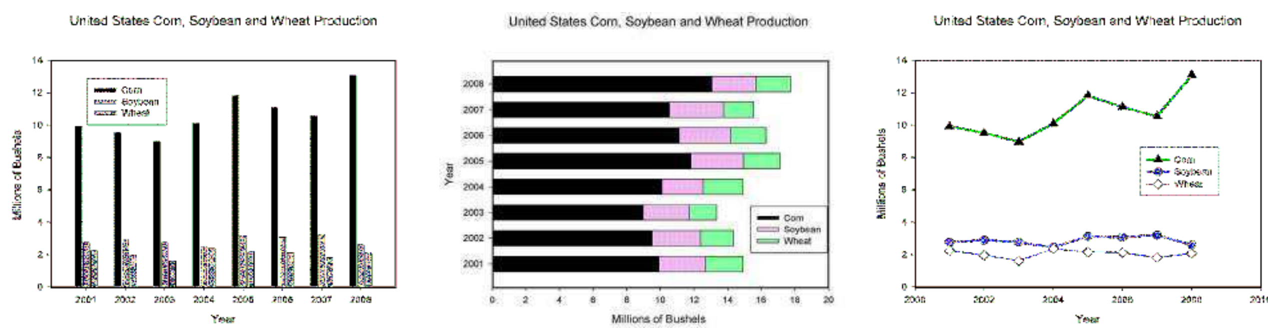


Figure 11-2. United States Corn, Soybean and Wheat Production

Table 11.1
Unite States Production of Corn, Soybeans and Wheat
(in millions of bushels)

Year	Corn	Soybean	Wheat
2001	9.92	2.76	2.23
2002	9.50	2.89	1.95
2003	8.97	2.76	1.61
2004	10.09	2.45	2.35
2005	11.81	3.12	2.16
2006	11.11	3.06	2.11
2007	10.54	3.19	1.81
2008	13.07	2.59	2.07

Tabular Representation of Data

A statistical experiment usually consists of collecting data from a random selection of the population. For example, suppose the systolic blood pressure of two hundred individuals are taken from a random sample of the population. The systolic blood pressure is measured in units of mmHg and is the blood pressure as the heart begins to pump. The diastolic blood pressure being a measure of the blood pressure between heart beats. The data set collected consists of 200 numbers, representing the sample size. A representative set of such numbers is presented in the table 11.2.

127	115	132	117	138	138	152	121	142	120	104	116	139	165	150	132	142	94	124	145
157	137	118	163	138	159	140	87	162	132	156	148	159	136	164	103	125	136	136	146
102	111	142	116	145	156	167	95	148	143	120	130	95	171	115	87	139	119	148	132
169	121	138	128	129	143	143	128	108	77	120	128	157	109	173	125	159	100	97	144
119	129	131	124	161	144	154	119	125	97	123	129	113	119	109	112	156	168	135	136
135	145	156	125	140	130	86	101	139	184	144	118	150	149	142	118	134	124	154	142
186	130	127	168	122	139	156	146	107	168	117	100	134	113	104	115	149	148	133	128
121	148	133	144	127	127	168	102	117	123	156	129	89	138	136	100	153	110	112	150
104	148	124	114	121	126	153	128	114	137	131	104	135	124	146	115	152	127	113	143
139	147	134	142	133	124	149	156	142	109	147	96	142	163	120	118	180	125	157	118

Table 11.2 Systolic Blood Pressure (mmHg) measurements taken from 200 Random Individuals

Examine the data in table 11.2 and order the data in **a tally sheet** to form **a frequency table**. Show that the smallest value is 74 and the largest value is 186. Divide the data into categories or **class intervals** of equal length by **defining an upper limit and lower limit and midpoint for each class interval**. This is called grouping the data. Examples of class intervals are given in the table 11.3 where 74 is the first midpoint with 16 midpoints to 186. If $74 + 16x = 186$, then $x = 7$ steps between midpoints or the class interval is of size 7. Go through the data and find the number of systolic blood pressures in each class interval. This is called **determining the class frequency** associated with the grouped data. Then calculate the relative frequency column, the cumulative frequency column and cumulative relative frequency column as illustrated in the table 11.3. The cumulative frequency associated with a value x is just the sum of the frequencies less than or equal to x. The cumulative relative frequency is obtained by dividing the cumulative frequency by the sample size. Note that **the cumulative frequency ends in the sample size** and **the cumulative relative frequency ends with 1**.

Table 11.3 Frequency Table						
Class Interval	Class Midpoint	Tallies	Frequency	Relative Frequency	Cumulative Frequency	Cumulative Relative Frequency
71-77	74	/	1	$\frac{1}{200} = 0.005$	1	0.005
78-84	81		0	$\frac{0}{200} = 0.000$	1	0.005
85-91	88	////	4	$\frac{4}{200} = 0.020$	5	0.025
92-98	95	///// /	6	$\frac{6}{200} = 0.030$	11	0.055
98-105	102	///// ///// /	11	$\frac{11}{200} = 0.055$	22	0.110
106-112	109	///// ////	9	$\frac{9}{200} = 0.045$	31	0.155
113-119	116	///// ///// /// ///// /////	23	$\frac{23}{200} = 0.115$	54	0.270
120-126	123	///// ///// /// ///// /////	23	$\frac{23}{200} = 0.115$	77	0.385
128-133	130	///// ///// ///// ///// ///// /	26	$\frac{26}{200} = 0.130$	103	0.515
134-140	137	///// ///// ///// ///// /////	25	$\frac{25}{200} = 0.125$	128	0.640
141-147	144	///// ///// //// ///// /////	24	$\frac{24}{200} = 0.120$	152	0.760
148-154	151	///// ///// ///// ///	18	$\frac{18}{200} = 0.090$	170	0.850
155-161	158	///// //// /////	14	$\frac{14}{200} = 0.070$	184	0.920
162-168	165	///// /////	10	$\frac{10}{200} = 0.050$	194	0.970
168-175	172	///	3	$\frac{3}{200} = 0.015$	197	0.985
176-182	179	/	1	$\frac{1}{200} = 0.005$	198	0.990
183-189	186	//	2	$\frac{2}{200} = 0.010$	200	1.00

A graphical representation of the data in table 11.3 can be presented by defining **a relative frequency function** $f(x)$ and **a cumulative relative frequency function** $F(x)$ associated with the sample. These functions are defined

$$\begin{aligned} f(x) &= \begin{cases} f_j & \text{when } x = X_j \\ 0 & \text{otherwise} \end{cases} \\ F(x) &= \sum_{x_j \le x} f(x_j) = \text{sum of all } f(x_j) \text{ for which } x_j \le x \end{aligned} \tag{11.1}$$

and are illustrated in the figure 11-3.

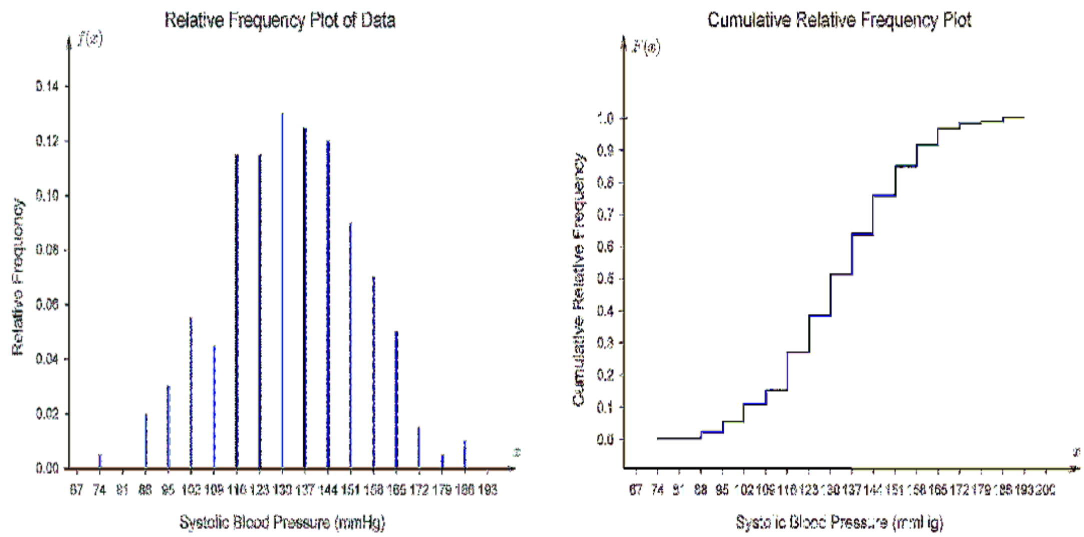


Figure 11-3.

The relative frequency function and cumulative relative frequency function

The results illustrated in the table 11.3 can be generalized. If $X_1, X_2, \ldots, X_k$ are k different numerical values in a sample of size N where X_1 occurs $\widetilde{f}_1$ times and X_2 occurs $\widetilde{f}_2$ times, ..., and X_k occurs $\widetilde{f}_k$ times, then $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_k$ are called the frequencies associated with the data set and satisfy

$$\widetilde{f}_1 + \widetilde{f}_2 + \cdots + \widetilde{f}_k = N = \text{sample size}$$

The **relative frequencies associated with the data** are defined by

$$f_1 = \frac{\widetilde{f}_1}{N}, \quad f_2 = \frac{\widetilde{f}_2}{N}, \quad \ldots, \quad f_k = \frac{\widetilde{f}_k}{N}$$

which satisfy the summation property

$$\sum_{i=1}^{k} f_i = f_1 + f_2 + \cdots + f_k = 1$$

Define **a frequency function** associated with the sample using

$$f(x) = \begin{cases} f_j, & \text{when } x = X_j \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, k$$

The frequency function determines **how the numbers in the sample are distributed**.

Also define **a cumulative frequency function** $F(x)$ for the sample, sometimes referred to as **a sample distribution function**. The cumulative frequency function is defined

$$F(x) = \sum_{t \le x} f(t) = \text{sum of all relative frequencies less than or equal to } x$$

Whenever the data has too many numerical values then one usually defines **class intervals** and **class midpoints** with class frequencies as in table 11.3. This is called **grouping of the data** and the corresponding frequency function and cumulative frequency function are associated with the grouped data.

The relative frequency distribution $f(x)$ is also called a **discrete probability distribution** for the sample and the cumulative relative frequency function $F(x)$ or distribution function represents **a probability**. In particular,

$$\begin{aligned} F(x) =& P(X \le x) = \text{Probability that population variable } X \text{ is less than or equal to } x \\ 1 - F(x) =& P(X > x) = \text{Probability that population variable } X \text{ is greater than } x \end{aligned} \tag{11.2}$$

Arithmetic Mean or Sample Mean

Given a set of data points $X_1, X_2, \dots, X_N$, define the **arithmetic mean or sample mean** of the data set by

$$\text{sample mean } = \overline{X} = \frac{X_1 + X_2 + \cdots + X_N}{N} = \frac{\sum_{j=1}^N X_j}{N} \tag{11.3}$$

If the frequency of the data points are known, say $X_1, X_2, \dots, X_k$ occur with frequencies $\widetilde{f}_1, \widetilde{f}_2, \dots, \widetilde{f}_k$, then the arithmetic mean is calculated

$$\overline{X} = \frac{\widetilde{f}_1 X_1 + \widetilde{f}_2 X_2 + \cdots + \widetilde{f}_k X_k}{\widetilde{f}_1 + \widetilde{f}_2 + \cdots + \widetilde{f}_k} = \frac{\sum_{j=1}^k \widetilde{f}_j X_j}{\sum_{j=1}^k \widetilde{f}_j} = \frac{\sum_{j=1}^k \widetilde{f}_j X_j}{N} \tag{11.4}$$

Note that the finite data collected is used to calculate an estimate of the **true population mean** μ associated with the total population.

Median, Mode and Percentiles

After arranging the data from low to high, **the median of the data set** is the middle value or **the arithmetic mean of the two middle values**. This value divides the data set into **two equal numbered parts**. In a similar fashion find those points which divide the data set, arranged in order of magnitude, into four equal parts. These values are usually denoted Q_1, Q_2, Q_3 and are called the **first, second and third**

quartiles. Note that Q_2 will be the same as the median. If the data set is divided into **ten equal parts** by numbers $D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9$, then these numbers are called **deciles**. If the data set is divided into one hundred equal parts by numbers $P_1, P_2, \ldots, P_{99}$, then these numbers are called **percentiles**. In general, if the data is divided up into quartiles, deciles, percentiles or some other equal subdivision, then the subdivisions created are called **quantiles**. The **mode** of the data set is that value which occurs with greatest frequency. Note that the mode may not exist or even if it does exist, it might not be a unique value. A **unimodal data set** is one which has a unique single mode.

The Geometric and Harmonic Mean

The **geometric mean** G associated with the data set $\{X_1, X_2, \ldots, X_N\}$ is the Nth root of the product of the numbers in the set. The geometric mean is denoted

$$G = \sqrt[N]{X_1 X_2 \cdots X_N} \tag{11.5}$$

The **harmonic mean** H associated with the above data set is obtain by first taking the arithmetic mean of the reciprocals and then taking the reciprocal of the result. The harmonic mean can be expressed using either of the relations

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{X_i}} \qquad \text{or} \qquad \frac{1}{H} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{X_i} \tag{11.6}$$

The arithmetic mean $\overline{X}$, geometric mean G and harmonic mean H, satisfy the inequalities

$$H \le G \le \overline{X} \tag{11.8}$$

The equality sign being used when all the numbers in the data set are equal to one another.

The Root Mean Square (RMS)

The **root mean square** (RMS), sometimes referred to as the **quadratic mean**, of the data set $\{X_1, X_2, \ldots, X_n\}$ is defined for a discrete set of values as

$$RMS = \sqrt{\frac{\sum_{j=1}^{N} X_j^2}{N}} \tag{11.8}$$

and for a continuous set of values $f(x)$ over an interval (a,b) it is defined

$$RMS = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx} \tag{11.9}$$

The root mean square is used to measure the **average magnitude** of a quantity that varies.

Mean Deviation and Sample Variance

The **mean deviation** (MD), sometimes called the **average deviation**, of a set of numbers $X_1, X_2, \ldots, X_N$ represents **a measure of the data spread from the mean** and is defined

$$MD = \frac{\sum_{j=1}^{N} \mid X_j - \overline{X} \mid}{N} = \frac{1}{N}\left[\mid X_1 - \overline{X} \mid + \mid X_2 - \overline{X} \mid + \cdots + \mid X_n - \overline{X} \mid\right] \tag{11.10}$$

where $\overline{X}$ is the arithmetic mean of the data. The mean deviation associated with numbers $X_1, X_2, \ldots, X_k$ occurring with frequencies $\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_k$ can be calculated

$$MD = \frac{\sum_{j=1}^{k} \widetilde{f}_j \mid X_j - \overline{X} \mid}{N} = \frac{1}{N}\left[\widetilde{f}_1 \mid X_1 - \overline{X} \mid + \widetilde{f}_2 \mid X_2 - \overline{X} \mid + \cdots + \widetilde{f}_k \mid X_k - \overline{X} \mid\right] \tag{11.11}$$

The **sample variance** of the data set $\{X_1, X_2, \ldots, X_n\}$ is denoted s^2 and is calculated using the relation

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2 = \frac{1}{n-1}\left[(x_1 - \overline{X})^2 + (x_2 - \overline{X})^2 + \cdots + (x_n - \overline{X})^2\right] \tag{11.12}$$

where $\overline{X}$ is the sample mean or arithmetic mean. The sample variance is a measure of how much **dispersion or spread** there is in the data. The positive square root of the sample variance is denoted s, which is called **the standard deviation** of the sample.

Note that some textbooks define the sample variance as

$$\mathcal{S}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X})^2 \tag{11.13}$$

where n is used as a divisor instead of $n-1$. This is confusing to beginning students who often ask, "Why the different definitions for sample variance in different textbooks?" The reason is that for small values for n, say $n < 30$, then equation (11.12) will produce a better estimate of **the true standard deviation σ associated with the total population** from which the sample is taken. For sample sizes larger than $n = 30$

there will be very little difference in calculation of the standard deviation from either definition. To convert the standard deviation $\mathcal{S}$, calculated using equation (11.13) one need only multiply $\mathcal{S}$ by $\sqrt{\dfrac{n}{n-1}}$ to obtain the value s as specified by equation (11.12).

The sample variance given by equation (11.12) requires that one first calculate $\overline{X}$, then one must calculate all the terms $X_j - \overline{X}$. All this preliminary calculation introduces **roundoff errors** into the final result. The sample variance can be calculated using a **short cut method** of computing without having to do preliminary calculations. The short cut method is derived using the expansion

$$(X_j - \overline{X})^2 = X_j^2 - 2X_j\overline{X} + \overline{X}^2$$

and substituting it into equation (11.12). A summation of terms gives

$$\sum_{j=1}^{n}(X_j - \overline{X})^2 = \sum_{j=1}^{n} X_j^2 - 2\overline{X}\sum_{j=1}^{n} X_j + \overline{X}^2\sum_{j=1}^{n}(1)$$

The substitution $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} X_j$ and using $\displaystyle\sum_{j=1}^{n}(1) = n$, gives the result

$$\begin{aligned}\sum_{j=1}^{n}(X_j - \overline{X})^2 =& \sum_{j=1}^{n} X_j^2 - 2\frac{1}{n}\sum_{j=1}^{n} X_j\sum_{j=1}^{n} X_j + \left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)^2 n\\ =& \sum_{j=1}^{n} X_j^2 - \frac{2}{n}\left(\sum_{j=1}^{n} X_j\right)^2 + \frac{1}{n}\left(\sum_{j=1}^{n} X_n\right)^2\\ =& \sum_{j=1}^{n} X_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} X_j\right)^2\end{aligned}$$

This produces the shortcut formula for the sample variance

$$s^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n} X_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} X_j\right)^2\right] \tag{11.14}$$

If $X_1, \ldots, X_m$ are m sample values occurring with frequencies $\widetilde{f}_1, \ldots, \widetilde{f}_m$, the equation (11.14) can be expressed in the form

$$s^2 = \frac{1}{n-1}\left[\sum_{j=1}^{m} X_j\widetilde{f}_j - \frac{1}{n}\left(\sum_{j=1}^{m} X_j\widetilde{f}_j\right)^2\right] \tag{11.15}$$

In general, if **the true population mean μ is known exactly**, so that $\mu = \frac{1}{N}\sum_{j=1}^{N} X_j$, where N is the population size, then the population standard deviation is given by

$$\sigma = \sqrt{\frac{\sum_{j=1}^{N}(X_j - \mu)^2}{N}} \tag{11.16}$$

Use N if the exact population mean is known and use $n-1$ if samples of size $n << N$ are selected from a population where the true mean μ is unknown.

Probability

An experiment or observation produces samples from a population where the outcome recorded either belongs or does not belong to a prescribed collection of events being studied. A **sample space S is the set of all possible outcomes from an experiment.** A sample space can be either finite or infinite. An example of a sample space with a finite collection of events is the roll of a single die. Here the sample space is $S = \{1, 2, 3, 4, 5, 6\}$ corresponding to the numbers on the six faces of the die. An example of an infinite sample space is that of an experiment where the outcome from an single event can be a real number within a specified range.

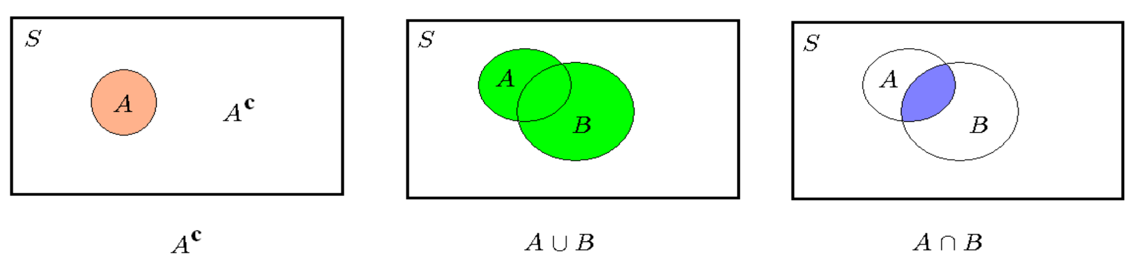


Figure 11-4. Venn diagrams

A Venn diagram consists of representing the sample space by a rectangle, then any event within S can be represented by the interior of a circle A within the rectangle. The set of all events not in A is called **the complement of A** and is denoted using the notation A^c. The **null set, empty set or impossible event** is denoted by the symbol $\emptyset$. The **union of two events** A and B is denoted $A \cup B$ and represents all events or experiments of S contained in A or B or both. The **intersection of two events** A and B is denoted $A \cap B$ and represents all events in S contained in both A and B. The concepts of a complement, union and intersection of sets is illustrated in the figure 11-4. If two sets A and B have no events in common, then this is denoted by $A \cap B = \emptyset$. In this case, the sets A and B are said to be **mutually exclusive events** or **disjoint events**. The notation $A \subset B$ is used to denote "all elements of A are contained in the set B" This can also be expressed $B \supset A$, which is read " B contains A". If the sample space contains n-sets $\{A_1, A_2, \ldots, A_n\}$, then the union of these sets is denoted

$$A_1 \cup A_2 \cup \cdots \cup A_n \qquad \text{or} \qquad \bigcup_{j=1}^{n} A_j$$

The intersection of these sets is denoted

$$A_1 \cap A_2 \cap \cdots \cap A_n \qquad \text{or} \qquad \bigcap_{j=1}^{n} A_j$$

If $A_j \cap A_k = \emptyset$, for all values of j and k, with $k \neq j$, then the sets $\{A_1, A_2, \ldots, A_n\}$ are said to represent **mutually exclusive events**.

Probability Fundamentals

Assuming that there are h **ways an event can happen** and f **ways for the event to fail** and these ways are **all equally likely to happen**, then the probability p that an event will happen in a given trial is

$$p = \frac{h}{h+f} \tag{11.17}$$

and the probability q that the event will fail is given by

$$q = \frac{f}{h+f} \tag{11.18}$$

These probabilities satisfy $p + q = 1$.

In general, given a finite sample space $S = \{e_1, e_2, \ldots, e_n\}$ containing n simple events $e_1, \ldots, e_n$, assign to each element of S a number $P(e_i)$, $i = 1, \ldots, n$ called the probability assigned to event e_i of S. The probability numbers $P(e_i)$ assigned must satisfy the following conditions.

1. Each probability is a nonnegative number satisfying $0 \le P(e_i) \le 1$
2. The sum of the probabilities assigned to all simple events of the sample space must sum to unity or

$$\sum_{j=1}^{n} P(e_j) = P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

3. If each event is **equally likely to happen**, then one usually assigns a probability value $P(e_i) = \frac{1}{n}$ to each event as then $\sum_{i=1}^{n} P(e_i) = 1$.
4. The probability assigned to the entire sample space is unity and one writes $P(S) = 1$.

Probability of an Event

After assigning probabilities to each simple event of S, it is then possible to determine the probability of any event E associated with events from S. Consider the following cases.

1. The event $E = \emptyset$ is the empty set.
2. The event E is one of the simple events e_i from S $(i$ fixed, with $1 \le i \le n)$
3. The event E is the union of two or more events from S.

For the case 1, define the probability of the empty set $\emptyset$ as zero and write $P(\emptyset) = 0$. In case 2, the probability of event E is the same as the probability $P(e_i)$ so that, $P(E) = P(e_i)$. Consider now the case 3. If E is an event associated with the sample space S and $\overline{E}$ is its complement, then

$$P(E) = 1 - P(\overline{E}) \tag{11.19}$$

This is known as the **complementation rule** for probabilities.

If E_1 and E_2 are **mutually exclusive events** associated with S, then $E_1 \cap E_2 = \emptyset$ and

$$P(E_1 \cup E_2) = P(E_1) + P(E_2), \qquad\qquad E_1 \cap E_2 = \emptyset \tag{11.20}$$

In general, if $E = E_1 \cup E_2 \cup \cdots \cup E_m$ is an event and $E_1, E_2, \ldots, E_m$ are mutually exclusive events associated with S, then the intersection gives $E_1 \cap E_2 \cap \cdots \cap E_m = \emptyset$ and the probability of E is

$$P(E) = P(E_1 \cup E_2 \cup \cdots \cup E_m) = P(E_1) + P(E_2) + \cdots + P(E_m) \tag{11.21}$$

This is known as the **addition rule** for mutually exclusive events.

If E_1 and E_2 are arbitrary events associated with a sample space S, and these events **are not mutually exclusive**, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \tag{11.22}$$

Here $P(E_1)$ is the sum of all the simple events defining E_1 and $P(E_2)$ is the sum of all the simple events defining E_2. If the events are not mutually exclusive, then the sum of the probabilities associated with the simple events common to both the events E_1 and E_2 are counted twice. The sum of these common probabilities of simple events which are counted twice is $P(E_1 \cap E_2)$ and so this value is subtracted from the sum $P(E_1) + P(E_2)$.

Example 11-1.

Two coins are tossed. What is the probability that at least one tail occurs? Assume the coins are not trick coins so that there are four equally likely events that can occur. Let H denote heads and T denote tails, then the sample space for the experiment is $S = \{HH, HT, TH, TT\}$. Because each event is equally likely, assign a probability of $1/4$ to each event. For this example, the event E to be investigated is

$$E = \{HT\} \cup \{TH\} \cup \{TT\}$$

That is, at least one tail occurs. Consequently,

$$P(E) = P(HT) + P(TH) + P(TT) = 1/4 + 1/4 + 1/4 = 3/4$$

■

Example 11-2.

A pair of fair dice are rolled. The sample space associated with this experiment is a representation of all possible outcomes.

$$\begin{array}{llllll} S = \{(1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) \\ (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ (1,6) & (2,6) & (3,6) & (4,6) & (5,6) & (6,6)\,\} \end{array}$$

There are 36 equally likely possible outcomes. Assign a probability of $1/36$ to each simple event.

If E_1 is the event that a 7 is rolled, then

$$P(E_1) = P((1,6)) + P((2,5)) + P((3,4)) + P((4,3)) + P((5,2)) + P((6,1)) = 6/36 = 1/6$$

If E_2 is the event that an 11 is rolled, then

$$P(E_2) = P((5,6)) + P((6,5)) = 2/36 = 1/18$$

If E_3 is the event doubles are rolled, then

$$P(E_3) = P((1,1)) + P((2,2)) + P((3,3)) + P((4,4)) + P((5,5)) + P((6,6)) = 6/36 = 1/6$$

If E_4 is the event that a 10 is rolled, then

$$P(E_4) = P((4,6)) + P((5,5)) + P((6,4)) = 3/36 = 1/12$$

If E_5 is the event a 10 is rolled or a double is rolled, the $E_5 = E_4 \cup E_3$. Note that the event $(5,5)$ is common to both events E_4 and E_3 with $E_4 \cap E_3 = (5,5)$ and $P(E_4 \cap E_3) = P((5,5)) = 1/36$. Hence,

$$P(E_5) = P(E_4 \cup E_3) = P(E_4) + P(E_3) - P(E_4 \cap E_3) = 3/36 + 6/36 - 1/36 = 8/36 = 2/9$$

If E_6 is the event that a 10 is rolled or a 7 is rolled, then $E_6 = E_4 \cup E_1$. Here $E_4 \cap E_1 = \emptyset$ and so these events are mutually exclusive. Consequently,

$$P(E_6) = P(E_4 \cup E_1) = P(E_4) + P(E_1) = 3/36 + 6/36 = 9/36 = 1/4$$

■

Note that there are many situations where **the sample space is not finite**. In such cases the probabilities assigned to the events in S are **based upon employment of relative frequencies observed from taking a large number of trials from the population being studied**. **For a large number of trials the relative frequency of an event happening is approximately the same as the probability of the event happening**. Observation of this fact should be made by examining the earlier development of relative frequency tables associated with our analysis of data collected. Also note that one can think of equation (11.20) is a special case of the more general equation (11.22), the special case occurring when E_1 and E_2 are mutually exclusive events.

If p is a probability of success assign to a single trial of an event, then the **expected number** of successes in n-trials is the product np.

Conditional Probability

If two events E_1 and E_2 are related in some manner such that the probability of occurrence of event E_1 depends upon whether E_2 has or has not occurred, then this is called **the conditional probability** of E_1 given E_2 and it is denoted using the notation $P(E_1 \mid E_2)$. Here the vertical line is read as "given"and events to the right of the vertical line are treated as events which have occurred. The conditional probability of E_1 given E_2 is

$$P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}, \qquad P(E_2) \neq 0 \tag{11.23}$$

The conditional probability of E_2 given E_1 is

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}, \qquad P(E_1) \neq 0 \tag{11.24}$$

The equations (11.23) and (11.24) imply that the probability of both events E_1 and E_2 occurring is given by

$$P(E_1 \cap E_2) = P(E_1)P(E_2 \mid E_1) = P(E_2)P(E_1 \mid E_2), \qquad P(E_1) \neq 0, \quad P(E_2) \neq 0 \tag{11.25}$$

If the events E_1 and E_2 are independent events, then

$$P(E_1 \cap E_2) = P(E_1)P(E_2) \tag{11.26}$$

and consequently

$$P(E_1 \mid E_2) = P(E_1), \qquad \text{and} \qquad P(E_2 \mid E_1) = P(E_2)$$

This condition occurs whenever the probability of E_1 **does not depend upon the event** E_2 and similarly, the probability of event E_2 **does not depend upon** E_1.

Two events E_1 and E_2 are said to be **independent events** if and only if the probability of occurrence of E_1 and E_2 is given by $P(E_1 \cap E_2) = P(E_1)P(E_2)$. That is, the probability of both E_1 and E_2 occurring is the product of the probabilities of occurrence of each event. Two events that are **not independent** are called **dependent events**.

In general, if $E_1, E_2, \ldots, E_m$ are all independent events, then

$$P(E_1 \cap E_2 \cap \cdots \cap E_m) = P(E_1)P(E_2)\cdots P(E_m) \tag{11.27}$$

This is sometime written in the form

$$P(\bigcap_{k=1}^{m} E_k) = \prod_{k=1}^{m} P(E_k) \tag{11.28}$$

and is known as **the multiplication principle for independent events**.

Example 11-3. Given an ordinary deck of 52 cards, suppose it is required to find the probability of selecting two cards and they are both aces. Here the probability of selecting an ace on the first draw is $P_1 = 4/52$. If the first card selected is an ace and it is not put back into the deck, then on the second draw there are only 3 aces left in the deck which now has only 51 cards. Consequently the probability of getting an ace on the second draw is $P_2 = 3/51$. The required probability of obtaining an ace on both the first and second drawing of cards is given by the multiplication principle for independent events so that one can write $P = P_1P_2 = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$

■

The above discussions can be generalized. In studying the occurrence or non-occurrence of three events E_1, E_2, E_3 the probability is denoted

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2) \tag{11.29}$$

and for independent events

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3) \tag{11.30}$$

with similar extensions to a higher number of events taking place.

Example 11-4.

The dice table is completely surrounded with players so that you can see only a part of the table. A player rolls the dice and you see one die comes up a 6, but you can't see the other die. What is the probability the player has rolled a 7 or 11?

Solution: Here there are two events E_1, E_2 with

$$E_1 = \text{event one die is a 6}$$

$$E_2 = \text{event sum of dice is 7 or 11}$$

and we are to find $P(E_2 \mid E_1)$. To solve this problem write down the simple events as

$$E_1 = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$E_2 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(5,6),(6,5)\}$$

$$\text{with} \quad E_1 \cap E_2 = \{(6,1),(6,5)\}$$

Recall the simple events all have equal probabilities of $1/36$ and consequently

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{2/36}{6/36} = 1/3$$

Observe that $P(E_2) = 8/36 = 2/9 \neq P(E_2 \mid E_1) = 1/3$. These events are not independent. That is, knowing one die is a 6 does effect the probability of the sum being 7 or 11.

■

Example 11-5.

Two cards are selected at random from an ordinary deck of 52 cards. Find the probability that both cards are spades. Find the probability that the first card is a spade and the second card is a heart. In performing this experiment assume that the first card selected is not replaced in the deck.

Solution: Examine the events

$$E_1 = \text{The event that the first card selected is a spade.}$$
$$E_2 = \text{The event that the second card selected is a spade.}$$
$$E_3 = \text{The event that the second card selected is a heart.}$$

Using elementary probability theory

$$P(E_1) = \frac{13}{52} = \frac{\text{Number of spades in deck}}{\text{Total number of cards in deck}}$$

Now if event E_1 has occurred, the deck now has only 51 cards with 12 spades. Consequently, the conditional probability is

$$P(E_2 \mid E_1) = \frac{12}{51} = \frac{\text{Number of spades in deck}}{\text{Total number of cards in deck}}$$
$$\text{and} \quad P(E_3 \mid E_1) = \frac{13}{51} = \frac{\text{Number of hearts in deck}}{\text{Total number of cards in deck}}$$

Calculate the probabilities

$$P(E_1 \cap E_2) = P(E_1)P(E_2 \mid E_1) = \frac{13}{52} \cdot \frac{12}{51} = \frac{3}{51}$$
$$\text{and} \quad P(E_1 \cap E_3) = P(E_1)P(E_3 \mid E_1) = \frac{13}{52} \cdot \frac{13}{51} = \frac{13}{204}$$

■

Permutations

Assume something can be done in ℓ different ways and one of these ℓ-ways has been done. Now if a second something can be done in m different ways, then the number of ways that the two somethings can be done is given by the product $\ell \cdot m$. If a third something can be done in n ways, then the three somethings can be done in $\ell \cdot m \cdot n$-ways. This **multiplication principle** can be extended if more than three somethings are involved in the study.

Example 11-6. How many three digit even numbers can be formed using the digits $\{1, 2, 3, 5, 7\}$ if repetition of any digits is not allowed?

Solution A three digit number has the representation (hundreds place)(tens place)(units place). If the three digit number is to be an even number, then the units place must be filled with the number 2. Hence there are $\ell = 1$ ways to perform this task. The tens place can be filled with any of the numbers 1,3,5,7 and so $m = 4$ ways to perform this task. Finally, if one of the numbers $1, 3, 5, 7$ is selected for the tens place and there is to be no repetition of numbers, then only 3 numbers are left for the hundreds place. This gives $n = 3$ ways for the hundreds place. This shows that there are $n \cdot m \cdot \ell = 3 \cdot 4 \cdot 1 = 12$ ways to complete the task.

■

Each arrangement in an ordered set of items is called **a permutation of the set of items**. For example, how many ways can you arrange three books on a shelf? There are 3 choices for the first book, 2 choices for the second book and 1 selection for the last book. This gives $3 \cdot 2 \cdot 1 = 3! = 6$ ways to arrange three books on a shelf. If the books are labeled a,b and c, then the 6 arrangements are

abc	cab	bca
bac	acb	cba

In general, the **number of permutations of n things is n-factorial**, written $n!$. That is, there are n choices for the first item, $(n-1)$ choices for the second item, $(n-2)$ choices for the third item, etc. This gives the number of permutations as

$$ {}_nP_n = n(n-1)(n-2)\cdots(3)(2)(1) = n! \tag{11.31} $$

The grouping of a selection of m items from a collection of n items, $m < n$, is called **the number of permutations of n items taken m at a time**. For example, to determine the number of permutations of the letters a,b,c,d taken two at a time, note that there are 4 choices for the first letter leaving three choices for the second letter. This gives 12 such arrangements. These arrangements are

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

In general, **the number of permutations of n things taken m at a time** is given by the formula

$$ {}_nP_m = n(n-1)(n-2)\cdots(n-m+1) \tag{11.32} $$

which is a product of m-factors. In the special case $m = n$, **the number of permutations of n things taken all the time** is ${}_nP_n$ as given by equation (11.31).

If in the collection of items there are n_1 repeats on one item, n_2 repeats of another item, ..., n_m repeats of still another item, then many of the total number of permutations will be repeats. To remove these repeats just divide by factorial associated with the number of repeats. This gives the permutation of n things taken all at a time as

$$ P = \frac{n!}{n_1!n_2!\cdots n_m!} \qquad \text{where } n = n_1 + n_2 + n_3 + \cdots + n_m \tag{11.33} $$

Combinations

A collection of items **without regard to the order of arrangement** is called **a combination**. For example, abc,bca,cab,bac,cba,acb all represent the same collection of the letters a,b and c, where the different permutations are ignored. The **number of combination of n things taken m at time** is denoted by using either of the notations $\binom{n}{m}$ or ${}_nC_m$ and is calculated as follows. If ${}_nC_m$ or $\binom{n}{m}$ is a collection of m items from a set of n items, then for each combination of m things there are $m!$ permutations so that one can write

$$
\begin{aligned}
{}_nP_m =& m!\binom{n}{m} = m!{}_nC_m \\
\text{or } {}_nC_m = \binom{n}{m} =& \frac{{}_nP_m}{m!} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!} \\
=& \frac{n!}{m!(n-m)!} \qquad n \ge 0, \quad 0 \le m \le n
\end{aligned} \tag{11.34}
$$

The term $\binom{n}{m}$ represents the mth binomial coefficient.

Note that binomial coefficients $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ and that $\binom{n}{m} = \binom{n}{n-m}$ The binomial coefficients also satisfy the recursive property

$$ \binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1} \qquad m \ge 0 \text{ and integral} $$

Binomial Coefficients

The binomial expansion $(p+q)^n$, for n an integer, can be expressed in the form

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{m}p^{n-m}q^m + \cdots + \binom{n}{n}q^n \qquad (11.35)$$

Note in the special case $p = q = 1$ one obtains

$$(1+1)^n = 2^n = 1 + \sum_{m=1}^{n}\binom{n}{m} = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \qquad (11.36)$$

and rearranging terms one finds

$$\sum_{m=1}^{n}\binom{n}{m} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1 \qquad (11.37)$$

Let p denote the probability that an event will happen and $q = 1-p$ denote the probability that the event will fail in a single trial and examine the probability that the event will happen in n-trials. In one trial $(p+q) = 1$. In two trials, the expansion

$$(p+q)^2 = \binom{2}{0}p^2 + \binom{2}{1}pq + \binom{2}{2}q^2 = p^2 + 2pq + q^2$$

represents all possible outcomes. For example, if the trial is flipping a coin and p is the probability of heads H and q is the probability of tail T, then p^2 is the probability of getting two successive heads HH, $2pq$ is the probability of getting a head and tail or tail and head HT or TH and q^2 is the probability of getting two tails TT. Similarly, in three trials of flipping a coin the expansion

$$(p+q)^3 = \binom{3}{0}p^3 + \binom{3}{1}p^2q + \binom{3}{2}pq^2 + \binom{3}{3}q^3$$

gives the probabilities

$\binom{3}{0}p^3 = p^3$ is probability of getting three successive heads HHH

$\binom{3}{1}p^2q = 3p^2q$ is the probability of getting HHT or HTH or THH
and represents the probability of getting two heads in 3 trials.

$\binom{3}{2}pq^2 = 3pq^2$ is the probability of getting HTT or THT or TTH
and represents the probability of getting one head in 3 trials.

$\binom{3}{3}q^3 = q^3$ is the probability of getting TTT
and represents the probability of not getting a head in 3 trials.

In general, in studying n-trials associate with a two event happening, one would examine the binomial expansion

$$(p+q)^n = \sum_{j=0}^{n} \binom{n}{j} p^{n-j} q^j$$

and the term within this expansion having the form

$$\binom{n}{m} p^m q^{n-m} = {}_nC_m\, p^m q^{n-m} = \frac{n!}{m!\,(n-m)!} p^m q^{n-m} \tag{11.38}$$

represents **the probability that the event will happen m times in n trials**.

Discrete and Continuous Probability Distributions

The estimated probability of an event is taken as the relative frequency of occurrence of the event. As the number of observations upon which the relative frequency is based increases, then the discrete probability is replace by a continuous function $f(x)$ called the probability function or probability density function of the distribution with the condition that the total area under the probability density function must equal unity. The figure 11-5 is a graphical representation illustrating this conversion.

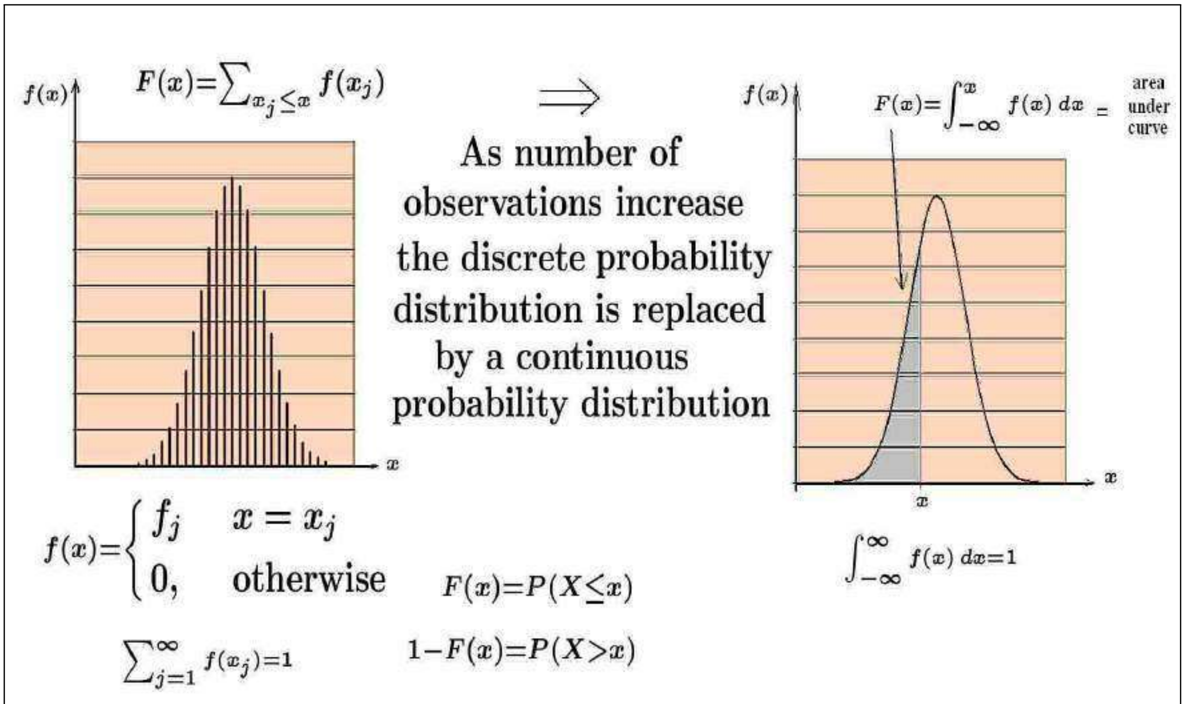


Figure 11-5.

Discrete frequency function replaced by continuous probability distribution.

The function $F(x) = \sum_{x_j \le x} f(x_j)$ is called the **cumulative frequency function** associated with the discrete sample. In the continuous case it is called the distribution function $F(x)$ and calculated by the integral $F(x) = \int_{-\infty}^{x} f(x)\,dx$ which represents the area from $-\infty$ to x under the probability density function. These summation processes are illustrated in the figure 11-6.

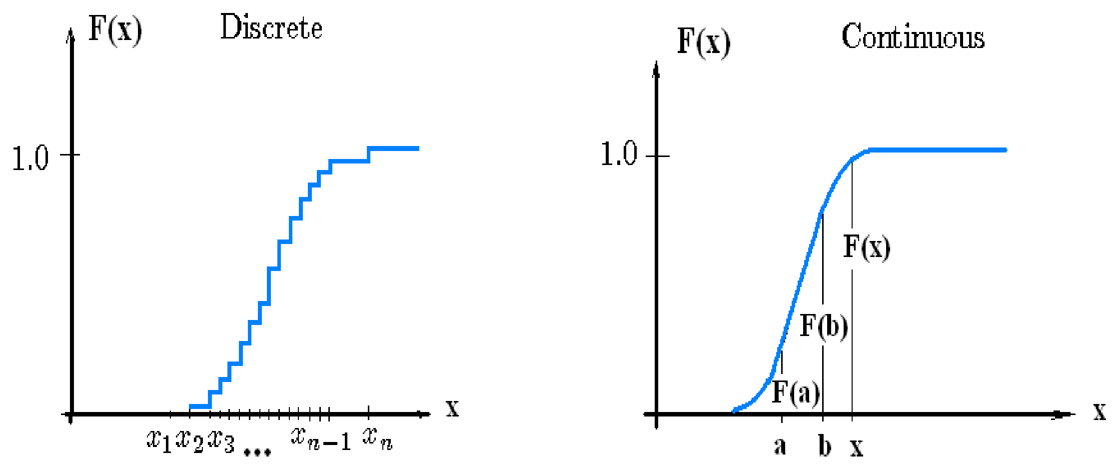


Figure 11-6. Cumulative frequency function for discrete and continuous case.

In both the discrete and continuous cases the cumulative frequency function represents the probability

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{with} \quad 1 - F(x) = P(X > x) = \int_{x}^{\infty} f(x)\,dx \tag{11.39}$$

with the property that if a, b are points x with $a < b$, then

$$P(a < x \le b) = P(X \le b) - P(X \le a) = F(b) - F(a) \tag{11.40}$$

which represents the probability that a random variable X lies between a and b. In the discrete case

$$P(a < X \le b) = \sum_{a < x_j \le b} f(x_j) = F(b) - F(a) \tag{11.41}$$

and in the continuous case

$$P(a < X \le b) = \int_{a}^{b} f(x)\,dx = F(b) - F(a) \tag{11.42}$$

Table 11.4 Mean and Variance for Discrete and Continuous Distributions		
Discrete		**Continuous**
$\mu = E[x] = \sum_{j=1}^{n} x_j f(x_j)$	**population mean** μ	$\mu = E[x] = \int_{-\infty}^{\infty} x f(x)\, dx$
$\sigma^2 = E[(x-\mu)^2] = \sum_{j=1}^{n}(x_j-\mu)^2 f(x_j)$	**population variance** σ^2	$\sigma^2 = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

The continuous cumulative frequency function satisfies the properties

$$\frac{dF(x)}{dx} = f(x), \qquad F(-\infty) = 0, \qquad F(+\infty) = 1, \quad \text{and} \quad F(a) < F(b) \text{ if } a < b \tag{11.43}$$

The table 11.4 illustrates the relationships of the mean and variance associated with the discrete and continuous probability densities.

If X is a real random variable and $g(X)$ is any continuous function of X, then the numbers

$$\begin{aligned} E[g(X)] &= \sum_{j=1}^{n} g(x_j) f(x_j) \qquad \text{discrete} \\ E[g(X)] &= \int_{-\infty}^{\infty} g(x) f(x)\, dx \qquad \text{continuous} \end{aligned} \tag{11.44}$$

associated with the probability density $f(x)$ are defined as **the mathematical expectation of the function** $g(X)$. In the special case $g(X) = X^k$ for $k = 1, 2, \dots, n$ an integer, the equations (11.44) become

$$\begin{aligned} E[X^k] &= \sum_{j} x_j^k f(x_j) \qquad \text{discrete} \\ E[X^k] &= \int_{-\infty}^{\infty} x^k f(x)\, dx \qquad \text{continuous} \end{aligned}$$

These expectation equations are referred to as **the kth moment of X**. In the special case $g(X) = (X-\mu)^k$, the equations (11.44) become

$$\begin{aligned} E[(X-\mu)^k] &= \sum_{j} (x_j-\mu)^k f(x_j) \quad \text{discrete} \\ E[(X-\mu)^k] &= \int_{-\infty}^{\infty} (x-\mu)^k f(x)\, dx \quad \text{continuous} \end{aligned}$$

and these quantities are called **the kth central moments of X**. Note the special cases

$$E[1] = 1, \qquad \mu = E[X], \qquad \sigma^2 = E[(X-\mu)^2] \tag{11.45}$$

The **expectation of a sum of random variables** $X_1, X_2, \ldots, X_n$ **equals the sum of the expectations** and consequently

$$E(X_1 + X_2 + \cdots X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) \tag{11.46}$$

The **expectation of a product of independent random variables equals the product of the expectations** which is expressed

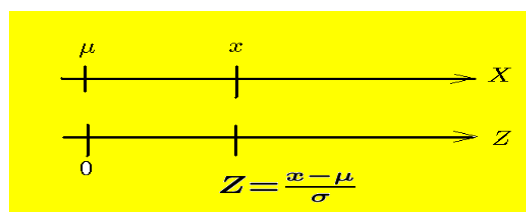
$$E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2)\cdots E(X_n) \tag{11.47}$$

Scaling

The probability density function $f(x)$ is said to be symmetric with respect a number $x = \mu$ if for all values of x the density function satisfies the relation

$$f(\mu + x) = f(\mu - x) \tag{11.48}$$

A random variable X having a mean μ and variance σ^2 can be **scaled** by introducing the new variable $Z = (X - \mu)/\sigma$. The variable Z is referred to as **the standardized variable** corresponding to X.



Let $f(x)$ denote the probability density function associated with the random variable X and define the function $f^*(z) = \sigma f(x) = \sigma f(\sigma z + \mu)$ as the probability function associated with the random variable Z. Using the scaling illustrated in the figure above, observe that $x = \sigma z + \mu$ with $dx = \sigma\, dz$ so that

$$f(x)\,dx = f(\sigma z + \mu)\sigma\,dz = f^*(z)\,dz$$

then the mean value on the Z-scale is given by

$$\begin{aligned}
\mu^* = &\int_{-\infty}^{\infty} z f^*(z)\,dz = \int_{-\infty}^{\infty} \left(\frac{x}{\sigma} - \frac{\mu}{\sigma}\right) f(x)\,dx \\
= &\frac{1}{\sigma}\int_{-\infty}^{\infty} x f(x)\,dx - \frac{\mu}{\sigma}\int_{-\infty}^{\infty} f(x)\,dx \\
= &\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}(1) = 0
\end{aligned}$$

and the variance on the Z-scale is given by

$$\sigma^{*2} = \int_{-\infty}^{\infty} (z-\mu^*)^2 f^*(z)\,dz = \int_{-\infty}^{\infty} z^2 f^*(z)\,dz \quad \text{since } \mu^*=0$$

$$\sigma^{*2} = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^2 f(x)\,dx = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \frac{1}{\sigma^2}\sigma^2 = 1$$

This demonstrates that the introduction of a scaled variable Z centers the mean at zero and introduces a variance of unity.

The Normal Distribution

The **normal probability distribution** is a continuous function with two parameters called μ and $\sigma > 0$. The parameter σ is called the **standard deviation** and σ^2 is called **the variance of the distribution**. The normal probability distribution has the form

$$f(x) = N(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-\mu)^2/\sigma^2\right], \qquad -\infty < x < \infty \tag{11.49}$$

and is illustrated in the figure 11-7. The parameter μ is known as **the mean of the distribution** and represents a location parameter for positioning the curve on the x-axis. Note the normal probability curve is **symmetric about the line** $x=\mu$. The parameter σ is sometimes called a scale parameter which is associated with the **spread and height** of the probability curve. The quantity σ^2 represents the variance of the distribution and σ represents the **standard deviation of the distribution**. The total area under this curve is 1 with approximately 68.27% of the area between the lines $\mu \pm \sigma$, 95.45% of the total area is between the lines $\mu \pm 2\sigma$ and 99.73% of the total area is between the lines $\mu \pm 3\sigma$. The area bounded by the curve $N(x;\mu,\sigma^2)$ and the x-axis is unity. The area under the curve $N(x;\mu,\sigma^2)$ between $X=b$ and $X=a<b$ represents the probability $P(a < X \le b)$. For example, one can write the probabilities

$$\begin{aligned} P(\mu-\sigma < X \le \mu+\sigma) =& .6827 \\ P(\mu-2\sigma < X \le \mu+2\sigma) =& .9545 \\ P(\mu-3\sigma < X \le \mu+3\sigma) =& .9973 \end{aligned} \tag{11.50}$$

The function

$$\phi(z) = N(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{11.51}$$

is called a normalized probability distribution with mean $\mu = 0$ and standard deviation of $\sigma = 1$.

The cumulative distribution function $F(x)$ associated with the normal probability density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$ is given by

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = P(X \le x) \tag{11.52}$$

and represents the area under the probability curve form $-\infty$ to x. Note that this integral cannot be evaluated in a closed form and one must use numerical methods to calculate the value of the integral for a given value of x. The area calculated represents the probability $P(X \le x)$. The total area under the normal probability density function is unity and so the area

$$1 - F(x) = \int_{x}^{\infty} f(x)\, dx = P(X > x) \tag{11.53}$$

represents the probability $P(X > x)$. These areas are illustrated in the figure 11-7.

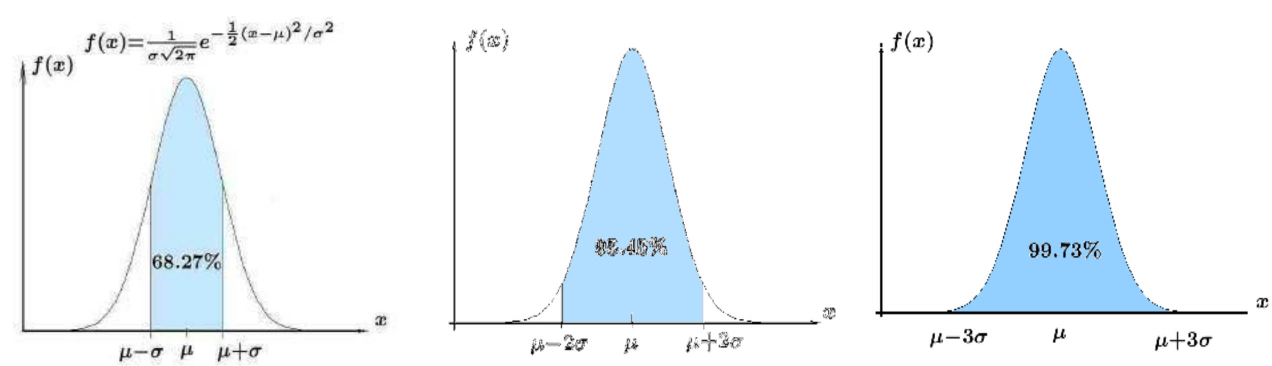


Figure 11-7. Percentage of total area under normal probability curve.

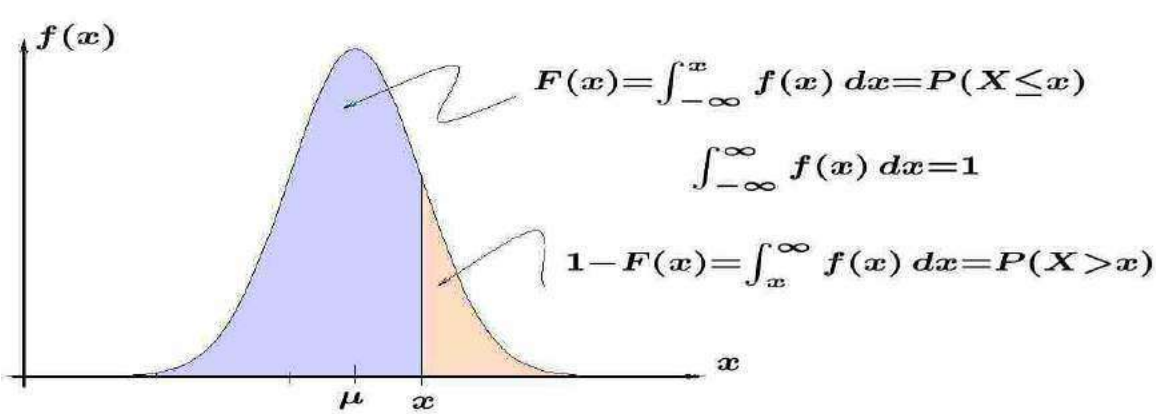


Figure 11-8. Area under normal probability curve representing probabilities.

Standardization

The normal probability density function, sometimes called **the Gaussian distribution**, has the form

$$N(x;\mu,\sigma^2) = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{11.54}$$

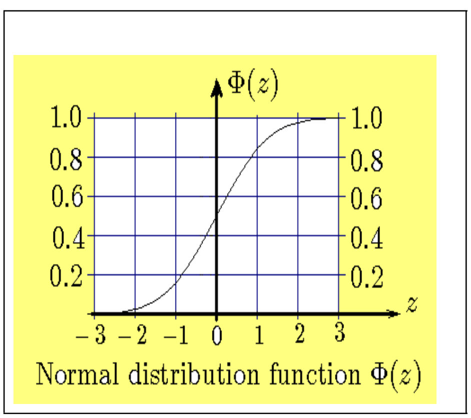
and the area under this curve between the values $x = a$ and $x = b$, where $a < b$, represents **the probability that a random variable X lies between the values a and b.** This probability is represented

$$P(\, a < X < b\,) = \int_a^b f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \tag{11.55}$$

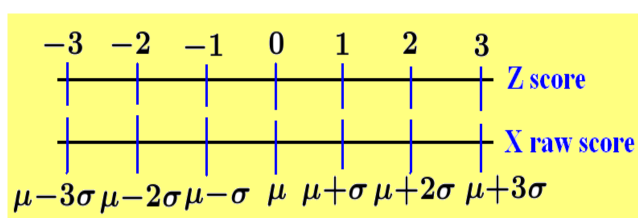
Note that this integral cannot be integrated in closed form and so numerical integration techniques are used to create tables for a normalized form or standard form associated with the above integral. See for example the table 11.5. The distribution function associated with the normal probability density function is given by

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{\xi-\mu}{\sigma}\right)^2}\,d\xi \tag{11.56}$$

Introducing the standardized variable $z = \dfrac{x-\mu}{\sigma}$, with $dz = \dfrac{dx}{\sigma}$, the distribution function, given by equation (11.56), with variable x is converted to a normalized form with variable z. The normalized form and associated scaling is illustrated below,



$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2}\,dz, \tag{11.57}$$



and equation (11.54) is replaced by the standard form for the probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{11.58}$$

which is the integrand of the integral given by equation (11.57). In the representations (11.58) and (11.57) the variable z is called a **random normal number**.

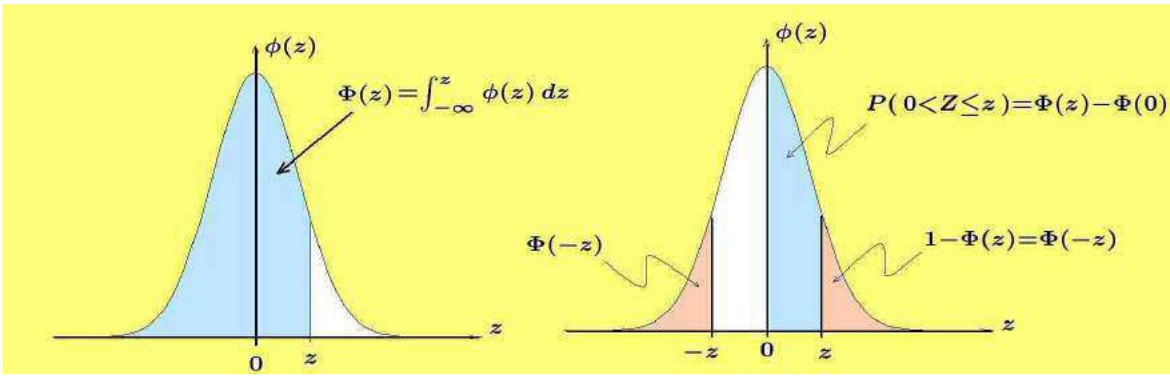


Figure 11-9.

Standard normal probability curve and distribution function as area.

Note that with a change of variable $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ so that in terms of probabilities

$$P(a < X \le b) = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \tag{11.59}$$

In figure 11-9, the normal probability curve is symmetric about $z = 0$ and so the area under the curve between 0 and z represents the probability $P(0 < Z \le z) = \Phi(z) - \Phi(0)$ and quantities like $\Phi(-z)$, by symmetry, have the value $\Phi(-z) = 1-\Phi(z)$. The standard normal curve has the properties that $\Phi(-\infty) = 0$, $\Phi(0) = 1/2$ and $\Phi(\infty) = 1$. The table 11.5 gives **the area under the standard normal curve** for values of $z \ge 0$, then it is possible to employ the **symmetry of the standard normal curve** to calculated specific areas associated with probabilities. For example, to find the area from $-\infty$ to -1.65 examine the table of values and find $\Phi(1.65) = .9505$ so that $\phi(-1.65) = 1 - .9505 = .0495$ or the area from -1.65 to $+\infty$ is .9505, then one can write the probability statement $P(Z > -1.65) = .9505$. As another example, to find the area under the standard normal curve between $z = -1.65$ and $z = 1$, first find the following values

$$\text{Area from } 0 \text{ to } 1 = \Phi(1) - \Phi(0) = .8413 - 0.5000 = .3413$$

$$\text{Area from } 0 \text{ to } 1.65 = \Phi(1.65) - \Phi(0) = .9505 - .5000 = .4505$$

$$\text{Area from } -1.65 \text{ to } 0 = .4505$$

$$\text{Area from } -1.65 \text{ to } 1 = .4505 + .3413 = .7918 = P(-1.65 < Z \le 1)$$

Sketch a graph of the above values as areas under the normal probability density function to get a better understanding of the values presented.

The **normal distribution function** $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\xi^2/2}\,d\xi$ and the **error function**[1] which is defined

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}\,du \tag{11.60}$$

can be related by writing

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\xi^2/2}\,d\xi + \frac{1}{\sqrt{2\pi}}\int_0^z e^{-\xi^2/2}\,d\xi = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^z e^{-\xi^2/2}\,d\xi$$

and then making the substitutions $u = \xi/\sqrt{2}$, $du = d\xi/\sqrt{2}$ to obtain

$$\Phi(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^{z/\sqrt{2}} e^{-u^2}\sqrt{2}\,du = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{z}{\sqrt{2}}\right) \tag{11.61}$$

The normal probability functions given by equations (11.49) and (11.57) are known by other names such as Gaussian distribution, normal curve, bell shaped curve, etc. The normal distribution occurs in the study of various types of errors such as measurements in the quality and precision control of tools and equipment. The normal distribution arises in many different applied areas of the physical and social sciences because of **the central limit theorem**. The central limit theorem, sometimes called **the law of large numbers**, involves consequences of taking large samples from any kind of distribution and can be described as follows. Perform an experiment and select n independent random variables X from some population. If $x_1, x_2, \ldots, x_n$ represents the set of n independent random variables selected, then the mean m_1 of this sample can be constructed. Perform the experiment again and calculate the mean m_2 of the second set of n random independent variables. Continue doing this same experiment a large number of times and collect all the mean values from each experiment. This gives a set of average values $S = \{m_1, m_2, \ldots, m_N\}$ created from performing the experiment N times. The central limit theorem says that the distribution of the set of average values S approaches a normal probability distribution with mean μ_s and variance σ_s^2 given by

$$\mu_s = \text{Mean of the set of averages from a large number of samples} = \mu$$
$$\sigma_s^2 = \text{Variance of set of averages from large number of samples} = \frac{\sigma^2}{n}$$

where μ and σ^2 represent the true mean and true variance of the population being sampled. The central limit theorem always holds and does not depend upon the

[1] There are alternative definitions of the error function due to scaling.

shape of the original distribution being sampled. The normal distribution is also related to **least-square estimation**. It is also used as the theoretical basis for **the chi-square, student-t and F-distributions**. The normal distribution is used in many Monte Carlo simulation computer programs.

The Binomial Distribution

The binomial probability distribution is given by

$$b(x;n,p) = f(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0,1,2,\dots,n \\ 0, & \text{otherwise} \end{cases} \qquad q = 1-p \tag{11.62}$$

It is a discrete probability distribution with parameters n and p where n represents the number of trials and p represents the probability of success in a single trial with $q = 1-p$ the probability of failure in a single trial. For large values of n the binomial distribution approaches the normal distribution. In equation (11.62), the function $f(x)$ represents the probability of x successes and $n-x$ failures in n-trials. The cumulative probabilities are given by

$$F(x) = B(x;n,p) = \sum_{k=0}^{x} b(k;n,p), \qquad \text{for } x = 0,1,2,\dots,n \tag{11.63}$$

As an exercise verify that

$$b(x;n,p) = b(n-x;n,1-p), \qquad B(x;n,p) = 1 - B(n-x-1;n,1-p) \tag{11.64}$$

The **binomial probability law**, sometimes called **the Bernoulli distribution**, occurs in those application areas where one of two possible outcomes can result in a single trial. For example, (yes, no), (success, failure), (left, right), (on, off), (defective, nondefective), etc. For example, if there are d defective items in a bin of N items and an item is selected at random from the bin, then the probability of obtaining a defective item in a single trial is $p = d/N$. The binomial probability distribution involves sampling with replacement. Consequently, each time a sample of n items is selected from the bin containing N items, the probability of obtaining x defective items is given by equation (11.62) with $p = d/N$ and $q = 1 - d/N$.

In the equation (11.62), the term

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}, \qquad \binom{n}{0} = 1, \quad \text{and} \quad \binom{0}{0} = 1 \tag{11.65}$$

represent the binomial coefficients in the binomial expansion

$$(p+q)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \binom{n}{2}p^{n-2}q^2 + \cdots + \binom{n}{x}p^x q^{n-x} + \cdots + \binom{n}{n}q^n = 1 \quad (11.66)$$

In equations (11.62) and (11.66) the term $\binom{n}{x}$ **represents the number of different way of selecting x-objects from a collection of n-objects** and the term

$$p^x q^{n-x} = \underbrace{pp\cdots p}_{\text{x times}}\ \underbrace{qq\cdots q}_{\text{n-x times}} \quad (11.67)$$

represents the probability of x successes and $n-x$ failures in n-trials without regard to any ordering of the arrangements of how the successes or failures occur. Consequently, the equation (11.67) must be multiplied by the number of different arrangements of the successes and failures and this is what produces the binomial probability distribution.

The binomial distribution has the following properties

$$\text{mean} = \mu = np \qquad \text{and} \qquad \text{variance} = \sigma^2 = npq \quad (11.68)$$

The figure 11-10 illustrates the binomial distribution for the parameter values $n = 10$ and $p = 0.2, 0.5$ and 0.9.

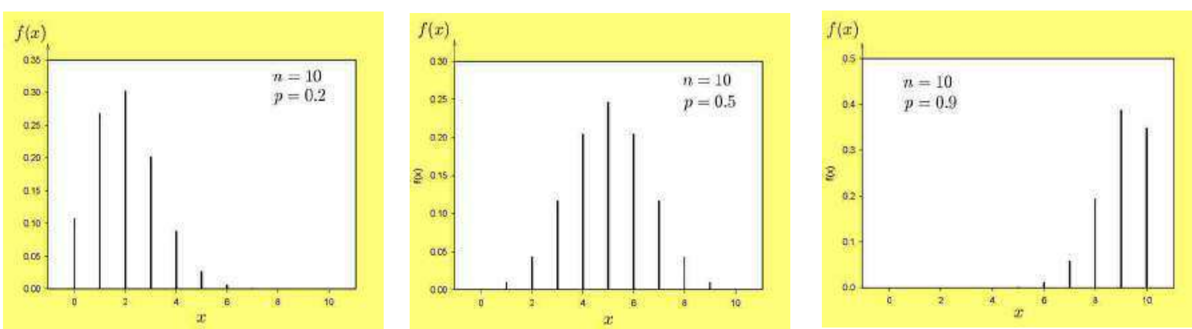


Figure 11-10. Selected binomial distributions for $n = 10$.

The Multinomial Distribution

The **multinomial distribution** occurs when **many events can happen during a single trial**. If only one event can result from m mutually exclusive events $E_1, E_2, \ldots, E_m$ occurring in a single trial, where $p_1, p_2, \ldots, p_m$ are the probabilities assigned to the m-events, then the probability of getting n_1 $E_1's$, n_2 $E_2's, \ldots, n_m$ $E_m's$ is given by the multinomial probability function

$$f(n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!n_2!\ldots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

where $n_1 + n_2 + \cdots + n_m = n$ and $p_1 + p_2 + \cdots + p_m = 1$.

The Poisson Distribution

The **Poisson probability distribution** has the form

$$f(x;\lambda) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0,1,2,3,\ldots \tag{11.69}$$

with parameter $\lambda > 0$. Here x is an integer which can increase without bound. The Poisson probability distribution has the following properties,

1. $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right) = e^{-\lambda}e^{\lambda} = 1$
2. mean$=\mu = \lambda$
3. variance $\sigma^2 = \lambda$

The cumulative probability function is given by

$$F(x;\lambda) = \sum_{k=0}^{x} f(k;\lambda) \tag{11.70}$$

The Poisson distribution occurs in application areas which record isolated events over a period of time. For example, the number of cars entering an intersection in a ten minute interval, the number of telephone lines in use during different periods of the day, the number of customers waiting in line, the life expectancy of a light bulb, the number of transistors that fail in one year, etc.

The figure 11-11 illustrates the Poisson distribution for the parameter values $\lambda = 1/2, 1, 2$ and 3.

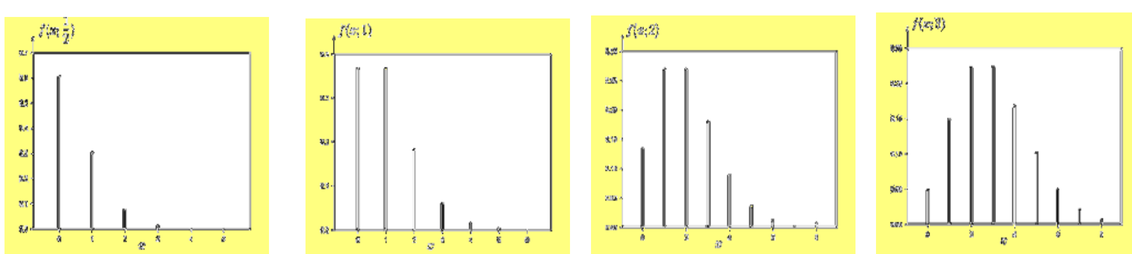


Figure 11-11. Selected Poisson distributions for $\lambda = \frac{1}{2}, 1, 2, 3$.

In general, the Poisson distribution is **a discrete probability distribution** used to determine **the number of events occurring in a fixed interval of time**.

The Hypergeometric Distribution

The **hypergeometric probability distribution** has the form

$$f(x) = h(x; n, n_1, n_2) = \frac{\binom{n_1}{x}\binom{n_2}{n-x}}{\binom{n_1+n_2}{n}}, \qquad x = 0, 1, 2, 3, \ldots, n \tag{11.71}$$

where x is an integer satisfying $0 \le x \le n$, n_1 represents the number of successes and n_2 represents the number of failures, where n items are selected from $(n_1 + n_2)$ items without replacement. This is **a probability distribution with three parameters**, n, n_1 and n_2. The hypergeometric probability distribution is used in quality control, estimates of animal population size from capture-recapture data, the spread of an infectious disease when a fixed number of individuals are exposed to an illness.

Note that the binomial distribution is used in **sampling with replacement** while the hypergeometric distribution is applicable for problems where there is **sampling without replacement**. The hypergeometric distribution has mean

$$\mu = \frac{nn_1}{n_1 + n_2}$$

and variance given by

$$\sigma^2 = \frac{nn_1n_2(n_1+n_2-n)}{(n_1+n_2)^2(n_1+n_2-1)}$$

The equation (11.71) represents the probability of x successes and $n-x$ failures selected from n_1+n_2 items where the sampling is without replacement. For example, to find the probability of selecting two aces from a standard deck of 52 playing cards in 6 draws with no replacement of cards selected one would select the following parameters for the hypergeometric distribution. Here there are 6 draws so that $n = 6$. There are 4 aces in the deck so $n_1 = 4$ is the number of successes in the deck and $n_2 = 48$ is the number of failures in the deck, with $n_1 + n_2 = 52$ the total number of cards in the deck. The hypergeometric distribution gives the probability of $x = 2$ successes in $n = 6$ draws as

$$h(2; 6, 4, 48) = \frac{\binom{4}{2}\binom{48}{4}}{\binom{52}{6}} = \frac{621}{10829} = 0.0573$$

The Exponential Distribution

The **exponential probability distribution** is a continuous probability distribution with parameter $\lambda > 0$ and is defined

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{11.72}$$

The exponential distribution is used in studying time to failure of a piece of equipment , waiting time for next event to occur, like waiting time for an elevator, or time waiting in line to be served. This distribution has the mean

$$\mu = \lambda$$

and the variance is given by

$$\sigma^2 = \lambda^2$$

Note that the area under the probability curve $f(x)$, for $-\infty < x < \infty$ is equal to 1 or

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} \lambda e^{-\lambda x}\, dx = 1$$

The Gamma Distribution

The **gamma probability distribution** is defined

$$f(x) = \begin{cases} \dfrac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1}\, e^{-x/\theta}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases} \tag{11.73}$$

where $\Gamma(\alpha)$ is the gamma function. This probability density function has the two parameters $\alpha > 0$ and $\theta > 0$. It is a continuous probability distribution with a shape parameter α and scale parameter θ. The gamma distribution is used frequently in econometrics.

This probability distribution arises in determining the waiting time for a given number of events to occur. For example, waiting for 10 calls to a switch board, or life testing until a failure occurs. It also occurs in weather prediction of precipitation processes. The gamma distribution has mean

$$\mu = \alpha\theta \qquad \text{and variance} \qquad \sigma^2 = \alpha\, \theta^2$$

The gamma distribution with parameters $\alpha = 1$ and $\theta = 1/\lambda$ produces the exponential distribution. The figure 11-12 illustrates the gamma distribution for selected values of the parameters α and θ.

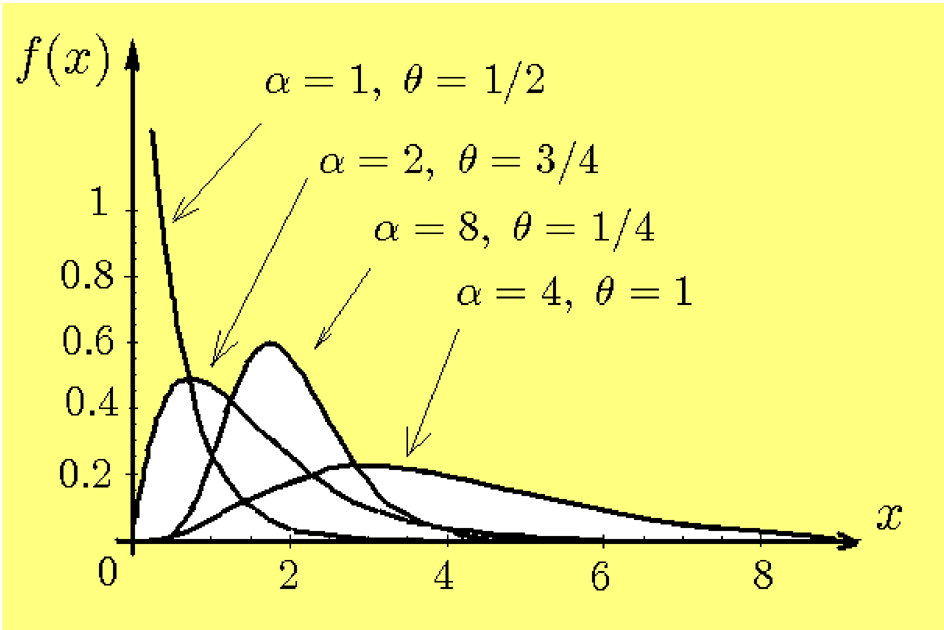


Figure 11-12. The gamma distribution for selected values of α and θ

Chi-Square χ^2 Distribution

The **chi-square probability distribution** has the form

$$f(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu-2)/2}\, e^{-x/2}, & \text{for } x > 0 \\ 0, & \text{elsewhere} \end{cases} \tag{11.74}$$

where $\Gamma(\ )$ represents the gamma function[2] and $\nu = 1, 2, 3, \ldots$ is a parameter called the number of degrees of freedom. Note that the chi-square distribution is sometimes written as the χ^2-distribution. It is a special case of the gamma distribution when the parameters of the gamma distribution take on the values $\alpha = \nu/2$ and $\theta = 2$. This distribution has the mean $\mu = \nu$ and variance $\sigma^2 = 2\nu$.

The chi-square distribution is used in testing of hypothesis, determining confidence intervals and testing differences in various statistics associated with independent samples. The tables 11.6(a) and (b) give values for areas under the probability density function.

[2] Recall the gamma function is defined $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ with the property $\Gamma(x+1) = x\Gamma(x)$.

Student's t-Distribution

The **student's[3] t-distribution** with n degrees of freedom is given by the probability density function

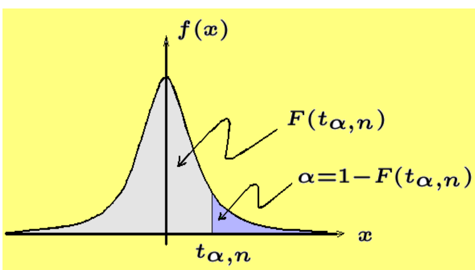
$$f(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \qquad -\infty < x < \infty \tag{11.75}$$

where $\Gamma(\ )$ denotes the gamma function and $n = 1, 2, 3, \ldots$ is a parameter. The student's t-distribution has the mean 0 for $n > 1$, otherwise the mean is undefined. Similarly, the variance is given by $\frac{n}{n-2}$ for $n > 2$, otherwise the variance is undefined.

The cumulative distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx \tag{11.76}$$

The table 11.6 contains values of $t_{\alpha,n}$ which satisfy the equation



$$\int_{t_{\alpha,n}}^{\infty} f(x)\,dx = \alpha = 1 - F(t_{\alpha,n}) \tag{11.77}$$

The normal distribution is related to the student's t-distribution as follows. If $\overline{x}$ and s are the mean and standard deviation associated with a random sample of size n from a normal distribution $N(x;\mu,\sigma^2)$, then the quantity $\frac{(\overline{x}-\mu)\sqrt{n}}{s}$ has a student-t-distribution with $n-1$ degrees of freedom.

The student's t-distribution is a continuous probability distribution used to estimate the mean of a population where (i) the population has a normal distribution (ii) the sample size from the population is small and (iii) the standard deviation of the population is unknown. The table 11.7 gives values for area under this probability density function.

[3] Developed by W.S. Gosset who used the name "Student" as a pseudonym.

The F-Distribution

The **F-distribution** has the probability density function

$$f(x) = f_{n,m}(x) = \begin{cases} \dfrac{\Gamma\left(\dfrac{m+n}{2}\right)}{\Gamma\left(\dfrac{m}{2}\right)\Gamma\left(\dfrac{n}{2}\right)} n^{n/2} m^{m/2} \dfrac{x^{n/2-1}}{(m+nx)^{(m+n)/2}}, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{11.78}$$

which is sometimes given in the form

$$f(x) = f_{n,m}(x) = \begin{cases} \dfrac{\Gamma\left(\dfrac{m+n}{2}\right)}{\Gamma\left(\dfrac{m}{2}\right)\Gamma\left(\dfrac{n}{2}\right)} (n/m)^{n/2} \dfrac{x^{n/2-1}}{(1+\frac{n}{m}x)^{(m+n)/2}}, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{11.79}$$

where $\Gamma(\ )$ denotes the gamma function. The F-distribution has the parameters $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$.

If X_1 and X_2 are independent random variables associated with a chi-square distribution having respectively the degrees of freedom n and m, then the quantity $Y = \dfrac{X_1/n}{X_2/m}$ will have a F-distribution with n and m degrees of freedom.

The tables 11.6 (a)(b)(c)(d)(e) contain values of $F_{\alpha,n,m}$ such that

$$\int_{F_{(\alpha,n,m)}}^{\infty} f_{n,m}(x)\, dx = \alpha$$

for α having the values $0.1, 0.05, .025, .01,$ and $.005$. Observe the symmetry of the F-distribution and note that in the use of the upper tail values from the tables it is customary to employ the relation

$$F(df_m, df_n, 1-\alpha/2) = \frac{1}{F(df_n, df_m, \alpha/2)} \tag{11.80}$$

where df_m and df_n denote the degrees of freedom for m and n.

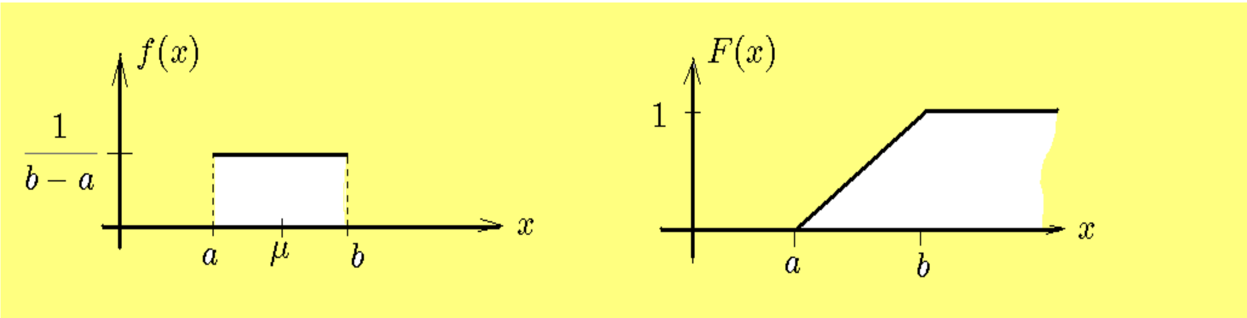
The chi-square, student t and F distributions are used in testing of hypothesis, confidence intervals and testing differences or ratios of various statistics associated with independent samples. The degrees of freedom associated with these distributions can be thought of as a parameter representing an increase in reliability of the calculated statistic. That is, a statistic associated with one degree of freedom is less reliable than the same statistic calculated using a higher degree of freedom. In some cases the degrees of freedom are related to the number of data points used to calculate the statistic. In some cases the degrees of freedom is obtained by subtracting 1 from the sample size n.

The Uniform Distribution

The uniform probability density function $f(x)$ and the associated distribution function $F(x)$ are given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{a}^{x} f(x)\,dx$$

It is sometimes referred to as the rectangular distribution on the interval $a < x < b$.



This distribution has the mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{a}^{b} x \frac{1}{b-a}\,dx = \frac{1}{2}(a+b)$$

and variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \frac{1}{12}(b-a)^2$$

The cumulative distribution function is given by $F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$

The uniform probability density function is used in pseudo-random number generators with sampling is over the interval $0 \le x \le 1$.

Confidence Intervals

Sampling theory is a study of the various relationships that exist between properties of a population and information obtained based upon samples from the population. For example, each sample collected from a population has associated with it a sample mean $\overline{x} = \mu_{\overline{x}}$ and sample variance $s^2 = \sigma^2_{\overline{x}}$. How do these quantities compare with the true population mean μ and true population variance σ^2? It would be nice to put limits, like numbers γ_1, γ_2, associated with the value $\mu_{\overline{x}}$ so that one can write a statement like

$$\mu_{\overline{x}} - \gamma_1 < \mu < \mu_{\overline{x}} + \gamma_2$$

It would also be nice to be able to adjust γ_1 and γ_2 so that one could say that there is a 90% probability that the true mean lies within the specified limits. It would be better still if one could change the 90% value to obtain limits for say a 95%,97%,or 99% probability that the true mean lie within the bounds specified. The probability values $90\%, 95\%, 97\%$ or 99% are called **confidence levels** associated with the calculated mean value. To determine such limits one can employ **the central limit theorem** from statistics which says that if (i) the number n of independent random variables in each sample (the sample size) is large with a finite mean and variance for each sample and (ii) the number of samples taken is large. Then the mean value associated with the large set of sample means will be normally distributed.

Another way to state the above is as follows. For X a continuous random variable which comes from some kind of probability distribution having a well defined mean μ and variance σ^2, the **central limit theorem** states that if a **large number of sample means are collected**, and one forms a table of these mean values and does an analysis of the collected set of n means and forms a frequency table, just like table 11.3, then one finds that these sample means are approximately normally distributed. The central limit theorem also states that the distribution of the sample means can be made as close to a normal distribution as desired, by taking larger and larger sample sizes. It can be shown that the distribution of the sample means $\bar{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$. The normal distribution can be scaled to standard form by making an appropriate change of variables.

To use the central limit theorem select a confidence level $\gamma = 1-\alpha$ which represents the area between the limits $-z_{\alpha/2}$ and $z_{\alpha/2}$ associated with the normalized normal probability density function as illustrated in the figure 11-13. This determines values α and $\alpha/2$ and the values $\pm z_{\alpha/2}$ can be obtained from the normalized probability table 11.5. Some example values for $1-\alpha$ are

$1-\alpha$	.90	.95	.99	.999
α	.10	.05	.01	.001
$\alpha/2$	.05	.025	.005	.0005
$z_{\alpha/2}$	1.645	1.960	2.576	3.291

If $\overline{x}$ is the mean of a sample $\{x_1, x_2, \ldots, x_n\}$ of size n, then confidence limits on the value $\overline{x}$ are determined as follows.

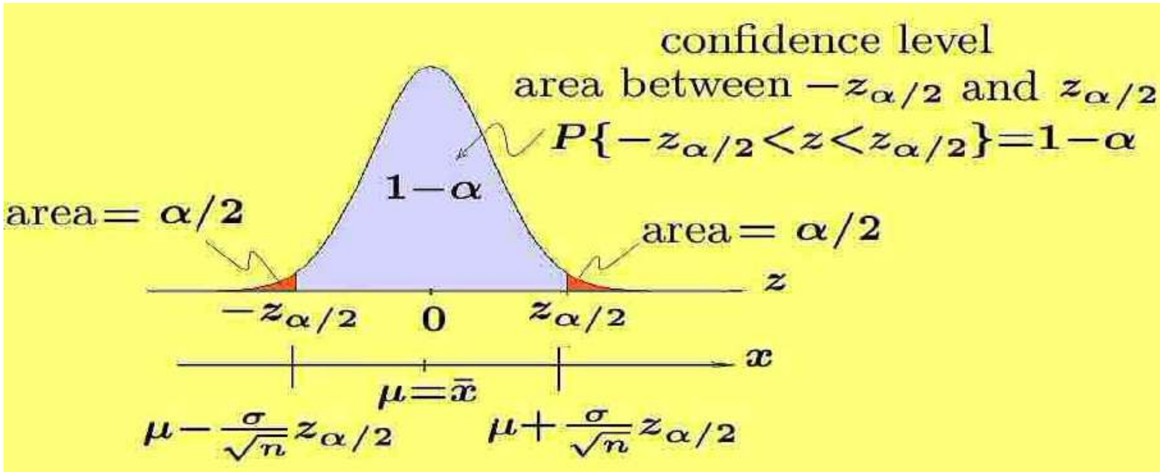


Figure 11-13. Raw scores scaled to normal probability density function.

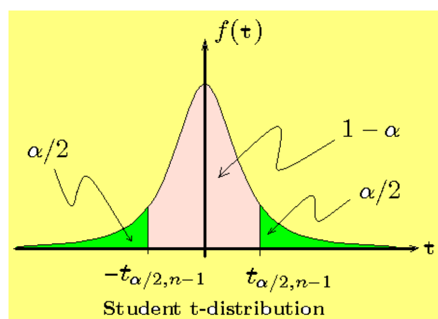
Normal distribution with known variance σ^2

If the variance of the population is known then use the central limit theorem to construct the following confidence interval for the mean μ of the population based upon a $1-\alpha=\gamma$ level of confidence

$$CONF\{\overline{x}-\frac{\sigma}{\sqrt{n}}z_{\alpha/2}\le\mu\le\overline{x}+\frac{\sigma}{\sqrt{n}}z_{\alpha/2}\} \tag{11.81}$$

Normal distribution with unknown variance σ^2

In the case where the population variance is unknown, then make use of the fact that $t=\dfrac{|\overline{x}-\mu|}{s/\sqrt{n}}$ follows **a student t-distribution** to construct a confidence interval. From the student t-distribution determine the value $t_{\alpha/2,n-1}$ based upon $n-1$ degrees of freedom, n being the sample size, such that the right tailed area under equals $\alpha/2$ as illustrated in the accompanying figure.



Some examples for a sample size of $n=11$ and degrees of freedom $n-1=10$ are given in the following table.

$1-\alpha$	.90	.95	.99	.999
α	.10	.05	.01	.001
$\alpha/2$	.05	.025	.005	.0005
$t_{\alpha/2,10}$	1.812	2.228	3.169	4.144

The confidence interval for the mean μ of the population uses the computed variance

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})^2 \tag{11.82}$$

to produce the $\gamma = 1 - \alpha$ confidence level

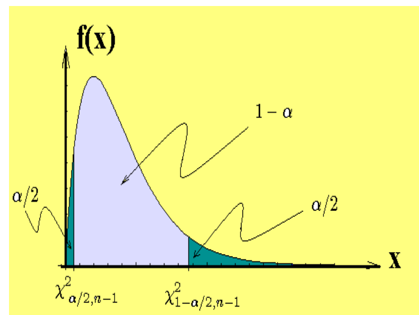
$$CONF\,\{\overline{x} - \frac{s}{\sqrt{n}}\,t_{\alpha/2,n-1} \le \mu \le \overline{x} + \frac{s}{\sqrt{n}}\,t_{\alpha/2,n-1}\} \tag{11.83}$$

where n is the sample size.

Confidence interval for the variance σ^2

The confidence interval for the variance σ^2 of the population having a normal distribution is based upon the fact that the variable $Y = (n-1)s^2/\sigma^2$ follows **a chi-square distribution with $n-1$ degrees of freedom**, where again n represents the sample size.

First select a level of confidence $\gamma = 1 - \alpha$ and then from a chi-square distribution table with $n-1$ degrees of freedom determine the $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ values which represent the points corresponding to the tail areas of the chi-square probability density function as illustrated.



Secondly, one must calculate the variance squared s^2 using equation (11.82), then construct the confidence interval for the variance of a normal distribution given by

$$CONF\,\{(n-1)\frac{s^2}{\chi^2_{1-\alpha/2,n-1}} \le \sigma^2 \le (n-1)\frac{s^2}{\chi^2_{\alpha/2,n-1}}\} \tag{11.84}$$

Least Squares Curve Fitting

A set of data points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_i, y_i), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ can be plotted on ordinary graph paper and then a line $y = \beta_0 + \beta_1 x$ can also be plotted to obtain a figure such as illustrated in the figure 11-14.

Assume that the data points are normally distributed about the straight line and that errors $e_1, e_2, \ldots, e_n$ occur in the y-variable, where the errors are defined as **the differences between the y-value on the line and the y-value of the data point**. What would be the "best" straight line to represent the given data points? There are many ways to define "best". By defining the error e_i associated with the ith data point (x_i, y_i) as

$$\begin{aligned} e_i =&(y \text{ of line at } x_i) - (y \text{ data value at } x_i) \\ e_i =&\beta_0 + \beta_1 x_i - y_i \end{aligned} \tag{11.85}$$

then associated with the given set of data are the errors

$$\begin{aligned} e_1 =&y(x_1) - y_1 = \beta_0 + \beta_1 x_1 - y_1 \\ e_2 =&y(x_2) - y_2 = \beta_0 + \beta_1 x_2 - y_2 \\ e_3 =&y(x_3) - y_3 = \beta_0 + \beta_1 x_3 - y_3 \\ &\vdots \\ e_n =&y(x_n) - y_n = \beta_0 + \beta_1 x_n - y_n. \end{aligned} \tag{11.86}$$

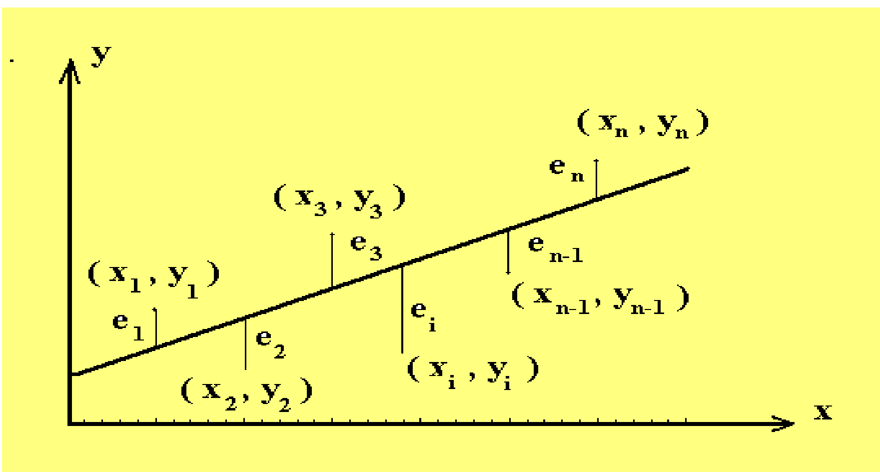


Figure 11-14. Straight line approximation to represent data points.

One way to define the "best" straight line $y = \beta_0 + \beta_1 x$ is to select the constants β_0 and β_1 which **minimize the sum of squares of the errors** associated with the data set. That is, if

$$E = E(\beta_0, \beta_1) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)^2 \tag{11.87}$$

denotes the sum of squares of the errors, then E has a minimum value when the conditions

$$\frac{\partial E}{\partial \beta_0} = 0 \quad \text{and} \quad \frac{\partial E}{\partial \beta_1} = 0 \tag{11.88}$$

are satisfied simultaneously. Hence, the constants β_0 and β_1 must be selected to satisfy the simultaneous equations

$$
\begin{aligned}
\frac{\partial E}{\partial \beta_0} &= 2\sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)\,(1) = 0 \\
\frac{\partial E}{\partial \beta_1} &= 2\sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)\,(x_i) = 0.
\end{aligned}
\tag{11.89}
$$

The equations (11.89) simplify to the 2×2 linear system of equations

$$
\begin{aligned}
n\beta_0 + \left(\sum_{i=1}^{n} x_i\right)\beta_1 &= \sum_{i=1}^{n} y_i \\
\left(\sum_{i=1}^{n} x_i\right)\beta_0 + \left(\sum_{i=1}^{n} x_i^2\right)\beta_1 &= \sum_{i=1}^{n} x_i y_i
\end{aligned}
\tag{11.90}
$$

which can then be solved for the coefficients β_0 and β_1. This gives the "best" least squares straight line $y = y(x) = \beta_0 + \beta_1 x$.

Alternatively, set all of the equations (11.86) equal to zero, to obtain a system of equations having the matrix form

$$
\begin{aligned}
A\overline{\beta} &= \overline{\mathbf{y}} \\
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}.
\end{aligned}
\tag{11.91}
$$

By doing this the data set of errors, calculated from the difference in the data set y values and the straight line y values, is represented as an over determined system of equations for determining the constants β_0 and β_1. That is, there are more equations than there are unknowns and so the unknowns β_0, β_1 are selected to minimize the sum of squares error associated with the over determined system of equations. Observe that left multiplying both sides of equation (11.91) by the transpose matrix A^T gives the new set of equations $A^T A\overline{\beta} = A^T\overline{\mathbf{y}}$ or

$$
\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}
$$

which simplifies to

$$\begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix} \tag{11.92}$$

which is the matrix form of the equations (11.90). This presents an alternative way to solve for the coefficients β_0 and β_1

Linear Regression

The previous least squares method applied to a straight line fit of data. The ideas presented can be generalized to fitting data to **any linear combination of functions**. Given a set of data points (x_i, y_i), for $i = 1, 2, \dots, n$, assume a curve fit function of the form

$$y = y(x) = \beta_0 f_0(x) + \beta_1 f_1(x) + \beta_2 f_2(x) + \cdots + \beta_k f_k(x) \tag{11.93}$$

where $\beta_0, \beta_1, \dots, \beta_k$ are unknown coefficients and $f_0(x), f_1(x), f_2(x), \dots, f_k(x)$ represent linearly independent functions, called the basis of the representation. Note that for the previous straight line fit the independent functions $f_0(x) = 1$ and $f_1(x) = x$ were used. In general, select any set of independent functions and select the β coefficients such that the sum of squares error

$$\begin{aligned} E = E(\beta_0, \beta_1, \dots, \beta_k) &= \sum_{i=1}^n (y(x_i) - y_i)^2 \\ E = E(\beta_0, \beta_1, \dots, \beta_k) &= \sum_{i=1}^n \left[\beta_0 f_0(x_i) + \beta_1 f_1(x_i) + \beta_2 f_2(x_i) + \cdots + \beta_k f_k(x_i) - y_i\right]^2 \end{aligned} \tag{11.94}$$

is a minimum. The determination of the β-values requires a solution be found from the set of **simultaneous least square equations**

$$\frac{\partial E}{\partial \beta_0} = 0, \quad \frac{\partial E}{\partial \beta_1} = 0, \quad \cdots, \quad \frac{\partial E}{\partial \beta_k} = 0. \tag{11.95}$$

Another way to obtain the system of equations (11.95) is to first represent the data in the matrix form

$$A\overline{\beta} = \overline{\mathbf{y}}$$

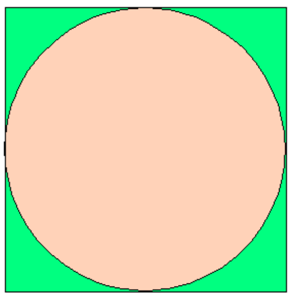
$$\begin{bmatrix} f_0(x_1) & f_1(x_1) & f_2(x_1) & \cdots & f_k(x_1) \\ f_0(x_2) & f_1(x_2) & f_2(x_2) & \cdots & f_k(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & f_2(x_n) & \cdots & f_k(x_n) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{11.96}$$

Both sides of the equation (11.96) can be left multiplied by the transpose matrix A^T and the resulting system can be solved for the unknown coefficients. In matrix notation write

$$\begin{aligned} A\overline{\beta} =& \overline{\mathbf{y}} \\ A^T A\overline{\beta} =& A^T\overline{\mathbf{y}} \\ \overline{\beta} =& (A^T A)^{-1} A^T \overline{\mathbf{y}}. \end{aligned} \tag{11.97}$$

The solution of the system of equations (11.95) or (11.97) will produce the coefficients β_i, $i = 0, 1, \ldots, k$, which minimizes the sum of square error.

Monte Carlo Methods



Monte Carlo methods is a term used to describe a wide variety of computer techniques which employ **random number generators** to simulate an event or events and then perform a statistical analysis of the results. Sometimes Monte Carlo methods are constructed to solve difficult problems where deterministic methods fail. If performed properly, Monte Carlo methods can give very accurate answers. The only drawback is that **some** Monte Carlo techniques take a very long time to run on the computer.

An example of a simple Monte Carlo method is the calculation of the area A of a circle using random numbers. Consider a circle with radius 1/2 which is placed inside the unit square having vertices $(0,0), (1,0), (1,1)$ and $(0,1)$. The area of this circle is $\pi/4 = 0.7853981634....$

Most computer languages have **a uniform random number generator** which generates pseudo-random numbers lying between 0 and 1. Construct a computer program which employs the uniform random number generator to generate two random numbers (x_r, y_r), where $0 < x_r < 1$ and $0 < y_r < 1$, then imagine the circle inside the unit square as a circular dart board and the random number generated by the computer program (x_r, y_r) is where the dart lands. Construct the computer program to perform a test as to whether the point (x_r, y_r) is on or off the circular dart board. Perform this test N-times and record the number of hits which land on or inside the circle. To calculate the area of the circle assume the ratio of hits inside circle to total number of points generated is in the same proportion as the area of the circle is to the area of the square. One can then use the ratio

$$\frac{\text{Number hits inside circle}}{\text{Total number of darts thrown}} = \frac{\text{Area of circle}}{\text{Area of square}}$$

to determine the area A of the circle. If H denotes the number of hits inside the circle and N represents the total number of darts thrown, the area of the circle is determined by $\frac{H}{N} = \frac{A}{1} = A$.

Class midpoint for Average Area	Frequency
0.78027	0
0.780783	0
0.781295	0
0.781808	2
0.782321	4
0.782834	12
0.783347	19
0.78386	44
0.784372	63
0.784885	69
0.785398	79
0.785911	62
0.786424	65
0.786937	44
0.78745	20
0.787962	9
0.788475	6
0.788988	2
0.789501	0
0.790014	0
0.790527	0

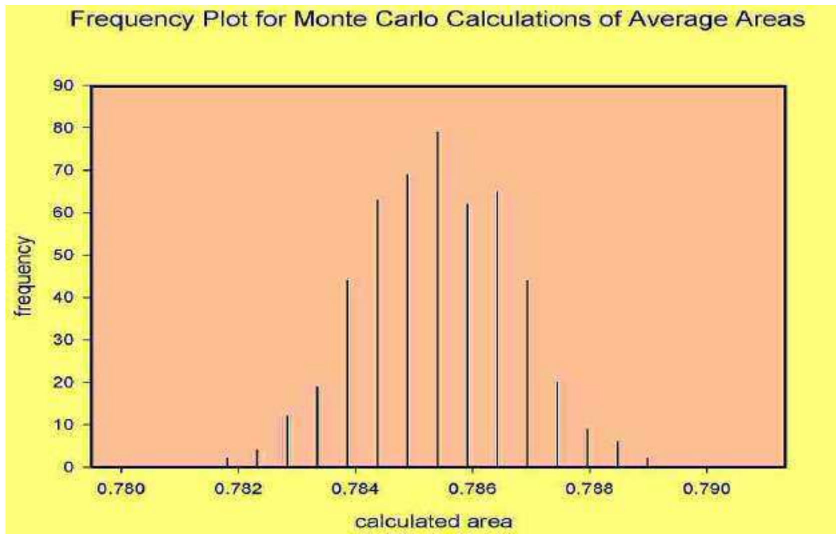


Figure 11-15. Average areas from Monte Carlo simulation.

Perform the above experiment K-times to calculate a set of approximate areas $\{A_1, A_2, \ldots, A_K\}$ having an average area $\overline{A} = \frac{1}{K}\sum_{i=1}^{K} A_i$. Put all of the above computer code in a loop and calculate M-averages $\{\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_M\}$. The central limit theorem tells us that the set of averages must be normally distributed. By calculating the mean and standard deviation associated with all these averages it is possible to determine very accurate bounds on the area of the circle. Using the values $N = 1000$ throws, $K = 100$ areas, and $M = 500$ area averages, modern laptop computers can calculate the results in less than one minute.

The data generated for the above values of N,K and M is presented in figure 11-15 as a bar chart having a mean 0.7853974 and standard deviation 0.001282.

Obtaining a Uniform Random Number Generator

Some form of a uniform random number generator, called a pseudorandom number generator and abbreviated (PRNG), can usually be found as an intrinsic function within many of the more popular computer programming languages. If the computer language you are using does not have a uniform random number generator, then you can obtain one from off the internet. Pseudorandom number generators generate a sequence of numbers $\{x_n\}$ satisfying $0 \le x_n \le 1$. The sequence of numbers generated is not truly random because of technical issues involving the mathematical methods used to generate the uniform random numbers. Most of the PRNG programs in use have passed many statistical tests which guarantee that the sequence generated is random enough for Monte Carlo studies and other statistical applications.

Linear Interpolation

In obtaining a specific numerical value from a table of (x, y) values it is sometimes necessary to use **linear interpolation** where a line is constructed between two known numerical values and then values along the line are used as estimates for the tabular values between the known values. In one dimension, one can say that if (x_1, y_1) and (x_2, y_2) are known values, then if x is a value between x_1 and x_2, the corresponding value for y is given by $y = y_1 + \left(\frac{y_2-y_1}{x_2-x_1}\right)(x - x_1)$. This result can be expressed in a variety of forms. One form is to make the substitution $x_2 - x_1 = h$ with $x = x_1 + \beta h$, then $y = y_1 + \beta(y_2 - y_1)$ or $y = (1 - \beta)y_1 + \beta y_2$.

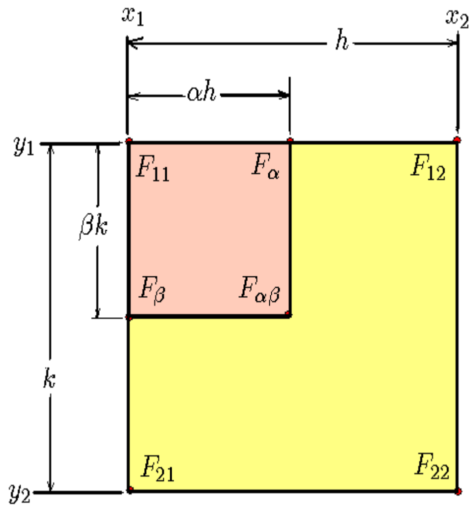
Interpolation in two-dimension

Consider the set of values in a table as illustrated in the accompanying figure. Let F_{11} denote the value in the table corresponding to the position (x_1, y_1). Similarly, define the values F_{12}, F_{21} and F_{22} corresponding respectively to the points (x_1, y_2), (x_2, y_1) and (x_2, y_2). Interpolation over this two dimensional array is the problem of determining the values F_α, F_β and $F_{\alpha,\beta}$ which are positioned on the boundaries and interior to the box connecting the known data values $F_{11}, F_{12}, F_{22}, F_{21}$.

		x_1	x_2	
	7.956	7.471	7.134	6.885
	7.343	6.872	6.545	6.302
	6.881	6.422	6.102	5.865
	6.521	6.071	5.757	5.525
y_1	6.233	5.791	5.482	5.253
y_2	5.998	5.562	5.257	5.031
	5.803	5.372	5.071	4.847
	5.638	5.212	4.913	4.692

Interpolate first in the x-direction and then in the y-direction or vice-versa and show that

$$F_\alpha =(1-\alpha)F_{11}+\alpha F_{12}, \qquad F_\beta=(1-\beta)F_{11}+\beta F_{21}$$
$$F_{\alpha,\beta} =(1-\alpha)(1-\beta)F_{11}+\alpha(1-\beta)F_{12}+\beta(1-\alpha)F_{21}+\alpha\beta F_{22}$$



Note how the F_α and F_β values vary as the parameters α and β vary from 0 to 1. This is a straight forward linear interpolation between the given values. The value $F_{\alpha,\beta}$ is obtained by first doing a linear interpolation in the y direction at the columns x_1 and x_2, which is then followed by a linear interpolation in the x-direction.

An alternative method of interpolation is to use the Taylor series expansion in both the x and y directions to obtain the alternative interpolation formula

$$F_{\alpha,\beta}=(1-\alpha-\beta)F_{11}+\beta F_{21}+\alpha F_{12}$$

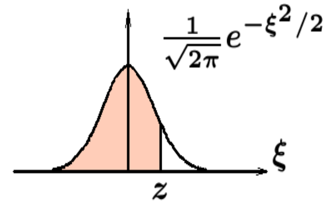
Sometimes it is necessary to modify the above interpolation formulas for application to entries in a three-dimensional array of numbers. The interpolation result is obtained by applying the one-dimensional interpolation formulas in each of the x, y and z directions.

Statistical Tables

This introduction to the study of statistics concludes with some well known statistical tables. These tables are employed in various types of statistics testing. Statistical tables in many forms where extensively used prior to the advent of computers. The internet provides the access to a much larger variety of statistical tables.

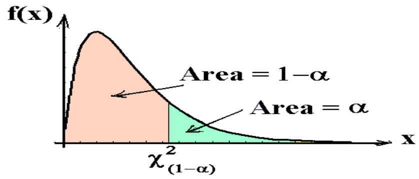
Table 11.5 Area Under Standard Normal Curve

$$Area = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$$



z	Area	z	Area	z	Area	z	Area	z	Area	z	Area	z	Area
0.00	.5000	.50	.6915	1.00	.8413	1.50	.9332	2.00	.9772	2.50	.9938	3.00	.9987
.01	.5040	.51	.6950	1.01	.8438	1.51	.9345	2.01	.9778	2.51	.9940	3.01	.9987
.02	.5080	.52	.6985	1.02	.8461	1.52	.9357	2.02	.9783	2.52	.9941	3.02	.9987
.03	.5120	.53	.7019	1.03	.8485	1.53	.9370	2.03	.9788	2.53	.9943	3.03	.9988
.04	.5160	.54	.7054	1.04	.8508	1.54	.9382	2.04	.9793	2.54	.9945	3.04	.9988
.05	.5199	.55	.7088	1.05	.8531	1.55	.9394	2.05	.9798	2.55	.9946	3.05	.9989
.06	.5239	.56	.7123	1.06	.8554	1.56	.9406	2.06	.9803	2.56	.9948	3.06	.9989
.07	.5279	.57	.7157	1.07	.8577	1.57	.9418	2.07	.9808	2.57	.9949	3.07	.9989
.08	.5319	.58	.7190	1.08	.8599	1.58	.9429	2.08	.9812	2.58	.9951	3.08	.9990
.09	.5359	.59	.7224	1.09	.8621	1.59	.9441	2.09	.9817	2.59	.9952	3.09	.9990
.10	.5398	.60	.7257	1.10	.8643	1.60	.9452	2.10	.9821	2.60	.9953	3.10	.9990
.11	.5438	.61	.7291	1.11	.8665	1.61	.9463	2.11	.9826	2.61	.9955	3.11	.9991
.12	.5478	.62	.7324	1.12	.8686	1.62	.9474	2.12	.9830	2.62	.9956	3.12	.9991
.13	.5517	.63	.7357	1.13	.8708	1.63	.9484	2.13	.9834	2.63	.9957	3.13	.9991
.14	.5557	.64	.7389	1.14	.8729	1.64	.9495	2.14	.9838	2.64	.9959	3.14	.9992
.15	.5596	.65	.7422	1.15	.8749	1.65	.9505	2.15	.9842	2.65	.9960	3.15	.9992
.16	.5636	.66	.7454	1.16	.8770	1.66	.9515	2.16	.9846	2.66	.9961	3.16	.9992
.17	.5675	.67	.7486	1.17	.8790	1.67	.9525	2.17	.9850	2.67	.9962	3.17	.9992
.18	.5714	.68	.7517	1.18	.8810	1.68	.9535	2.18	.9854	2.68	.9963	3.18	.9993
.19	.5753	.69	.7549	1.19	.8830	1.69	.9545	2.19	.9857	2.69	.9964	3.19	.9993
.20	.5793	.70	.7580	1.20	.8849	1.70	.9554	2.20	.9861	2.70	.9965	3.20	.9993
.21	.5832	.71	.7611	1.21	.8869	1.71	.9564	2.21	.9864	2.71	.9966	3.21	.9993
.22	.5871	.72	.7642	1.22	.8888	1.72	.9573	2.22	.9868	2.72	.9967	3.22	.9994
.23	.5910	.73	.7673	1.23	.8907	1.73	.9582	2.23	.9871	2.73	.9968	3.23	.9994
.24	.5948	.74	.7704	1.24	.8925	1.74	.9591	2.24	.9875	2.74	.9969	3.24	.9994
.25	.5987	.75	.7734	1.25	.8944	1.75	.9599	2.25	.9878	2.75	.9970	3.25	.9994
.26	.6026	.76	.7764	1.26	.8962	1.76	.9608	2.26	.9881	2.76	.9971	3.26	.9994
.27	.6064	.77	.7794	1.27	.8980	1.77	.9616	2.27	.9884	2.77	.9972	3.27	.9995
.28	.6103	.78	.7823	1.28	.8997	1.78	.9625	2.28	.9887	2.78	.9973	3.28	.9995
.29	.6141	.79	.7852	1.29	.9015	1.79	.9633	2.29	.9890	2.79	.9974	3.29	.9995
.30	.6179	.80	.7881	1.30	.9032	1.80	.9641	2.30	.9893	2.80	.9974	3.30	.9995
.31	.6217	.81	.7910	1.31	.9049	1.81	.9649	2.31	.9896	2.81	.9975	3.31	.9995
.32	.6255	.82	.7939	1.32	.9066	1.82	.9656	2.32	.9898	2.82	.9976	3.32	.9995
.33	.6293	.83	.7967	1.33	.9082	1.83	.9664	2.33	.9901	2.83	.9977	3.33	.9996
.34	.6331	.84	.7995	1.34	.9099	1.84	.9671	2.34	.9904	2.84	.9977	3.34	.9996
.35	.6368	.85	.8023	1.35	.9115	1.85	.9678	2.35	.9906	2.85	.9978	3.35	.9996
.36	.6406	.86	.8051	1.36	.9131	1.86	.9686	2.36	.9909	2.86	.9979	3.36	.9996
.37	.6443	.87	.8078	1.37	.9147	1.87	.9693	2.37	.9911	2.87	.9979	3.37	.9996
.38	.6480	.88	.8106	1.38	.9162	1.88	.9699	2.38	.9913	2.88	.9980	3.38	.9996
.39	.6517	.89	.8133	1.39	.9177	1.89	.9706	2.39	.9916	2.89	.9981	3.39	.9997
.40	.6554	.90	.8159	1.40	.9192	1.90	.9713	2.40	.9918	2.90	.9981	3.40	.9997
.41	.6591	.91	.8186	1.41	.9207	1.91	.9719	2.41	.9920	2.91	.9982	3.41	.9997
.42	.6628	.92	.8212	1.42	.9222	1.92	.9726	2.42	.9922	2.92	.9982	3.42	.9997
.43	.6664	.93	.8238	1.43	.9236	1.93	.9732	2.43	.9925	2.93	.9983	3.43	.9997
.44	.6700	.94	.8264	1.44	.9251	1.94	.9738	2.44	.9927	2.94	.9984	3.44	.9997
.45	.6736	.95	.8289	1.45	.9265	1.95	.9744	2.45	.9929	2.95	.9984	3.45	.9997
.46	.6772	.96	.8315	1.46	.9279	1.96	.9750	2.46	.9931	2.96	.9985	3.46	.9997
.47	.6808	.97	.8340	1.47	.9292	1.97	.9756	2.47	.9932	2.97	.9985	3.47	.9997
.48	.6844	.98	.8365	1.48	.9306	1.98	.9761	2.48	.9934	2.98	.9986	3.48	.9997
.49	.6879	.99	.8389	1.49	.9319	1.99	.9767	2.49	.9936	2.99	.9986	3.49	.9998

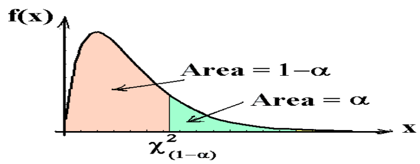
Table 11.6(a). Critical Values for the Chi-square Distribution with ν Degrees of Freedom



$$\int_0^{\chi^2_{(1-\alpha)}} \frac{x^{(\nu/2)-1}}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2}\, dx = 1-\alpha$$

α	0.995	0.990	0.975	0.950	0.900
$1-\alpha$	0.005	0.010	0.025	0.050	0.100
ν	$\chi^2_{0.005}$	$\chi^2_{0.010}$	$\chi^2_{0.025}$	$\chi^2_{0.050}$	$\chi^2_{0.100}$
1	0.0000	0.0002	0.0010	0.0039	0.0158
2	0.0100	0.0201	0.0506	0.1026	0.2107
3	0.0717	0.1148	0.2158	0.3518	0.5844
4	0.2070	0.2971	0.4844	0.7107	1.0636
5	0.4117	0.5543	0.8312	1.1455	1.6103
6	0.6757	0.8721	1.2373	1.6354	2.2041
7	0.9893	1.2390	1.6899	2.1673	2.8331
8	1.3444	1.6465	2.1797	2.7326	3.4895
9	1.7349	2.0879	2.7004	3.3251	4.1682
10	2.1559	2.5582	3.2470	3.9403	4.8652
11	2.6032	3.0535	3.8157	4.5748	5.5778
12	3.0738	3.5706	4.4038	5.2260	6.3038
13	3.5650	4.1069	5.0088	5.8919	7.0415
14	4.0747	4.6604	5.6287	6.5706	7.7895
15	4.6009	5.2293	6.2621	7.2609	8.5468
16	5.1422	5.8122	6.9077	7.9616	9.3122
17	5.6972	6.4078	7.5642	8.6718	10.0852
18	6.2648	7.0149	8.2307	9.3905	10.8649
19	6.8440	7.6327	8.9065	10.1170	11.6509
20	7.4338	8.2604	9.5908	10.8508	12.4426
21	8.0337	8.8972	10.2829	11.5913	13.2396
22	8.6427	9.5425	10.9823	12.3380	14.0415
23	9.2604	10.1957	11.6886	13.0905	14.8480
24	9.8862	10.8564	12.4012	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1602	12.1981	13.8439	15.3792	17.2919
27	11.8076	12.8785	14.5734	16.1514	18.1139
28	12.4613	13.5647	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7084	19.7677
30	13.7867	14.9535	16.7908	18.4927	20.5992
40	20.7065	22.1643	24.4330	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7643	37.6886
60	35.5345	37.4849	40.4817	43.1880	46.4589
70	43.2752	45.4417	48.7576	51.7393	55.3289
80	51.1719	53.5401	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2911
100	67.3276	70.0649	74.2219	77.9295	82.3581

Table 11.6(b). Critical Values for the Chi-square Distribution with ν Degrees of Freedom

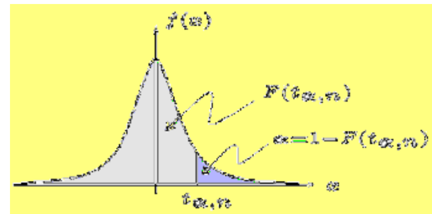


$$\int_0^{\chi^2_{(1-\alpha)}} \frac{x^{(\nu/2)-1}}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2}\, dx = 1-\alpha$$

α	0.100	0.050	0.025	0.010	0.005
$1-\alpha$	0.900	0.950	0.975	0.990	0.995
ν	$\chi^2_{0.900}$	$\chi^2_{0.950}$	$\chi^2_{0.975}$	$\chi^2_{0.990}$	$\chi^2_{0.995}$
1	2.7055	3.8415	5.0239	6.6349	7.8794
2	4.6052	5.9915	7.3778	9.2103	10.5966
3	6.2514	7.8147	9.3484	11.3449	12.8382
4	7.7794	9.4877	11.1433	13.2767	14.8603
5	9.2364	11.0705	12.8325	15.0863	16.7496
6	10.6446	12.5916	14.4494	16.8119	18.5476
7	12.0170	14.0671	16.0128	18.4753	20.2777
8	13.3616	15.5073	17.5345	20.0902	21.9550
9	14.6837	16.9190	19.0228	21.6660	23.5894
10	15.9872	18.3070	20.4832	23.2093	25.1882
11	17.2750	19.6751	21.9200	24.7250	26.7568
12	18.5493	21.0261	23.3367	26.2170	28.2995
13	19.8119	22.3620	24.7356	27.6882	29.8195
14	21.0641	23.6848	26.1189	29.1412	31.3193
15	22.3071	24.9958	27.4884	30.5779	32.8013
16	23.5418	26.2962	28.8454	31.9999	34.2672
17	24.7690	27.5871	30.1910	33.4087	35.7185
18	25.9894	28.8693	31.5264	34.8053	37.1565
19	27.2036	30.1435	32.8523	36.1909	38.5823
20	28.4120	31.4104	34.1696	37.5662	39.9968
21	29.6151	32.6706	35.4789	38.9322	41.4011
22	30.8133	33.9244	36.7807	40.2894	42.7957
23	32.0069	35.1725	38.0756	41.6384	44.1813
24	33.1962	36.4150	39.3641	42.9798	45.5585
25	34.3816	37.6525	40.6465	44.3141	46.9279
26	35.5632	38.8851	41.9232	45.6417	48.2899
27	36.7412	40.1133	43.1945	46.9629	49.6449
28	37.9159	41.3371	44.4608	48.2782	50.9934
29	39.0875	42.5570	45.7223	49.5879	52.3356
30	40.2560	43.7730	46.9792	50.8922	53.6720
40	51.8051	55.7585	59.3417	63.6907	66.7660
50	63.1671	67.5048	71.4202	76.1539	79.4900
60	74.3970	79.0819	83.2977	88.3794	91.9517
70	85.5270	90.5312	95.0232	100.4252	104.2149
80	96.5782	101.8795	106.6286	112.3288	116.3211
90	107.5650	113.1453	118.1359	124.1163	128.2989
100	118.4980	124.3421	129.5612	135.8067	140.1695

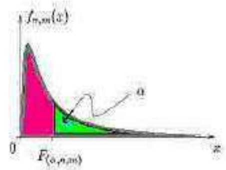
Table 11.7 Critical Values $t_{\alpha,n}$ for the Student's t Distribution with n Degrees of Freedom

$$\int_{t_{\alpha,n}}^{\infty} \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n}\right)^{-(n+1)/2} dx = \alpha = 1 - F(t_{\alpha,n})$$



n	$t_{0.1,n}$	$t_{0.05,n}$	$t_{0.025,n}$	$t_{0.01,n}$	$t_{0.005,n}$	$t_{0.001}$
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
40	1.303	1.684	2.021	2.423	2.704	3.307
50	1.299	1.676	2.009	2.403	2.678	3.261
60	1.296	1.671	2.000	2.390	2.660	3.232
70	1.294	1.667	1.994	2.381	2.648	3.211
80	1.292	1.664	1.990	2.374	2.639	3.195
90	1.291	1.662	1.987	2.368	2.632	3.183
100	1.290	1.660	1.984	2.364	2.626	3.174
∞	1.282	1.645	1.960	2.326	2.576	3.098

Table 11.8 (a) Critical Values of the F-Distribution for $\alpha = 0.1$

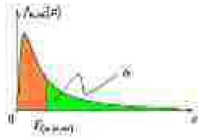


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	39.863	49.500	53.593	55.833	57.240	58.204	58.906	59.439	59.858	60.195	61.220	61.740	62.055	62.265	62.529	62.794	63.061
2	8.526	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.381	9.392	9.425	9.441	9.451	9.458	9.466	9.475	9.483
3	5.538	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.240	5.230	5.200	5.184	5.175	5.168	5.160	5.151	5.143
4	4.545	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.936	3.920	3.870	3.844	3.828	3.817	3.804	3.790	3.775
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297	3.238	3.207	3.187	3.174	3.157	3.140	3.123
6	3.776	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.958	2.937	2.871	2.836	2.815	2.800	2.781	2.762	2.742
7	3.589	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.725	2.703	2.632	2.595	2.571	2.555	2.535	2.514	2.493
8	3.458	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.561	2.538	2.464	2.425	2.400	2.383	2.361	2.339	2.316
9	3.360	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.440	2.416	2.340	2.298	2.272	2.255	2.232	2.208	2.184
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323	2.244	2.201	2.174	2.155	2.132	2.107	2.082
11	3.225	2.860	2.660	2.536	2.451	2.389	2.342	2.304	2.274	2.248	2.167	2.123	2.095	2.076	2.052	2.026	2.000
12	3.177	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.214	2.188	2.105	2.060	2.031	2.011	1.986	1.960	1.932
13	3.136	2.763	2.560	2.434	2.347	2.283	2.234	2.195	2.164	2.138	2.053	2.007	1.978	1.958	1.931	1.904	1.876
14	3.102	2.726	2.522	2.395	2.307	2.243	2.193	2.154	2.122	2.095	2.010	1.962	1.933	1.912	1.885	1.857	1.828
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059	1.972	1.924	1.894	1.873	1.845	1.817	1.787
16	3.048	2.668	2.462	2.333	2.244	2.178	2.128	2.088	2.055	2.028	1.940	1.891	1.860	1.839	1.811	1.782	1.751
17	3.026	2.645	2.437	2.308	2.218	2.152	2.102	2.061	2.028	2.001	1.912	1.862	1.831	1.809	1.781	1.751	1.719
18	3.007	2.624	2.416	2.286	2.196	2.130	2.079	2.038	2.005	1.977	1.887	1.837	1.805	1.783	1.754	1.723	1.691
19	2.990	2.606	2.397	2.266	2.176	2.109	2.058	2.017	1.984	1.956	1.865	1.814	1.782	1.759	1.730	1.699	1.666
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937	1.845	1.794	1.761	1.738	1.708	1.677	1.643
21	2.961	2.575	2.365	2.233	2.142	2.075	2.023	1.982	1.948	1.920	1.827	1.776	1.742	1.719	1.689	1.657	1.623
22	2.949	2.561	2.351	2.219	2.128	2.060	2.008	1.967	1.933	1.904	1.811	1.759	1.726	1.702	1.671	1.639	1.604
23	2.937	2.549	2.339	2.207	2.115	2.047	1.995	1.953	1.919	1.890	1.796	1.744	1.710	1.686	1.655	1.622	1.587
24	2.927	2.538	2.327	2.195	2.103	2.035	1.983	1.941	1.906	1.877	1.783	1.730	1.696	1.672	1.641	1.607	1.571
25	2.918	2.528	2.317	2.184	2.092	2.024	1.971	1.929	1.895	1.866	1.771	1.718	1.683	1.659	1.627	1.593	1.557
30	2.881	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.849	1.819	1.722	1.667	1.632	1.606	1.573	1.538	1.499
40	2.835	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.793	1.763	1.662	1.605	1.568	1.541	1.506	1.467	1.425
50	2.809	2.412	2.197	2.061	1.966	1.895	1.840	1.796	1.760	1.729	1.627	1.568	1.529	1.502	1.465	1.424	1.379
60	2.791	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.738	1.707	1.603	1.543	1.504	1.476	1.437	1.395	1.348
100	2.756	2.356	2.139	2.002	1.906	1.834	1.778	1.732	1.695	1.663	1.557	1.494	1.453	1.423	1.382	1.336	1.282
120	2.748	2.347	2.130	1.992	1.896	1.824	1.767	1.722	1.684	1.652	1.545	1.482	1.440	1.409	1.368	1.320	1.265

Table 11.8 (b) Critical Values of the F-Distribution for $\alpha = 0.05$

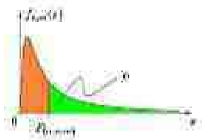


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	161.448	199.500	215.707	224.583	230.162	233.986	236.768	238.883	240.543	241.882	245.950	248.013	249.260	250.095	251.143	252.196	253.253
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396	19.429	19.446	19.456	19.462	19.471	19.479	19.487
3	10.128	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786	8.703	8.660	8.634	8.617	8.594	8.572	8.549
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.858	5.803	5.769	5.746	5.717	5.688	5.658
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735	4.619	4.558	4.521	4.496	4.464	4.431	4.398
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	3.938	3.874	3.835	3.808	3.774	3.740	3.705
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.511	3.445	3.404	3.376	3.340	3.304	3.267
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347	3.218	3.150	3.108	3.079	3.043	3.005	2.967
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.006	2.936	2.893	2.864	2.826	2.787	2.748
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978	2.845	2.774	2.730	2.700	2.661	2.621	2.580
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.719	2.646	2.601	2.570	2.531	2.490	2.448
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.617	2.544	2.498	2.466	2.426	2.384	2.341
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671	2.533	2.459	2.412	2.380	2.339	2.297	2.252
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.463	2.388	2.341	2.308	2.266	2.223	2.178
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544	2.403	2.328	2.280	2.247	2.204	2.160	2.114
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.352	2.276	2.227	2.194	2.151	2.106	2.059
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.308	2.230	2.181	2.148	2.104	2.058	2.011
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.269	2.191	2.141	2.107	2.063	2.017	1.968
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.234	2.155	2.106	2.071	2.026	1.980	1.930
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.203	2.124	2.074	2.039	1.994	1.946	1.896
21	4.325	3.467	3.072	2.840	2.685	2.573	2.488	2.420	2.366	2.321	2.176	2.096	2.045	2.010	1.965	1.916	1.866
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.151	2.071	2.020	1.984	1.938	1.889	1.838
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275	2.128	2.048	1.996	1.961	1.914	1.865	1.813
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.108	2.027	1.975	1.939	1.892	1.842	1.790
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.236	2.089	2.007	1.955	1.919	1.872	1.822	1.768
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.015	1.932	1.878	1.841	1.792	1.740	1.683
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077	1.924	1.839	1.783	1.744	1.693	1.637	1.577
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026	1.871	1.784	1.727	1.687	1.634	1.576	1.511
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.836	1.748	1.690	1.649	1.594	1.534	1.467
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927	1.768	1.676	1.616	1.573	1.515	1.450	1.376
120	3.920	3.072	2.680	2.447	2.290	2.175	2.087	2.016	1.959	1.910	1.750	1.659	1.598	1.554	1.495	1.429	1.352

Table 11.8 (c) Critical Values of the F-Distribution for $\alpha = 0.025$

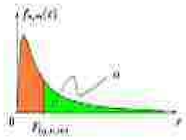


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	647.789	799.500	864.163	899.583	921.848	937.111	948.217	956.656	963.285	968.627	984.867	993.103	998.081	1001.414	1005.598	1009.800	1014.020
2	38.506	39.000	39.165	39.248	39.298	39.331	39.355	39.373	39.387	39.398	39.431	39.448	39.458	39.465	39.473	39.481	39.490
3	17.443	16.044	15.439	15.101	14.885	14.735	14.624	14.540	14.473	14.419	14.253	14.167	14.115	14.081	14.037	13.992	13.947
4	12.218	10.649	9.979	9.605	9.364	9.197	9.074	8.980	8.905	8.844	8.657	8.560	8.501	8.461	8.411	8.360	8.309
5	10.007	8.434	7.764	7.388	7.146	6.978	6.853	6.757	6.681	6.619	6.428	6.329	6.268	6.227	6.175	6.123	6.069
6	8.813	7.260	6.599	6.227	5.988	5.820	5.695	5.600	5.523	5.461	5.269	5.168	5.107	5.065	5.012	4.959	4.904
7	8.073	6.542	5.890	5.523	5.285	5.119	4.995	4.899	4.823	4.761	4.568	4.467	4.405	4.362	4.309	4.254	4.199
8	7.571	6.059	5.416	5.053	4.817	4.652	4.529	4.433	4.357	4.295	4.101	3.999	3.937	3.894	3.840	3.784	3.728
9	7.209	5.715	5.078	4.718	4.484	4.320	4.197	4.102	4.026	3.964	3.769	3.667	3.604	3.560	3.505	3.449	3.392
10	6.937	5.456	4.826	4.468	4.236	4.072	3.950	3.855	3.779	3.717	3.522	3.419	3.355	3.311	3.255	3.198	3.140
11	6.724	5.256	4.630	4.275	4.044	3.881	3.759	3.664	3.588	3.526	3.330	3.226	3.162	3.118	3.061	3.004	2.944
12	6.554	5.096	4.474	4.121	3.891	3.728	3.607	3.512	3.436	3.374	3.177	3.073	3.008	2.963	2.906	2.848	2.787
13	6.414	4.965	4.347	3.996	3.767	3.604	3.483	3.388	3.312	3.250	3.053	2.948	2.882	2.837	2.780	2.720	2.659
14	6.298	4.857	4.242	3.892	3.663	3.501	3.380	3.285	3.209	3.147	2.949	2.844	2.778	2.732	2.674	2.614	2.552
15	6.200	4.765	4.153	3.804	3.576	3.415	3.293	3.199	3.123	3.060	2.862	2.756	2.689	2.644	2.585	2.524	2.461
16	6.115	4.687	4.077	3.729	3.502	3.341	3.219	3.125	3.049	2.986	2.788	2.681	2.614	2.568	2.509	2.447	2.383
17	6.042	4.619	4.011	3.665	3.438	3.277	3.156	3.061	2.985	2.922	2.723	2.616	2.548	2.502	2.442	2.380	2.315
18	5.978	4.560	3.954	3.608	3.382	3.221	3.100	3.005	2.929	2.866	2.667	2.559	2.491	2.445	2.384	2.321	2.256
19	5.922	4.508	3.903	3.559	3.333	3.172	3.051	2.956	2.880	2.817	2.617	2.509	2.441	2.394	2.333	2.270	2.203
20	5.871	4.461	3.859	3.515	3.289	3.128	3.007	2.913	2.837	2.774	2.573	2.464	2.396	2.349	2.287	2.223	2.156
21	5.827	4.420	3.819	3.475	3.250	3.090	2.969	2.874	2.798	2.735	2.534	2.425	2.356	2.308	2.246	2.182	2.114
22	5.786	4.383	3.783	3.440	3.215	3.055	2.934	2.839	2.763	2.700	2.498	2.389	2.320	2.272	2.210	2.145	2.076
23	5.750	4.349	3.750	3.408	3.183	3.023	2.902	2.808	2.731	2.668	2.466	2.357	2.287	2.239	2.176	2.111	2.041
24	5.717	4.319	3.721	3.379	3.155	2.995	2.874	2.779	2.703	2.640	2.437	2.327	2.257	2.209	2.146	2.080	2.010
25	5.686	4.291	3.694	3.353	3.129	2.969	2.848	2.753	2.677	2.613	2.411	2.300	2.230	2.182	2.118	2.052	1.981
30	5.568	4.182	3.589	3.250	3.026	2.867	2.746	2.651	2.575	2.511	2.307	2.195	2.124	2.074	2.009	1.940	1.866
40	5.424	4.051	3.463	3.126	2.904	2.744	2.624	2.529	2.452	2.388	2.182	2.068	1.994	1.943	1.875	1.803	1.724
50	5.340	3.975	3.390	3.054	2.833	2.674	2.553	2.458	2.381	2.317	2.109	1.993	1.919	1.866	1.796	1.721	1.639
60	5.286	3.925	3.343	3.008	2.786	2.627	2.507	2.412	2.334	2.270	2.061	1.944	1.869	1.815	1.744	1.667	1.581
100	5.179	3.828	3.250	2.917	2.696	2.537	2.417	2.321	2.244	2.179	1.968	1.849	1.770	1.715	1.640	1.558	1.463
120	5.152	3.805	3.227	2.894	2.674	2.515	2.395	2.299	2.222	2.157	1.945	1.825	1.746	1.690	1.614	1.530	1.433

Table 11.8 (d) Critical Values of the F-Distribution for $\alpha = 0.01$

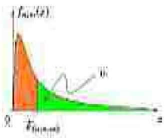


$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

$m \backslash n$	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	4052.181	4999.500	5403.352	5624.583	5763.650	5858.986	5928.356	5981.070	6022.473	6055.847	6157.285	6208.730	6239.825	6260.649	6286.782	6313.030	6339.391
2	98.503	99.000	99.166	99.249	99.299	99.333	99.356	99.374	99.388	99.399	99.433	99.449	99.459	99.466	99.474	99.482	99.491
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345	27.229	26.872	26.690	26.579	26.505	26.411	26.316	26.221
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659	14.546	14.198	14.020	13.911	13.838	13.745	13.652	13.558
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158	10.051	9.722	9.553	9.449	9.379	9.291	9.202	9.112
6	13.745	10.925	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874	7.559	7.396	7.296	7.229	7.143	7.057	6.969
7	12.246	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620	6.314	6.155	6.058	5.992	5.908	5.824	5.737
8	11.259	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.515	5.359	5.263	5.198	5.116	5.032	4.946
9	10.561	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	4.962	4.808	4.713	4.649	4.567	4.483	4.398
10	10.044	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849	4.558	4.405	4.311	4.247	4.165	4.082	3.996
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539	4.251	4.099	4.005	3.941	3.860	3.776	3.690
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296	4.010	3.858	3.765	3.701	3.619	3.535	3.449
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.191	4.100	3.815	3.665	3.571	3.507	3.425	3.341	3.255
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	4.030	3.939	3.656	3.505	3.412	3.348	3.266	3.181	3.094
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.522	3.372	3.278	3.214	3.132	3.047	2.959
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.780	3.691	3.409	3.259	3.165	3.101	3.018	2.933	2.845
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593	3.312	3.162	3.068	3.003	2.920	2.835	2.746
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.227	3.077	2.983	2.919	2.835	2.749	2.660
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.523	3.434	3.153	3.003	2.909	2.844	2.761	2.674	2.584
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.088	2.938	2.843	2.778	2.695	2.608	2.517
21	8.017	5.780	4.874	4.369	4.042	3.812	3.640	3.506	3.398	3.310	3.030	2.880	2.785	2.720	2.636	2.548	2.457
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258	2.978	2.827	2.733	2.667	2.583	2.495	2.403
23	7.881	5.664	4.765	4.264	3.939	3.710	3.539	3.406	3.299	3.211	2.931	2.781	2.686	2.620	2.535	2.447	2.354
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168	2.889	2.738	2.643	2.577	2.492	2.403	2.310
25	7.770	5.568	4.675	4.177	3.855	3.627	3.457	3.324	3.217	3.129	2.850	2.699	2.604	2.538	2.453	2.364	2.270
30	7.562	5.390	4.510	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.700	2.549	2.453	2.386	2.299	2.208	2.111
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.522	2.369	2.271	2.203	2.114	2.019	1.917
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698	2.419	2.265	2.167	2.098	2.007	1.909	1.803
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.352	2.198	2.098	2.028	1.936	1.836	1.726
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503	2.223	2.067	1.965	1.893	1.797	1.692	1.572
120	6.851	4.787	3.949	3.480	3.174	2.956	2.792	2.663	2.559	2.472	2.192	2.035	1.932	1.860	1.763	1.656	1.533

Table 11.8 (e) Critical Values of the F-Distribution for $\alpha = 0.005$



$$\int_{F_{(\alpha,n,m)}}^{\infty} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} n^{n/2} m^{m/2} \frac{x^{n/2-1}\,dx}{(m+nx)^{(m+n)/2}} = \alpha$$

n is degrees of freedom for numerator and m is degrees of freedom for denominator

m \ n	1	2	3	4	5	6	7	8	9	10	15	20	25	30	40	60	120
1	16210.723	19999.500	21614.741	22499.583	23055.798	23437.111	23714.566	23925.406	24091.004	24224.487	24630.205	24835.971	24960.340	25043.628	25148.153	25253.137	25358.573
2	198.501	199.000	199.166	199.250	199.300	199.333	199.357	199.375	199.388	199.400	199.433	199.450	199.460	199.466	199.475	199.483	199.491
3	55.552	49.799	47.467	46.195	45.392	44.838	44.434	44.126	43.882	43.686	43.085	42.778	42.591	42.466	42.308	42.149	41.989
4	31.333	26.284	24.259	23.155	22.456	21.975	21.622	21.352	21.139	20.967	20.438	20.167	20.002	19.892	19.752	19.611	19.468
5	22.785	18.314	16.530	15.556	14.940	14.513	14.200	13.961	13.772	13.618	13.146	12.903	12.755	12.656	12.530	12.402	12.274
6	18.635	14.544	12.917	12.028	11.464	11.073	10.786	10.566	10.391	10.250	9.814	9.589	9.451	9.358	9.241	9.122	9.001
7	16.236	12.404	10.882	10.050	9.522	9.155	8.885	8.678	8.514	8.380	7.968	7.754	7.623	7.534	7.422	7.309	7.193
8	14.688	11.042	9.596	8.805	8.302	7.952	7.694	7.496	7.339	7.211	6.814	6.608	6.482	6.396	6.288	6.177	6.065
9	13.614	10.107	8.717	7.956	7.471	7.134	6.885	6.693	6.541	6.417	6.032	5.832	5.708	5.625	5.519	5.410	5.300
10	12.826	9.427	8.081	7.343	6.872	6.545	6.302	6.116	5.968	5.847	5.471	5.274	5.153	5.071	4.966	4.859	4.750
11	12.226	8.912	7.600	6.881	6.422	6.102	5.865	5.682	5.537	5.418	5.049	4.855	4.736	4.654	4.551	4.445	4.337
12	11.754	8.510	7.226	6.521	6.071	5.757	5.525	5.345	5.202	5.085	4.721	4.530	4.412	4.331	4.228	4.123	4.015
13	11.374	8.186	6.926	6.233	5.791	5.482	5.253	5.076	4.935	4.820	4.460	4.270	4.153	4.073	3.970	3.866	3.758
14	11.060	7.922	6.680	5.998	5.562	5.257	5.031	4.857	4.717	4.603	4.247	4.059	3.942	3.862	3.760	3.655	3.547
15	10.798	7.701	6.476	5.803	5.372	5.071	4.847	4.674	4.536	4.424	4.070	3.883	3.766	3.687	3.585	3.480	3.372
16	10.575	7.514	6.303	5.638	5.212	4.913	4.692	4.521	4.384	4.272	3.920	3.734	3.618	3.539	3.437	3.332	3.224
17	10.384	7.354	6.156	5.497	5.075	4.779	4.559	4.389	4.254	4.142	3.793	3.607	3.492	3.412	3.311	3.206	3.097
18	10.218	7.215	6.028	5.375	4.956	4.663	4.445	4.276	4.141	4.030	3.683	3.498	3.382	3.303	3.201	3.096	2.987
19	10.073	7.093	5.916	5.268	4.853	4.561	4.345	4.177	4.043	3.933	3.587	3.402	3.287	3.208	3.106	3.000	2.891
20	9.944	6.986	5.818	5.174	4.762	4.472	4.257	4.090	3.956	3.847	3.502	3.318	3.203	3.123	3.022	2.916	2.806
21	9.830	6.891	5.730	5.091	4.681	4.393	4.179	4.013	3.880	3.771	3.427	3.243	3.128	3.049	2.947	2.841	2.730
22	9.727	6.806	5.652	5.017	4.609	4.322	4.109	3.944	3.812	3.703	3.360	3.176	3.061	2.982	2.880	2.774	2.663
23	9.635	6.730	5.582	4.950	4.544	4.259	4.047	3.882	3.750	3.642	3.300	3.116	3.001	2.922	2.820	2.713	2.602
24	9.551	6.661	5.519	4.890	4.486	4.202	3.991	3.826	3.695	3.587	3.246	3.062	2.947	2.868	2.765	2.658	2.546
25	9.475	6.598	5.462	4.835	4.433	4.150	3.939	3.776	3.645	3.537	3.196	3.013	2.898	2.819	2.716	2.609	2.496
30	9.180	6.355	5.239	4.623	4.228	3.949	3.742	3.580	3.450	3.344	3.006	2.823	2.708	2.628	2.524	2.415	2.300
40	8.828	6.066	4.976	4.374	3.986	3.713	3.509	3.350	3.222	3.117	2.781	2.598	2.482	2.401	2.296	2.184	2.064
50	8.626	5.902	4.826	4.232	3.849	3.579	3.376	3.219	3.092	2.988	2.653	2.470	2.353	2.272	2.164	2.050	1.925
60	8.495	5.795	4.729	4.140	3.760	3.492	3.291	3.134	3.008	2.904	2.570	2.387	2.270	2.187	2.079	1.962	1.834
100	8.241	5.589	4.542	3.963	3.589	3.325	3.127	2.972	2.847	2.744	2.411	2.227	2.108	2.024	1.912	1.790	1.652
120	8.179	5.539	4.497	3.921	3.548	3.285	3.087	2.933	2.808	2.705	2.373	2.188	2.069	1.984	1.871	1.747	1.606

Exercises

▶ **11-1.** A box contains 10 white balls, 3 black balls and 2 red balls.

(a) What is the probability of drawing a white ball?
(b) What is the probability of drawing a black ball?
(c) What is the probability of drawing a red ball?

▶ **11-2.** A box contains 10 white balls, 3 black balls and 2 red balls.

(a) What is the probability of drawing a white ball and then drawing a black ball?
(b) What is the probability of drawing two white balls?
(c) What is the probability of drawing a red ball and then a black ball?

▶ **11-3.** A bowl of fruit contains 3 apples, 5 oranges and 3 pears. If two fruits are selected at random,

(a) What is the probability of getting 2 pears?
(b) What is the probability of getting 2 apples?
(c) What is the probability of getting 2 oranges?

▶ **11-4.** Calculate the mean, variance and standard deviations of the following.

(a) G=grade of student on 6 exams. $G = \{84, 91, 72, 68, 87, 78\}$
(b) T=test scores for class of 17 students.

$T = \{71, 82, 66, 88, 100, 97, 96, 100, 77, 77, 84, 89, 93, 98, 100, 100, 75\}$

(c) A= Average absenteeism rate in days missed per 100 working days over 6 year period taken from a certain factory. $A = \{8.05, 13.35, 5.10, 4.43, 6.22, 7.81\}$

▶ **11-5.** Show that $s^2 = \frac{1}{n-1}\sum_{j=1}^{m}(x_j - \bar{x})^2 n f_j$ can be written

$$s^2 = \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - \frac{1}{n}\left(\sum_{j=1}^{m} x_j n f_j\right)^2\right\}$$

▶ **11-6.** If a pair of fair dice are rolled and X denote the sum of the upward numbers, then find the probabilities of rolling a 2,3,4,5,6,7,8,9,10,11 and 12.

▸ **11-7.** Find the arithmetic mean, geometric mean and harmonic mean of the given numbers.

$$(a)\quad 1,2,3,4,5,6,7 \qquad (b)\quad 1,3,5,7,9,11,13 \qquad (c)\quad 2,4,6,8,10,12,14$$

▸ **11-8.** What is the probability of getting 10 consecutive heads in the toss of a fair coin?

▸ **11-9.** Given the following sample of ball bearing diameters, in inches, taken over one production cycle.

Ball bearing diameters in inches									
.738	.729	.743	.740	.736	.728	.735	.741	.737	.740
.735	.730	.736	.733	.745	.736	.742	.735	.734	.738
.734	.737	.732	.744	.741	.738	.732	.737	.742	.746
.739	.740	.735	.730	.744	.733	.727	.732	.734	.735
.724	.730	.739	.739	.733	.726	.735	.746	.731	.737
.738	.739	.735	.727	.735	.736	.744	.740	.736	.740

(a) Make a tally sheet and use the class marks $\{.725, .728, .731, .734, .737, .740, .743, .746\}$ with class intervals of length $\pm.0015$ added to the class marks.

(b) Determine and sketch the frequency distribution and cumulative frequency distribution as well as the relative frequency distribution and cumulative relative frequency distribution.

(c) Find the mean and variance directly.

(d) Find the mean and variance using the class marks and frequencies.

(e) Using the results from parts (a) and (b), approximate the following probabilities if X represents a random variable representing the diameter of a ball bearing.

$$(i)\ P(X \le .737) \qquad (ii)\ P(.728 < X \le .734) \qquad (iii)\ P(X > .734)$$

(f) In the absence of other information how can the relative frequency distribution be interpreted?

▸ **11-10.** A box contains 10 identical balls. Six of the balls are white and 4 of the balls are black.

(a) What is the probability of drawing a white ball from the box?

(b) What is the probability of drawing a black ball from the box?

▶ **11-11. (Binomial Distribution)**

The discrete binomial distribution is given by $f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$

If $q = 1 - p$ show that

$$(a)\ (p+q)^n = \sum_{x=0}^{n} f(x) = 1, \quad (b)\ \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x} = (p+q)^{n-1}, \quad (c)\ x\binom{n}{x} = n\binom{n-1}{x-1}$$

(d) Use parts (b) and (c) to show the mean of the binomial distribution is given by

$$\mu = E[x] = \sum_{x=0}^{n} x f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np$$

▶ **11-12.**

(a) Show that $s^2 = \dfrac{1}{N-1} \sum_{j=1}^{N} (x_j - \bar{x})^2$ can be written in the shortcut form

$$s^2 = \frac{1}{N(N-1)} \left\{ N \sum_{j=1}^{N} x_j^2 - \left(\sum_{j=1}^{N} x_j \right)^2 \right\}$$

(b) Illustrate the use of the above two formulas by completing the table below and evaluating s^2 by two different methods.

x	x^2	$x - \bar{x}$	$(x - \bar{x})^2$
6			
3			
8			
5			
2			
$\sum_{j=1}^{5} x_j =$	$\sum_{j=1}^{5} x^2 =$		$\sum_{j=1}^{5} (x - \bar{x})^2 =$
$\bar{x} =$			
$\left(\sum_{j=1}^{5} x_j \right)^2 =$			

▶ **11-13.** For the given probability density functions $f(x)$, find the cumulative distribution function $F(x) = \int_{-\infty}^{x} f(x)\,dx$ and then plot graphs of both $f(x)$ and $F(x)$.

(a) $f(x) = \begin{cases} \alpha\, e^{-\alpha x}, & x > 0 \text{ and } \alpha > 0 \text{ constant} \\ 0, & x \le 0 \end{cases}$

(b) $f(x) = \begin{cases} 0, & x \le -x_0 \\ \frac{1}{2x_0}, & -x_0 < x < x_0 \\ 0, & x \ge x_0 \quad \text{where } x_0 > 0 \text{ is a constant} \end{cases}$

(c) $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \; -\infty < x < \infty$ Leave $F(x)$ in integral form.

(d) $f(x) = \begin{cases} \frac{1}{2}\, e^{x}, & -\infty < x < 0 \\ \frac{1}{2}\, e^{-x}, & 0 < x < \infty \end{cases}$

▶ **11-14.** Use factorials to show

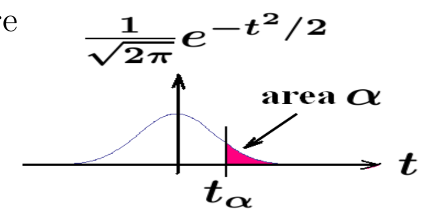
(a) $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$

(b) $\binom{n}{m+1} = \dfrac{n-m}{m+1}\binom{n}{m}$

▶ **11-15.**

(a) Use a table of areas to find values of t_α given the are

Do for $\alpha = 0.001,\ 0.01,\ 0.025,\ 0.05,\ 0.1$



(b) Explain how you would use the table of areas to calculate the probability $P(\alpha < X < \beta)$ associated with a normal distribution $(\mu = 0, \sigma = 1)$.

(c) Use the table of areas to verify (i) $P(-1 < X < 1) \approx 0.68$, (ii) $P(-2 < x < 2) \approx 0.955$, (iii) $P(-3 < X < 3) \approx 0.997$

▶ **11-16.** Given an ordinary deck of 52 playing cards.

(a) What is the probability of drawing a black ace?

(b) What is the probability of drawing an ace or a king?

▶ **11-17.** Given an ordinary deck of 52 playing cards. Let E_1 denote the event of drawing an ace and E_2 the event of drawing a heart.

(a) Are the events E_1 and E_2 mutually exclusive?

(b) What is the probability of drawing either an ace or a heart or both?

▶ **11-18. (Computer Problem for Normal Distribution)**

There are numerous web sites which use numerical methods to calculate the area $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ under the normalized probability curve $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Use one of these web sites to verify the values given in the table 11.4.

▶ **11-19. (Binomial Distribution)**

Show the variance of the binomial distribution $f(x)$, given in the problem 11-11, is $\sigma^2 = npq$ by verifying the following relations.

$$(i) \quad \text{Show } \sigma^2 = E[(x-\mu)^2 f(x)] = E[x^2] - (E[x])^2$$

$$(ii) \quad \text{Show } E[x^2] = \sum_{x=1}^{n} [x(x-1)+x] f(x) = n(n-1)p^2 + np$$

$$(iii) \quad \text{Show } \sigma^2 = npq$$

▶ **11-20.** Sketch the bell shaped probability density curve $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ associated with the normalized normal distribution. Find and show sketches of the given probabilities as areas of shaded regions on this curve.

(a) $P(\, z \le 0.5\,)$	(d) $P(\, z > 0.5\,)$	(g) $P(\, z < 2.3\,)$
(b) $P(\, z \le -2.2\,)$	(e) $P(\, -1.3 \le x \le 0.76\,)$	(h) $P(\, z > 2.3)$
(c) $P(\, \lvert z \rvert \le 2\,)$	(f) $P(\, \lvert z \rvert \le 3)$	(i) $P(\, \lvert z \rvert \le 1\,)$

▶ **11-21.** (Hypergeometric distribution)

A certain shipment of transistors contains 12 transistors, 3 of which are defective. If a batch of 5 transistors is drawn from the shipment, then

(a) determine $f(x)$, $x = 0, 1, 2, 3$ which represents the probability that x of the 5 items selected are defective.

(b) determine the minimum number of transistors that must be drawn to make the probability of obtaining at least 5 nondefective transistors greater than 0.8.

▶ **11-22.** Let X denote a random variable normally distributed with mean $\mu = 12$ and standard deviation $\sigma = 4$. Find values of and show with sketches the probabilities as areas of shaded regions for representing the probabilities

(a) $P(X \le 14)$	(c) $P(X \le 10)$
(b) $P(8 \le X \le 16)$	(d) $P(0 \le X \le 24)$

▶ **11-23.** Assume that a given State has regulations specifying that the fluoride levels in water may not exceed 1.5 milligrams per liter. Your are given the assignment to analyze the following sample of fluoride levels, in milligrams per liter, taken over a 45 day period.

.753	.945	.883	.721	.812	.731	.833	.891	.792
.860	.890	.782	.923	.858	1.01	.842	.890	.825
.843	.849	.772	1.05	.972	.910	.732	.799	.897
.855	.830	.761	.934	.942	.890	.782	.835	.899
.977	.891	.824	.837	.792	.843	.844	.803	.943

(a) Use class intervals about the class marks $M = \{.73, .78, .83, .88, .93, .98, 1.03\}$ where $\pm .025$ is added to each class mark to form the class interval. Find and plot the frequency and cumulative frequency distribution for this data.

(b) Find the mean and variance associated with the given data.

(c) If X is a random variable representing the fluoride level from the above sample, then approximate the following probabilities.

$$(i)\ P(\,X \le .88\,) \qquad (ii)\ P(\,.78 < X \le .93\,) \qquad (iii)\ P(\,X > .83\,)$$

▶ **11-24.** (Binomial distribution)

Let p denote the probability of an event happening (success) and $q = 1 - p$ denote the probability of an event not happening (failure) in a single trial. To study success or failure of an event in n-trials one usually first calculates $(p+q)^n$.

(a) Show that

$$(p+q)^n = \binom{n}{0} q^n + \binom{n}{1} p q^{n-1} + \cdots + \binom{n}{x} p^x q^{n-x} + \cdots + \binom{n}{n} p^n$$

where the term $f(x) = \binom{n}{x} p^x q^{n-x}$ denotes the probability density function representing the probability that the event will happen exactly x times in n trials and there are $n - x$ failures, with $x = 0, 1, 2, \ldots, n$ an integer.

(b) Find the probability of getting exactly 2 heads in 5 tosses of a fair coin.

(c) Find the probability of getting at least 2 heads in 5 tosses of a fair coin.

(d) Find the probability of getting at least 4 heads in 6 tosses of a fair coin.

▶ **11-25.** (Binomial distribution)

Forty identical transistors are placed on life tests simultaneously and are operated for T hours. The probability that any transistor survives to time T is 0.8. Let X denote a random variable that represents the number of transistors which are operational at time T. The distribution function for X is needed to compute probabilities. The binomial distribution is applicable if (a) each transistor is identical and has the same chance of failure as any other and (b) life testing of each transistor is identical and is accomplished under separate independent conditions. Let success mean survival of transistor to time T, then $p = 0.8$ and $n = 40$. The probability density function for the random variable X is

$$f(x) = \binom{40}{x}(0.8)^x(0.2)^{40-x}, \qquad x = 0, 1, 2, \ldots, 40$$

(a) Find the probability that exactly 33 transistors are operational at time T.
(b) Find the probability that 3 transistors have failed by time T.
(c) Find the probability that at least 3 transistors have failed by time T (i.e. the probability that no more than 37 have survived equals $\sum_{k=0}^{37} f(k)$)
(d) Find the probability that 80% of transistors survive.

▶ **11-26.** (Poisson distribution)

Many random experiments involve time. In an experiment, at any instant of time, either something happens or it does not happen and only one thing can occur. The number of these things that happen in a prescribed time interval is observed and recorded. The Poisson distribution describes such situations. Let events which occur randomly in time be called random points. A random variable X will then represent the number of random points that occur in the interval between times $t = 0$ and time $t > 0$. The probability of observing exactly x random points between 0 and t is given by the probability density function

$$f(x) = f(x; t) = \frac{(\lambda t)^x \, e^{-\lambda t}}{x!}, \qquad x = 0, 1, 2, \ldots$$

where $\lambda t > 0$ and $\lambda > 0$ represents the average number of random points per unit of time. If $\lambda t = 9$ and X denotes a random variable
(a) Show that $f(x) = \frac{9^x e^{-9}}{x!}$ and $f(x+1) = \frac{9}{x+1} f(x)$
(b) Find $P(X > 4)$ i.e. 5 or more random points occur.
(c) Find $P(X \le 8)$ i.e. no more than 8 random points occur.
(d) Find $P(8 \le X \le 12)$ i.e. between 8 and 12 random points occur.

▶ **11-27.** (Poisson distribution)

Let X denote a random variable representing the number of light bulbs which fail during a specified time interval T. The random variable X is assumed to have a Poisson probability density function where an average of 2 bulbs fail during the time interval T. Use the Poisson probability density function

$$f(x) = \frac{2^x}{x!}e^{-2}, \quad x = 0, 1, 2, \ldots$$

and find the probability

(a) of no failures during time interval T

(b) of more than one failure during time interval T

(c) of more than five failures during time interval T

▶ **11-28.** The error function or Gauss error function is defined[1]

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

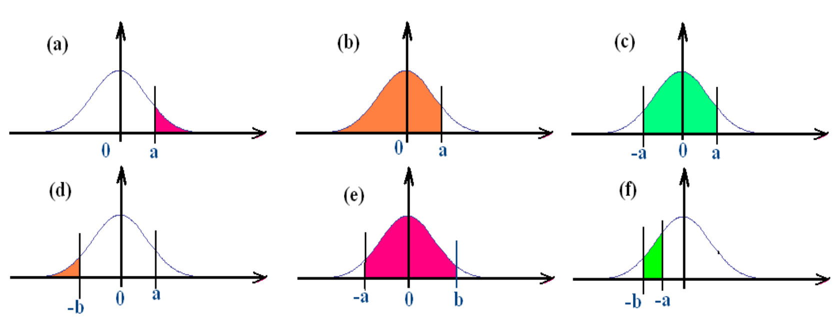
(a) Show that

$$\int_{-\infty}^x N(x;0,1)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,dt = \frac{1}{2}\left[1+\text{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

(b) Show that

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^x e^{-\left(\frac{t-\mu}{\sigma}\right)^2}\,dt = \frac{1}{2}\left[1+\text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

▶ **11-29.** Each of the curves below represent graphs of the normal probability distribution $N(x;0,1)$. Explain how one would use areas associated with the normal probability tables to find the areas of the shaded regions.



Find the area associated with the shaded areas.

[1] There are alternative definitions for the error function.

▶ **11-30.** If $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$, find the value of and illustrate with sketches the representations of the following probabilities as shaded area under the normal probability density curve.

(a)	$P(Z \le 1)$	(d)	$P(Z \le -3.2)$	(g)	$P(Z \le 2)$
(b)	$P(Z > 1)$	(e)	$P(-1.2 \le Z \le 0.75)$	(h)	$P(\vert Z\vert \le 2$
(c)	$P(Z \le 3.2)$	(f)	$P(\vert Z\vert \le z)$	(i)	$P(\vert Z\vert \le 3)$

▶ **11-31.** (Monte Carlo computer problem)

(a) Give a physical interpretation to the integral $I_1 = \frac{1}{b-a} \int_a^b f(x)\, dx$

(b) Give a physical interpretation to the summation $I_2 = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$ where $a \le x_i \le b$ for all integers i.

(c) If $I_1 = I_2$, show estimate for $I = \int_a^b f(x)\, dx$ is given by $I = \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$

(d) Calculate 500 random numbers[2] x_i with $x_i \in (-1, 2)$ and estimate the integral $I = \frac{1}{\sqrt{2\pi}} \int_{-1}^{2} e^{-x^2/2}\, dx$. Do this over and over again and calculate 1000 estimates for I and then do descriptive statistics on your results and compare your computer answer with the answer obtained from table lookup.

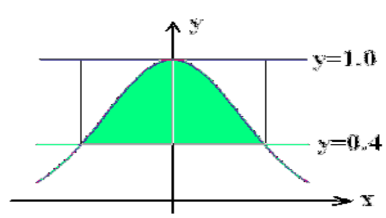
▶ **11-32.** (Monte Carlo computer problem)

Write a Monte Carlo computer program to calculate the area bounded by the curves $y = e^{-x^2/2}$ and $y = 0.4$ as illustrated below. Set up as an integral (see previous problem) or throw darts at area.

Hint 1: $y = e^{-x^2/2} = 0.4$ when $x = x^* = \pm\sqrt{-2\ln(0.4)}$

Hint 2: Construct a rectangle where $-x^* < x < x^*$ and $0.4 \le y \le 1.0$ about area to be calculated by Monte Carlo method.

Hint 3: Generate random numbers (x_r, y_r) with $-x^* \le x_r \le x^*$ and $0.4 \le y_r \le 1.0$ and determine if the point (x_r, y_r) is inside or outside the area to be determined.



Be sure to perform descriptive statistics on your results and if your instructor gives you extra credit, put confidence intervals on your answer for the area.

Chapter 12
Introduction to more Advanced Material

The following is a potpourri of selected topics involving mathematical applications of calculus together with an introduction to advanced calculus techniques and mathematical methods related to calculus. The material selected presents applications and topics that you might encounter in your scientific investigations in other courses. The material presented will also give you some idea of what to expect in more advanced mathematical courses beyond calculus.

An integration method

To integrate the definite integral $\int_a^b f(x)\,dx$ you can use the following integration method **if you know the inverse function** $f^{-1}(x)$ **associated with** $f(x)$.

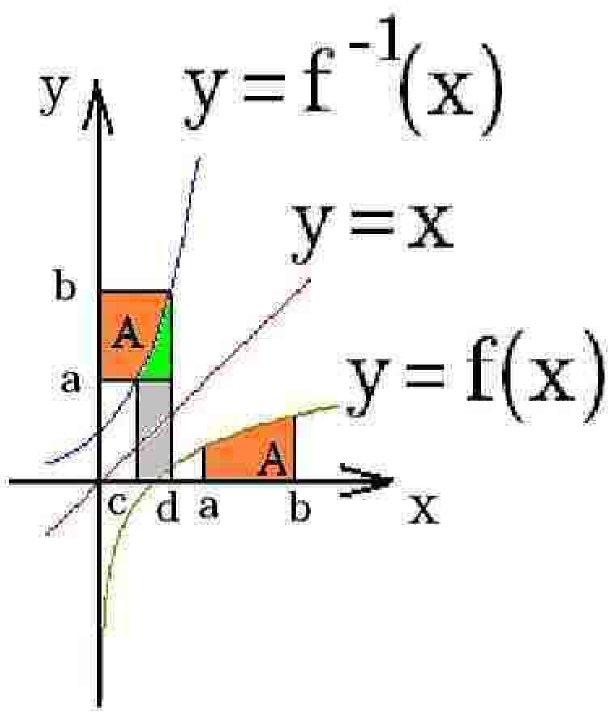


Figure 12-1. A function and its inverse function.

We know that the functions $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y=x$ as illustrated in the figure 12-1. Examine the figure 12-1 and note that one can express the area $A=\int_a^b f(x)\,dx$ as

$$\begin{aligned} A=\int_a^b f(x)\,dx =& \underbrace{(b-a)d}_{orange+green\ area} - \underbrace{\int_c^d [f^{-1}(x)-a]\,dx}_{green\ area} \\ =& (b-a)d - \int_c^d f^{-1}(x)\,dx + (d-c)a \end{aligned} \tag{12.1}$$

which shows that the area A is given by the rectangular area (orange plus green area) minus the area under the inverse curve (green plus grey area) corrected by the rectangular grey area. Hence, if you know $\displaystyle\int_a^b f(x)\,dx$, then you can find $\displaystyle\int_c^d f^{-1}(x)\,dx$ and vice-versa.

The use of integration to sum infinite series

There is a definite relation between certain infinite series and definite integrals. For example, consider the relationship between the problem of finding the sum of the alternating infinite series

$$\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\frac{1}{a+4b}-\cdots+(-1)^n\frac{1}{a+nb}+\cdots \tag{12.2}$$

where $a>0$, $b>0$ and the associated problem of evaluating the definite integral

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt \tag{12.3}$$

Use the well known series expansion

$$\frac{1}{1+x}=1-x+x^2+x^3-x^4+\cdots \tag{12.4}$$

with x replaced by t^b to write the equation (12.3) in the form

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt=\int_0^1 t^{a-1}\left[1-t^b+t^{2b}-t^{3b}+t^{4b}-\cdots\right]\,dt \tag{12.5}$$

and then integrate each term to produce the result

$$\begin{aligned} \int_0^1 \frac{t^{a-1}}{1+t^b}\,dt =& \left[\frac{t^a}{a}-\frac{t^{a+b}}{a+b}+\frac{t^{a+2b}}{a+2b}+\cdots+(-1)^n\frac{t^{a+nb}}{a+nb}+\cdots\right]_0^1 \\ \int_0^1 \frac{t^{a-1}}{1+t^b}\,dt =& \frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}+\cdots+(-1)^n\frac{1}{a+nb}+\cdots \end{aligned} \tag{12.6}$$

This demonstrates that if the infinite series on the right-hand side converges, then it can be evaluated by calculating the integral on the left-hand side.

Example 12-1. (**Sum of series**)

Find the sum of the infinite series

$$\frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \cdots \tag{12.7}$$

which is an example of the series (12.2) when $a = 1$ and $b = 3$.

Solution

Use the equation (12.6) to find the sum of the given infinite series by evaluating the integral

$$I = \int_0^1 \frac{dt}{1+t^3} \tag{12.8}$$

Use partial fractions and write

$$\frac{1}{1+t^3} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2}$$

and show that $A = 1/3$, $B = -1/3$ and $C = 2/3$. Using some algebra and completing the square on the denominator term the required integral can be reduced to the following standard forms

$$\begin{aligned} I = \int_0^1 \frac{dt}{1+t^3} &= \frac{1}{3}\int_0^1 \frac{dt}{1+t} + \int_0^1 \frac{-t/3 + 2/3}{1-t+t^2}\,dt \\ &= \frac{1}{3}\int_0^1 \frac{dt}{1+t} - \frac{1}{6}\int_0^1 \frac{2t-1}{1-t+t^2}\,dt + \frac{1}{2}\int_0^1 \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + (t-1/2)^2} \end{aligned}$$

where each integral is in a standard form which can be easily integrated. If you don't recognize these integrals then look them up in an integration table. Integration of each term produces

$$I = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + \frac{1}{3}\ln(1+t) - \frac{1}{6}\ln(1-t+t^2)\right]_0^1 = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln(2)\right) \tag{12.9}$$

giving the final result

$$I = \frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \cdots = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln(2)\right) \tag{12.10}$$

■

Example 12-2. (**Sum of series**)

Show that

$$\frac{1}{2\cdot 5}+\frac{1}{8\cdot 11}+\frac{1}{14\cdot 17}+\frac{1}{20\cdot 23}+\cdots=\frac{1}{9}(\frac{\pi}{3}+\ln 2)$$

Solution

Let S denote the sum of the series and use partial fractions to write

$$\frac{1}{n\cdot(n+3)}=\frac{A}{n}+\frac{B}{n+3}$$

for $n=2,8,14,20,\ldots$ to show that

$$S=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\frac{1}{14}-\frac{1}{17}+\frac{1}{20}-\frac{1}{23}+\cdots\right]$$

Note that the sum S is a special case of the Taylor's series

$$S(x)=\frac{1}{3}\left[\frac{x^2}{2}-\frac{x^5}{5}+\frac{x^8}{8}-\frac{x^{11}}{11}+\frac{x^{14}}{14}-\frac{x^{17}}{17}+\frac{x^{20}}{20}-\frac{x^{23}}{23}+\cdots\right]$$

with $S=S(1)$ the desired sum. The derivative of $S(x)$ produces

$$\frac{dS}{dx}=\frac{1}{3}\left[x-x^4+x^7-x^{10}+x^{13}-x^{16}+x^{19}-x^{22}+\cdots\right]$$

The derivative series is recognized as a geometric series with sum $\frac{x}{x^3+1}$ so that one can write

$$\frac{dS}{dx}=\frac{1}{3}\frac{x}{x^3+1}$$

The desired series sum can now be expressed in terms of an integral

$$S=S(1)=\frac{1}{3}\int_0^1\frac{x}{x^3+1}\,dx$$

As an exercise, use partial fractions and show

$$S=S(1)=\frac{1}{3}\int_0^1\frac{x}{x^3+1}\,dx=\frac{1}{3}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)-\frac{1}{3}\ln(x+1)+\frac{1}{6}\ln(1-x+x^2)\right]_0^1$$

which simplifies to

$$S=S(1)=\frac{1}{9}\left(\frac{\pi}{\sqrt{3}}-\ln 2\right)$$

■

Refraction through a prism

Refraction through a prism is often encountered in physics courses and the calculus needed to analyze the physical problem is messy. Let's investigate this problem.

Consider the prism illustrated in the figure 12-2. The angle A of a prism is known as the apex angle or refracting angle. The angles associated with a ray of light entering or leaving the sides of a prism are measured with respect to a normal line constructed to a side of the prism and the ray of light is governed by Snell's law[1]

$$\begin{array}{ll} \text{entering} & \text{leaving} \\ n_a \sin x = n_g \sin \alpha & n_g \sin \beta = n_a \sin \xi \end{array} \tag{12.11}$$

where n_m denotes the refractive index for light in medium m ($m = a$ air and $m = g$ glass). Here x is called the angle of incidence and α is called the angle of refraction. The refracted ray travels across the prism and again undergoes Snell's law and becomes the exiting ray. There is a reversibility principle whereby light can travel in either direction along the path illustrated in the figure 12-2. In figure 12-2 the incident ray is extended and the exiting ray also has been extended and these extended rays intersect in the angle y called **the angle of deviation**.

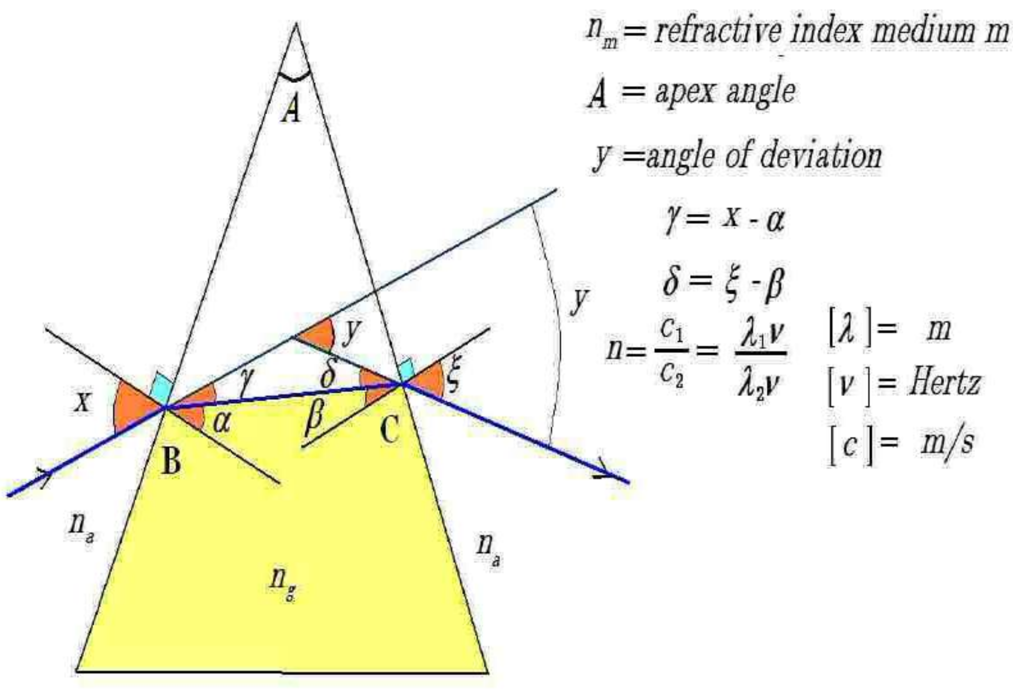


Figure 12-2. Ray of light through a prism.

[1] See Chapter 2 , pages 122-124.

Observe that the sum of the angles of the triangle ABC with top angle A and base angles $\frac{\pi}{2}-\alpha$ and $\frac{\pi}{2}-\beta$ is π radians so that one can sum the angles of the triangle and write

$$\frac{\pi}{2}-\alpha+\frac{\pi}{2}-\beta+A=\pi \quad \text{or} \quad A=\alpha+\beta \tag{12.12}$$

Here the deviation angle y is the exterior angle of a triangle with γ and δ the two opposite interior angles. Consequently one can write

$$y=\gamma+\delta=(x-\alpha)+(\xi-\beta) \quad \text{or} \quad y=x-A+\xi \tag{12.13}$$

Use equation (12.12) to show

$$\sin\beta=\sin(A-\alpha)=\sin A\cos\alpha-\sin\alpha\cos A \tag{12.14}$$

and then use the equations (12.11) in the form

$$\sin\beta=\frac{n_g}{n_a}\sin\xi, \qquad \sin\alpha=\frac{n_a}{n_g}\sin x, \qquad \cos\alpha=\sqrt{1-\left(\frac{n_a}{n_g}\right)^2\sin^2 x}$$

to express equation (12.14) in the form

$$\begin{aligned}\frac{n_a}{n_g}\sin\xi=&\sin A\sqrt{1-\sin^2\alpha}-\frac{n_a}{n_g}\sin x\cos A\\ \frac{n_a}{n_g}\sin\xi=&\sin A\sqrt{1-\left(\frac{n_a}{n_g}\right)^2\sin^2 x}-\frac{n_a}{n_g}\sin x\cos A\\ \sin\xi=&\frac{1}{n_a}\sin A\sqrt{n_g^2-n_a^2\sin^2 x}-\sin x\cos A\end{aligned} \tag{12.15}$$

The equations (12.15) together with equation (12.13) can be used to express y as a function of x. One finds that the angle of deviation in terms of the incident angle x can be expressed in the form

$$y=x-A+\sin^{-1}\left[\frac{1}{n_a}\sin A\sqrt{n_g^2-n_a^2\sin^2 x}-\cos A\sin x\right] \tag{12.16}$$

The figure 12-3 illustrates a graph of the angle of deviation as a function of the incident value for selected nominal values of A, n_a, n_g. Observe that there is some incident angle where the angle of deviation is a minimum. To find out where this minimum value occurs one can differentiate equation (12.16) to obtain

$$\frac{dy}{dx}=1-\frac{\frac{n_a\sin(A)\sin(x)\cos(x)}{\sqrt{{n_g}^2-{n_a}^2\sin^2(x)}}+\cos(A)\cos(x)}{\sqrt{1-\left(\cos(A)\sin(x)-\frac{\sin(A)\sqrt{{n_g}^2-{n_a}^2\sin^2(x)}}{n_a}\right)^2}} \tag{12.17}$$

At a minimum value the derivative must equal zero and so equation (12.17) must be set equal to zero and x must be solved for. This is not an easy task and so numerical methods must be resorted to.

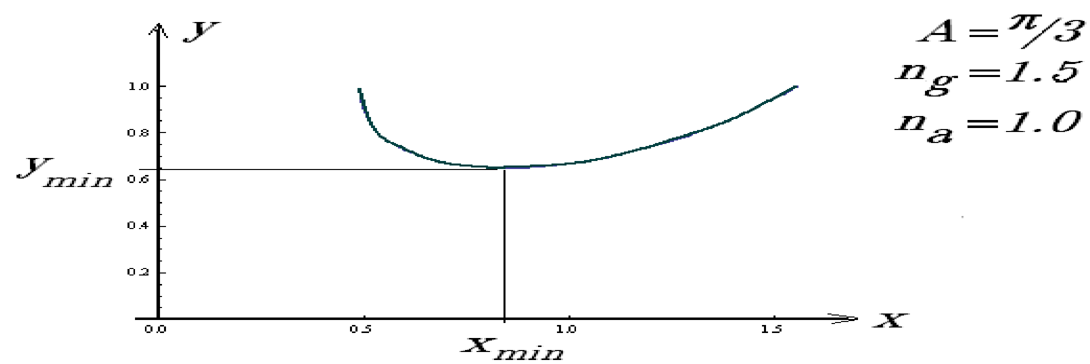


Figure 12-3.

Deviation angle versus incident angle for prism using nominal values above.

An alternative approach of finding the value of x which produces a minimum value is as follows. Observe that when $y = y_{min}$, then $\frac{dy}{dx} = 0$. Assume that y has the value $y = y_{min}$ and differentiate the equations (12.12) and (12.13) with respect to x and show

$$\frac{dA}{dx} = \frac{d\alpha}{dx} + \frac{d\beta}{dx} = 0 \qquad \text{or} \qquad \frac{d\beta}{dx} = -\frac{d\alpha}{dx} \tag{12.18}$$

because the angle A is a constant. One also finds that when $y = y_{min}$, then

$$\frac{dy}{dx} = 1 + \frac{d\xi}{dx} = 0 \qquad \text{or} \qquad \frac{d\xi}{dx} = -1 \tag{12.19}$$

because of our assumption that $y = y_{min}$ and hence $\frac{dy}{dx} = 0$. Next differentiate the equations (12.11) with respect to x and show

$$n_a \cos x = n_g \cos\alpha \frac{d\alpha}{dx} \tag{12.20}$$

and

$$n_g \cos\beta \frac{d\beta}{dx} = n_a \cos\xi \frac{d\xi}{dx} \tag{12.21}$$

The form of the equations (12.20) and (12.21) can be changed by multiplying equation (12.20) by $\cos\beta$ and multiplying equation (12.21) by $\cos\alpha$. The resulting equations are further simplified by using the results from equations (12.18) and (12.19) to produce the equations

$$n_a \cos x \cos\beta = n_g \cos\alpha \cos\beta \frac{d\alpha}{dx} \tag{12.22}$$

$$-n_a \cos\xi \cos\alpha = -\, n_g \cos\alpha \cos\beta \frac{d\alpha}{dx} \tag{12.23}$$

which must hold when $y = y_{min}$.

Addition of the equations (12.22) and (12.23) after simplification produces the condition

$$\cos x \cos\beta = \cos\xi \cos\alpha \tag{12.24}$$

Now square both sides of equation (12.24) and verify that

$$\begin{aligned} \cos^2 x \cos^2\beta =& \cos^2\xi \cos^2\alpha \\ (1-\sin^2 x)(1-\sin^2\beta) =& (1-\sin^2\xi)(1-\sin^2\alpha) \\ (1-\frac{n_g^2}{n_a^2}\sin^2\alpha)(1-\sin^2\beta) =& (1-\frac{n_a^2}{n_g^2}\sin^2\beta)(1-\sin^2\alpha) \end{aligned} \tag{12.25}$$

Expand the last equation from equation (12.25) and simplify the result to show equation (12.25) reduces to the condition that

$$|\sin\beta| = |\sin\alpha| \tag{12.26}$$

under the condition that $y = y_{min}$. The equation (12.26) implies that under the condition that $y = y_{min}$ there must be symmetry in the light ray moving through the prism. That is, the conditions

$$\beta = \alpha = \frac{A}{2} \qquad \text{and} \qquad x = \frac{y_{min} + A}{2} \tag{12.27}$$

must be satisfied when a minimum angular deviation is achieved.

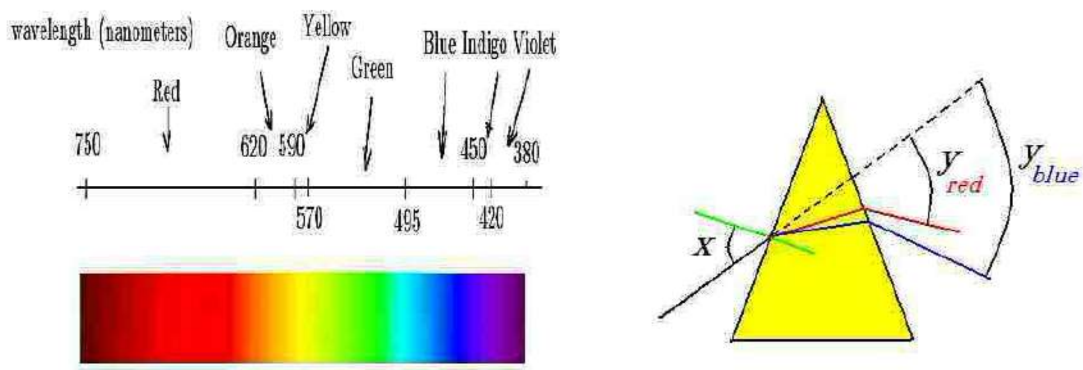


Figure 12-4.
White light splits into colors.

Recall that white light gets split into the colors Red, Orange, Yellow, Green, Blue, Indigo, Violet, remembered using the acronym (ROY G BIV). This is because the deviation angle for red light is less than the deviation angle for violet light. One can observe this by expressing the index of refraction in terms of the frequency (ν) and wavelength (λ) of the light. The refraction index being given by

$$n = c_1/c_2 = \lambda_1\nu/\lambda_2\nu$$

See the figures 12-2 and 12-3.

Differentiation of Implicit Functions

The following is a prsentation of various ways of representing implicit functions and the resulting techniques used to obtain derivatives from these representations. The ideas presented can be extended to cover systems of m-equations in n-unknowns, with $m < n$. The given m-equations define implicitly m of the variables in terms of the remaining $n - m$ variables. Make note in the following representations that if you are given m-equations, then there will always be m dependent variables.

one equation, two unknowns

Any equation of the form

$$F(x,y) = 0 \tag{12.28}$$

implicitly defines y as one or more functions of x. Treating y as a function of x differentiate the equation (12.28) with respect to x and show

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \tag{12.29}$$

from which one can solve for the first derivative to obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \tag{12.30}$$

provided that $\frac{\partial F}{\partial y} \neq 0$. Higher derivatives are obtained by differentiating the first derivative. For example, differentiate the equation (12.29) with respect to x and show

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x\,\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y}\frac{d^2 y}{dx^2} + \frac{dy}{dx}\left[\frac{\partial^2 F}{\partial y\,\partial x} + \frac{\partial^2 F}{\partial y^2}\frac{dy}{dx}\right] = 0 \tag{12.31}$$

One can then solve for the second derivative term. Higher ordered derivatives are obtained by differentiating the equation (12.31).

one equation, three unknowns

An equation of the form

$$F(x,y,z) = 0 \tag{12.32}$$

implicitly defines z as one or more functions of x and y provided that $\frac{\partial F}{\partial z} \neq 0$. Treating z as a function of x and y one can differentiate the equation (12.32) with respect to x and obtain

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \tag{12.33}$$

Solving for $\frac{\partial z}{\partial x}$ one finds

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{12.34}$$

An alternative representation of the derivative of z with respect to x can be obtained as follows. Take the differential of equation (12.32) to obtain

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = 0 \tag{12.35}$$

If y is held constant, then dy is zero and equation (12.35) yields the result

$$\left(\frac{dz}{dx}\right)_y = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \tag{12.36}$$

Here the symbol on the left of equation (12.36) is used to emphasize that y is being held constant during the differentiation process. Some engineering texts feel this notation is less ambiguous than the use of the partial derivative symbol occurring in equation (12.34).

Differentiating the equation (12.32) with respect to y produces the result

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \tag{12.37}$$

from which one finds the partial derivative

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial F}{\partial z} \neq 0 \tag{12.38}$$

Alternatively, set x equal to a constant so that $dx = 0$ in equation (12.35), then equation (12.35) produces the result

$$\left(\frac{dz}{dy}\right)_x = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial F}{\partial z} \neq 0 \tag{12.39}$$

Here the derivative is represented using the alternative notation $\left(\frac{dz}{dy}\right)_x$ emphasizing the derivative is obtained holding x constant.

one equation, four unknowns

Any equation of the form

$$F(x,y,z,w) = 0 \tag{12.40}$$

implicitly defines w as one or more functions of x, y and z. Differentiate equation (12.40) with respect to x and show

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial w}} \tag{12.41}$$

Differentiate equation (12.40) with respect to y and show

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial w}} \tag{12.42}$$

Differentiate equation (12.40) with respect to z and show

$$\frac{\partial F}{\partial z} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} = 0 \qquad \text{or} \qquad \frac{\partial w}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial w}} \tag{12.43}$$

provided $\frac{\partial F}{\partial w} \neq 0$.

one equation, n-unknowns

Any equation of the form

$$F(x_1, x_2, \ldots, x_n, w) = 0 \tag{12.44}$$

implicitly defines w as one or more functions of the n-variables $(x_1, x_2, \ldots, x_n)$. It is left as an exercise to show that for a fixed integer value of j between 1 and n that

$$\frac{\partial w}{\partial x_j} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial w}}, \quad \text{provided} \quad \frac{\partial F}{\partial w} \neq 0 \tag{12.45}$$

two equations, three unknowns

Given two equations having the form

$$F(x,y,z) = 0 \quad \text{and} \quad G(x,y,z) = 0 \tag{12.46}$$

then these equations define implicitly (a) z as a function of x and (b) y as a function of x. Treat $z = z(x)$ and $y = y(x)$ and differentiate each of the equations (12.46) with respect to x and show

$$\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} &= 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} + \frac{\partial G}{\partial z}\frac{dz}{dx} &= 0 \end{aligned} \tag{12.47}$$

The equations (12.47) represent two equations in the two unknowns $\frac{dy}{dx}$ and $\frac{dz}{dx}$ which can be solved. Use Cramers rule[2] and show these equations have the solutions

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} \tag{12.48}$$

where

$$\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} \neq 0$$

[2] See Cramers rule in Appendix B

The determinants in the equations (12.48) are called Jacobian determinants of F and G and are often expressed using the shorthand notation

$$\frac{\partial(F,G)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} = \frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial G}{\partial y}\frac{\partial F}{\partial z} \tag{12.49}$$

In terms of Jacobian determinants the derivatives represented by the equations (12.48) can be represented

$$\frac{dy}{dx} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}} \tag{12.50}$$

where $\dfrac{\partial(F,G)}{\partial(y,z)} \neq 0$. Note the patterns associated with the partial derivatives and the Jacobian determinants. We will make use of these patterns to calculate derivatives directly from the given transformation equations in later presentations and examples.

two equations, four unknowns

Two equations of the form

$$\begin{aligned} F(x,y,u,v) =& 0 \\ G(x,y,u,v) =& 0 \end{aligned} \tag{12.51}$$

implicitly define u and v as functions of x and y so that one can write $u = u(x,y)$ and $v = v(x,y)$. The derivatives of the equations(12.51) with respect to x can then be expressed

$$\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} =& 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} =& 0 \end{aligned} \tag{12.54}$$

The equations (12.54) represent two equations in the two unknowns $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. One can use Cramers rule to solve this system of equations and obtain the solutions

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}} \tag{12.53}$$

This solution is valid provided the Jacobian determinant $\dfrac{\partial(F,G)}{\partial(u,v)}$ is different from zero.

In a similar fashion one can differentiate the equations (12.51) with respect to the variable y and obtain

$$\begin{aligned} \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} =&0 \\ \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} =&0 \end{aligned} \tag{12.54}$$

which produces two equations in the two unknowns $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$. Solving this system of equations using Cramers rule produces the solutions

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}} \tag{12.55}$$

This solution is valid provided the Jacobian determinant $\dfrac{\partial(F,G)}{\partial(u,v)}$ is different from zero.

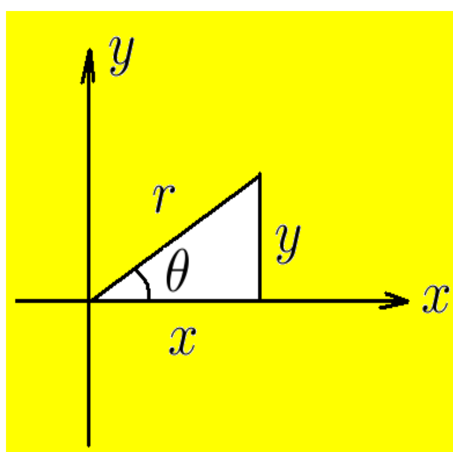
Example 12-3. (Conversion of the Laplace equation)

Transform the Laplace equation

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \tag{12.56}$$

from rectangular (x,y) coordinates to (r,θ) polar coordinates.

Solution 1



$x = r\cos\theta$
$y = r\sin\theta$

Figure 12-5.

Rectangular (x,y) and polar (r,θ) coordinates

If $U = U(x, y)$ is converted to polar coordinates to become $U = U(r, \theta)$ one can treat $r = r(x, y)$ and $\theta = \theta(x, y)$ to calculate the following derivatives of U with respect x and y

$$\begin{aligned}\frac{\partial U}{\partial x} =& \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x}\\ \frac{\partial^2 U}{\partial x^2} =& \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial r\,\partial \theta}\frac{\partial \theta}{\partial x}\right]\\ &+ \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x}\left[\frac{\partial^2 U}{\partial \theta\,\partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 U}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right]\end{aligned} \tag{12.57}$$

and

$$\begin{aligned}\frac{\partial U}{\partial y} =& \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y}\\ \frac{\partial^2 U}{\partial y^2} =& \frac{\partial U}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial r}{\partial x}\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 U}{\partial r\,\partial \theta}\frac{\partial \theta}{\partial y}\right]\\ &+ \frac{\partial U}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial \theta}{\partial y}\left[\frac{\partial^2 U}{\partial \theta\,\partial r}\frac{\partial r}{\partial y} + \frac{\partial^2 U}{\partial \theta^2}\frac{\partial \theta}{\partial y}\right]\end{aligned} \tag{12.58}$$

The transformation equations from rectangular to polar coordinates is performed using the transformation equations

$$x = r\,\cos\theta \qquad \text{and} \qquad y = r\,\sin\theta \tag{12.59}$$

One can solve for r and θ in terms of x and y to obtain

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{12.60}$$

Differentiate the equations (12.60) with respect to x and show

$$2r\frac{\partial r}{\partial x} = 2x \qquad \text{and} \qquad \sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \tag{12.61}$$

which simplifies using the transformation equations (12.59) to the values

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \text{and} \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \tag{12.62}$$

Differentiate the the equations (12.60) with respect to y and show

$$2r\frac{\partial r}{\partial y} = 2y \qquad \text{and} \qquad \sec^2\theta\frac{\partial \theta}{\partial y} = \frac{1}{x} \tag{12.63}$$

Use the transformation equations (12.59) and simplify the equations (12.63) and show

$$\frac{\partial r}{\partial y} = \sin\theta \qquad \text{and} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \tag{12.64}$$

It is now possible to differentiate the first derivatives given by equations (12.62) and (12.64) to obtain the second derivatives

$$
\begin{aligned}
\frac{\partial^2 r}{\partial x^2} &= -\sin\theta \frac{\partial \theta}{\partial x} \\
\frac{\partial^2 r}{\partial x^2} &= \frac{\sin^2\theta}{r}
\end{aligned}
, \qquad
\begin{aligned}
\frac{\partial^2 r}{\partial y^2} &= \cos\theta \frac{\partial \theta}{\partial y} \\
\frac{\partial^2 r}{\partial y^2} &= \frac{\cos^2\theta}{r}
\end{aligned}
\tag{12.65}
$$

and

$$
\begin{aligned}
\frac{\partial^2 \theta}{\partial x^2} &= -\frac{\cos\theta}{r}\frac{\partial \theta}{\partial x} + \frac{\sin\theta}{r}\frac{\partial r}{\partial x} \\
\frac{\partial^2 \theta}{\partial x^2} &= \frac{2\sin\theta\cos\theta}{r^2}
\end{aligned}
, \qquad
\begin{aligned}
\frac{\partial^2 \theta}{\partial y^2} &= -\frac{\sin\theta}{r}\frac{\partial \theta}{\partial y} - \frac{\cos\theta}{r^2}\frac{\partial r}{\partial y} \\
\frac{\partial^2 \theta}{\partial y^2} &= -\frac{2\sin\theta\cos\theta}{r^2}
\end{aligned}
\tag{12.66}
$$

Substitute the first and second derivatives from equations (12.62), (12.64), (12.66) into the equations (12.57) and (12.58) to show that after simplification the equation (12.56) becomes

$$
\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} = \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0 \tag{12.67}
$$

Solution 2

Write the transformation equations (12.59) as

$$
F(x, y, r, \theta) = x - r\cos\theta = 0 \qquad \text{and} \qquad G(x, y, r, \theta) = y - r\sin\theta = 0 \tag{12.68}
$$

and use the notation for Jacobian determinants to write

$$
\frac{\partial r}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,\theta)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} 1 & r\sin\theta \\ 0 & -r\cos\theta \end{vmatrix}}{\begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix}} = \frac{r\cos\theta}{r} = \cos\theta
$$

$$
\frac{\partial \theta}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(r,x)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} -sin\theta & -r\cos\theta \\ -\sin\theta & 0 \end{vmatrix}}{r} = -\frac{\sin\theta}{r}
$$

$$
\frac{\partial r}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,\theta)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} 0 & r\sin\theta \\ 1 & -r\cos\theta \end{vmatrix}}{r} = \sin\theta
$$

$$
\frac{\partial \theta}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(r,y)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} -\cos\theta & 0 \\ -\sin\theta & 1 \end{vmatrix}}{r} = \frac{\cos\theta}{r}
$$

These derivatives can be compared with the previous results given in the equations (12.62) and (12.64).

■

three equations, five unknowns

A set of equations having the form

$$\begin{aligned} F(x,y,u,v,w) =& 0 \\ G(x,y,u,v,w) =& 0 \\ H(x,y,u,v,w) =& 0 \end{aligned} \tag{12.69}$$

implicitly defines u, v, w as functions of x and y so that one can write

$$u = u(x,y) \qquad v = v(x,y) \qquad w = w(x,y) \tag{12.70}$$

The partial derivatives of u, v and w with respect to x and y are calculated in a manner similar to the previous representations presented.

In order to save space in typesetting sometimes the notation for partial derivatives is shortened to the use of subscripts. For example, one can define

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad F_x = \frac{\partial F}{\partial x}, \quad F_{xx} = \frac{\partial^2 F}{\partial x^2}, \quad F_w = \frac{\partial F}{\partial w}, \quad F_{xy} = \frac{\partial^2 F}{\partial x\, \partial y}, \quad \text{etc}$$

We now employ this notation and take the partial derivatives of equations F, G and H with respect to x and write

$$\begin{aligned} F_x + F_u u_x + F_v v_x + F_w w_x =& 0 \\ G_x + G_u u_x + G_v v_x + G_w w_x =& 0 \\ H_x + H_u u_x + H_v v_x + H_w w_x =& 0 \end{aligned} \tag{12.71}$$

The equations (12.71) represent three equations in the three unknowns u_x, v_x and w_x which can be solved using Cramers rule. This can be accomplished by defining the 3 by 3 Jacobian determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix} \tag{12.72}$$

If this Jacobian determinant is different from zero, the system of equations (12.71) has a unique solution for u_x, v_x, w_x given by

$$u_x = -\frac{\frac{\partial(F,G,H)}{\partial(x,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad v_x = -\frac{\frac{\partial(F,G,H)}{\partial(u,x,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad w_x = -\frac{\frac{\partial(F,G,H)}{\partial(u,v,x)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}} \tag{12.73}$$

DIfferentiate the equations (12.69) with respect to y to obtain

$$\begin{aligned} F_y + F_u u_y + F_v v_y + F_w w_y =& 0 \\ G_y + G_u u_y + G_v v_y + G_w w_y =& 0 \\ H_y + H_u u_y + H_v v_y + H_w w_y =& 0 \end{aligned} \tag{12.74}$$

This produces three equations in the three unknowns u_y, v_y, w_y which can be solved using Cramers rule. If the Jacobian determinant of F, G, H is different from zero, then the unique solution is given by

$$u_y = -\frac{\frac{\partial(F,G,H)}{\partial(y,v,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad v_y = -\frac{\frac{\partial(F,G,H)}{\partial(u,y,w)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}}, \quad w_y = -\frac{\frac{\partial(F,G,H)}{\partial(u,v,y)}}{\frac{\partial(F,G,H)}{\partial(u,v,w)}} \tag{12.75}$$

Generalization

The system of equations

$$\begin{aligned} F_1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) =& 0 \\ F_2(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) =& 0 \\ \vdots \qquad\qquad & \vdots \\ F_m(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) =& 0 \end{aligned} \tag{12.76}$$

in $m+n$ unknowns implicitly defines the functions

$$\begin{aligned} y_1 =& y_1(x_1, x_2, \ldots, x_n) \\ y_2 =& y_2(x_1, x_2, \ldots, x_n) \\ \vdots \qquad & \vdots \\ y_m =& y_m(x_1, x_2, \ldots, x_n) \end{aligned} \tag{12.77}$$

If the Jacobian determinant

$$\frac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, y_2, \ldots, y_m)} = \begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_2} & \cdots & \frac{\partial F_m}{\partial y_m} \end{vmatrix}$$

is different from zero at a point $(x_1^0, x_2^0, \ldots, x_n^0, y_1^0, y_2^0, \ldots, y_m^0)$, then one can calculate the partial derivatives $\dfrac{\partial y_i}{\partial x_j}$ at this point for any combination of integer values for i, j satisfying $1 \le i \le m$ and $1 \le j \le n$. The above is true because if one calculates the derivatives of the functions in equation (12.76) with respect to say x_j, $1 \le j \le n$, one finds

$$\begin{aligned}
\frac{\partial F_1}{\partial x_j} + \frac{\partial F_1}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial F_1}{\partial y_m}\frac{\partial y_m}{\partial x_j} =& 0\\
\frac{\partial F_2}{\partial x_j} + \frac{\partial F_2}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial F_2}{\partial y_m}\frac{\partial y_m}{\partial x_j} =& 0\\
\vdots \qquad \vdots \qquad\quad & \vdots\\
\frac{\partial F_n}{\partial x_j} + \frac{\partial F_n}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial F_n}{\partial y_m}\frac{\partial y_m}{\partial x_j} =& 0
\end{aligned} \tag{12.78}$$

The system of equations (12.78) can be solved by Cramers rule and because the Jacobian determinant is different from zero the system of equations (12.78) has a unique solution for the various first partial derivatives.

Higher derivatives can be obtained by differentiating the first order partial derivatives.

The Gamma Function

One definition of the Gamma function is given by the integral

$$\Gamma(x) = \int_0^\infty \xi^{x-1} e^{-\xi}\, d\xi \tag{12.79}$$

The value $x = 1$ substituted into the equation (12.79) produces the result

$$\Gamma(1) = \int_0^\infty e^{-\xi}\, d\xi = -e^{-\xi}\Big]_0^\infty = 1 \tag{12.80}$$

so that $\Gamma(1) = 1$. Substitute $x = n+1$, a positive integer, into equation (12.79) and then integrate by parts to obtain

$$\Gamma(n+1) = \int_0^\infty e^{-\xi}\xi^n\, d\xi = \left[-\xi^n e^{-\xi}\right]_0^\infty + \int_0^\infty n\xi^{n-1}e^{-\xi}\, d\xi$$

which simplifies to the recurrence relation

$$\Gamma(n+1) = n\Gamma(n), \qquad n > 0, \text{ an integer} \tag{12.81}$$

In general, for any real positive value for x which is less than unity, one can show that $\Gamma(x)$ is a particular solution of the functional equation

$$\Gamma(x+1) = x\Gamma(x), \qquad x > 0 \tag{12.82}$$

Replacing x by $-x$, the equation (12.82) is sometimes represented

$$\Gamma(1-x) = -x\Gamma(-x) \tag{12.83}$$

Using the recurrence relation (12.81) one can show that

$$\begin{aligned}
\Gamma(n) =&(n-1)\Gamma(n-1)\\
\Gamma(n-1) =&(n-2)\Gamma(n-2)\\
\Gamma(n-2) =&(n-3)\Gamma(n-3)\\
\vdots \quad =& \quad \vdots\\
\Gamma(3) =&2\,\Gamma(2)\\
\Gamma(2) =&1\,\Gamma(1) = 1
\end{aligned} \tag{12.84}$$

The equations (12.81) and (12.84) demonstrate that

$$\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n! \tag{12.85}$$

Observe that when $n = 0$ the equation (12.85) becomes $\Gamma(1) = 0!$, but we know $\Gamma(1) = 1$, hence this is one of the reasons for the convention of defining $0!$ as 1.

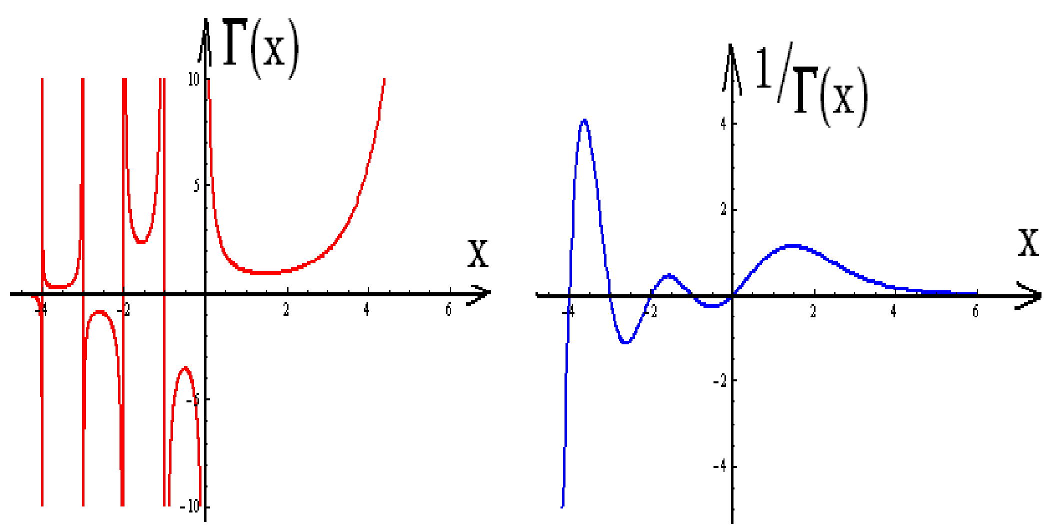


Figure 12-6.

The Gamma function $\Gamma(x)$ and $1/\Gamma(x)$.

Write equation (12.81) in the form $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ to show that for $n = 0, -1, -2, \ldots$ the function $\Gamma(n)$ becomes infinite. The function $\Gamma(x)$ and $1/\Gamma(x)$ are illustrated in the figure 12-6.

Example 12-4.

$$\text{Show that } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Solution

Substitute $x = 1/2$ into equation (12.79) and show

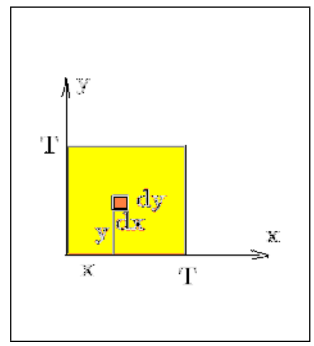
$$\Gamma(\frac{1}{2}) = \int_0^\infty \xi^{-1/2} e^{-\xi}\, d\xi = \int_0^\infty \frac{e^{-\xi}}{\sqrt{\xi}}\, d\xi \tag{12.86}$$

In equation (12.86) make the substitution $\xi = x^2$ with $d\xi = 2x\, dx$ to obtain

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-x^2}}{x} 2x\, dx = 2\int_0^\infty e^{-x^2}\, dx \tag{12.87}$$

Let I denote the integrals

$$I = \int_0^\infty e^{-x^2}\, dx \qquad \text{and} \qquad I = \int_0^\infty e^{-y^2}\, dy$$



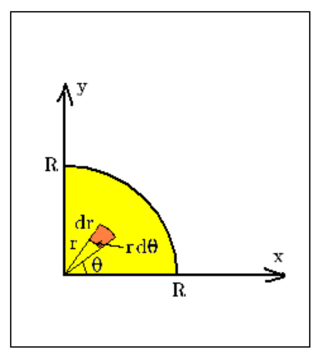
and form the double integral

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \lim_{T\to\infty} \int_0^T \int_0^T e^{-(x^2+y^2)}\, dxdy$$

and observe that as T increases without bound the area of integration fills up the first quadrant.

Change the double integral for I^2 from rectangular to polar coordinates where

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dxdy = r dr d\theta$$



and write

$$I^2 = \lim_{R\to\infty} \int_{r=0}^R \int_0^{\pi/2} e^{-r^2}\, r dr d\theta$$

and observe that as R increases without bound the area of integration is still over the first quadrant. Now integrate with respect to θ and then integrate with respect to r to show

$$I^2 = \frac{\pi}{4} \qquad \text{or} \qquad I = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

Substitute this result into the equation (12.87) to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

■

Product of odd and even integers

One can now apply the previous results

$$\Gamma(n) = (n-1)\Gamma(n-1), \qquad \Gamma(1) = 1, \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{12.88}$$

to show that for n a positive integer

$$\Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\cdots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi} \tag{12.89}$$

This result demonstrates that an alternative representation for **the product of the odd integers** is given by

$$1\cdot 3\cdot 5\cdots(2n-5)(2n-3)(2n-1) = \frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right) \tag{12.90}$$

Using the equations (12.88) one can demonstrate

$$\Gamma\left(\frac{2n+2}{2}\right) = \left(\frac{2n}{2}\right)\left(\frac{2n-2}{2}\right)\left(\frac{2n-4}{2}\right)\cdots\left(\frac{6}{2}\right)\left(\frac{4}{2}\right)\left(\frac{2}{2}\right) \tag{12.91}$$

which shows that **the product of the even integers** can be represented in the form

$$2\cdot 4\cdot 6\cdots(2n-4)(2n-2)(2n) = 2^n\Gamma(n+1) \tag{12.92}$$

Example 12-5.

Let $S_n = \int_0^{\pi/2} \sin^n x\, dx$ and integrate by parts using

$$\begin{aligned} U &= \sin^{n-1} x & dV &= \sin x\, dx \\ dU &= (n-1)\sin^{n-2} x\, \cos x\, dx & V &= -\cos x \end{aligned}$$

to obtain

$$\begin{aligned} S_n &= -\sin^{n-1} x\, \cos x\Big]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\sin^{n-2} x\, \cos^2 x\, dx \\ S_n &= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)\, dx = (n-1)S_{n-2} - (n-1)S_n \end{aligned}$$

which simplifies to the recurrence formula

$$S_n = \frac{n-1}{n} S_{n-2} \tag{12.93}$$

The above recurrence relation implies that

$$S_n = \frac{n-1}{n} S_{n-2} = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) S_{n-4} = \cdots \tag{12.94}$$

If n is odd, say $n = 2m-1$, then equation (12.94) eventually becomes

$$S_{2m-1} = \left(\frac{2m-2}{2m-1}\right)\left(\frac{2m-4}{2m-3}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\cdot S_1 \tag{12.95}$$

where

$$S_1 = \int_0^{\pi/2} \sin x\, dx = -\cos x\big]_0^{\pi/2} = 1$$

The numerator of equation (12.95) is a **product of even integers** and the denominator of equation (12.95) is a **product of odd integers** so that one can employ the results from equations (12.90) and (12.92) to write equation (12.95) in the form

$$S_{2m-1} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(m)}{\Gamma\left(\frac{2m+1}{2}\right)} \tag{12.96}$$

If n is even, say $n = 2m$, then equation (12.94) eventually becomes

$$S_{2m} = \left(\frac{2m-1}{2m}\right)\left(\frac{2m-3}{2m-2}\right)\cdots\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\, S_0 \tag{12.97}$$

where

$$S_0 = \int_0^{\pi/2} dx = \frac{\pi}{2} \tag{12.98}$$

Note that the numerator in equation (12.97) is a **product of odd integers** and the denominator is a **product of even integers**. Using the results from equations (12.90) and (12.92) the above result can be expressed in the form

$$S_{2m} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{2m+1}{2}\right)}{\Gamma(m+1)}\,\frac{\pi}{2} \tag{12.99}$$

■

Example 12-6.

Let $C_n = \displaystyle\int_0^{\pi/2} \cos^n x\, dx$ and follow the step-by-step analysis as in the previous example and demonstrate that

$$C_n = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{\sqrt{\pi}}{2}\frac{\Gamma(m)}{\Gamma\left(\frac{2m+1}{2}\right)} & \text{if } n = 2m-1 \text{ is odd} \\ \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{2m+1}{2}\right)}{\Gamma(m+1)}\,\frac{\pi}{2} & \text{if } n = 2m \text{ is even} \end{cases} \tag{12.100}$$

■

Various representations for the Gamma function

The integral representation of the Gamma function

$$\Gamma(x) = \int_0^\infty \xi^{x-1}\, e^{-\xi}\, d\xi \qquad x > 0 \tag{12.101}$$

can be transformed to many alternative representations for use in special situations.

(i) The substitution $\xi = \ln(\frac{1}{y})$ or $y = e^{-\xi}$, converts the equation (12.101) to the form

$$\Gamma(x) = \int_0^1 \left(\ln\frac{1}{y}\right)^{x-1} dy \tag{12.102}$$

which is the form Euler originally studied.

(ii) The substitution $\xi = zt$ converts equation (12.101) to the form

$$\Gamma(x) = \int_0^\infty z^{x-1} t^{x-1} e^{-zt}\, z\, dt$$

and replacing x by $x+1$ there results

$$\Gamma(x+1) = z^{x+1} \int_0^\infty t^x e^{-zt}\, dt \tag{12.103}$$

The above change of variables are just a sampling of forms for obtaining alternative integral representations of the Gamma function.

Euler's constant γ, defined by the limit

$$\gamma = \lim_{n\to\infty}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \ln(n)\right] = 0.5772156649\cdots \tag{12.104}$$

occurs in many alternative representations of the Gamma function. When γ is represented as a continued fraction (See chapter 4, page 337) one finds the list notation given by

$$\gamma = [0\,; 1, 1, 2, 1, 2, 1, , 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 40, 1, \ldots]$$

The first nine convergents are

$$\begin{array}{lll} \gamma_1 = 1 & \gamma_4 = \frac{4}{7} = 0.571428571 & \gamma_7 = \frac{71}{123} = 0.577235772 \\ \gamma_2 = \frac{1}{2} = 0.5 & \gamma_5 = \frac{11}{19} = 0.578947368 & \gamma_8 = \frac{228}{395} = 0.577215190 \\ \gamma_3 = \frac{3}{5} = 0.6 & \gamma_6 = \frac{15}{26} = 0.5769233077 & \gamma_9 = \frac{3035}{5258} = 0.577215671 \end{array} \tag{12.105}$$

where γ_9 is accurate to seven decimal places.

Sometime around 1729 Euler defined the Gamma function in the form

$$\Gamma(x) = \lim_{n\to\infty} \frac{(n-1)!\, n^x}{x(1+x)(2+x)(3+x)\cdots(n-1+x)} = \lim_{n\to\infty} \frac{n^x}{x(1+\frac{x}{1})(1+\frac{x}{2})\cdots(1+\frac{x}{n-1})} \tag{12.106}$$

Karl Weierstrass modified Euler's form for the Gamma function and represented it in the form

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right] \tag{12.107}$$

where γ is Euler's constant from equation (12.104). Here the infinite product $\prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right]$ is convergent for all values of z positive, negative, real or complex.

Other forms of the Gamma function can be found in the mathematical literature. The representation of the Gamma function in the complex plane provides new incites into properties of the Gamma function. As an interesting exercise check out some textbooks on the Gamma function to see how all of the above forms of the Gamma function are equivalent. This type of exercise is one example illustrating the concept that functions and ideas, which occur in mathematical studies, can be presented in a variety of ways.

Euler formula for the Gamma function

Having a variety of forms for representing the Gamma function provides one the opportunity to seek out and discover other properties of the Gamma function. For example, employ the Weirstrass representation

$$\frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{n=1}^{\infty} \left[\left(1+\frac{x}{n}\right) e^{-x/n}\right] \tag{12.108}$$

and show

$$\frac{1}{\Gamma(x)} \frac{1}{\Gamma(-x)} = -x^2 e^{\gamma x} e^{-\gamma x} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}\right) e^{-x/n} e^{x/n} \tag{12.109}$$

This equation simplifies using the property $\Gamma(1-x) = -x\Gamma(-x)$. One can verify equation (12.109) simplifies to

$$\frac{1}{\Gamma(x)} \frac{1}{\Gamma(1-x)} = x \prod_{n=1}^{\infty} \left(1-\frac{x^2}{n^2}\right)$$

or

$$\frac{1}{\Gamma(x)} \frac{1}{\Gamma(1-x)} = x \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots \tag{12.110}$$

Recall Euler's infinite product formula for $\sin\theta$ (see Example 4-38)

$$\sin(\pi x) = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots \tag{12.111}$$

and compare this infinite product with the one occurring in equation (12.110) to show

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{12.112}$$

which is known as Euler's reflection formula for the Gamma function. Make note of the fact that if the value $x = 1/2$ is substituted into equation (12.112) one obtains

$$\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \pi \qquad \text{or} \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{12.114}$$

which agrees with our previous result.

Using the previous result $n\Gamma(n) = \Gamma(1+n)$, the equation (12.112) is sometimes written in the form

$$\Gamma(1+n)\Gamma(1-n) = \frac{n\pi}{\sin n\pi} \tag{12.114}$$

The Zeta function related to the Gamma function

The Gamma function $\Gamma(z)$ and the Riemann Zeta function $\zeta(z)$ are related. Recall that one definition of the Zeta function is (see Example 4-38)

$$\zeta(z) = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^z}, \qquad z>1 \tag{12.115}$$

and the integral form for representing the Gamma function is given by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad z>0 \tag{12.116}$$

In equation (12.116) make the change of variable $t = rx$ and show

$$\Gamma(z) = \int_0^\infty (rx)^{z-1}e^{-rx}\,r\,dx = r^z\int_0^\infty x^{z-1}e^{-rx}\,dx \tag{12.117}$$

One can express equation (12.117) in the form

$$\frac{1}{r^z} = \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}e^{-rx}\,dx \tag{12.118}$$

A summation of equation (12.118) over integer values for r produces the result

$$\zeta(z) = \sum_{r=1}^{\infty}\frac{1}{r^z} = \frac{1}{\Gamma(z)}\sum_{r=1}^{\infty}\int_0^\infty x^{z-1}e^{-rx}\,dx \tag{12.119}$$

Now interchange the roles of summation and integration on the right hand side of equation (12.119) to obtain

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} x^{z-1} \sum_{r=1}^{\infty} e^{-rx}\, dx \tag{12.120}$$

where now the summation on the right hand side of equation (12.120) is the geometric series

$$\sum_{r=1}^{\infty} e^{-rx} = e^{-x} + e^{-2x} + e^{-3x} + \cdots = \frac{e^{-x}}{1-e^{-x}}$$

Consequently, the equation (12.120) simplifies to the form

$$\zeta(z)\Gamma(z) = \int_0^{\infty} x^{z-1} \frac{e^{-x}}{1-e^{-x}}\, dx \tag{12.121}$$

Observe that the Gamma function and Zeta function properties dictate that z be restricted such that $z \neq 1, 0, -1, -2, -3, \ldots$ in using equation (12.121).

Product property of the Gamma function

The Gamma function satisfies the product property that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}} \tag{12.122}$$

In order to derive this result we develop the following background material.

Example 12-7.

Show that

$$\sin n\theta = 2^{n-1} \sin\theta \sin(\theta + \frac{\pi}{n}) \sin(\theta + \frac{2\pi}{n}) \sin(\theta + \frac{3\pi}{n}) \cdots \sin(\frac{\theta + (n-1)\pi}{n}) \tag{12.123}$$

and

$$\lim_{\theta\to 0} \frac{\sin n\theta}{\sin\theta} = n = 2^{n-1} \sin(\frac{\pi}{n}) \sin(\frac{2\pi}{n}) \sin(\frac{3\pi}{n}) \cdots \sin(\frac{(n-1)\pi}{n}) \tag{12.124}$$

Solution

The following proof follows that presented in the reference Hobson[3]

First show that the function

$$x^{2n} - 2x^n \cos n\theta + 1 \tag{12.125}$$

[3] Ernest William Hobson, **A treatise on plane trigonometry**, 5th Edition, Cambridge University Press, 1921, Pages 117-119.

can be written as a product of factors having the form

$$x^{2n} - 2x^n \cos n\theta + 1 = 2^{n-1} \prod_{r=0}^{n-1} (x^2 - 2x \cos(\theta + \frac{2r\pi}{n}) + 1) \tag{12.126}$$

This is accomplished by considering the function $x^{2n} - 2x^n \cos n\theta + 1$ and then dividing it by x^n and defining

$$u_n = x^n - 2\cos n\theta + x^{-n} \tag{12.127}$$

and then verifying that u_n can be written as

$$\begin{aligned} u_n =&(x^{n-1} + x^{-n+1})(x - 2\cos\theta + x^{-1}) \\ &+ 2\cos\theta(x^{n-1} - 2\cos[(n-1)\theta] + x^{-n+1}) - (x^{n-2} - 2\cos[(n-2)\theta] + x^{-(n-2)}) \end{aligned} \tag{12.128}$$

or in terms of the u_n definition

$$u_n = (x^{n-1} + x^{-n+1})u_1 + 2u_{n-1}\cos\theta - u_{n-2} \tag{12.129}$$

Observe that u_n is divisible by u_1 if both u_{n-1} and u_{n-2} are also divisible by u_1. To show this is true verify that

$$u_2 = x^2 - 2\cos 2\theta + x^{-2} = (x - 2\cos\theta + x^{-1})(x + 2\cos\theta + x^{-1})$$

and consequently u_2 is divisible by u_1. Using equation (12.129) write

$$u_3 = (x^2 + x^{-2})u_1 + 2u_2\cos\theta - u_1$$

to show u_3 is divisible by u_1. Continuing in this fashion $u_4, u_5, u_6, \ldots, u_{n-2}, u_{n-1}$, are all divisible by u_1. This demonstrates that $x^2 - 2x\cos\theta + 1$ is a factor of $x^{2n} - 2x^n \cos n\theta + 1$. Since θ is an arbitrary angle, replace θ by $\theta + 2r\pi/n$, r an integer constant, to show

$$x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1 \quad \text{is a factor of} \quad x^{2n} - 2x^n \cos[n\left(\theta + \frac{2r\pi}{n}\right)] + 1$$

for $r = 0, 1, 2, \ldots, n-1$.

Using the trigonometric identity

$$\cos n\theta = \cos[n(\theta + \frac{2r\pi}{n})]$$

for r an integer, one can say that the factors of

$$x^{2n} - 2x^n \cos n\theta + 1$$

are given by $x^2 - 2x\cos(\theta + \dfrac{2r\pi}{n}) + 1$ for $r = 0, 1, 2, \ldots, n-1$. This implies $x^{2n} - 2x^n \cos n\theta + 1$ can be expressed as

$$x^{2n} - 2x^n \cos n\theta + 1 = \prod_{r=0}^{n-1} (x^2 - 2x\cos(\theta + \frac{2r\pi}{n}) + 1) \tag{12.130}$$

In the special case $x = 1$ the equation (12.130) simplifies to

$$1 - \cos n\theta = 2^{n-1} \prod_{r=0}^{n-1} \left(1 - \cos\left(\theta + \frac{2r\pi}{n}\right)\right) \tag{12.131}$$

Replacing θ by 2θ in equation (12.131) and simplifying one obtains

$$2\sin^2 n\theta = 2^{n-1} 2^n \sin^2\theta \sin^2(\theta + \frac{\pi}{n}) \sin^2(\theta + \frac{2\pi}{n}) \cdots \sin^2(\theta + \frac{(n-1)\pi}{n}) \tag{12.132}$$

Further simplify equation (12.132) and then take the square root of both sides to obtain

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1} \sin(\theta + \frac{\pi}{n}) \sin(\theta + \frac{2\pi}{n}) \cdots \sin(\theta + \frac{(n-1)\pi}{n}) \tag{12.133}$$

where the positive square root is taken when each term is positive. In equation (12.133) take the limit as $\theta \to 0$ and verify

$$n = 2^{n-1} \sin(\frac{\pi}{n}) \sin(\frac{2\pi}{n}) \cdots \sin(\frac{(n-1)\pi}{n}) \tag{12.134}$$

■

Now consider the product

$$y = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right)$$

and reverse the terms within the product to show

$$y^2 = \left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)\right]\left[\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right)\right]\cdots\left[\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right]$$

followed by writing equation (12.112) in the form

$$\Gamma\left(\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right) = \frac{\pi}{\sin(\frac{m\pi}{n})}$$

for $m = 1, 2, \ldots, n-1$. This identity produces

$$y^2 = \frac{\pi}{\sin(\frac{\pi}{n})} \frac{\pi}{\sin(\frac{2\pi}{n})} \frac{\pi}{\sin(\frac{3\pi}{n})} \cdots \frac{\pi}{\sin(\frac{(n-1)\pi}{n})}$$

One can now employ the result from Example 12-6, equation (12.134), to show

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$$

Derivatives of $\ln \Gamma(z)$

Using the Weierstrass definition of the Gamma function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{z}\right) e^{-z/n} \right] \tag{12.135}$$

take the natural logarithm of both sides and show

$$\ln \Gamma(z) = -\ln z - \gamma z - \sum_{k=1}^{\infty} \left[\ln\left(1 + \frac{z}{k}\right) - \frac{z}{k} \right] \tag{12.136}$$

Take the derivative of each term in this equation to show

$$\frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{k=1}^{\infty} \left(\frac{1}{z+k} - \frac{1}{k} \right) \tag{12.137}$$

Differentiate equation (12.137) to obtain

$$\frac{d^2}{dz^2} \ln \Gamma(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{1}{(z+k)^2} = \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots \tag{12.138}$$

Continuing differentiating the function $\ln \Gamma(z)$ and use the fact that

$$\begin{aligned}
\frac{d}{dz}(z+k)^{-2} =& (-2)(z+k)^{-3} \\
\frac{d^2}{dz^2}(z+k)^{-2} =& (-2)(-3)(z+k)^{-4} \\
\frac{d^3}{dz^3}(z+k)^{-2} =& (-2)(-3)(-4)(z+k)^{-5} \\
\vdots \qquad & \qquad \vdots
\end{aligned}$$

and demonstrate that

$$\begin{aligned}
\frac{d^n}{dz^n} \ln \Gamma(z) =& (-1)^n (n-1)! \sum_{k=0}^{\infty} \frac{1}{(z+k)^n} \\
\frac{d^{n+1}}{d^{n+1}} \ln \Gamma(z) =& (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}
\end{aligned} \tag{12.139}$$

Make note of the following definitions. The function $\zeta(n,z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^n}$, **where any term where** $(z+k) = 0$ **is understood to be excluded from the summation process**, is defined as **the Hurwitz[4] Zeta function** and satisfies $\zeta(n,0)=\zeta(n)$, the Zeta function. The function

$$\psi_n(z)=(-1)^{n+1}n!\zeta(n+1,z)=\frac{d^{n+1}}{d^{n+1}}\ln[\Gamma(z)] \tag{12.140}$$

is referred to as **the polygamma function of order n.** Observe that

$$\psi_0(z)=\frac{d}{dz}\ln\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{and} \qquad \psi_n(z)=\frac{d^n}{dz^n}\psi_0(z)$$

The polygamma function of order zero $\psi_0(x)$ is called **the digamma function** and is illustrated in the figure 12-7.

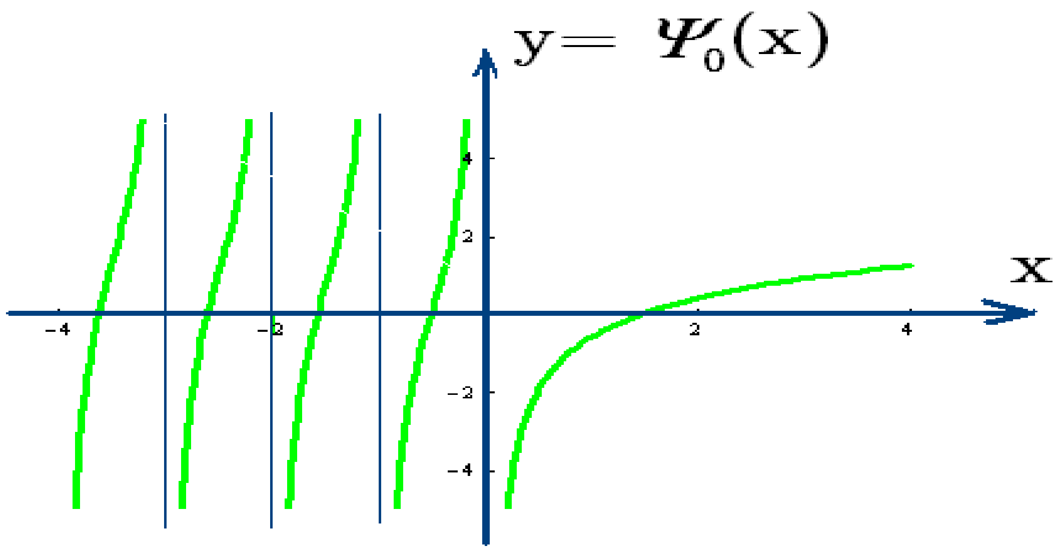


Figure 12-7. The polygamma function of order zero.

[4] Adolf Hurwitz (1859-1919) German professor of mathematics.

Taylor series expansion for $\ln\Gamma(x+1)$

Make reference to the equations (12.136), (12.137), (12.138), (12.139), and verify that when z is replaced by $(x+1)$ in these equations, one obtains the following values.

$$\begin{aligned}
\ln\Gamma(x+1)\Big|_{x=0} =& \ln\Gamma(1) = 0 \\
\frac{d}{dx}\ln\Gamma(x+1)\Big|_{x=0} =& -\gamma \quad \text{because (12.137) is a telescoping series} \\
\frac{d^2}{dx^2}\ln\Gamma(x+1)\Big|_{x=0} =& \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \zeta(2) \\
\vdots \qquad & \qquad \vdots \\
\frac{d^n}{dx^n}\ln\Gamma(x+1)\Big|_{x=0} =& (-1)^n(n-1)!\zeta(n) \qquad n \ge 2
\end{aligned} \tag{12.141}$$

where

$$\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$$

is the Riemann Zeta function. The above values of the derivatives of $\ln\Gamma(x+1)$, evaluated at $x=0$, produces the Taylor series expansion for $\ln\Gamma(x+1)$ as

$$\ln\Gamma(x+1) = -\gamma x + \zeta(2)\frac{x^2}{2} - \zeta(3)\frac{x^3}{3} + \zeta(4)\frac{x^4}{4} + \cdots + (-1)^n\zeta(n)\frac{x^n}{n} + \cdots \tag{12.142}$$

which converges if x is less than unity.

Another product formula

Define the function

$$\phi(z) = \frac{n^z\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{(n-1)}{n}\right)}{n\Gamma(nz)} \tag{12.143}$$

and use the equation (12.106) to write

$$\Gamma\left(z+\frac{r}{n}\right) = \lim_{m\to\infty}\frac{(m-1)!\,m^{z+r/n}}{\left(z+\frac{r}{n}\right)\left(z+\frac{r}{n}+1\right)\left(z+\frac{r}{n}+2\right)\cdots\left(z+\frac{r}{n}+m-1\right)} \tag{12.144}$$

for $r=0,1,2,\ldots,n-1$ and

$$\Gamma(nz) = \lim_{m\to\infty}\frac{(nm-1)!(nm)^{nz}}{(nz)(nz+1)(nz+2)\cdots(nz+nm-1)} \tag{12.145}$$

to express equation (12.143) in the form

$$\phi(z) = \frac{n^{nz}\displaystyle\prod_{r=0}^{n-1}\lim_{m\to\infty}\frac{(m-1)!\,m^{z+r/n}}{\left(z+\frac{r}{n}\right)\left(z+\frac{r}{n}+1\right)\left(z+\frac{r}{n}+2\right)\cdots\left(z+\frac{r}{n}+m-1\right)}}{n\displaystyle\lim_{m\to\infty}\frac{(nm-1)!(nm)^{nz}}{nz(nz+1)(nz+2)\cdots(nz+nm-1)}} \tag{12.146}$$

The product nm is used in the definition of $\Gamma(nz)$ to show that the equation (12.146) simplifies after a lot of careful algebra to

$$\phi(z) = \lim_{m\to\infty} \frac{n^{nz-1}\,[(m-1)!]^n\, m^{\frac{n-1}{2}}\, m^{mn}}{(nm-1)!\,(nm)^{nz}} = \lim_{m\to\infty} \frac{[(m-1)!]^n\, m^{\frac{n-1}{2}}\, m^{mn-1}}{(nm-1)!} \tag{12.147}$$

The equation (12.147) shows that $\phi(z)$ is independent of z and is a constant. To find the value of the constant, select a value of z where equation (12.143) can be evaluated. Selecting the value $z = 1/n$ one finds after simplification the product formula first derived by Gauss[5] and Legendre[6]

$$n^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right) = n^{1/2}\,(2\pi)^{\frac{n-1}{2}}\Gamma(nz) \tag{12.148}$$

Using equation (12.148) one can produce the special cases

$$\begin{aligned} n=2 \qquad\qquad & \Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \Gamma(2z)\,(2\pi)^{1/2}\,2^{1/2-2z} \\ n=3 \qquad\qquad & \Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right) = \Gamma(3z)(2\pi)\,3^{1/2-3z} \end{aligned} \tag{12.149}$$

Example 12-8. (Summation)

The function $\Psi(x)$ defined by $\Psi(x) = \dfrac{d}{dx}\ln\Gamma(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ occurs in representing the summation of many finite and infinite convergent series. The Gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x)$$

so that

$$\ln\Gamma(x+1) = \ln x + \ln\Gamma(x) \tag{12.150}$$

Differentiate equation (12.150) and show

$$\Psi(x+1) = \frac{1}{x} + \Psi(x) \qquad \text{or} \qquad \Psi(x+1) - \Psi(x) = \frac{1}{x} \tag{12.151}$$

This demonstrates that $\Psi(x)$ satisfies the difference equation

$$\Delta\Psi(x) = \frac{1}{x} \qquad \text{or} \qquad \Delta\Psi(a+n) = \frac{1}{a+n}, \quad \text{a is constant} \tag{12.152}$$

---

[5] Carl Friedrich Gauss (1777-1855) A famous German mathematician.
[6] Adrien-Marie Legendre (1752-1833) A famous French mathematician.

Using the results from pages 361-362, one can write

$$\sum_{i=1}^{n} \Delta\Psi(a+i) = \sum_{i=1}^{n} \frac{1}{a+i} = \Psi(a+n+1) - \Psi(a+1) \tag{12.153}$$

As an example of how equation (12.153) can be employed, examine the finite sum

$$S = \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+nb} \tag{12.154}$$

This finite series can be expressed

$$S = \frac{1}{b}\sum_{i=1}^{n} \frac{1}{\frac{a}{b}+i} = \frac{1}{b}\sum_{i=1}^{n} \Delta\Psi(\frac{a}{b}+i) = \frac{1}{b}\left[\Psi(\frac{a}{b}+n+1) - \Psi(\frac{a}{b}+1)\right] \tag{12.155}$$

■

Use differential equations to find series

Another way to find the series representation of a given function is to first differentiate the function and then form a differential equation satisfied by the given function. One can then substitute a power series into the differential equation and determine the coefficients of the power series by comparing like terms as illustrated in the following examples.

Example 12-9. Determination of series

Find a power series expansion to represent the function $y = a^x$, where a is a constant.

Solution Write $y = y(x) = a^x = e^{x\ln a}$ and differentiate this function to obtain $\frac{dy}{dx} = e^{x\ln a}\ln a = a^x \ln a$, so that $y = a^x$ is a solution of the differential equation

$$\frac{dy}{dx} = y\,\ln a \tag{12.156}$$

satisfying the initial condition at $x = 0$, $y(0) = a^0 = 1$.

Assume that y has the power series representation

$$y = y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots \tag{12.157}$$

with derivative

$$\frac{dy}{dx} = y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \cdots \tag{12.158}$$

where $c_0, c_1, c_2, c_3, \dots$ are constants to be determined. Substitute the representations (12.157) and (12.158) into the differential equation (12.156) to obtain

$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots = \ln a \left[c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \right] \tag{12.159}$$

In equation (12.159) **equate the coefficients of like powers of** x and show

$$\begin{aligned} c_1 =& c_0 \ \ln a \\ 2c_2 =& c_1 \ \ln a \\ 3c_3 =& c_2 \ \ln a \\ \vdots \quad & \quad \vdots \\ (n+1)c_{n+1} =& c_n \ \ln a \\ \vdots \quad & \quad \vdots \end{aligned} \tag{12.160}$$

The general equation

$$(n+1)c_{n+1} = c_n \ \ln a \qquad \text{or} \qquad c_{n+1} = \frac{c_n \ \ln a}{(n+1)} \tag{12.161}$$

which holds for $n = 0, 1, 2, 3, \dots$ is called **a recurrence relation** or **recurrence formula** associated with the given series and tells one how to select the coefficients in order to satisfy the differential equation. Recall that $c_0 = y(0) = 1$ is determined from the initial value $x = 0$. Using the recurrence formula (12.161) and the equations (12.160) one finds

$$\begin{aligned} n =& 0 & c_1 =& \ln a \\ n =& 1 & c_2 =& \frac{1}{2!}(\ln a)^2 \\ n =& 2 & c_3 =& \frac{1}{3!}(\ln a)^3 \\ \vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots \\ n =& m & c_m =& \frac{1}{m!}(\ln a)^m \end{aligned} \tag{12.162}$$

and consequently the power series expansion for $y = a^x$ is given by

$$y = a^x = 1 + x \ \ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \cdots + \frac{x^m}{m!}(\ln a)^m + \cdots \tag{12.163}$$

■

Example 12-10. Determination of series

Find a power series expansion to represent the function $y = y(x) = (h+x)^n$, where h is a constant.

Solution Differentiate the function $y = y(x) = (h+x)^n$ and show

$$\frac{dy}{dx} = y'(x) = n(h+x)^{n-1} \tag{12.164}$$

Multiply equation (12.164) by $(h+x)$ and show y is a solution of the differential equation

$$(h+x)\frac{dy}{dx} = n\,y \tag{12.165}$$

with initial condition at $x = 0$ given by $y(0) = h^n$. Assume $y = (h+x)^n$ has the power series representation

$$y = (h+x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_m x^m + \cdots \tag{12.166}$$

with derivative

$$\frac{dy}{dx} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + mc_m x^m + \cdots \tag{12.167}$$

Note that the index m has been selected for the general term of the series as the value n occurs in the differential equations and we don't want these values to become confused with one another. Substitute the power series (12.166) and (12.167) into the differential equation (12.165) to obtain

$$(h+x)\left[c_1 + 2c_2 x + 3c_3 x^2 + \cdots + mc_m x^{m-1} + \cdots\right] = n[c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m + \cdots] \tag{12.168}$$

Expand the lefthand side of equation (12.168) and show

$$\begin{aligned} hc_1 + &2hc_2 x + 3hc_3 x^2 + \cdots + hmc_m x^{m-1} + \cdots \\ &+ c_1 x + 2c_2 x^3 + 3c_3 x^3 + \cdots + mc_m x^m + \cdots \\ &= \; n\,c_0 + n\,c_1 x + n\,c_2 x^2 + n\,c_3 x^3 + \cdots + n\,c_m x^m + \cdots \end{aligned} \tag{12.169}$$

In equation (12.169) **equate the coefficients of like powers of** x to obtain a **recurrence relation** or **recurrence formula**. One finds

$$\begin{aligned} hc_1 =& n\,c_0 \\ (2hc_2 + c_1) =& n\,c_1 \\ (3hc_2 + 2c_2) =& n\,c_2 \\ \vdots \quad & \quad \vdots \\ [(m+1)hc_{m+1} + nc_m] =& n\,c_m \end{aligned} \tag{12.170}$$

Here the recurrence formula is

$$(m+1)h\, c_{m+1} + m\, c_m = n\, c_m \qquad \text{or} \qquad c_{m+1} = \frac{(n-m)}{(m+1)h}\, c_m \tag{12.171}$$

for $m = 0, 1, 2, \ldots$. If $y = y(x) = (h+x)^n$, then $y(0) = c_0 = h^n$. Substitute the values $m = 0, 1, 2, 3, \ldots, n, n+1, \ldots$ into the recurrence formula (12.171) to obtain

$$\begin{aligned}
m =&0 \qquad & c_1 =&\frac{n}{h}\, c_0 = nh^{n-1} = \binom{n}{1} h^{n-1}\\
m =&1 & c_2 =&\frac{(n-1)}{2h}\, c_1 = \frac{n(n-1)}{2!} h^{n-2} = \binom{n}{2} h^{n-2}\\
m =&2 & c_3 =&\frac{(n-2)}{3h}\, c_2 = \frac{n(n-1)(n-2)}{3!} h^{n-3} = \binom{n}{3} h^{n-3}\\
\vdots\quad & \vdots & \vdots \quad & \vdots\\
m =&n-1 & c_n =&\frac{1}{nh}\, c_{n-1} = \frac{n!}{n!} h^0 = 1 = \binom{n}{n} h^0\\
m =&n & c_{n+1} =&0\\
m =&n+1 & c_{n+2} =&0
\end{aligned} \tag{12.172}$$

and $c_n = 0$ for all integer values of m satisfying $m \geq n$. In the equations (12.172) the terms

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!}, & m \leq n \\ 0, & m > n \end{cases} \tag{12.173}$$

are the binomial coefficients. Substituting the values given by equations (12.172) into the power series (12.166) one obtains the finite series of terms

$$y = (h+x)^n = \binom{n}{0} h^n + \binom{n}{1} h^{n-1}x + \binom{n}{2} h^{n-2}x^2 + \cdots + \binom{n}{n-1} hx^{n-1} + \binom{n}{n} x^n \tag{12.175}$$

which is the well-known binomial expansion.

■

The Laplace Transform

Consider the mathematical operator box labeled $\mathcal{L}\{f(t)\}$ or $\mathcal{L}\{\quad ; t \to s\}$ as illustrated in the figure 12-8. This mathematical operator $\mathcal{L}$ is called **the Laplace transform operator and is defined**

$$\begin{aligned}
&\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\, dt = F(s)\\
\text{or} \qquad &\mathcal{L}\{f(t);\ t \to s\} = \lim_{T\to\infty} \int_0^T f(t)e^{-st}\, dt = F(s), \qquad s > 0
\end{aligned} \tag{12.175}$$

and represents a transformation from a function $f(t)$ in the t-domain to a function $F(s)$ in the s-domain (frequently called the frequency domain). The Laplace transform[7] has many applications in mathematics, statistics, physics and engineering.

[7] This operator is named after Pierre Simon Laplace (1749-1857) A famous French mathematician.

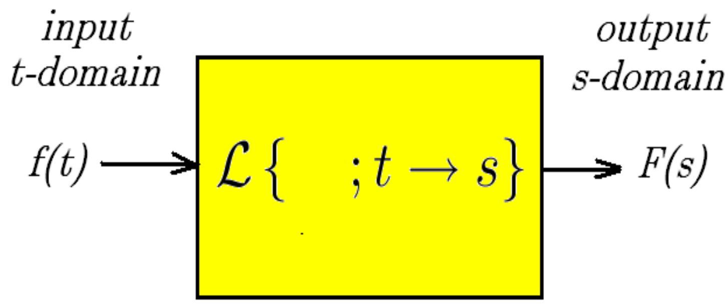


Figure 12-8. **The Laplace transform operator.**

In the defining equation (12.175) the parameter s is selected such that the integral exists and many times it is expressed as a complex variable $s=\sigma+i\,\omega$ where σ and ω are real and i is an imaginary component satisfying $i^2=-1$. The Laplace transform can be studied with or without employing knowledge of complex variables.

Example 12-11. (Laplace transform)

Find the Laplace transform of $\sin(\alpha\,t)$ where α is a nonzero constant.

Solution

By definition $\mathcal{L}\{\sin(\alpha\,t);\ t\to s\}=\int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt$

Integrate by parts with $u=\sin(\alpha\,t)$ and $dv=e^{-st}\,dt$ to obtain

$$I=\int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt=-\sin(\alpha\,t)\,\frac{e^{-st}}{s}\Big|_0^\infty+\frac{\alpha}{s}\int_0^\infty\cos(\alpha\,t)\,e^{-st}\,dt$$

Integrate by parts again with $u=\cos(\alpha\,t)$ and $dv=e^{-st}\,dt$ and show

$$I=\int_0^\infty \sin(\alpha\,t)\,e^{-st}\,dt=\frac{\alpha}{s}\left[\cos(\alpha\,t)\,\frac{e^{-st}}{-s}\Big|_0^\infty-\frac{\alpha}{s}\,I\right]$$

This last equation simplifies to

$$(1+\frac{\alpha^2}{s^2})I=\frac{\alpha}{s^2}\qquad\text{or}\qquad I=\int_0^\infty\sin(\alpha\,t)\,e^{-st}\,dt=\mathcal{L}\{\sin(\alpha\,t);t\to s\}=\frac{\alpha}{s^2+\alpha^2}$$

■

Example 12-12. **Laplace transform**

Find the Laplace transform of $e^{\alpha t}$

Solution

By definition $\mathcal{L}\left\{e^{\alpha t}\ ;\ t \to s\right\} = \int_0^\infty e^{\alpha t} e^{-st}\, dt = \int_0^\infty e^{-(s-\alpha)t}\, dt$ Scale the integral to obtain

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha}\int_0^\infty e^{-u}\, du, \qquad u = (s-\alpha)\, t$$

to obtain

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha}\left[-e^{-u}\right]_0^\infty = \frac{1}{s-\alpha}\lim_{T\to\infty}\left[-e^{-T} - (-e^{0})\right]$$

which produces the result

$$\mathcal{L}\left\{e^{\alpha t}\right\} = \frac{1}{s-\alpha} \qquad \text{provided that } s > \alpha$$

■

Using various integration techniques one can verify the following short table of Laplace transforms.

Short Table of Laplace Transforms	
Function $f(t)$	**Laplace Transform $\mathcal{L}\{f(t)\} = F(s)$**
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2!}{s^3}$
t^{n-1}	$\frac{(n-1)!}{s^n} \qquad n = 1, 2, 3, \ldots$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$t\, e^{\alpha t}$	$\frac{1}{(s-\alpha)^2}$
$t^{n-1}\, e^{\alpha t}$	$\frac{(n-1)!}{(s-\alpha)^n} \qquad n = 1, 2, 3, \ldots$
$t^{k-1}\, e^{\alpha t}$	$\frac{\Gamma(k)}{(s-\alpha)^k}, \qquad k > 0$
$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
$\sinh(\alpha t)$	$\frac{\alpha}{s^2-\alpha^2}$
$\cosh(\alpha t)$	$\frac{s}{s^2-\alpha^2}$

Inverse Laplace Transformation $\mathcal{L}^{-1}$

The symbol $\mathcal{L}^{-1}$ is used to denote the inverse Laplace transform operator with the property that $\mathcal{L}^{-1}$ undoes what $\mathcal{L}$ does. That is, if the inverse Laplace transform operator is applied to both sides of the equation $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}^{-1}\mathcal{L}\{f(t)\} = f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{12.176}$$

This indicates that the table of Laplace transforms given above is to be interpreted in either of two ways. Reading the above table left to right indicates $\mathcal{L}\{f(t)\} = F(s)$ and reading the table from right to left indicates $f(t) = \mathcal{L}^{-1}\{F(s)\}$. In general, one can say

$$\mathcal{L}\{f(t)\} = F(s) \quad \text{if and only if} \quad f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{12.177}$$

One use of the Laplace transform operator is to take a difficult problem in the $t-$domain and transform it to an easier problem in the $s-$domain. Solve the easier problem in the $s-$domain and convert the answer back to the $t-$domain. There are two ways by which one can convert a Laplace transform back to the $t-$domain. One conversion method is to have an extensive table of Laplace transforms so that one can use table lookup to convert a function $F(s)$ back to the correct function $f(t)$ by using the property (12.176) or (12.177). Table lookup is the preferred method for now.

Another method used to find the inverse Laplace transform is more advanced and requires knowledge of complex variable theory. This more advanced method is expressed in the language of complex variables as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F(s);\, s \to t\ \} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds$$

where the integration is part of a line integral in the complex plane. Once you learn all the theory involved, the inverse Laplace transform is really simple to use. The difficulty in using the complex form for the inverse transform is that you will need to take a complete course in complex variable theory to use and understand how to employ it. The inverse Laplace transformation techniques often used are illustrated in the figure 12-9.

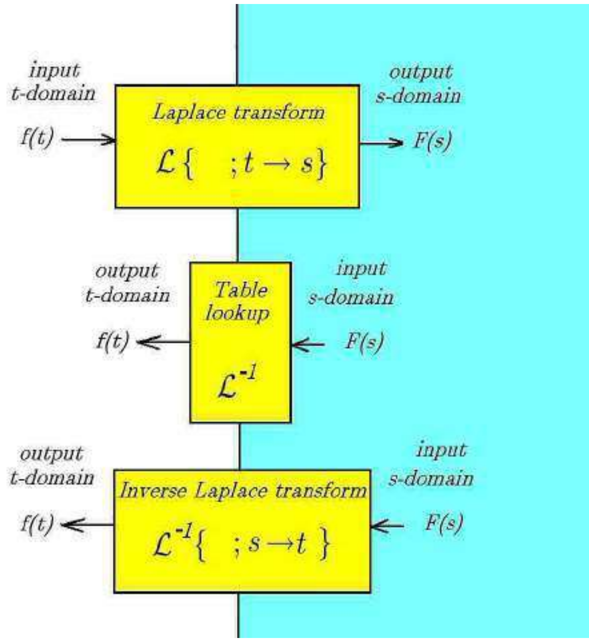


Figure 12-9. Inverting the Laplace transform.

Properties of the Laplace transform

Using the definition of the Laplace transform and applying various integration techniques one can develop a table of Laplace transform properties.

Example 12-13. (Laplace transform of derivative)

Show that if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^+) \text{ or } f'(t) = \mathcal{L}^{-1}\left\{sF(s) - f(0^+)\right\}$$

Solution

By definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integrate by parts with $u = e^{-st}$ and $dv = f'(t)\, dt$ to obtain

$$\mathcal{L}\{f'(t)\} = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f(t) e^{-st}\, dt = sF(s) - f(0^+)$$

Note that $f(t)$ need only be defined for $t \geq 0$ so that the 0^+ is to remind you that the function $f(t)$ evaluated at zero is just the right-hand limit as $t \to 0$.

■

Example 12-14. First shift property

If $F(s) = \mathcal{L}\{f(t)\}$, show that $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$ or $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$

Solution

By definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

so that if one replaces s by $s-a$ there results

$$F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}\,dt = \int_0^\infty e^{at}f(t)\,e^{-st}\,dt = \mathcal{L}\left\{e^{at}f(t)\right\}$$

or

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

■

Example 12-15. (Second shift property)

If $F(s) = \mathcal{L}\{f(t)\}$, show $\mathcal{L}\{f(t-\alpha)H(t-\alpha)\} = e^{-\alpha s}F(s)$ or

$$f(t-\alpha)H(t-\alpha) = \mathcal{L}^{-1}\left\{e^{-\alpha s}F(s)\right\}$$

where $H(t)$ is the Heaviside step function defined by

$$H(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$

Solution

By definition

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

so that replacing t by u and then multiplying by $e^{-\alpha s}$ one finds

$$e^{-\alpha s}F(s) = \int_0^\infty f(u)e^{-su}e^{-\alpha s}\,du = \int_0^\infty f(u)e^{-s(u+\alpha)}\,du$$

Make the change of variables $t = u+\alpha$ with $dt = du$ and then calculate the appropriate limits of integration to show

$$e^{-\alpha s}F(s) = \int_{t=\alpha}^\infty f(t-\alpha)e^{-st}\,dt = \int_0^a (0)\cdot f(t-\alpha)e^{-st}\,dt + \int_\alpha^\infty (1)\cdot f(t-\alpha)\,dt$$

This last integral has the more compact form

$$e^{-\alpha s}F(s) = \int_0^\infty f(t-\alpha)H(t-\alpha)e^{-st}\,dt = \mathcal{L}\{f(t-\alpha)H(t-\alpha)\}$$

or

$$\mathcal{L}^{-1}\left\{e^{-\alpha s}F(s)\right\} = f(t-\alpha)H(t-\alpha)$$

■

As an integration exercise one can verify the various transform properties listed on the next page.

Properties of the Laplace Transform		
Function $f(t)$	**Laplace Transform $F(s)$**	**Comment**
$c_1 f(t)$	$c_1 F(s)$	linearity property
$f'(t)$	$sF(s) - f(0^+)$	Derivative property
$f''(t)$	$s^2 F(s) - sf(0^+) - f'(0+)$	Derivative property
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0^+) - \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$	Derivative property
$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s} F(s) \quad s > 0$	Integration transform
$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$	linearity property
$tf(t)$	$-F'(s)$	multiplication by t property
$t^2 f(t)$	$(-1)^2 F''(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
$e^{\alpha t} f(t)$	$F(s-\alpha)$	First shift property
$\frac{1}{t} f(t)$	$\int_s^\infty F(s)\, ds$	Division by t property
$f(t-\alpha)H(t-\alpha)$	$e^{-\alpha s} F(s)$	Second shift property
$\frac{1}{\alpha} f(\frac{t}{\alpha})$	$F(\alpha s)$	scaling property
$\frac{1}{\alpha} e^{\beta t/\alpha} f(\frac{t}{\alpha})$	$f(\alpha s - \beta)$	shifting scaling
$f(t+p) = f(t)$	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$	periodic property
$\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$	Convolution property

Note the following repetitive properties exhibited by the above table.

(i) The derivative property, when expressed in words states, **Differentiation of function in the t-domain is represented in the s-domain by multiplication of the transform function by s and subtracting the initial value of the function differentiated.** Note that the second derivative and higher derivatives follow this rule. For example, if $f'(t)$ and $f''(t)$ are the functions differentiated, then

$$\mathcal{L}\{f''(t)\} = \underbrace{s[sF(s) - f(0^+)] - f'(0^+)}_{\substack{\text{s times transform of}\\ \text{function differentiated}\\ \text{minus initial value}\\ \text{of function differentiated}}}$$

$$\mathcal{L}\{f'''(t)\} = \underbrace{s[s^2F(s) - sf(0^+) - f'(0^+)] - f''(0^+)}_{\substack{\text{s times transform of}\\ \text{function differentiated}\\ \text{minus initial value}\\ \text{of function differentiated}}}$$

(ii) Multiplication by t in the t-domain corresponds to a differentiation in the s-domain multiplied by a -1.

(iii) Division by t in the t-domain corresponds to an integration from s to ∞ in the s-domain.

Example 12-16. Laplace transform

Use Laplace transform techniques to solve the differential equation $\frac{dy}{dt} = \alpha y$ with initial condition $y(0) = 1$, where α is a known constant.

Solution

Here $y = y(t)$ is a function of time t and taking the Laplace transform of both sides of the given differential equation produces

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{\alpha\, y\}$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ denote the transform in the s-domain and make note that

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) \quad \text{and} \quad \mathcal{L}\{\alpha\, y(t)\} = \alpha\, \mathcal{L}\{y(t)\} = \alpha\, Y(s)$$

so that the differential equation in the t-domain becomes **an algebraic equation** in the s-domain. The resulting algebraic equation is

$$sY(s) - 1 = \alpha\, Y(s)$$

One can now solve this algebraic equation for the transform function $Y(s)$ to obtain

$$Y(s) = \frac{1}{s-\alpha}$$

Using table lookup one finds the inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = e^{\alpha t}$$

One can verify the correctness of the solution by showing the function $y(t) = e^{\alpha t}$ satisfies the given differential equation and given initial condition.

Warning—The Laplace transform technique for solving differential equations only works on linear differential equations. It is not applicable in dealing with nonlinear differential equations. Also note that when dealing with more difficult linear equations one needs to develop more advanced methods for obtaining an inverse Laplace transform.

■

Introduction to Complex Variable Theory

Consider the figure 12-10, where S represents a nonempty set of points in the $z = x + i\,y$ complex z-plane, where i is an imaginary unit with the property that $i^2 = -1$. If there exists a rule f which assigns to each value $z = x + i\,y$ belonging to S, one and only one complex number $\omega = u + i\,v$, then the correspondence is called a function or mapping of the point z to the point ω and this correspondence is denoted using the notation

$$\omega = f(z) = f(x + i\,y) = u + i\,v = u(x,y) + i\,v(x,y)$$

Here $\omega = u + i\,v$ is the image point of $z = x + i\,y$ and is represented in a plane called the $\omega-$plane. Functions of a complex variable $\omega = f(z)$ are represented as mappings from the $z-$plane to the $\omega-$plane as illustrated in the figure 12-10. Note that if S denotes a region in the $z-$plane and $f(z)$ is a single-valued, then the image of S under the mapping $\omega = f(z)$ is the region S' in the $\omega-$plane. The boundary curve C of S in the $z-$plane has the image curve C' in the $\omega-$plane.

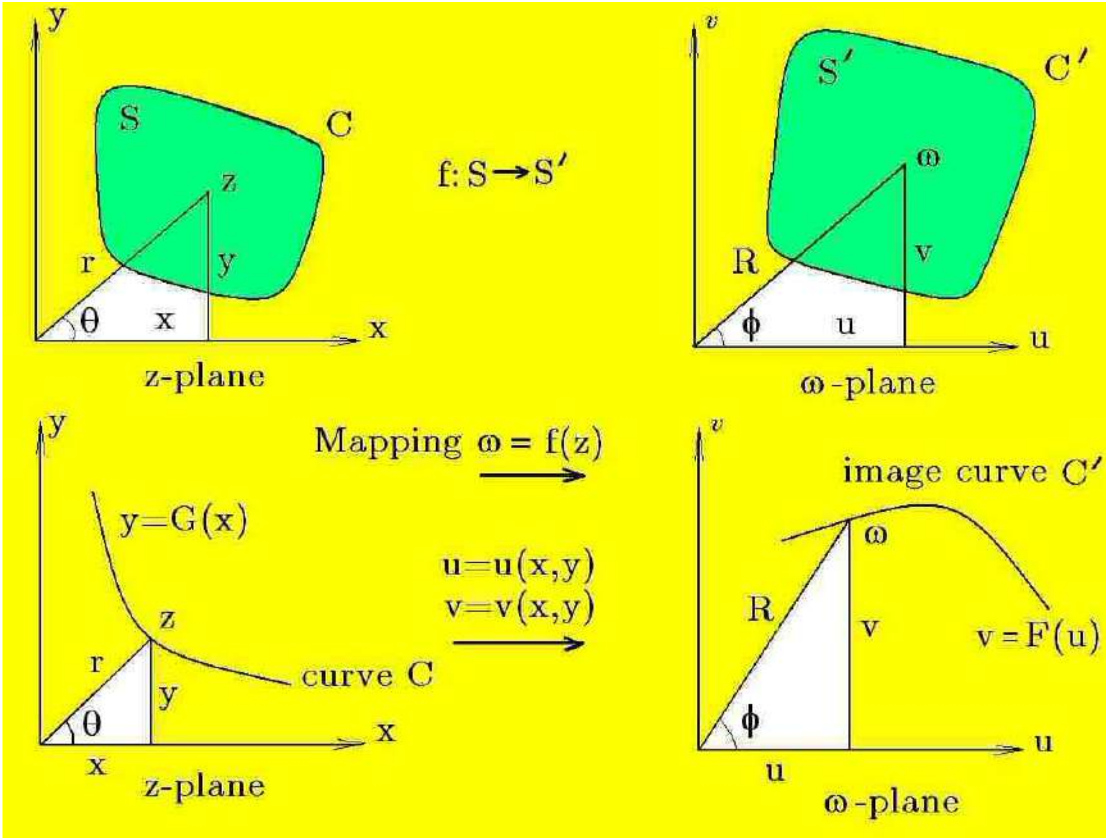


Figure 12-10. **Representing function of complex variable as mapping.**

Polar coordinates (r, θ) in the $z-$plane correspond to polar coordinates (R, ϕ) in the $\omega-$plane, where $x = r\cos\theta,\ y = r\sin\theta$ in the $z-$plane and $u = R\cos\phi,\ v = R\sin\phi$ in the $\omega-$plane. A curve $y = G(x)$ in the $z-$plane has the image curve with parametric form

$$u = u(x, G(x)), \quad v = v(x, G(x))$$

in the $\omega-$plane, which produces the image curve $v = F(u)$.

Functions of a complex variable $\omega = f(z)$ represent a mapping from the $z-$plane to the $\omega-$plane. One usually selects special regions S and curves C to illustrate the mappings. For example, circles, squares, triangles, etc.

Example 12-17. **(Representing function of a complex variable as mapping)**

Consider the complex function

$$\omega = f(z) = z^2 = (x + i\,y)^2 = x^2 + 2ixy + i^2 y^2 = (x^2 - y^2) + i\,(2xy) = u + i\,v$$

The complex function $\omega = f(z) = z^2$ defines the mapping

$$u = u(x, y) = x^2 - y^2 \quad \text{and} \quad v = v(x, y) = 2xy$$

In polar coordinates with $z = re^{i\theta}$, the image point is $\omega = f(z) = z^2 = r^2 e^{i2\theta} = Re^{i\phi}$ so that the mapping in polar form is given by

$$R = r^2, \quad \text{and} \quad \phi = 2\theta$$

Knowing the transformation equations one can then construct special figures to give various interpretations of this mapping. The figure 12-11 illustrates four selected interpretations to illustrate the mapping $\omega = f(z) = z^2$.

The first mapping illustrates two circles with radius r_1 and r_2 and their image circles r_1^2 and r_2^2. The rays $\theta = \alpha$ and $\theta = \beta$ map to the image rays $\phi = 2\alpha$ and $\phi = 2\beta$. The green region S maps to the green region S'.

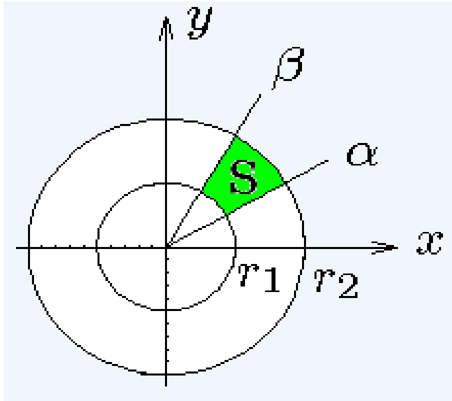
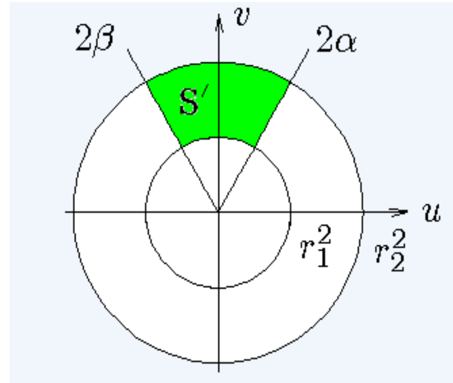
The second mapping illustrates the hyperpola $x^2 - y^2 = u$ and $2xy = v$ for the selected values of u and v given by u_0, u_1 and v_0, v_1. These hyperbola map to straight lines in the $\omega-$plane.

The third mapping illustrates the line $x = 0$ mapping to the line segment $\overline{A'B'}$, where $v = 0$, with $u = -y^2$, $B < y < A$. The line $y = 0$ maps to the line segment $\overline{B'C'}$,where $v = 0$ and $u = x^2$,for $0 < x < x_1$. The line $x = x_1$, $C < y < D$ maps to the parabola with parametric equations $u = x_1^2 - y^2$, $v = 2x_1 y$, $C < y < D$.

The fourth mapping illustrates the hyperbola $x^2 - y^2 = u$, for $u = 0, 2, 4, 6$ mapping to the lines $u = 0, 2, 4, 6$ and the hyperbola $2xy = v$, for $v = 2, 4, 6$ are mapped to the lines $v = 2, 4, 6$.

It is left as an exercise for you to make up additional figures to illustrate the mapping $\omega = f(z) = z^2$.

■

<table>
<tr><th></th><th>z-plane</th><th>Mapping</th><th>ω-plane</th></tr>
<tr><td>1.</td><td>
</td><td>$\omega = z^2$</td><td>
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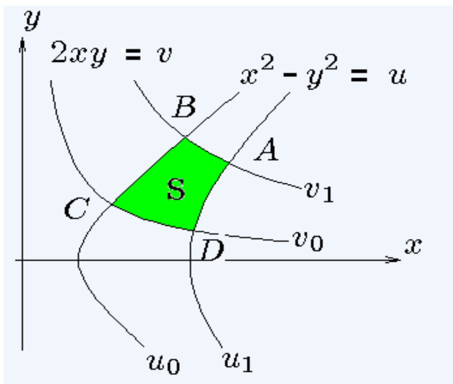
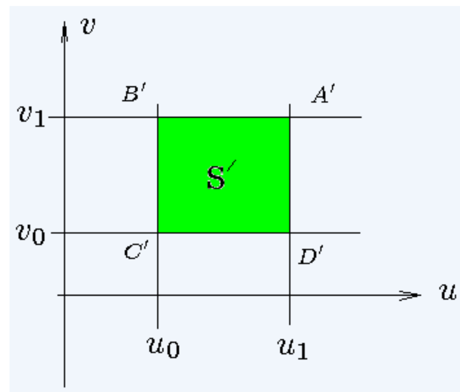
<tr><td>2.</td><td>
</td><td>$\omega = z^2$</td><td>
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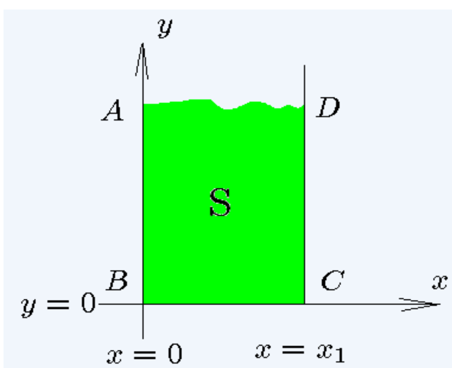
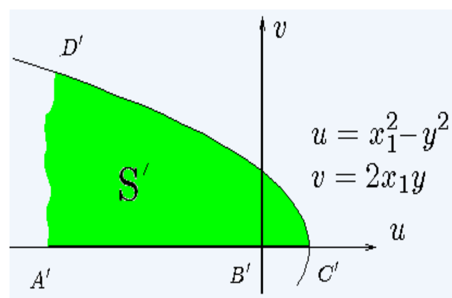
<tr><td>3.</td><td>
</td><td>$\omega = z^2$</td><td>
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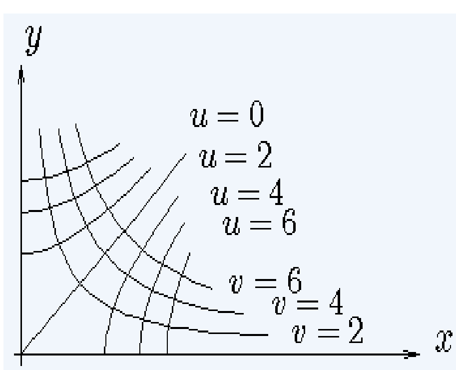
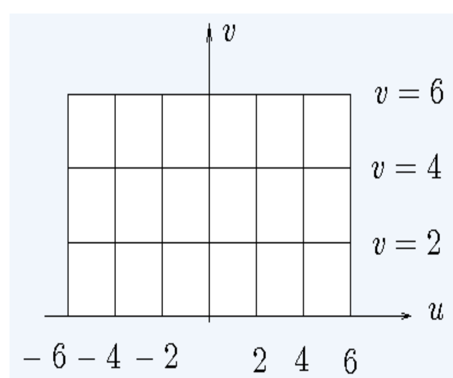
<tr><td>4.</td><td>
</td><td>$\omega = z^2$</td><td>
</td></tr>
</table>

Figure 12-11. Selected images illustrating the mapping $\omega = f(z) = z^2$

Derivative of a Complex Function

The derivative of a complex function $\omega = f(z) = u(x,y) + i\,v(x,y)$, where $z = x + i\,y$, is defined in the exact same way as that of a real function $y = f(x)$. That is,

$$\begin{aligned}\frac{d\omega}{dz} &= f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}\\ \frac{d\omega}{dz} &= f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta z, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - [u(x,y) + i\,v(x,y)]}{\Delta x + i\,\Delta y}\end{aligned} \tag{12.178}$$

if this limit exists. In the definition of a derivative, equation (12.178), **the limit must be independent of the path taken as Δz tends toward zero.**

Consider the points $z = x + i\,y$ and $z + \Delta z = (x + \Delta x) + i\,(y + \Delta y)$ in the $z-$plane as illustrated in the figure 12-11.

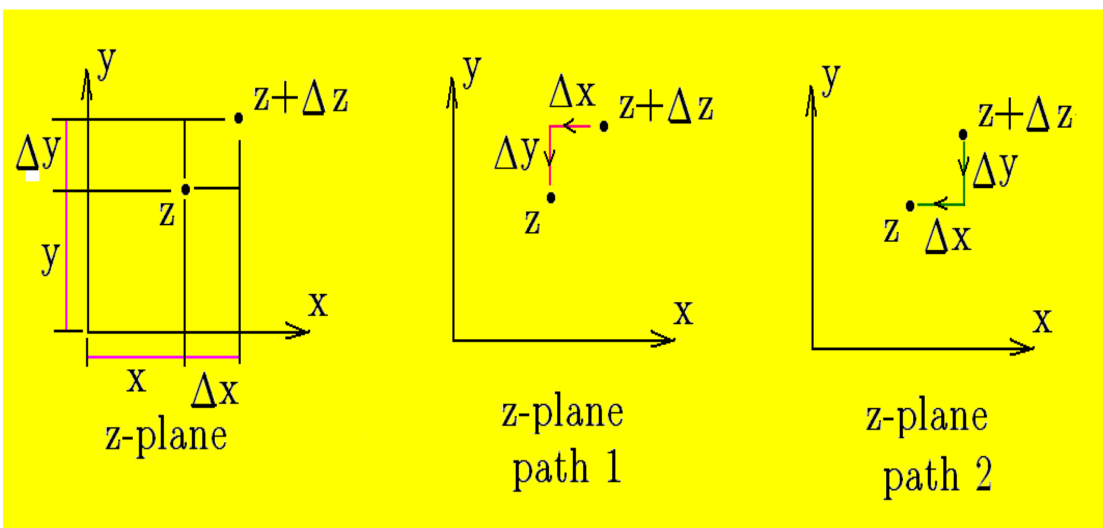


Figure 12-12. Delta z approaching zero along different paths.

If the limit in equation (12.178) approaches zero along the path 1 of figure 12-12, then the definition given by equation (12.178) becomes, after first setting $\Delta x = 0$

$$\begin{aligned}\frac{d\omega}{dz} &= f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - [u(x,y) + i\,v(x,y)]}{i\,\Delta y}\\ \frac{d\omega}{dz} &= f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}\\ \frac{d\omega}{dz} &= f'(z) = \; -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{See page 158, Volume I}\end{aligned} \tag{12.179}$$

If the limit in equation (12.178) approaches zero along the path 2 of figure 12-12, then the definition given by equation (12.178) becomes, after first setting $\Delta y = 0$

$$\begin{aligned}
\frac{d\omega}{dz} &= f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - [u(x,y) + i\,v(x,y)]}{\Delta x} \\
\frac{d\omega}{dz} &= f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \\
\frac{d\omega}{dz} &= f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad \text{See page 158, Volume I}
\end{aligned} \tag{12.180}$$

If the derivative of the complex function is to exist with the limit of equation (12.178) being independent of how Δz approaches zero, then it is necessary that the derivative results from the equations (12.179) and (12.180) must equal one another or

$$\frac{d\omega}{dz} = f'(z) = \frac{\partial u}{\partial z} + i\,\frac{\partial v}{\partial x} = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{12.181}$$

Equating the real and imaginary parts in equation (12.181) one finds that a necessary condition for the existence of a complex derivative associated with the complex function given by $\omega = f(z) = u(x,y) + i\,v(x,y)$ is that the following equations must be satisfied simultaneously

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{12.182}$$

These simultaneous conditions are known as the **Cauchy-Riemann equations** for the existence of a derivative associated with a function of a complex variable.

A function $\omega = f(z) = u(x,y) + i\,v(x,y)$ which is both single-valued and differential at a point z_0 and all points in some small region around the point z_0, is said to be **analytic or regular at the point** z_0**.**

Integration of a Complex Function

An introductory calculus course introduces the concepts of differentiation and integration associated with functions of a real variable. Indefinite integration of a real function was defined as the inverse operation of differentiation of the real function. A definite integral was defined as the limit of a summation process representing area under a curve. In complex variable theory one can have indefinite and definite integrals but their physical interpretation is not quite the same as when dealing with real quantities. In addition to indefinite and definite integrations one must know properties of contour integrals.

Contour integration

Let C denote a curve in the z-plane connecting two points $z = a$ and $z = b$ as illustrated in figure 12-13.

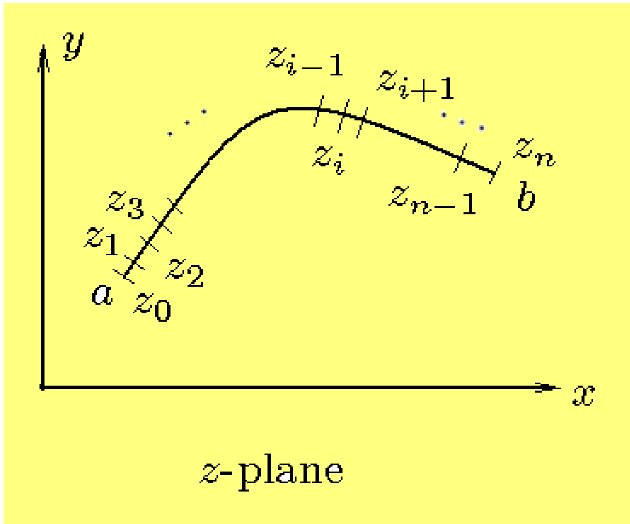


Figure 12-13. Curve in the z-plane.

The curve C is assumed to be a smooth curve represented by a set of parametric equations

$$x = x(t), \quad y = y(t), \quad t_a \le t \le t_b$$

The equation $z = z(t) = x(t) + iy(t)$, for $t_a \le t \le t_b$ represents points on the curve C with the end points given by

$$z(t_a) = x(t_a) + iy(t_a) = a \quad \text{and} \quad z(t_b) = x(t_b) + iy(t_b) = b.$$

Divide the interval (t_a, t_b) into n parts by defining a step size $h = \frac{t_b - t_a}{n}$ and letting $t_0 = t_a,\ t_1 = t_a + h,\ t_2 = t_a + 2h, \ldots, t_n = t_a + nh = t_a + n\frac{(t_b - t_a)}{n} = t_b$. Each of the values $t_i,\ i = 0, 1, 2, \ldots, n$, gives a point $z_i = z(t_i)$ on the curve C. For $f(z)$, a continuous function at all points z on the curve C, let $\Delta z_i = z_{i+1} - z_i$ and form the sum

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\Delta z_i = \sum_{i=0}^{n-1} f(\xi_i)(z_{i+1} - z_i), \tag{12.183}$$

where ξ_i is an arbitrary point on the curve C between the points z_i and z_{i+1}. Now let n increase without bound, while $|\Delta z_i|$ approaches zero. The limit of the summation in equation (12.183) is called **the complex line integral of $f(z)$ along the curve C** and is denoted

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(\xi_i)\Delta z_i. \tag{12.184}$$

If $f(z) = u(x,y) + i\,v(x,y)$ is a function of a complex variable, then we can express the complex line integral of $f(z)$ along a curve C in the form of **a real line integral** by writing

$$\begin{aligned}\int_C f(z)\,dz =& \int_C [u(x,y) + i\,v(x,y)]\,(dx + i\,dy)\\ =& \int_C [u(x,y)\,dx - v(x,y)\,dy] + i\int_C [v(x,y)\,dx + u(x,y)\,dy]\end{aligned} \tag{12.185}$$

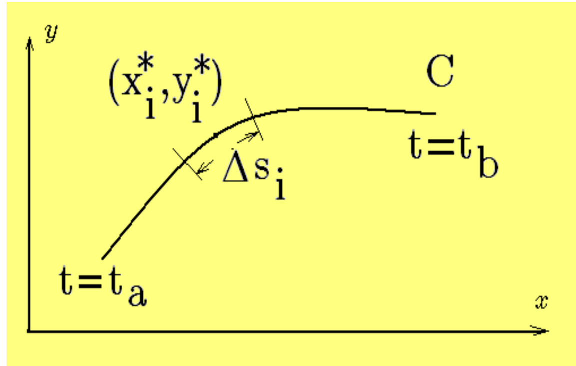
where $x = x(t),\ y = y(t),\ dx = x'(t)\,dt$ and $dy = y'(t)\,dt$ are substituted for the $x,\ y,\ dx$ and dy values and the limits of integration on the parameter t go from t_a to t_b. This gives the integral

$$\int_C f(z)\,dz = \int_{t_a}^{t_b} \Big[u(x(t),y(t))\,x'(t) - v(x(t),y(t))y'(t)\Big]\,dt + i\int_{t_a}^{t_b} \Big[v(x(t),y(t))\,x'(t) + u(x(t),y(t))\,y'(t)\Big]\,dt$$

Now both the real part and imaginary parts are evaluated just like the real integrals you studied in calculus of real variables.

If the **parametric equations defining the curve C are not given, then you must construct the parametric equations defining the contour C over which the integration occurs.** Complex line integrals along a curve C involve a summation process where values of the function being integrated must be known on a specified path C connecting points a and b. In special cases the value of the complex integral is very much dependent upon the path of integration while in other special circumstances the value of the line integral is independent of the path of integration joining the end points. In some special circumstances the path of integration C can be continuously deformed into other paths C^* without changing the value of the complex integral. If you take a course in complex variable theory you will be introduced to various theorems and properties associated with integration involving analytic functions $f(z)$ which are well defined over specific regions of the z-plane.

The integration of a function along a curve is called a line integral. A familiar line integral is the calculation of arc length between two points on a curve. Let $ds^2 = dx^2 + dy^2$ denote an element of arc length squared and let C denote a curve defined by the parametric equations $x = x(t),\ y = y(t)$, for $t_a \le t \le t_b$, then the arc length L between two points $a=[x(t_a),y(t_a)]$ and $b=[x(t_b),y(t_b)]$ on the curve is given by the integral



$$L=\int_C ds=\int_{t_a}^{t_b}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

If the curve C represents a wire with variable density $f(x,y)$ [gm/cm], then the total mass m of the wire between the points a and b is given by $m=\int_C f(x,y)\,ds$ which can be thought of as the limit of a summation process. If the curve C is partitioned into n pieces of lengths $\Delta s_1,\Delta s_2,\dots,\Delta s_i,\dots$, then in the limit as n increases without bound and Δs_i approaches zero, one can express the total mass m of the wire as the limiting process

$$m=\int_C f(x,y)\,ds=\lim_{\substack{\Delta s_i\to 0\\ n\to\infty}}\sum_{i=1}^n f(x_i^*,y_i^*)\,\Delta s_i=\int_{t_a}^{t_b} f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

where (x_i^*,y_i^*) is a general point on the Δs_i arc length.

Whenever the values of x and y are restricted to lie on a given curve defined by $x=x(t)$ and $y=y(t)$ for $t_a\le t\le t_b$, then integrals of the form

$$I=\int_C P(x,y)\,dx+Q(x,y)\,dy=\int_{t_a}^{t_b} P(x(t),y(t))x'(t)\,dt+Q(x(t),y(t))y'(t)\,dt \tag{12.186}$$

are **called line integrals and are defined by a limiting process such as above.** Line integrals are reduced to ordinary integrals by substituting the parametric values $x=x(t)$ and $y=y(t)$ associated with the curve C and integrating with respect to the parameter t. The above line integral is sometimes written in the form

$$\int_C f(z)\,dz=\int_{t_a}^{t_b} f(z(t))\,z'(t)\,dt \tag{12.187}$$

where $z=z(t)$ is a parametric representation of the curve C over the range $t_a\le t\le t_b$.

Whenever the curve C is not a smooth curve, but is composed of a finite number of arcs which are smooth, then the curve C is called piecewise smooth. If $C_1,C_2,\dots,C_m$ denote the finite number of arcs over which the curve is smooth and

$C = C_1 \cup C_2 \cup \cdots \cup C_m$, then the line integral can be broken up and written as a summation of the line integrals over each section of the curve which is smooth and one would express this by writing

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_m} f(z)\,dz \tag{12.188}$$

Indefinite integration

If $F(z)$ is a function of a complex variable such that

$$\frac{dF(z)}{dz} = F'(z) = f(z)$$

then $F(z)$ is called an anti-derivative of $f(z)$ or an indefinite integral of $f(z)$. The indefinite integral is denoted using the notation $\int f(z)\,dz = F(z) + c$ where c is a constant and $F'(z) = f(z)$ Note that the addition of a constant of integration is included because the derivative of a constant is zero. Consequently, any two functions which differ by a constant will have the same derivatives.

Example 12-18. (Indefinite integration)

Let $F(z) = 3\sin z + z^3 + 5z^2 - z$ with $\dfrac{dF}{dz} = F'(z) = 3\cos z + 3z^2 + 10z - 1$, then one can write $\int (3\cos z + 3z^2 + 10z - 1)\,dz = 3\sin z + z^3 + 5z^2 - z + c$ where c is an arbitrary constant of integration

■

The table 12.1 gives a short table of indefinite integrals associated with selected functions of a complex variable. Note that the results are identical with those derived in a standard calculus course.

Table 12.1 Short Table of Integrals			
1.	$\int z^n\,dz = \frac{z^{n+1}}{n+1} + c, \quad n \neq -1$	11.	$\int \sinh z\,dz = \cosh z + c$
2.	$\int \frac{dz}{z} = \log z + c$	12.	$\int \cosh z\,dz = \sinh z + c$
3.	$\int e^z\,dz = e^z + c$	13.	$\int \tanh z\,dz = \log(\cosh z) + c$
4.	$\int k^z\,dz = \frac{k^z}{\log k} + c, \quad k$ is a constant	14.	$\int \operatorname{sech}^2 z\,dz = \tanh z + c$
5.	$\int \sin z\,dz = -\cos z + c$	15.	$\int \frac{dz}{\sqrt{z^2+\alpha^2}} = \log(z + \sqrt{z^2+\alpha^2}) + c$
6.	$\int \cos z\,dz = \sin z + c$	16.	$\int \frac{dz}{z^2+\alpha^2} = \frac{1}{\alpha}\tan^{-1}\frac{z}{\alpha} + c$
7.	$\int \tan z\,dz = \log \sec z + c = -\log \cos z + c$	17.	$\int \frac{dz}{z^2-\alpha^2} = \frac{1}{2\alpha}\log\left(\frac{z-\alpha}{z+\alpha}\right) + c$
8.	$\int \sec^2 z\,dz = \tan z + c$	18.	$\int \frac{dz}{\sqrt{\alpha^2-z^2}} = \sin^{-1}\frac{z}{\alpha} + c$
9.	$\int \sec z \tan z\,dz = \sec z + c$	19.	$\int e^{\alpha z}\sin\beta z\,dz = e^{\alpha z}\frac{\alpha\sin\beta z - \beta\cos\beta z}{\alpha^2+\beta^2} + c$
10.	$\int \csc z \cot z\,dz = -\csc z + c$	20.	$\int e^{\alpha z}\cos\beta z\,dz = e^{\alpha z}\frac{\alpha\cos\beta z + \beta\sin\beta z}{\alpha^2+\beta^2} + c$
c denotes an arbitrary constant of integration			

Definite integrals

The definite integral of a complex function $f(t) = u(t) + i\,v(t)$ which is continuous for $t_a \le t \le t_b$ has the form

$$\int_{t_a}^{t_b} f(t)\,dt = \int_{t_a}^{t_b} u(t)\,dt + i\int_{t_a}^{t_b} v(t)\,dt \tag{12.189}$$

and has the following properties.

1. The integral of a linear combination of functions is a linear combination of the integrals of the functions or

$$\int_{t_a}^{t_b} [c_1 f(t) + c_2 g(t)]\,dt = c_1\int_{t_a}^{t_b} f(t)\,dt + c_2\int_{t_a}^{t_b} g(t)\,dt$$

where c_1 and c_2 are complex constants.

2. If $f(t)$ is continuous and $t_a < t_c < t_b$, then

$$\int_{t_a}^{t_b} f(t)\,dt = \int_{t_a}^{t_c} f(t)\,dt + \int_{t_c}^{t_b} f(t)\,dt$$

3. The modulus of the integral is less than or equal to the integral of the modulus

$$\left|\int_{t_a}^{t_b} f(t)\,dt\right| \le \int_{t_a}^{t_b} |f(t)|\,dt$$

4. If $F = F(t)$ is such that $\dfrac{dF}{dt} = F'(t) = f(t)$ for $t_a \le t \le t_b$, then

$$\int_{t_a}^{t_b} f(t)\,dt = F(t)]_{t_a}^{t_b} = F(t_b) - F(t_a)$$

5. If $G(t)$ is defined $G(t) = \displaystyle\int_{t_a}^{t} f(t)\,dt$, then $\dfrac{dG}{dt} = G'(t) = f(t)$
6. The conjugate of the integral is equal to the integral of the conjugate

$$\overline{\int_{t_a}^{t_b} f(t)\,dt} = \int_{t_a}^{t_b} \overline{f(t)}\,dt$$

7. Let $f(t,\tau)$ denote a function of the two variables t and τ which is defined and continuous everywhere over the rectangular region $R = \{(t,\tau)\,|\,t_a \le t \le t_b,\ \tau_c \le \tau \le \tau_d\}$. If $g(\tau) = \displaystyle\int_{t_a}^{t_b} f(t,\tau)\,dt$ and the partial derivatives of f exist and are continuous on R, then

$$\frac{dg}{d\tau} = \int_{t_a}^{t_b} \frac{\partial f(t,\tau)}{\partial \tau}\,dt$$

which shows that differentiation under the integral sign is permissible.

Assume $F(z)$ is an analytic function with derivative $f(z) = \frac{dF}{dz}$ and $z = z(t)$ for $t_1 \le t \le t_2$ is a piecewise smooth arc C in a region R of the z-plane, then one can write

$$\int_C f(z)\,dz = \int_{t_1}^{t_1} \frac{dF}{dz}\,dz = F(z(t))\Big|_{t_1}^{t_2} = F(z(t_2)) - F(z(t_1))$$

This is a fundamental integration property in the z-plane. Note that if $F(z) = U + iV$ is analytic and $f(z) = u + iv$, then one can write

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y}$$

and consequently,

$$\begin{aligned}\int_C f(z)\,dz =& \int_C (u+i\,v)(dx+i\,dy)\\ =& \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)\\ =& \int_{t_1}^{t_2} \frac{\partial U}{\partial x}x'(t)\,dt + \frac{\partial U}{\partial y}y'(t)\,dt + i\int_{t_1}^{t_2} \frac{\partial V}{\partial x}x'(t)\,dt + \frac{\partial V}{\partial y}y'(t)\,dt\\ =& \int_{z(t_1)}^{z(t_2)} dU + i\int_{z(t_1)}^{z(t_2)} dV = \int_{z(t_1)}^{z(t_2)} dF = F(z)\Big|_{z(t_1)}^{z(t_2)} = F(z(t_2)) - F(z(t_1))\end{aligned}$$

Example 12-19. Evaluate the integral $I = \int_C (z-z_0)^n\,dz$ where n is an integer and C is the arc of the circle defined by $z = z(t) = z_0 + r\,e^{it}$ for $t_1 \le t \le t_2$ and $r > 0$ constant. For $n \ne -1$ we have

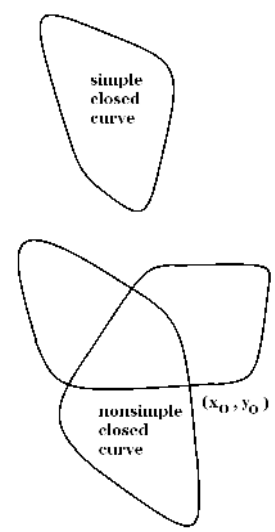
$$I = \int_{z(t_1)}^{z(t_2)} (z-z_0)^n\,dz = \frac{(z-z_0)^{n+1}}{n+1}\Big|_{z(t_1)}^{z(t_2)} \qquad n \ne -1$$

For $n = -1$, substitute $z - z_0 = r\,e^{it}$ with $dz = r\,e^{it}\,i\,dt$ to obtain

$$I = \int_C \frac{dz}{z-z_0} = \int_{t_1}^{t_2} \frac{r\,e^{it}\,i\,dt}{r\,e^{it}} = \int_{t_1}^{t_2} i\,dt = i\,(t_2 - t_1)$$

Note that a special case $z(t_1) = z(t_2)$ occurs when the arc of the circle closes on itself.

Closed curves



A plane curve C in the z-plane can be defined by a set of parametric equations $\{x(t), y(t)\}$ for the parameter t ranging over a set of values $t_1 \le t \le t_2$. A curve C is called a closed curve if the end points coincide. If C is a closed curve, then the end conditions satisfy $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. If (x_0, y_0) is a point on the curve C, which is not an end point, and there exists more than one value of the parameter t such that $\{x(t), y(t)\} = \{x_0, y_0\}$, then the point (x_0, y_0) is called a multiple point or point where the curve C crosses itself. A curve C is called a simple closed curve if the end points meet and it has no multiple points.

Whenever a curve C is a simple closed curve, the line integral of $f(z)$ around C or contour integral around the curve C is represented by an integral having one of the forms

$$\circlearrowleft\!\!\!\!\!\oint_C f(z)\,dz. \qquad \text{or} \qquad \circlearrowright\!\!\!\!\!\oint_C f(z)\,dz \tag{12.190}$$

The arrow on the circle indicating the direction of integration as being clockwise or counterclockwise as viewed looking down on the z-plane. A simple closed curve C is said to be traversed in the positive sense if the direction of integration is in a counterclockwise direction around the boundary and it is said to be traversed in the negative sense if the direction of integration is in the clockwise direction around the boundary. One can write $\oint_C f(z)\,dz = -\oint_C f(z)\,dz$. Observe that by changing the direction of integration one changes the sign of the integral.

Example 12-20. **(Contour integration)**

For n an integer, and z_0, ρ constants, integrate the function $f(z) = (z-z_0)^n$ around a circle of radius ρ centered at the point z_0. Perform the integration in the positive sense.

Solution: Let C denote the circle $|z - z_0| = \rho$ of radius ρ centered at the point z_0. The curve C can be represented in the parametric form

$$z = z(t) = z_0 + \rho e^{it}, \quad 0 \le t \le 2\pi \quad \text{with} \quad dz = \rho e^{it} i dt.$$

We then have

$$\oint_C f(z)\,dz = \oint_C (z-z_0)^n\,dz = \int_0^{2\pi} \rho^n e^{int} i\rho e^{it}\,dt = i\rho^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt.$$

For n different from -1 we have $\oint_C (z-z_0)^n\,dz = i\rho^{n+1}\left[\dfrac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi} = 0$. For n equal to -1 we have $\oint_C (z-z_0)^n\,dz = \oint_C \dfrac{dz}{z-z_0} = i\int_0^{2\pi} dt = 2\pi i$. Hence, the line or contour integral of the function $f(z) = (z-z_0)^n$, with n an integer, which is taken around a circle C centered at z_0 with radius ρ, can be expressed

$$\oint_C (z-z_0)^n\,dz = \begin{cases} 2\pi i & \text{when } n = -1 \\ 0 & \text{when } n \ne -1 \text{ and an integer.} \end{cases} \tag{12.191}$$

This result will be used quite frequently throughout the remainder of this text.

■

The Laurent series

For $z = x + iy$ a complex variable and $z_0 = x_0 + iy_0$ a fixed point in the complex z-plane, one must deal with the following quantities. (i) The magnitude of z denoted by $|z| = \sqrt{x^2+y^2}$ which represents the distance of the point z from the origin in the $z-$plane. (ii) The quantity $|z-z_0| = R$ represents a circle of radius R since

$$|z-z_0| = |x+iy-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2} = R$$
$$\text{or} \quad (x-x_0)^2+(y-y_0)^2 = R^2$$

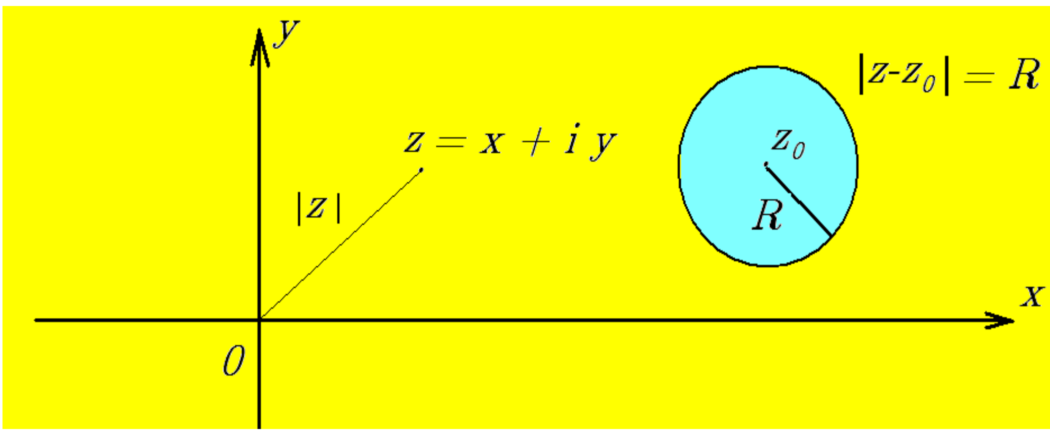


Figure 12-14. The complex $z-$plane.

In the theory of complex variables there is an important type of series called **the Laurent[8] series** which represents a function $f(z)$ in an expansion about a singular point z_0 having the form of a power series having both positive and negative powers of $(z-z_0)$. **The point z_0 is called the center of the Laurent series.** The Laurent series has the form

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n \tag{12.192}$$

where the quantities $c_1, c_2, \ldots$ and $\alpha_0, \alpha_1, \ldots$ are constants. In expanded form the Laurent series (12.192) becomes

$$f(z) = \cdots + \frac{c_3}{(z-z_0)^3} + \frac{c_2}{(z-z_0)^2} + \frac{c_1}{(z-z_0)} + \alpha_0 + \alpha_1(z-z_0) + \alpha_2(z-z_0)^2 + \cdots \tag{12.193}$$

The Laurent series converges in some annular region $R_2 < |z-z_0| < R_1$. It can be shown that the series $\sum_{n=0}^{\infty} \alpha_n(z-z_0)^n$ converges for z in the circular region $|z-z_0| < R_1$ and the series $\sum_{n=1}^{\infty} \frac{c_n}{(z-z_0)^n}$ converges for the circular region $|z-z_0| > R_2$ Here R_1 and

[8] Pierre Alphonse Laurent (1813-1854) A French mathematician who studied complex analysis.

R_2 are positive constants with $R_2 < R_1$. The annular region of convergence is the intersection of these two regions.

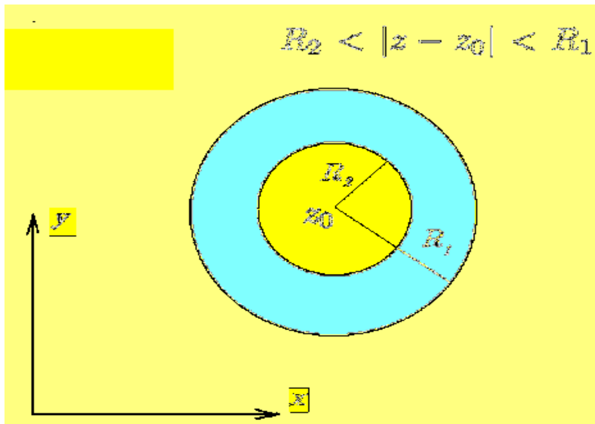


Figure 12-15. Annular region of convergence for Laurent series.

The special Laurent series where the point z_0 is the only singular point of $f(z)$ inside the disk $|z-z_0| < R_2$ is of extreme importance in the study of complex variables. In this special case the point z_0 is **called an isolated singular point.** This special series with negative powers of $z - z_0$ having the form

$$\sum_{n=1}^{\infty} \frac{c_n}{(z-z_0)^n} = \cdots + \frac{c_m}{(z-z_0)^m} + \cdots + \frac{c_2}{(z-z_0)^2} + \frac{c_1}{(z-z_0)} \tag{12.194}$$

is called **the principal part of the Laurent series** and associated with the principal part of the series is the following terminology

(i) The term c_1 is called **the residue of $f(z)$ at the isolated singular point z_0.**

(ii) If there is only one term in the principal part of the Laurent series, then the singular point z_0 is called **a pole of order 1 or a simple pole.**

(iii) If there are only two terms in the principal part of the Laurent series, then the singular point z_0 is called **a pole of order 2.**

(iv) If there are $m-$terms in the principal part of the Laurent series, then the singular point z_0 is called **a pole of order m.**

(v) If there is an infinite number of terms in the principal part of the Laurent series, then the singular point z_0 is called an **an essential singularity.**

If you get involved with complex variable theory, the binomial expansion

$$(a+b)^{-1} = a^{-1} + (-1)\,a^{-2}b + (-1)(-2)\,a^{-3}b^2/2! + \cdots \quad \text{converges for } |b| < |a| \tag{12.195}$$

will be of great assistance in dealing with Laurent series.

Example 12-21. (**Laurent series**)

Express the function $f(z) = \dfrac{z}{(z-1)(z-3)}$ as a Laurent series centered at the singular point $z = 1$.

Solution

The problem is to express $f(z)$ as a series involving both negative and positive powers of $(z-1)$. To accomplish this task analyze the following two representations of $f(z)$

$$(i) \quad f(z) = \frac{[(z-1)+1]}{(z-1)[(z-1)-2]} = \left[1 + \frac{1}{z-1}\right][(z-1)-2]^{-1}$$

$$(ii) \quad f(z) = \frac{[(z-1)+1]}{(z-1)[(z-1)-2]} = -\left[1 + \frac{1}{z-1}\right][2-(z-1)]^{-1}$$

The last term in the representations (i) and (ii) have binomial expansions having the form of equation (12.195). The binomial expansion for the last term in representation (i) converges for $2 < |z-1|$ and the binomial expansion for the last term in representation (ii) converges for $|z-1| < 2$. The term $(z-1) > 0$ is assumed to hold in both the representations (i) and (ii). Therefore, to get the annular region isolating the singular point at $z = 1$ the representation (ii) is used. Expanding the last term in the representation (ii) gives

$$f(z) = -\left[1 + \frac{1}{z-1}\right]\left[\frac{1}{2} + \frac{1}{2^2}(z-1) + \frac{1}{2^3}(z-1)^2 + \frac{1}{2^4}(z-1)^3 + \cdots + \frac{1}{2^{n+1}}(z-1)^n + \cdots\right]$$

(a) Multiply the first term in the representation (ii) by the expanded second term and then collect like terms to obtain the Laurent series expansion

$$f(z) = \frac{-1/2}{z-1} - \frac{3}{2^2} - \frac{3}{2^3}(z-1) - \frac{3}{2^4}(z-1)^2 - \frac{3}{2^5}(z-1)^3 - \frac{3}{2^6}(z-1)^4 - \cdots \tag{12.196}$$

which converges in the annular region defined by **the intersection** of the regions (1) and (2) defined by

$$Region(1) = |z-1| > 0 \qquad \text{and} \qquad Region(2) = |z-1| < 2$$

The region(1) represents region of convergence of **the principal part of the Laurent series** and the region(2) represents the **region of convergence of that part of the Laurent series with terms** $(z-1)$ **with positive exponents.** The intersection of these two regions is given by $Region(1) \cap Region(2)$ produces the annular region $0 < |z-1| < 2$

illustrated in the figure 12-16.
(b) If one had used the binomial expansion on the last term in the representation (i) for $f(z)$, one obtains a series which converges for $|z-1| > 2$. After multiplication by the first term, the resulting series would converge in the annular region $r_1 < |z-1| < r_2$ where $r_1 = 2$ and $r_2 = lim_{r\to\infty} r$. This is **not the annular region** which **isolates the singular point at** $z = 1$ because the singular point $z = 3$ is also inside this region.

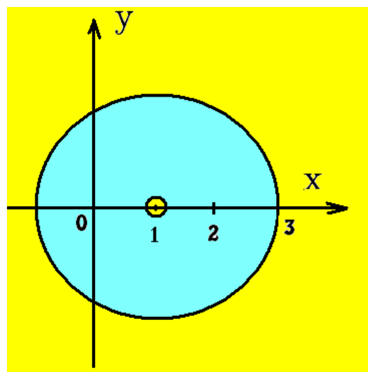


Figure 12-16. Annular region of convergence.

In general, the **radius of convergence for the power series with positive powers** or non-principal part of the Laurent series, is represented by the **distance from the center of the series to the nearest other singular point of the function being studied.** The correct Laurent series expansion, which isolates the singular point at $z = 1$, shows that $f(z)$ has a simple pole at $z = 1$ and the residue of $f(z)$ at $z = 1$ has the value $-1/2$.

■

The above examples represent a small fraction of the many concepts presented in the study of functions of a complex variable.

APPENDIX A
Units of Measurement

The following units, abbreviations and prefixes are from the Système International d'Unitès (designated SI in all Languages.)

Prefixes.

Abbreviations		
Prefix	Multiplication factor	Symbol
exa	10^{18}	W
peta	10^{15}	P
tera	10^{12}	T
giga	10^{9}	G
mega	10^{6}	M
kilo	10^{3}	K
hecto	10^{2}	h
deka	10	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

Basic Units.

Basic units of measurement		
Unit	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	$^\circ$K
Luminous intensity	candela	cd

Supplementary units		
Unit	Name	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

DERIVED UNITS		
Name	Units	Symbol
Area	square meter	m^2
Volume	cubic meter	m^3
Frequency	hertz	Hz (s^{-1})
Density	kilogram per cubic meter	kg/m^3
Velocity	meter per second	m/s
Angular velocity	radian per second	rad/s
Acceleration	meter per second squared	m/s^2
Angular acceleration	radian per second squared	rad/s^2
Force	newton	N $(kg \cdot m/s^2)$
Pressure	newton per square meter	N/m^2
Kinematic viscosity	square meter per second	m^2/s
Dynamic viscosity	newton second per square meter	$N \cdot s/m^2$
Work, energy, quantity of heat	joule	J $(N \cdot m)$
Power	watt	W (J/s)
Electric charge	coulomb	C $(A \cdot s)$
Voltage, Potential difference	volt	V (W/A)
Electromotive force	volt	V (W/A)
Electric force field	volt per meter	V/m
Electric resistance	ohm	Ω (V/A)
Electric capacitance	farad	F $(A \cdot s/V)$
Magnetic flux	weber	Wb $(V \cdot s)$
Inductance	henry	H $(V \cdot s/A)$
Magnetic flux density	tesla	T (Wb/m^2)
Magnetic field strength	ampere per meter	A/m
Magnetomotive force	ampere	A

Physical Constants:

- $4 \arctan 1 = \pi = 3.14159\,26535\,89793\,23846\,2643\,\ldots$
- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\,18284\,59045\,23536\,0287\,\ldots$
- Euler's constant $\gamma = 0.57721\,56649\,01532\,86060\,6512\,\ldots$
- $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$ Euler's constant
- Speed of light in vacuum $= 2.997925(10)^8\ m\ s^{-1}$
- Electron charge $= 1.60210(10)^{-19}\ C$
- Avogadro's constant $= 6.0221415(10)^{23}\ mol^{-1}$
- Plank's constant $= 6.6256(10)^{-34}\ J\,s$
- Universal gas constant $= 8.3143\ J\,K^{-1}\,mol^{-1} = 8314.3\ J\,Kg^{-1}\,K^{-1}$
- Boltzmann constant $= 1.38054(10)^{-23}\ J\,K^{-1}$
- Stefan–Boltzmann constant $= 5.6697(10)^{-8}\ W\,m^{-2}\,K^{-4}$
- Gravitational constant $= 6.67(10)^{-11}\ N\,m^2 kg^{-2}$

APPENDIX B

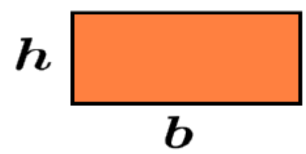
Background Material

Geometry

Rectangle

Area = (base)(height) = bh

Perimeter = $2b + 2h$

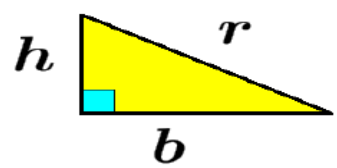


Right Triangle

Area = $\frac{1}{2}$(base)(height) = $\frac{1}{2}bh$

Perimeter = $b + h + r$

where $r^2 = b^2 + h^2$ is the Pythagorean theorem



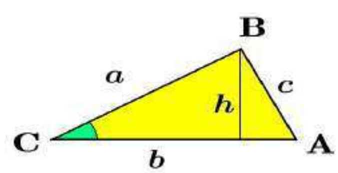
Triangle with sides a, b, c and angles A, B, C

Area = $\frac{1}{2}$(base)(height) = $\frac{1}{2}bh = \frac{1}{2}b(a\sin C)$

Perimeter = $a + b + c$

Law of Sines $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos C$

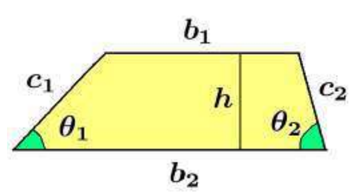


Trapezoid

Area = $\frac{1}{2}(b_1 + b_2)h$

Perimeter = $b_1 + b_2 + c_1 + c_2$

$c_1 = \frac{h}{\sin\theta_1}$ $c_2 = \frac{h}{\sin\theta_2}$

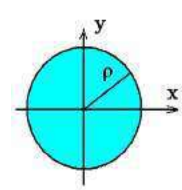


Circle

Area = $\pi\rho^2$

Perimeter = $2\pi\rho$

Equation $x^2 + y^2 = \rho^2$

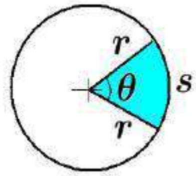


Sector of Circle

Area $= \frac{1}{2}r^2\theta$, $\quad \theta$ in radians

$s = \text{arclength} = r\theta$, $\quad \theta$ in radians

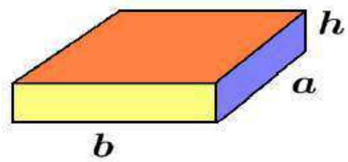
Perimeter $= 2r + s$



Rectangular Parallelepiped

$V = \text{Volume} = abh$

$S = \text{Surface area} = 2(ab + ah + bh)$



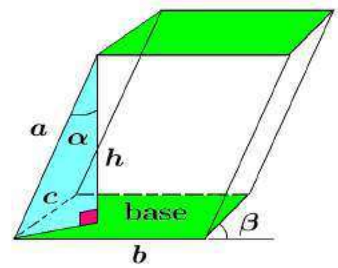
Parallelepiped

Composed of 6 parallelograms

$V = \text{Volume} = (Area\ of\ base)(height)$

$A = \text{Area of base} = bc\sin\beta$

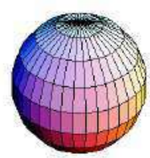
$\text{height} = h = a\cos\alpha$



Sphere of radius ρ

$V = \text{Volume} = \frac{4}{3}\pi\rho^3$

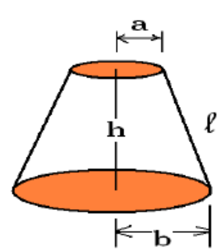
$S = \text{Surface area} = 4\pi\rho^2$



Frustum of right circular cone

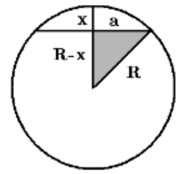
$V = \text{Volume} = \frac{\pi}{3}(a^2 + ab + b^2)h$

Lateral surface area $= \pi\ell(a + b)$



Chord Theorem for circle

$a^2 = x(2R - x)$

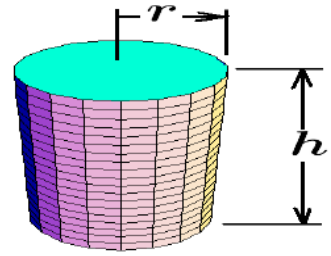


Right Circular Cylinder

$V = \text{Volume} = (\text{Area of base})(\text{height}) = (\pi r^2)h$

Lateral surface area $= 2\pi r h$

Total surface area $= 2\pi r h + 2(\pi r^2)$

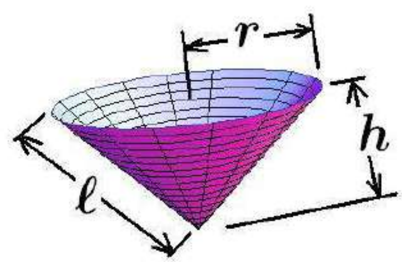


Right Circular Cone

$V = \text{Volume} = \frac{1}{3}\pi r^2 h$

Lateral surface area $= \pi r \ell = \pi r \sqrt{h^2 + r^2}$

height $= h$, base radius r



Algebra

Products and Factors
$(x+a)(x+b) = x^2 + (a+b)x + ab$
$(x+a)^2 = x^2 + 2ax + a^2$
$(x-b)^2 = x^2 - 2bx + b^2$
$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ac+bc+ab)x + abc$
$x^2 - y^2 = (x-y)(x+y)$
$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
$x^4 - y^4 = (x-y)(x+y)(x^2 + y^2)$
If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Binomial Expansion

For $n = 1, 2, 3, \dots$ an integer, then

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n$$

where $n!$ is read n factorial and is defined

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \text{ and } 0! = 1 \text{ by definition.}$$

Binomial Coefficients

The binomial coefficients can also be defined by the expression

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where } n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

where for $n = 1, 2, 3, \dots$ is an integer. The binomial expansion has the alternative representation

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 \cdots + \binom{n}{n}y^n$$

Laws of Exponents

Let s and t denote real numbers and let m and n denote positive integers. For nonzero values of x and y

$$x^0 = 1, \quad x \neq 0 \qquad (x^s)^t = x^{st} \qquad x^{1/n} = \sqrt[n]{x}$$

$$x^s x^t = x^{s+t} \qquad (xy)^s = x^s y^s \qquad x^{m/n} = \sqrt[n]{x^m}$$

$$\frac{x^s}{x^t} = x^{s-t} \qquad x^{-s} = \frac{1}{x^s} \qquad \left(\frac{x}{y}\right)^{1/n} = \frac{x^{1/n}}{y^{1/n}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Laws of Logarithms

If $x = b^y$ and $b \neq 0$, then one can write $y = \log_b x$, where y is called the logarithm of x to the base b. For $P > 0$ and $Q > 0$, logarithms satisfy the following properties

$$\log_b(PQ) = \log_b P + \log_b Q$$

$$\log_b \frac{P}{Q} = \log_b P - \log_b Q$$

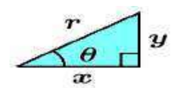
$$\log_b Q^P = P \log_b Q$$

Trigonometry

Pythagorean identities

Using the Pythagorean theorem $x^2 + y^2 = r^2$ associated with a right triangle with sides x, y and hypotenuse r, there results the following trigonometric identities, known as the Pythagorean identities.

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \qquad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \qquad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2,$$



$$\cos^2\theta + \sin^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta,$$

Angle Addition and Difference Formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B, \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Double angle formulas

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\cot A}{\cot^2 A - 1}$$

Half angle formulas

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

+	+
−	−

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

−	+
−	+

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

−	+
+	−

The sign depends upon the quadrant $A/2$ lies in.

Multiple angle formulas

$$\sin 3A = 3\sin A - 4\sin^3 A, \qquad \sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$$

$$\cos 3A = 4\cos^3 A - 3\cos A, \qquad \cos 4A = 8\cos^4 A - 8\cos^2 A + 1$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \qquad \tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

Multiple angle formulas

$$\begin{aligned}
\sin 5A =& 5\sin A - 20\sin^3 A + 16\sin^5 A\\
\cos 5A =& 16\cos^5 A - 20\cos^3 A + 5\cos A\\
\tan 5A =& \frac{\tan^5 A - 10\tan^3 A + 5\tan A}{1 - 10\tan^2 A + 5\tan^4 A}\\
\sin 6A =& 6\cos^5 A\sin A - 20\cos^3 A\sin^3 A + 6\cos A\sin^5 A\\
\cos 6A =& \cos^6 A - 15\cos^4 A\sin^2 A + 15\cos^2 A\sin^4 A - \sin^6 A\\
\tan 6A =& \frac{6\tan A - 20\tan^3 A + 6\tan^5 A}{1 - 15\tan^2 A + 15\tan^4 A - \tan^6 A}
\end{aligned}$$

Summation and difference formula

$$\begin{aligned}
\sin A + \sin B =& 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}), & \sin A - \sin B =& 2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})\\
\cos A + \cos B =& 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2}), & \cos A - \cos B =& -2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2})\\
\tan A + \tan B =& \frac{\sin(A+B)}{\cos A\cos B}, & \tan A - \tan B =& \frac{\sin(A-B)}{\cos A\cos B}
\end{aligned}$$

Product formula

$$\begin{aligned}
\sin A\sin B =& \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\
\cos A\cos B =& \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)\\
\sin A\cos B =& \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)
\end{aligned}$$

Additional relations

$$\begin{aligned}
\sin(A+B)\sin(A-B) &= \sin^2 A - \sin^2 B,\\
-\sin(A+B)\sin(A-B) &= \cos^2 A - \cos^2 B,\\
\cos(A+B)\cos(A-B) &= \cos^2 A - \sin^2 B,
\end{aligned}$$

$$\begin{aligned}
\frac{\sin A \pm \sin B}{\cos A + \cos B} &= \tan(\frac{A \pm B}{2})\\
\frac{\sin A \pm \sin B}{\cos A - \cos B} &= -\cot(\frac{A \mp B}{2})\\
\frac{\sin A + \sin B}{\sin A - \sin B} &= \frac{\tan(\frac{A+B}{2})}{\tan(\frac{A-B}{2})}
\end{aligned}$$

Powers of trigonometric functions

$$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A, \qquad \cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$$
$$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A, \qquad \cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$$
$$\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A, \qquad \cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$$

Inverse Trigonometric Functions

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x \qquad \sin^{-1}\frac{1}{x} = \csc^{-1} x$$
$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \qquad \cos^{-1}\frac{1}{x} = \sec^{-1} x$$
$$\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x \qquad \tan^{-1}\frac{1}{x} = \cot^{-1} x$$

Symmetry properties of trigonometric functions

$$\sin\theta = -\sin(-\theta) = \cos(\pi/2-\theta) = -\cos(\pi/2+\theta) = +\sin(\pi-\theta) = -\sin(\pi+\theta)$$
$$\cos\theta = +\cos(-\theta) = \sin(\pi/2-\theta) = +\sin(\pi/2+\theta) = -\cos(\pi-\theta) = -\cos(\pi+\theta)$$
$$\tan\theta = -\tan(-\theta) = \cot(\pi/2-\theta) = -\cot(\pi/2+\theta) = -\tan(\pi-\theta) = +\tan(\pi+\theta)$$
$$\cot\theta = -\cot(-\theta) = \tan(\pi/2-\theta) = -\tan(\pi/2+\theta) = -\cot(\pi-\theta) = +\cot(\pi+\theta)$$
$$\sec\theta = +\sec(-\theta) = \csc(\pi/2-\theta) = +\csc(\pi/2+\theta) = -\sec(\pi-\theta) = -\sec(\pi+\theta)$$
$$\csc\theta = -\csc(-\theta) = \sec(\pi/2-\theta) = +\sec(\pi/2+\theta) = +\csc(\pi-\theta) = -\csc(\pi+\theta)$$

Transformations

The following transformations are sometimes useful in simplifying expressions.

1. If $\tan\frac{u}{2} = A$, then

$$\sin u = \frac{2A}{1+A^2}, \qquad \cos u = \frac{1-A^2}{1+A^2}, \qquad \tan u = \frac{2A}{1-A^2}$$

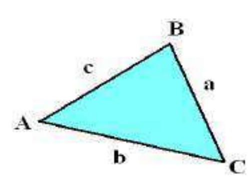
2. The transformation $\sin v = y$, requires $\cos v = \sqrt{1-y^2}$, and $\tan v = \dfrac{y}{\sqrt{1-y^2}}$

Law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Special Numbers

Rational Numbers

All those numbers having the form p/q, where p and q are integers and q is understood to be different from zero, are called rational numbers.

Irrational Numbers

Those numbers that cannot be written as the ratio of two numbers are called irrational numbers.

The Number π

The Greek letter π (pronounced pi) is an irrational number and can be defined as the limiting sum[1] of the infinite series

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots + \frac{(-1)^n}{2n+1} + \cdots\right)$$

Using a computer one can verify that the numerical value of π to 50 decimal places is given by

$$\pi = 3.14159265358979323846264338327950288419716939937510\ldots$$

The number π has the physical significance of representing the circumference C of a circle divided by its diameter D. The symbol π for the ratio C/D was introduced by William Jones (1675-1749), a Welsh mathematician. It became a standard notation for representing C/D after Euler also started using the symbol π for this ratio sometime around 1737.

The Number e

The limiting sum

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

is an irrational number which by agreement called the number e. Using a computer this number, to 50 decimal places, has the numerical value

$$e = 2.71828182845904523536028747135266249775724709369996\ldots$$

The number e is referred to as the base of the natural logarithm and the function $f(x) = e^x$ is called the exponential function.

[1] Limits are very important in the study of calculus.

Greek Alphabet

Letter		Name
A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu

Letter		Name
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

Notation

By convention letters from the beginning of an alphabet, such as $a, b, c, \ldots$ or the Greek letters $\alpha, \beta, \gamma, \ldots$ are often used to denote quantities which have a constant value. Subscripted quantities such as $x_0, x_1, x_2, \ldots$ or $y_0, y_1, y_2, \ldots$ can also be used to represent constant quantities. A variable is a quantity which is allowed to change its value. The letters u, v, w, x, y, z or the Greek letters ξ, η, ζ are most often used to denote variable quantities.

Inequalities

The mathematical symbols = (equals), $\neq$ (not equal), $<$ (less than), $<<$ (much less than), $\leq$ (less than or equal), $>$ (greater than), $>>$ (much greater than) $\geq$ (greater than or equal), and $|\ |$ (absolute value) occur frequently in mathematics to compare real numbers $a, b, c, \ldots$. The law of trichotomy states that if a and b are real numbers, then exactly one of the following must be true. Either a equals b, a is less than b or a is greater than b. These statements are expressed using the mathematical notations[2]

$$a = b, \qquad a < b, \qquad a > b$$

[2] In mathematical notation, the statement $b > a$, read "b is greater than a", can also be represented $a < b$ or "a is less than b"depending upon your way of looking at things.

Inequalities can be defined in terms of addition or subtraction. For example, one can define

$a < b$ if and only if $a - b < 0$

$a > b$ if and only if $a - b > 0$, or alternatively

$a > b$ if and only if there exists a positive number x such that $b + x = a$.

In dealing with inequalities be sure to observe the following properties associated with real numbers $a, b, c, \ldots$

1. A constant can be added to both sides of an inequality without changing the inequality sign.

   If $a < b$, then $a + c < b + c$ for all numbers c

2. Both sides of an inequality can be multiplied or divided by a positive constant without changing the inequality sign.

   If $a < b$ and $c > 0$, then $ac < bc$ or $a/c < b/c$

3. If both sides of an inequality are multiplied or divided by a negative quantity, then the inequality sign changes.

   If $b > a$ and $c < 0$, then $bc < ac$ or $b/c < a/c$

4. The transitivity law

   If $a < b$, and $b < c$, then $a < c$

   If $a = b$ and $b = c$, then $a = c$

   If $a > b$, and $b > c$, then $a > c$

5. If $a > 0$ and $b > 0$, then $ab > 0$
6. If $a < 0$ and $b < 0$, then $ab > 0$ or $0 < ab$
7. If $a > 0$ and $b > 0$ with $a < b$, then $\sqrt{a} < \sqrt{b}$

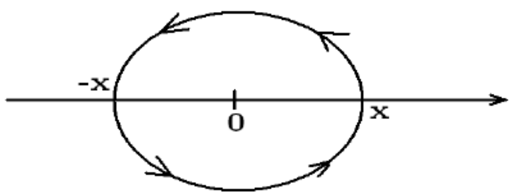
A negative times a negative is a positive

To prove that a real negative number multiplied by another real negative number gives a positive number start by assuming a and b are real numbers satisfying $a < 0$ and $b < 0$, then one can write

$$-a + a < -a \quad \text{or } 0 < -a \qquad \text{and} \qquad -b + b < -b \quad \text{or } 0 < -b$$

since equals can be added to both sides of an inequality without changing the inequality sign. Using the fact that both sides of an inequality can be multiplied by a positive number without changing the inequality sign, one can write

$$0 < (-a)(-b) \qquad \text{or} \qquad (-a)(-b) > 0$$



Another way to show a negative times a negative is a positive is as follows. Think of a number line with the number 0 separating the positive numbers and negative numbers. By agreement, if a number on this number line is multiplied by -1, then the number is to be rotated counterclockwise 180 degrees. If the positive number x is multiplied by -1, then it is rotated counterclockwise 180 degrees to produce the number $-x$. If the number $-x$ is multiplied by -1, then it is to be rotated 180 degrees counterclockwise to produce the positive number x. If $a > 0$ and $b > 0$, then the product $a(-b)$ scales the number $-b$ to produce the negative number $-ab$. If the number $-ab$ is multiplied by -1, which is equivalent to the product $(-a)(-b)$, one obtains by rotation the number $+ab$.

Absolute Value

The absolute value of a number x is defined

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The symbol $\iff$ is often used to represent equivalence of two equations. For example, if a and b are real numbers the statements

$$|x-a| \leq b \quad \iff \quad -b \leq x-a \leq b \quad \iff \quad a-b \leq x \leq a+b$$

are all equivalent statements involving restrictions on the real number x.

An important inequality known as the triangle inequality is written

$$|x+y| \leq |x| + |y| \tag{1.1}$$

where x and y are real numbers. To prove this inequality observe that $|x|$ satisfies $-|x| \leq x \leq |x|$ and also $-|y| \leq y \leq |y|$, so that by adding these results one obtains

$$-(|x|+|y|) \leq x+y \leq |x|+|y| \qquad \text{or} \qquad |x+y| \leq |x|+|y| \tag{1.2}$$

Related to the inequality (1.2) is the reverse triangle inequality

$$|x-y| \geq |x| - |y| \tag{1.3}$$

a proof of which is left as an exercise.

Cramer's Rule

The system of two equations in two unknowns

$$\begin{aligned} \alpha_1 x + \beta_1 y =& \gamma_1 \\ \alpha_2 x + \beta_2 y =& \gamma_2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

has a unique solution if $\alpha_1\beta_2 - \alpha_2\beta_1$ is nonzero. The unique solution is given by

$$x = \frac{\begin{vmatrix} \gamma_1 & \beta_1 \\ \gamma_2 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}} \quad \text{where} \quad \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - \alpha_2\beta_1$$

$-\alpha_2\beta_1$

$+\alpha_1\beta_2$

is a single number called the determinant of the coefficients.

The system of three equations in three unknowns

$$\begin{aligned} \alpha_1 x + \beta_1 y + \gamma_1 z =& \delta_1 \\ \alpha_2 x + \beta_2 y + \gamma_2 z =& \delta_2 \\ \alpha_3 x + \beta_3 y + \gamma_3 x =& \delta_3 \end{aligned}$$

has a unique solution if the determinant of the coefficients

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \alpha_3\beta_2\gamma_1 - \beta_3\gamma_2\alpha_1 - \gamma_3\alpha_2\beta_2$$

is nonzero. A mnemonic device to aid in calculating the determinant of the coefficients is to append the first two columns of the coefficients to the end of the array and then draw diagonals through the coefficients. Multiply the elements along an arrow and place a plus sign on the products associated with the down arrows and a minus sign associated with the products of the up arrows. This gives the figure

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \alpha_3\beta_2\gamma_1 - \beta_3\gamma_2\alpha_1 - \gamma_3\alpha_2\beta_2$$

The solution of the three equations, three unknown system of equations is given by the determinant ratios

$$x = \frac{\begin{vmatrix} \delta_1 & \beta_1 & \gamma_1 \\ \delta_2 & \beta_2 & \gamma_2 \\ \delta_3 & \beta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \alpha_1 & \delta_1 & \gamma_1 \\ \alpha_2 & \delta_2 & \gamma_2 \\ \alpha_3 & \delta_3 & \gamma_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} \alpha_1 & \beta_1 & \delta_1 \\ \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}}$$

and is known as Cramer's rule for solving a system of equations.

Appendix C
Table of Integrals
Indefinite Integrals

General Integration Properties

1. If $\dfrac{dF(x)}{dx} = f(x)$, then $\displaystyle\int f(x)\,dx = F(x) + C$

2. If $\displaystyle\int f(x)\,dx = F(x) + C$, then the substitution $x = g(u)$ gives $\displaystyle\int f(g(u))\,g'(u)\,du = F(g(u)) + C$
For example, if $\displaystyle\int \frac{dx}{x^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\frac{x}{\beta} + C$, then $\displaystyle\int \frac{du}{(u+\alpha)^2+\beta^2} = \frac{1}{\beta}\tan^{-1}\frac{u+\alpha}{\beta} + C$

3. Integration by parts. If $v_1(x) = \displaystyle\int v(x)\,dx$, then $\displaystyle\int u(x)v(x)\,dx = u(x)v_1(x) - \int u'(x)v_1(x)\,dx$

4. Repeated integration by parts or generalized integration by parts.
If $v_1(x) = \int v(x)\,dx,\quad v_2(x) = \int v_1(x)\,dx, \ldots, v_n(x) = \int v_{n-1}(x)\,dx$, then

$$\int u(x)v(x)\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots + (-1)^{n-1}u^{n-1}v_n + (-1)^n \int u^{(n)}(x)v_n(x)\,dx$$

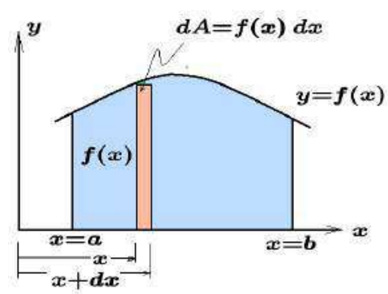
5. If $f^{-1}(x)$ is the inverse function of $f(x)$ and if $\displaystyle\int f(x)\,dx$ is known, then

$$\int f^{-1}(x)\,dx = zf(z) - \int f(z)\,dz, \qquad \text{where} \quad z = f^{-1}(x)$$

6. Fundamental theorem of calculus.
If the indefinite integral of $f(x)$ is known, say $\displaystyle\int f(x)\,dx = F(x) + C$, then the definite integral

$$\int_a^b dA = \int_a^b f(x)\,dx = F(x)\big]_a^b = F(b) - F(a)$$

represents the area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = a$ and $x = b$.



7. Inequalities.
(i) If $f(x) \le g(x)$ for all $x \in (a,b)$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
(ii) If $|f(x) \le M|$ for all $x \in (a,b)$ and $\int_a^b f(x)\,dx$ exists, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b f(x)\,dx \le M(b-a)$$

8. $\displaystyle\int \frac{u'(x)\,dx}{u(x)} = \ln|u(x)| + C$

9. $\displaystyle\int (\alpha u(x)+\beta)^n u'(x)\,dx = \frac{(\alpha u(x)+\beta)^{n+1}}{\alpha(n+1)} + C$

10. $\displaystyle\int \frac{u'(x)v(x) - v'(x)u(x)}{v^2(x)}\,dx = \frac{u(x)}{v(x)} + C$

11. $\displaystyle\int \frac{u'(x)v(x) - u(x)v'(x)}{u(x)v(x)}\,dx = \ln|\frac{u(x)}{v(x)}| + C$

12. $\displaystyle\int \frac{u'(x)v(x) - u(x)v'(x)}{u^2(x)+v^2(x)}\,dx = \tan^{-1}\frac{u(x)}{v(x)} + C$

13. $\displaystyle\int \frac{u'(x)v(x) - u(x)v'(x)}{u^2(x)-v^2(x)}\,dx = \frac{1}{2}\ln|\frac{u(x)-v(x)}{u(x)+v(x)}| + C$

14. $\displaystyle\int \frac{u'(x)\,dx}{\sqrt{u^2(x)+\alpha}} = \ln|u(x) + \sqrt{u^2(x)+\alpha}| + C$

15. $\displaystyle\int \frac{u(x)\,dx}{(u(x)+\alpha)(u(x)+\beta)} = \begin{cases} \dfrac{\alpha}{\alpha-\beta}\displaystyle\int \frac{dx}{u(x)+\alpha} - \frac{\beta}{\alpha-\beta}\int \frac{dx}{u(x)+\beta}, & \alpha \neq \beta \\ \displaystyle\int \frac{dx}{u(x)+\alpha} - \alpha\int \frac{dx}{(u(x)+\alpha)^2}, & \beta = \alpha \end{cases}$

16. $\displaystyle\int \frac{u'(x)\,dx}{\alpha u^2(x)+\beta u(x)} = \frac{1}{\beta}\ln|\frac{u(x)}{\alpha u(x)+\beta}| + C$

17. $\displaystyle\int \frac{u'(x)\,dx}{u(x)\sqrt{u^2(x)-\alpha^2}} = \frac{1}{\alpha}\sec^{-1}\frac{u(x)}{\alpha} + C$

18. $\displaystyle\int \frac{u'(x)\,dx}{\alpha^2+\beta^2u^2(x)} = \frac{1}{\alpha\beta}\tan^{-1}\frac{\beta u(x)}{\alpha} + C$

19. $\displaystyle\int \frac{u'(x)\,dx}{\alpha^2u^2(x)-\beta^2} = \frac{1}{2\alpha\beta}\ln|\frac{\alpha u(x)-\beta}{\alpha u(x)+\beta}| + C$

20. $\displaystyle\int f(\sin x)\,dx = 2\int f\left(\frac{2u}{1+u^2}\right)\frac{du}{1+u^2}, \quad u = \tan\frac{x}{2}$

21. $\displaystyle\int f(\sin x)\,dx = \int f(u)\frac{du}{\sqrt{1-u^2}}, \quad u = \sin x$

22. $\displaystyle\int f(\cos x)\,dx = 2\int f\left(\frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}, \quad u = \tan\frac{x}{2}$

23. $\displaystyle\int f(\cos x)\,dx = -\int f(u)\frac{du}{\sqrt{1-u^2}}, \quad u = \cos x$

24. $\displaystyle\int f(\sin x, \cos x)\,dx = \int f(u, \sqrt{1-u^2})\frac{du}{\sqrt{1-u^2}}, \quad u = \sin x$

25. $\displaystyle\int f(\sin x, \cos x)\,dx = 2\int f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{du}{1+u^2}, \quad u = \tan\frac{x}{2}$

26. $\displaystyle\int f(x, \sqrt{\alpha+\beta x})\,dx = \frac{2}{\beta}\int f\left(\frac{u^2-\alpha}{\beta}, u\right) u du, \qquad u^2 = \alpha + \beta x$

27. $\displaystyle\int f(x, \sqrt{\alpha^2-x^2})\,dx = \alpha\int f(\alpha\sin u, a\cos u)\,\cos u\,du, \qquad x = \alpha\sin u$

General Integrals

28. $\int c\,u(x)\,dx = c\int u(x)\,dx$

29. $\int [u(x)+v(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx$

30. $\int u(x)\,u'(x)\,dx = \frac{1}{2}\mid u(x)\mid^2 + C$

31. $\int [u(x)-v(x)]\,dx = \int u(x)\,dx - \int v(x)\,dx]$

32. $\int u^n(x)\,u'(x)\,dx = \frac{[u(x)]^{n+1}}{n+1} + C$

33. $\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$

34. $\int F'[u(x)]\,u'(x)\,dx = F[u(x)] + C$

35. $\int \frac{u'(x)}{u(x)}\,dx = \ln\mid u(x)\mid + C$

36. $\int \frac{u'}{2\sqrt{u}}\,dx = \sqrt{u} + C$

37. $\int 1\,dx = x + C$

38. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

39. $\int \frac{1}{x}\,dx = \ln\mid x\mid + C$

40. $\int e^{au}\,u'\,dx = \frac{1}{a}e^{au} + C$

41. $\int a^u\,u'\,dx = \frac{1}{\ln a}a^u + C$

42. $\int \sin u\,u'\,dx = \cos u + C$

43. $\int \cos u\,u'\,dx = -\sin u + C$

44. $\int \tan u\,u'\,dx = \ln\mid \sec u\mid + C$

45. $\int \cot u\,u'\,dx = \ln\mid \sin u\mid + C$

46. $\int \sec u\,u'\,dx = \ln\mid \sec u + \tan u\mid + C$

47. $\int \csc u\,u'\,dx = \ln\mid \csc u - \cot u\mid + C$

48. $\int \sinh u\,u'\,dx = \cosh u + C$

49. $\int \cosh u\,u'\,dx = \sinh u + C$

50. $\int \tanh u\,u'\,dx = \ln\cosh u + C$

51. $\int \coth u\,u'\,dx = \ln\sinh u + C$

52. $\int \operatorname{sech} u\,u'\,dx = \sin^{-1}(\tanh u) + C$

53. $\int \operatorname{csch} u\,u'\,dx = \ln\tanh\frac{u}{2} + C$

54. $\int \sin^2 u\,u'\,dx = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

55. $\int \cos^2 u\,u'\,dx = \frac{u}{2} + \frac{1}{4}\sin 2u + C$

56. $\int \tan^2 u\,u'\,dx = \tan u - u + C$

57. $\int \cot^2 u\,u'\,dx = -\cot u - u + C$

58. $\int \sec^2 u\,u'\,dx = \tan u + C$

59. $\int \csc^2 u\,u'\,dx = -\cot u + C$

60. $\int \sinh^2 u\,u'\,dx = \frac{1}{4}\sinh 2u - \frac{1}{2}u + C$

61. $\int \cosh^2 u\,u'\,dx = \frac{1}{4}\sinh 2u + \frac{1}{2}u + C$

62. $\int \tanh^2 u\,u'\,dx = u - \tanh u + C$

63. $\int \coth^2 u\,u'\,dx = u - \coth u + C$

64. $\int \operatorname{sech}^2 u\,u'\,dx = \tanh u + C$

65. $\int \operatorname{csch}^2 u\,u'\,dx = -\coth u + C$

66. $\int \sec u\tan u\,u'\,dx = \sec u + C$

67. $\int \csc u\cot u\,u'\,dx = -\csc u + C$

68. $\int \operatorname{sech} u\tanh u\,u'\,dx = -\operatorname{sech} u + C$

69. $\int \operatorname{csch} u\coth u\,u'\,dx = -\operatorname{csch} u + C$

Integrals containing $X = a + bx$, $a \neq 0$ and $b \neq 0$

70. $\int X^n\,dx = \frac{X^{n+1}}{b(n+1)} + C, \quad n \neq -1$

71. $\int xX^n\,dx = \frac{X^{n+2}}{b^2(n+2)} - \frac{aX^{n+1}}{b^2(n+1)} + C, \quad n \neq -1, n \neq -2$

72. $\int X(x+c)^n\,dx = \frac{b}{n+2}(x+c)^{n+2} + \frac{a-bc}{n+1}(x+c)^{n+1} + C$

73. $$\int x^2 X^n\,dx = \frac{1}{b^3}\left[\frac{X^{n+3}}{n+3} - \frac{2aX^{n+2}}{n+2} + \frac{a^2X^{n+1}}{n+1}\right] + C$$

74. $$\int x^{n-1}X^m\,dx = \frac{1}{n+m}x^nX^m + \frac{am}{m+n}\int x^{n-1}X^{m-1}\,dx$$

75. $$\int \frac{X^m}{x^{n+1}}\,dx = -\frac{1}{na}\frac{X^{m+1}}{x^n} + \frac{m-n+1}{n}\frac{b}{a}\int \frac{X^m}{x^n}\,dx$$

76. $$\int \frac{dx}{X} = \frac{1}{b}\ln X + C$$

77. $$\int \frac{x\,dx}{X} = \frac{1}{b^2}\left(X - a\ln|\,X\,|\right) + C$$

78. $$\int \frac{x^2\,dx}{X} = \frac{1}{2b^3}\left(X^2 - 4aX + 2a^2\ln|\,X\,|\right) + C$$

79. $$\int \frac{dx}{xX} = \frac{1}{a}\ln\left|\,\frac{x}{X}\,\right| + C$$

80. $$\int \frac{dx}{x^3X} = -\frac{a+2bx}{a^2xX} + \frac{2b}{a^3}\ln\left|\frac{X}{x}\right| + C$$

81. $$\int \frac{dx}{X^2} = -\frac{1}{bX} + C$$

82. $$\int \frac{x\,dx}{X^2} = \frac{1}{b^2}\left[\ln|\,X\,| + \frac{a}{X}\right] + C$$

83. $$\int \frac{x^2\,dx}{X^2} = \frac{1}{b^3}\left[X - 2a\ln|\,X\,| - \frac{a^2}{X}\right] + C$$

84. $$\int \frac{dx}{xX^2} = \frac{1}{aX} - \frac{1}{a^2}\ln\left|\,\frac{X}{x}\,\right| + C$$

85. $$\int \frac{dx}{x^2X^2} = -\frac{a+2bx}{a^2xX} + \frac{2b}{a^3}\ln\left|\,\frac{X}{x}\,\right| + C$$

86. $$\int \frac{dx}{X^3} = -\frac{1}{2bX^2} + C$$

87. $$\int \frac{x\,dx}{X^3} = \frac{1}{b^2}\left[\frac{-1}{X} + \frac{a}{2X^2}\right] + C$$

88. $$\int \frac{x^2\,dx}{X^3} = \frac{1}{b^3}\left[\ln|\,X\,| + \frac{2a}{X} - \frac{a^2}{2X^2}\right] + C$$

89. $$\int \frac{dx}{xX^3} = \frac{1}{2aX^2} + \frac{1}{aX} - \ln\left|\,\frac{X}{x}\,\right| + C$$

90. $$\int \frac{dx}{x^2X^3} = \frac{-b}{2a^2X} - \frac{2b}{a^3X} - \frac{1}{a^3x} + \frac{3b}{a^4}\ln\left|\frac{X}{x}\right|$$

91. $$\int \frac{x\,dx}{X^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)X^{n-2}} + \frac{a}{(n-1)X^{n-1}}\right] + C, \quad n \neq 1, 2$$

92. $$\int \frac{x^2\,dx}{X^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)X^{n-3}} + \frac{2a}{(n-2)X^{n-2}} - \frac{a^2}{(n-1)X^{n-1}}\right] + C, \quad n \neq 1, 2, 3$$

93. $$\int \sqrt{X}\,dx = \frac{2}{3b}X^{3/2} + C$$

94. $$\int x\sqrt{X}\,dx = \frac{2}{15b^2}(3bx - 2a)X^{3/2} + C$$

95. $$\int x^2\sqrt{X}\,dx = \frac{2}{105b^3}(8a^2 - 12abx + 15b^2x^2)X^{3/2} + C$$

96. $$\int \frac{\sqrt{X}}{x}\,dx = 2\sqrt{X} + a\int \frac{dx}{x\sqrt{X}}$$

97. $$\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$$

98. $$\int \frac{dx}{\sqrt{X}} = \frac{2}{b}\sqrt{X} + C$$

99. $$\int \frac{x\,dx}{\sqrt{X}} = \frac{2}{3b^2}(bx - 2a)\sqrt{X} + C$$

100. $$\int \frac{x^2\,dx}{\sqrt{X}} = \frac{2}{15b^3}(8a^2 - 4abx + 3b^2x^2)\sqrt{X} + C$$

101. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{X}-\sqrt{a}}{\sqrt{X}+\sqrt{a}}\right| + C_1, & a > 0 \\ \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{X}{-a}} + C_2, & a < 0 \end{cases}$$

102. $$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}$$

103. $$\int x^n\sqrt{X}\,dx = \frac{2}{(2n+3)b}x^nX^{3/2} - \frac{2na}{(2n+3)b}\int x^{n-1}\sqrt{X}\,dx$$

104. $$\int \frac{\sqrt{X}}{x^n}\,dx = \frac{-1}{(n-1)a}\frac{X^{3/2}}{x^{n-1}} - \frac{(2n-5)b}{2(n-1)a}\int \frac{\sqrt{X}}{x^{n-1}}\,dx$$

105. $$\int x^{m-1}X^n\,dx = \frac{x^mX^n}{m+n} + \frac{an}{m+n}\int x^{m-1}X^{n-1}\,dx + C$$

106. $$\int \frac{X^n}{x^{m+1}}\,dx = -\frac{X^{n+1}}{ma\,x^m} + \frac{n-m+1}{m}\frac{b}{a}\int \frac{X^n}{x^m}\,dx$$

107. $$\int \frac{X^n}{x}\,dx = \frac{X^n}{n} + a\int \frac{X^{n-1}}{x}\,dx$$

Integrals containing $X = a + bx$ and $Y = \alpha + \beta x$, $(b \neq 0,\ \beta \neq 0,\ \Delta = a\beta - \alpha b \neq 0\ )$

108. $$\int \frac{dx}{XY} = \frac{1}{\Delta}\ln\left|\ \frac{Y}{X}\ \right| + C$$

109. $$\int \frac{x\,dx}{XY} = \frac{1}{\Delta}\left[\frac{a}{b}\ln|\ X\ | - \frac{\alpha}{\beta}\ln|\ Y\ |\right] + C$$

110. $$\int \frac{x^2\,dx}{XY} = \frac{x}{b\beta} = \frac{a^2}{b^2\Delta}\ln|X| + \frac{\alpha^2}{\beta^2\Delta}\ln|Y| + C$$

111. $$\int \frac{dx}{X^2Y} = \frac{1}{\Delta}\left(\frac{1}{X} + \frac{\beta}{\Delta}\ln\left|\frac{Y}{X}\right|\right) + C$$

112. $$\int \frac{x\,dx}{X^2Y} = -\frac{a}{b\Delta X} - \frac{\alpha}{\Delta^2}\ln\left|\frac{Y}{X}\right| + C$$

113. $$\int \frac{x^2\,dx}{X^2Y} = \frac{a^2}{b^2\Delta X} + \frac{1}{\Delta^2}\left[\frac{\alpha^2}{\beta}\ln|Y| + \frac{a(a\beta - 2\alpha b)}{b^2}\ln|X|\right] + C$$

114. $$\int \frac{X}{Y}\,dx = \frac{b}{\beta}x + \frac{\Delta}{\beta^2}\ln\left|\frac{Y}{X}\right| + C$$

115. $$\int \sqrt{XY}\,dx = \frac{\Delta + 2bY}{4b\beta}\sqrt{XY} - \frac{\Delta^2}{8b\beta}\int \frac{dx}{\sqrt{XY}}$$

116. $$\int \frac{dx}{X^nY^m} = \frac{-1}{(m-1)\Delta X^{n-1}Y^{m-1}} + \frac{(m+n-2)b}{(m-1)\Delta}\int \frac{dx}{X^nY^{m-1}}, \qquad m \neq 1$$

117. $$\int \frac{dx}{Y\sqrt{X}} = \begin{cases} \dfrac{2}{\sqrt{-\Delta\beta}}\tan^{-1}\dfrac{\beta\sqrt{X}}{\sqrt{-\Delta\beta}}, +C_1 & \Delta\beta < 0 \\ \dfrac{1}{\sqrt{\Delta\beta}}\ln\left|\dfrac{\beta\sqrt{X} - \sqrt{\Delta\beta}}{\beta\sqrt{X} + \sqrt{\Delta\beta}}\right| + C_2, & \Delta\beta > 0 \end{cases}$$

118. $$\int \frac{dx}{\sqrt{XY}} = \begin{cases} \dfrac{2}{\sqrt{-b\beta}}\tan^{-1}\sqrt{\dfrac{-\beta X}{bY}} + C_1, & b\beta < 0,\ bY > 0 \\ \dfrac{2}{\sqrt{b\beta}}\tanh^{-1}\sqrt{\dfrac{\beta X}{bY}} + C_2, & b\beta > 0,\ bY > 0 \end{cases}$$

119. $$\int \frac{x\,dx}{\sqrt{XY}} = \frac{1}{b\beta}\sqrt{XY} - \frac{(b\alpha + a\beta)}{2b\beta}\int \frac{dx}{\sqrt{XY}}$$

120. $$\int \frac{\sqrt{Y}}{\sqrt{X}}\,dx = \frac{1}{b}\sqrt{XY} - \frac{\Delta}{2b}\int \frac{dx}{\sqrt{XY}}$$

121. $$\int \frac{\sqrt{X}}{Y}\,dx = \frac{2}{\beta}\sqrt{X} + \frac{\Delta}{\beta}\int \frac{dx}{Y\sqrt{X}}$$

Integrals containing terms of the form $a + bx^n$

122. $$\int \frac{dx}{a+bx^2} = \begin{cases} \dfrac{1}{\sqrt{ab}} \tan^{-1}\left(\sqrt{\dfrac{b}{a}}x\right) + C, & ab > 0 \\ \dfrac{1}{2\sqrt{-ab}} \ln\left|\dfrac{a+\sqrt{-ab}x}{a-\sqrt{-ab}x}\right| + C, & ab < 0 \end{cases}$$

123. $$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln|x^2 + \frac{a}{b}| + C$$

124. $$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}$$

125. $$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}$$

126. $$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right| + C$$

127. $$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$$

128. $$\int \frac{dx}{(a+bx^2)^{n+1}} = \frac{1}{2na}\frac{x}{(a+bx^2)^n} + \frac{2n-1}{2na}\int \frac{dx}{(a+bx^2)^n}$$

129. $$\int \frac{dx}{\alpha^3+\beta^3x^3} = \frac{1}{6\alpha^2\beta}\left[2\sqrt{3}\tan^{-1}\left(\frac{2\beta x-\alpha}{\sqrt{3}\alpha}\right) + \ln\left|\frac{(\alpha+\beta x)^2}{\alpha^2-\alpha\beta x+\beta^2x^2}\right|\right] + C$$

130. $$\int \frac{x\,dx}{\alpha^3+\beta^3x^3} = \frac{1}{6\alpha\beta^2}\left[2\sqrt{3}\tan^{-1}\left(\frac{2\beta x-\alpha}{\sqrt{3}\alpha}\right) - \ln\left|\frac{(\alpha+\beta x)^2}{\alpha^2-\alpha\beta x+\beta^2x^2}\right|\right] + C$$

If $X = a + bx^n$, then

131. $$\int x^{m-1}X^p\,dx = \frac{x^mX^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx$$

132. $$\int x^{m-1}X^p\,dx = -\frac{x^mX^{p+1}}{an(p+1)} + \frac{m+pn+n}{an(p+1)}\int x^{m-1}X^{p+1}\,dx$$

133. $$\int x^{m-1}X^P\,dx = \frac{x^{m-n}X^{p+1}}{b(m+pn)} - \frac{(m-n)\,a}{b(m+pn)}\int x^{m-n-1}X^p\,dx$$

134. $$\int x^{m-1}X^p\,dx = \frac{x^mX^{p+1}}{am} - \frac{(m+pn+n)\,b}{am}\int x^{m+n-1}X^p\,dx$$

135. $$\int x^{m-1}X^p\,dx = \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}X^{p+1}\,dx$$

136. $$\int x^{m-1}X^p\,dx = \frac{x^mX^p}{m} - \frac{bpn}{m}\int x^{m+n-1}X^{p-1}\,dx$$

Integrals containing $X = 2ax - x^2, \quad a \neq 0$

137. $\int \sqrt{X}\,dx = \frac{(x-a)}{2}\sqrt{X} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

138. $\int \frac{dx}{\sqrt{X}} = \sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

139. $\int x\sqrt{X}\,dx = \sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

140. $\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X} + a\sin^{-1}\left(\frac{x-a}{|a|}\right) + C$

141. $\int \frac{dx}{X^{3/2}} = \frac{x-a}{a^2\sqrt{X}} + C$

142. $\int \frac{x\,dx}{X^{3/2}} = \frac{x}{a\sqrt{X}} + C$

143. $\int \frac{dx}{X} = \frac{1}{2a}\ln|\frac{x}{x-2a}| + C$

144. $\int \frac{x\,dx}{X} = -\ln|x-2a| + C$

145. $\int \frac{dx}{X^2} = -\frac{1}{4ax} - \frac{1}{4a^2(x-2a)} + \frac{1}{4a^2}\ln|\frac{x}{x-2a}| + C$

146. $\int \frac{x\,dx}{X^2} = -\frac{1}{2a(x-2a)} + \frac{1}{4a^2}\ln|\frac{x}{x-2a}| + C$

147. $\int x^n\sqrt{X}\,dx = -\frac{1}{n+2}x^{n-1}X^{3/2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{X}\,dx, \quad n \neq -2$

148. $\int \frac{\sqrt{X}\,dx}{x^n} = \frac{1}{(3-2n)a}\frac{X^{3/2}}{x^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{X}}{x^{n-1}}\,dx, \quad n \neq 3/2$

Integrals containing $X = ax^2 + bx + c$ with $\Delta = 4ac - b^2, \quad \Delta \neq 0, \quad a \neq 0$

149. $$\int \frac{dx}{X} = \begin{cases} \frac{1}{\sqrt{-\Delta}}\ln\left(\frac{2ax+b-\sqrt{-\Delta}}{2ax+b+\sqrt{-\Delta}}\right) + C_1, & \Delta < 0 \\ \frac{2}{\sqrt{\Delta}}\tan^{-1}\frac{2ax+b}{\sqrt{\Delta}} + C_2, & \Delta > 0 \\ -\frac{1}{a(x+b/2a)} + C_3, & \Delta = 0 \end{cases}$$

150. $\int \frac{x\,dx}{X} = \frac{1}{2c}\ln|\,X\,| - \frac{b}{2a}\int \frac{1}{X}\,dx$

151. $\int \frac{x^2\,dx}{X} = \frac{x}{a} - \frac{b}{2a^2}\ln|X| + \frac{2ac-\Delta}{2a^2}\int \frac{dx}{X}$

152. $\displaystyle \int \frac{dx}{xX} = \frac{1}{2c}\ln|\frac{x^2}{X}| - \frac{b}{2c}\int \frac{dx}{X}$

153. $\displaystyle \int \frac{dx}{x^2X} = \frac{b}{2c^2}\ln|\frac{X}{x^2}| - \frac{1}{cx} + \frac{2ac-\Delta}{2c^2}\int \frac{dx}{X}$

154. $\displaystyle \int \frac{dx}{X^2} = \frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int \frac{dx}{X}$

155. $\displaystyle \int \frac{x\,dx}{X^2} = -\frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int \frac{dx}{X}$

156. $\displaystyle \int \frac{x^2\,dx}{X^2} = \frac{(2ac-\Delta)x+bc}{a\Delta X} + \frac{2c}{\Delta}\int \frac{dx}{X}$

157. $\displaystyle \int \frac{dx}{xX^2} = \frac{1}{2cX} - \frac{b}{2c}\int \frac{dx}{X^2} + \frac{1}{c}\int \frac{dx}{xX}$

158. $\displaystyle \int \frac{dx}{x^2X^2} = -\frac{1}{cxX} - \frac{3a}{c}\int \frac{dx}{X^2} - \frac{2b}{c}\int \frac{dx}{xX^2}$

159. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln|2\sqrt{aX}+2ax+b| + C_1, & a>0 \\ \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{2ax+b}{\sqrt{\Delta}}\right) + C_2, & a>,\ \Delta>0 \\ -\dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) + C_3, & a<0,\ \Delta<0 \end{cases}$$

160. $\displaystyle \int \frac{x\,dx}{\sqrt{X}} = \frac{1}{a}\sqrt{X} - \frac{b}{2a}\int \frac{dx}{\sqrt{X}}$

161. $\displaystyle \int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2}\right)\sqrt{X} + \frac{2b^2-\Delta}{8a^2}\int \frac{dx}{\sqrt{X}}$

162. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\dfrac{1}{\sqrt{c}}\ln|\dfrac{2\sqrt{cX}}{x} + \dfrac{2c}{x} + b| + C_1, & c>0 \\ -\dfrac{1}{\sqrt{c}}\sinh^{-1}\left(\dfrac{bx+2c}{x\sqrt{\Delta}}\right) + C_2, & c>0,\ \Delta>0 \\ \dfrac{1}{\sqrt{-c}}\sin^{-1}\left(\dfrac{bx+2c}{x\sqrt{-\Delta}}\right) + C_3, & c<0,\ \Delta<0 \end{cases}$$

163. $\displaystyle \int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{X}}$

164. $\displaystyle \int \sqrt{X}\,dx = \frac{1}{4a}(2ax+b)\sqrt{X} + \frac{\Delta}{8a}\int \frac{dx}{\sqrt{X}}$

165. $\displaystyle \int x\sqrt{X}\,dx = \frac{1}{3a}X^{3/2} - \frac{b(2ax+b)}{8a^2}\sqrt{X} - \frac{b\Delta}{16a^2}\int \frac{dx}{\sqrt{X}}$

166. $\displaystyle \int x^2\sqrt{X}\,dx = \frac{6ax-5b}{24a^2}X^{3/2} + \frac{4b^2-\Delta}{16a^2}\int \sqrt{X}\,dx$

167. $$\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + c\int \frac{dx}{x\sqrt{X}}$$

168. $$\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + a\int \frac{dx}{\sqrt{X}} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$$

169. $$\int \frac{dx}{X^{3/2}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} + C$$

170. $$\int \frac{x\,dx}{X^{3/2}} = \frac{-2(bx+2c)}{\Delta\sqrt{X}} + C$$

171. $$\int \frac{x^2\,dx}{X^{3/2}} = \frac{(b^2-\Delta)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$$

172. $$\int \frac{dx}{xX^{3/2}} = \frac{1}{x\sqrt{X}} + \frac{1}{c}\int \frac{dx}{x\sqrt{X}} - \frac{b}{2c}\int \frac{dx}{X^{3/2}}$$

173. $$\int \frac{dx}{x^2X^{3/2}} = -\frac{ax^2+2bx+c}{c^2x\sqrt{X}} + \frac{b^2-2ac}{2c^2}\int \frac{dx}{X^{3/2}} - \frac{3b}{2c^2}\int \frac{dx}{x\sqrt{X}}$$

174. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax+b)}{\Delta\sqrt{X}} + C$$

175. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax+b)}{3\Delta\sqrt{X}}\left(\frac{1}{X} + \frac{8a}{\Delta}\right) + C$$

176. $$\int X\sqrt{X}\,dx = \frac{(2ax+b)}{8a}\sqrt{X}\left(X + \frac{3\Delta}{8a}\right) + \frac{3\Delta^2}{128a^2}\int \frac{dx}{\sqrt{X}}$$

177. $$\int X^2\sqrt{X}\,dx = \frac{(2ax+b)}{8a}\sqrt{X}\left(X^2 + \frac{5\Delta}{16a}X + \frac{15\Delta^2}{128a^2}\right) + \frac{5\Delta^3}{1024a^3}\int \frac{dx}{\sqrt{X}}$$

178. $$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2c)}{\Delta\sqrt{X}} + C$$

179. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(b^2-\Delta)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$$

180. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a}\int X\sqrt{X}\,dx$$

181. $\int f(x, \sqrt{ax^2+bx+c})\,dx$ Try substitutions (i) $\sqrt{ax^2+bx+c} = \sqrt{a}(x+z)$

(ii) $\sqrt{ax^2+bx+c} = xz + \sqrt{c}$ and if $ax^2+bx+c = a(x-x_1)(x-x_2)$, then (iii) let $(x-x_2) = z^2(x-x_1)$

Integrals containing $X = x^2 + a^2$

182. $$\int \frac{dx}{X} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad \text{or} \quad \frac{1}{a}\cos^{-1}\frac{a}{\sqrt{x^2+a^2}} + C \quad \text{or} \quad \frac{1}{a}\sec^{-1}\frac{\sqrt{x^2+a^2}}{a} + C$$

183. $$\int \frac{x\,dx}{X} = \frac{1}{2}\ln X + C$$

184. $\displaystyle\int \frac{x^2\,dx}{X} = x - a\tan^{-1}\frac{x}{a} + C$

185. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{x^2}{2} - \frac{a^2}{2}\ln|x^2+a^2| + C$

186. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2a^2}\ln|\frac{x^2}{X}| + C$

187. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} - \frac{1}{a^3}\tan^{-1}\frac{x}{a} + C$

188. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

189. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

190. $\displaystyle\int \frac{x\,dx}{X^2} = -\frac{1}{2X} + C$

191. $\displaystyle\int \frac{x^2\,dx}{X^2} = -\frac{x}{2X} + \frac{1}{2a}\tan^{-1}\frac{x}{a} + C$

192. $\displaystyle\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X| + C$

193. $\displaystyle\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x}{X}| + C$

194. $\displaystyle\int \frac{dx}{x^2X^2} = -\frac{1}{a^4X} - \frac{x}{2a^4X} - \frac{3}{2a^5}\tan^{-1}\frac{x}{a} + C$

195. $\displaystyle\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4X} - \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

196. $\displaystyle\int \frac{dx}{X^3} = \frac{x}{4a^2X^2} + \frac{3x}{8a^4X} + \frac{3}{8a^5}\tan^{-1}\frac{x}{a} + C$

197. $\displaystyle\int \frac{dx}{X^n} = \frac{x}{2(n-1)a^2X^{n-1}} + \frac{2n-3}{(2(n-1)a^2}\int\frac{dx}{X^{n-1}}, \qquad n > 1$

198. $\displaystyle\int \frac{x\,dx}{X^n} = -\frac{1}{2(n-1)X^{n-1}} + C$

199. $\displaystyle\int \frac{dx}{xX^n} = \frac{1}{2(n-1)a^2X^{n-1}} + \frac{1}{a^2}\int\frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = x^2 + a^2$

200. $\displaystyle\int \sqrt{X}\,dx = \frac{1}{2}xX + \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

201. $\int x\sqrt{X}\,dx = \frac{1}{3}X^{3/2} + C$

202. $\int x^2\sqrt{X}\,dx = \frac{1}{4}xX^{3/2} - \frac{1}{8}a^2x\sqrt{X} - \frac{a^2}{8}\ln|x+\sqrt{X}| + C$

203. $\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} - \frac{a^2}{3}X^{3/2} + C$

204. $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln|\frac{a+\sqrt{X}}{x}| + C$

205. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \ln|x+\sqrt{X}| + C$

206. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\ln|\frac{a+\sqrt{X}}{x}| + C$

207. $\int \frac{dx}{\sqrt{X}} = \ln|x+\sqrt{X}| + C \quad \text{or} \quad \sinh^{-1}\frac{x}{a} + C$

208. $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X} + C$

209. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

210. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + C$

211. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln|\frac{a+\sqrt{X}}{x}| + C$

212. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x} + C$

213. $\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

214. $\int X^{3/2}\,dx = \frac{1}{4}X^{3/2} + \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\ln|x+\sqrt{X}| + C$

215. $\int xX^{3/2}\,dx = \frac{1}{5}X^{5/2} + C$

216. $\int x^2X^{3/2}\,dx = \frac{1}{6}X^{5/2} - \frac{1}{24}a^2xX^{3/2} - \frac{1}{16}a^4x\sqrt{X} - \frac{1}{16}a^6\ln|x+\sqrt{X}| + C$

217. $\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} - \frac{1}{5}a^2X^{5/2} + C$

218. $\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2} + a^2\sqrt{X} - a^3 \ln|\frac{a+\sqrt{X}}{x}| + C$

219. $\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2 \ln|x+\sqrt{X}| + C$

220. $\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a \ln|\frac{a+\sqrt{x}}{x}| + C$

221. $\int \frac{dx}{X^{3/2}} = \frac{x}{a^2\sqrt{X}} + C$

222. $\int \frac{x\,dx}{X^{3/2}} = \frac{-1}{\sqrt{X}} + C$

223. $\int \frac{x^2\,dx}{X^{3/2}} = \frac{-x}{\sqrt{X}} + \ln|x+\sqrt{X}| + C$

224. $\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \frac{a^2}{\sqrt{X}} + C$

225. $\int \frac{dx}{xX^{3/2}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

226. $\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} - \frac{x}{a^4\sqrt{X}} + C$

227. $\int \frac{dx}{x^3X^{3/2}} = \frac{-1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\ln|\frac{a+\sqrt{X}}{x}| + C$

228. $\int f(x,\sqrt{X})\,dx = a\int f(a\tan u, a\sec u)\,\sec^2 u\,du, \quad x = a\tan u$

Integrals containing $X = x^2 - a^2$ with $x^2 > a^2$

229. $\int \frac{dx}{X} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C$ or $-\frac{1}{a}\coth^{-1}\frac{x}{a} + C$ or $-\frac{1}{a}\tanh^{-1}\frac{a}{x} + C$

230. $\int \frac{x\,dx}{X} = \frac{1}{2}\ln X + C$

231. $\int \frac{x^2\,dx}{X} = x + \frac{a}{2}\ln|\frac{x-a}{x+a}| + C$

232. $\int \frac{x^3\,dx}{X} = \frac{x^2}{2} + \frac{a^2}{2}\ln|X| + C$

233. $\int \frac{dx}{xX} = \frac{1}{2a^2}\ln|\frac{X}{x^2}| + C$

234. $\int \frac{dx}{x^2X} = \frac{1}{a^2x} + \frac{1}{2a^3}\ln|\frac{x-a}{x+a}| + C$

235. $\int \frac{dx}{x^3X} = \frac{1}{2a^2x} - \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

236. $\int \frac{dx}{X^2} = \frac{-x}{2a^2X} - \frac{1}{4a^3}\ln|\frac{x-a}{x+a}| + C$

237. $\int \frac{x\,dx}{X^2} = \frac{-1}{2X} + C$

238. $\int \frac{x^2\,dx}{X^2} = \frac{-x}{2X} + \frac{1}{4a}\ln|\frac{x-a}{x+a}| + C$

239. $\int \frac{x^3\,dx}{X^2} = \frac{-a^2}{2X} + \frac{1}{2}\ln|X| + C$

240. $\int \frac{dx}{xX^2} = \frac{-1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

241. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} - \frac{x}{2a^4X} - \frac{3}{4a^5}\ln|\frac{x-a}{x+a}| + C$

242. $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4X} + \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

243. $\int \frac{dx}{X^n} = \frac{-x}{2(n-1)a^2X^{n-1}} - \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{X^{n-1}}, \qquad n > 1$

244. $\int \frac{x\,dx}{X^n} = \frac{-1}{2(n-1)X^{n-1}} + C$

245. $\int \frac{dx}{xX^n} = \frac{-1}{2(n-1)a^2X^{n-1}} - \frac{1}{a^2}\int \frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = x^2 - a^2$ with $x^2 > a^2$

246. $\int \sqrt{X}\,dx = \frac{1}{2}x\sqrt{X} - \frac{a^2}{2}\ln|x + \sqrt{X}| + C$

247. $\int x\sqrt{X}\,dx = \frac{1}{3}X^{3/2} + C$

248. $\int x^2\sqrt{X}\,dx = \frac{1}{4}xX^{3/2} + \frac{1}{8}a^2x\sqrt{X} - \frac{a^4}{8}\ln|x + \sqrt{X}| + C$

249. $\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} + \frac{1}{3}a^2X^{3/2} + C$

250. $\int \frac{X}{x}\,dx = \sqrt{X} - a\sec^{-1}|\frac{x}{a}| + C$

251. $\displaystyle\int \frac{X}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \ln|x+\sqrt{X}| + C$

252. $\displaystyle\int \frac{X}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\sec^{-1}|\frac{x}{a}| + C$

253. $\displaystyle\int \frac{dx}{\sqrt{X}} = \ln|x+\sqrt{X}| + C$

254. $\displaystyle\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X} + C$

255. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\ln|x+\sqrt{X}| + C$

256. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} + a^2\sqrt{X} + C$

257. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\sec^{-1}|\frac{x}{a}| + C$

258. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2x} + C$

259. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}|\frac{x}{a}| + C$

260. $\displaystyle\int X^{3/2}\,dx = \frac{x}{4}X^{3/2} - \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\ln|x+\sqrt{X}| + C$

261. $\displaystyle\int xX^{3/2}\,dx = \frac{1}{5}X^{5/2} + C$

262. $\displaystyle\int x^2X^{3/2}\,dx = \frac{1}{6}xX^{5/2} + \frac{1}{24}a^2xX^{3/2} - \frac{1}{16}a^4x\sqrt{X} + \frac{a^6}{16}\ln|x+\sqrt{X}| + C$

263. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} + \frac{1}{5}a^2X^{5/2} + C$

264. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + a^3\sec^{-1}|\frac{x}{a}| + C$

265. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\ln|x+\sqrt{X}| + C$

266. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a\sec^{-1}|\frac{x}{a}| + C$

267. $\displaystyle\int \frac{dx}{X^{3/2}} = -\frac{x}{a^2\sqrt{X}} + C$

268. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{-1}{\sqrt{X}} + C$

269. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = -\frac{x}{\sqrt{X}} - \frac{a^2}{\sqrt{X}} + C$

270. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \ln|x + \sqrt{X}| + C$

271. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{-1}{a^2\sqrt{X}} - \frac{1}{a^3}\sec^{-1}|\frac{x}{a}| + C$

272. $\displaystyle\int \frac{dx}{x^2X^{3/2}} = -\frac{\sqrt{X}}{a^4x} - \frac{x}{a^4\sqrt{X}} + C$

273. $\displaystyle\int \frac{dx}{x^3X^{3/2}} = \frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\sec^{-1}|\frac{x}{a}| + C$

Integrals containing $X = a^2 - x^2$ with $x^2 < a^2$

274. $\displaystyle\int \frac{dx}{X} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{or} \quad \frac{1}{a}\tanh^{-1}\frac{x}{a} + C$

275. $\displaystyle\int \frac{x\,dx}{X} = -\frac{1}{2}\ln X + C$

276. $\displaystyle\int \frac{x^2\,dx}{X} = -x + \frac{a}{2}\ln|\frac{a+x}{a-x}| + C$

277. $\displaystyle\int \frac{x^3\,dx}{X} = -\frac{1}{2}x^2 - \frac{a^2}{2}\ln|X| + C$

278. $\displaystyle\int \frac{d}{xX} = \frac{1}{2a^2}\ln|\frac{x^2}{X}| + C$

279. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{a^2x} + \frac{1}{2a^3}\ln|\frac{a+x}{a-x}| + C$

280. $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

281. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{4a^3}\ln|\frac{a+x}{a-x}| + C$

282. $\displaystyle\int \frac{x\,dx}{X^2} = \frac{1}{2X} + C$

283. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{x}{2X} - \frac{1}{4a}\ln|\frac{a+x}{a-x}| + C$

284. $\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X| + C$

285. $\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\ln|\frac{x^2}{X}| + C$

286. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} + \frac{x}{2a^4X} + \frac{3}{4a^5}\ln|\frac{a+x}{a-x}| + C$

287. $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} + \frac{1}{2a^4X} + \frac{1}{a^6}\ln|\frac{x^2}{X}| + C$

288. $\int \frac{dx}{X^n} = \frac{x}{2(n-1)a^2X^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{X^{n-1}}$

289. $\int \frac{x\,dx}{X^n} = \frac{1}{2(n-1)X^{n-1}} + C$

290. $\int \frac{dx}{xX^n} = \frac{1}{2(n-1)a^2X^{n-1}} + \frac{1}{a^2}\int \frac{dx}{xX^{n-1}}$

Integrals containing the square root of $X = a^2 - x^2$ with $x^2 < a^2$

291. $\int \sqrt{X}\,dx = \frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

292. $\int x\sqrt{X}\,dx = -\frac{1}{3}X^{3/2} + C$

293. $\int x^2\sqrt{X}\,dx = -\frac{1}{4}xX^{3/2} + \frac{1}{8}a^2x\sqrt{X} + \frac{1}{8}a^4\sin^{-1}\frac{x}{a} + C$

294. $\int x^3\sqrt{X}\,dx = \frac{1}{5}X^{5/2} - \frac{1}{3}a^2X^{3/2} + C$

295. $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln|\frac{a+\sqrt{X}}{x}| + C$

296. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} - \sin^{-1}\frac{x}{a} + C$

297. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\ln|\frac{a+\sqrt{X}}{x}| + C$

298. $\int \frac{dx}{\sqrt{X}} = \sin^{-1}\frac{x}{a} + C$

299. $\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X} + C$

300. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = -\frac{1}{2}x\sqrt{X} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

301. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \frac{1}{3}X^{3/2} - a^2\sqrt{X} + C$

302. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln|\frac{a+\sqrt{X}}{x}| + C$

303. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x} + C$

304. $\displaystyle\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} - \frac{1}{2a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

305. $\displaystyle\int X^{3/2}\,dx = \frac{1}{4}xX^{3/2} + \frac{3}{8}a^2x\sqrt{X} + \frac{3}{8}a^4\sin^{-1}\frac{x}{a} + C$

306. $\displaystyle\int xX^{3/2}\,dx = -\frac{1}{5}X^{5/2} + C$

307. $\displaystyle\int x^2X^{3/2}\,dx = -\frac{1}{6}xX^{5/2} + \frac{1}{24}a^2xX^{3/2} + \frac{1}{16}a^4x\sqrt{X} + \frac{a^6}{16}\sin^{-1}\frac{x}{a} + C$

308. $\displaystyle\int x^3X^{3/2}\,dx = \frac{1}{7}X^{7/2} - \frac{1}{5}a^2X^{5/2} + C$

309. $\displaystyle\int \frac{X^{3/2}}{x}\,dx = \frac{1}{3}X^{3/2}a^2\sqrt{X} - a^3\ln|\frac{a+\sqrt{X}}{x}| + C$

310. $\displaystyle\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{3/2}}{x} - \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\sin^{-1}\frac{x}{a} + C$

311. $\displaystyle\int \frac{X^{3/2}}{x^3}\,dx = -\frac{X^{3/2}}{2x^2} - \frac{3}{2}\sqrt{X} + \frac{3}{2}a\ln|\frac{a+\sqrt{X}}{x}| + C$

312. $\displaystyle\int \frac{dx}{X^{3/2}} = \frac{x}{a^2\sqrt{X}} + C$

313. $\displaystyle\int \frac{x\,dx}{X^{3/2}} = \frac{1}{\sqrt{X}} + C$

314. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = \frac{x}{\sqrt{X}} - \sin^{-1}\frac{x}{a} + C$

315. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \sqrt{X} + \frac{a^2}{\sqrt{X}} + C$

316. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln|\frac{a+\sqrt{X}}{x}| + C$

317. $\int \frac{dx}{x^2 X^{3/2}} = -\frac{\sqrt{X}}{a^4 x} + \frac{x}{a^4 \sqrt{X}} + C$

318. $\int \frac{dx}{x^3 X^{3/2}} = -\frac{1}{2a^2 x^2 \sqrt{X}} + \frac{3}{2a^4 \sqrt{X}} - \frac{3}{2a^5} \ln|\frac{a+\sqrt{X}}{x}| + C$

Integrals Containing $X = x^3 + a^3$

319. $\int \frac{dx}{X} = \frac{1}{6a^2} \ln|\frac{(x+a)^3}{X}| + \frac{1}{\sqrt{3}a^2} \tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

320. $\int \frac{x\,dx}{X} = \frac{1}{6a} \ln|\frac{X}{(x+a)^3}| + \frac{1}{\sqrt{3}a} \tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

321. $\int \frac{x^2\,dx}{X} = \frac{1}{2} \ln|X| + C$

322. $\int \frac{dx}{xX} = \frac{1}{3a^3} \ln|\frac{x^3}{X}| + C$

323. $\int \frac{dx}{x^2 X} = -\frac{1}{a^2 x} - \frac{1}{6a^4} \ln|\frac{X}{(x+a)^3}| - \frac{1}{\sqrt{3}a^4} \tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

324. $\int \frac{dx}{X^2} = \frac{x}{3a^3 X} + \frac{1}{9a^5} \ln|\frac{(x+a)^3}{X}| + \frac{2}{3\sqrt{3}a^5} \tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

325. $\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^3 X} + \frac{1}{18a^4} \ln|\frac{X}{(x+a)^3}| + \frac{1}{3\sqrt{3}a^4} \tan^{-1}\left(\frac{2x-a}{\sqrt{3}a}\right) + C$

326. $\int \frac{x^2\,dx}{X^2} = -\frac{1}{3X} + C$

327. $\int \frac{dx}{xX^2} = \frac{1}{3a^2 X} + \frac{1}{3a^6} \ln|\frac{x^3}{X}| + C$

328. $\int \frac{dx}{x^2 X^2} = -\frac{1}{a^6 x} - \frac{x^2}{3a^6 X} - \frac{4}{3a^6} \int \frac{x\,dx}{X}$

329. $\int \frac{dx}{X^3} = \frac{1}{54a^3} \left[\frac{9a^5 x}{X^2} + \frac{15a^2 x}{X} + 10\sqrt{3} \tan^{-1}(\frac{2x-a}{\sqrt{3}a}) + 10 \ln|x+a| - 5 \ln|x^2 - ax + a^2|\right] + C$

Integrals containing $X = x^4 + a^4$

330. $\int \frac{dx}{X} = \frac{1}{4\sqrt{2}a^3} \ln|\frac{X}{(x^2 - \sqrt{2}ax + a^2)^2}| - \frac{1}{2\sqrt{2}a^3} \tan^{-1}\left(\frac{\sqrt{2}ax}{x^2 - a^2}\right) + C$

331. $\int \frac{x\,dx}{X} = \frac{1}{2a^2} \tan^{-1}\left(\frac{x^2}{a^2}\right) + C$

332. $\int \frac{x^2\,dx}{X} = \frac{1}{4\sqrt{2}a} \ln|\frac{X}{(x^2 + \sqrt{2}ax + a^2)^2}| - \frac{1}{2\sqrt{2}a} \tan^{-1}\left(\frac{\sqrt{2}ax}{x^2 - a^2}\right) + C$

333. $\int \frac{x^3\,dx}{X} = \frac{1}{4} \ln|X| + C$

334. $\int \frac{dx}{xX} = \frac{1}{4a^4} \ln|\frac{x^4}{X}| + C$

335. $\int \frac{dx}{x^2 X} = -\frac{1}{a^4 x} - \frac{1}{\sqrt{24}a^5} \ln|\frac{(x^2 - \sqrt{2}ax + a^2)^2}{X}| + \frac{1}{2\sqrt{2}a^5} \tan^{-1}\left(\frac{\sqrt{2}ax}{x^2 - a^2}\right) + C$

336. $\int \frac{dx}{x^3 X} = -\frac{1}{2a^4 x^2} - \frac{1}{2a^6} \tan^{-1}\left(\frac{x^2}{a^2}\right) + C$

Integrals containing $X = x^4 - a^4$

337. $\displaystyle\int \frac{dx}{X} = \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right| - \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + C$

338. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C$

339. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{1}{4a}\ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right) + C$

340. $\displaystyle\int \frac{x^3\,dx}{X} = \frac{1}{4}\ln|X| + C$

341. $\displaystyle\int \frac{dx}{xX} = \frac{1}{4a^4}\ln\left|\frac{X}{x^4}\right| + C$

342. $\displaystyle\int \frac{dx}{x^2X} = \frac{1}{a^4x} + \frac{1}{4a^5}\ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a^5}\tan^{-1}\left(\frac{x}{a}\right) + C$

343. $\displaystyle\int \frac{dx}{x^3X} = \frac{1}{2a^4x^2} + \frac{1}{4a^6}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C$

Miscelaneous algebraic integrals

344. $\displaystyle\int \frac{dx}{b^2+(x+a)^2} = \frac{1}{b}\tan^{-1}\frac{x+a}{b} + C$

345. $\displaystyle\int \frac{dx}{b^2-(x+a)^2} = \frac{1}{b}\tanh^{-1}\frac{x+a}{b} + C$

346. $\displaystyle\int \frac{dx}{(x+a)^2-b^2} = -\frac{1}{b}\coth^{-1}\frac{x+a}{b} + C$

347. $\displaystyle\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}} + C$

348. $\displaystyle\int \frac{dx}{\sqrt{x(a+x)}} = 2\sinh^{-1}\sqrt{\frac{x}{a}} + C$

349. $\displaystyle\int \frac{dx}{\sqrt{x(x-a)}} = 2\cosh^{-1}\sqrt{\frac{x}{a}} + C$

350. $\displaystyle\int \frac{dx}{(b+x)(a-x)} = 2\tan^{-1}\sqrt{\frac{b+x}{a-x}} + C, \qquad a > x$

351. $\displaystyle\int \frac{dx}{(x-b)(a-x)} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}} + C, \qquad a > x > b$

352. $\displaystyle\int \frac{dx}{(x+b)(x+a)} = \begin{cases} 2\tanh^{-1}\sqrt{\frac{x+b}{x+a}} + C_1, & a > b \\ 2\tanh^{-1}\sqrt{\frac{x+a}{x+b}} + C_2, & a < b \end{cases}$

353. $$\int \frac{dx}{x\sqrt{x^{2n}-a^{2n}}} = -\frac{1}{na^n}\sin^{-1}\left(\frac{a^n}{x^n}\right) + C$$

354. $$\int \sqrt{\frac{x+a}{x-a}}\,dx = \sqrt{x^2-a^2} + a\cosh^{-1}\frac{x}{a} + C$$

355. $$\int \sqrt{\frac{a+x}{a-x}}\,dx = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$$

356. $$\int x\sqrt{\frac{a-x}{a+x}}\,dx = \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{(x-2a)}{2}\sqrt{a^2-x^2} + C, \quad a > x$$

357. $$\int x\sqrt{\frac{a+x}{a-x}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{x+2a}{2}\sqrt{a^2-x^2} + C$$

358. $$\int (x+a)\sqrt{\frac{x+b}{x-b}}\,dx = (x+a+b)\sqrt{x^2-b^2} + \frac{b}{2}(2a+b)\cosh^{-1}\frac{x}{b} + C$$

359. $$\int \frac{dx}{\sqrt{2ax+x^2}} = \ln|x+a+\sqrt{2ax+x^2}| + C$$

360. $$\int \sqrt{ax^2+c}\,dx = \begin{cases} \frac{1}{2}x\sqrt{ax^2+c} + \frac{c}{2\sqrt{a}}\ln|\sqrt{a}x + \sqrt{ax^2+c}| + c, & a > 0 \\ \frac{1}{2}x\sqrt{ax^2+c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(\sqrt{\frac{-a}{c}}\,x\right) + C, & a < 0 \end{cases}$$

361. $$\int \sqrt{\frac{1+ax}{1-ax}}\,dx = \frac{1}{a}\sin^{-1}x - \frac{1}{a}\sqrt{1-x^2} + C$$

362. $$\int \frac{dx}{(ax+b)^2+(cx+d)^2} = \frac{1}{ad-bc}\tan^{-1}\left[\frac{(a^2+c^2)x+(ab+cd)}{ad-bc}\right] + C, \quad ad-bc \neq 0$$

363. $$\int \frac{dx}{(ax+b)^2-(cx+d)^2} = \frac{1}{2(bc-ad)}\ln\left|\frac{(a+c)x+(b+d)}{(a-c)x+(b-d)}\right| + C, \quad ad-bc \neq 0$$

364. $$\int \frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2} = \frac{1}{2(ad-bc)}\tan^{-1}\left[\frac{(a^2+c^2)x^2+(ab+cd)}{ad-bc}\right] + C, \quad ad-bc \neq 0$$

365. $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$$

366. $$\int \frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}\,dx = x + \frac{1}{d^2-c^2}\left[\frac{(a^2-c^2)(b^2-c^2)}{c}\tan^{-1}\frac{x}{c} - \frac{(a^2-d^2)(b^2-d^2)}{d}\tan^{-1}\frac{x}{d}\right] + C$$

367. $$\int \frac{ax^2+b}{(cx^2+d)(ex^2+f)}\,dx = \frac{1}{\sqrt{cd}}\left(\frac{ad-bc}{ed-fc}\right)\tan^{-1}\left(\sqrt{\frac{c}{d}}\,x\right) + \frac{1}{\sqrt{ef}}\left(\frac{af-be}{fc-ed}\right)\tan^{-1}\left(\sqrt{\frac{e}{f}}\,x\right) + C$$

368. $$\int \frac{x\,dx}{(ax^2+bx+c)^2+(ax^2-bx+c)^2} = \frac{1}{4b\sqrt{b^2+4ac}}\ln\left|\frac{2a^2x^2+2ac+b^2-b\sqrt{b^2+4ac}}{2a^2x^2+2ac+b^2+b\sqrt{b^2+4ac}}\right| + C, \quad b^2+4ac > 0$$

369. $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right) + C$$

370. $$\int \frac{(x^2+\alpha^2)(x^2+\beta^2)}{(x^2+\gamma^2)(x^2+\delta^2)}\,dx = x + \frac{1}{\delta^2-\gamma^2}\left[\frac{(\alpha^2-\gamma^2)(\beta^2-\gamma^2)}{\gamma}\tan^{-1}\frac{x}{\gamma} - \frac{(\alpha^2-\delta^2)(\beta^2-\delta^2)}{\delta}\tan^{-1}\frac{x}{\delta}\right] + C$$

371. $$\int \frac{\alpha x^2+\beta}{(\gamma x^2+\delta)(\epsilon x^2+\zeta)}\,dx = \frac{1}{\sqrt{\gamma\delta}}\frac{\alpha\delta-\beta\gamma}{\epsilon\delta-\zeta\gamma}\tan^{-1}\left(\sqrt{\frac{\gamma}{\delta}}\,x\right) + \frac{1}{\sqrt{\epsilon\zeta}}\frac{\alpha\zeta-\beta\epsilon}{\zeta\gamma-\epsilon\delta}\tan^{-1}\left(\sqrt{\frac{\epsilon}{\zeta}}\,x\right) + C$$

372. $$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \cosh^{-1}\left(\frac{2x+a+b}{a-b}\right) + C, \qquad a \neq b$$

373. $$\int \frac{dx}{\sqrt{(x-b)(a-x)}} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}} + C$$

374. $$\int \frac{dx}{(\alpha x+\beta)^2+(\gamma x+\delta)^2} = \frac{1}{\alpha\delta-\beta\gamma}\tan^{-1}\left[\frac{(\alpha^2+\gamma^2)x+(\alpha\beta+\gamma\delta)}{\alpha\delta-\beta\gamma}\right] + C$$

375. $$\int \frac{x\,dx}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{2ab}\sin^{-1}\left[\frac{(a^2+b^2)x^2-(a^4+b^4)}{(a^2-b^2)(a^2+b^2-x^2)}\right] + C$$

376. $$\int \frac{(x+b)\,dx}{(x^2+a^2)\sqrt{x^2+c^2}} = \frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{x^2+c^2}{x^2+a^2}} + \frac{b}{a\sqrt{a^2-c^2}}\cosh^{-1}\left[\frac{a}{c}\sqrt{\frac{x^2+c^2}{x^2+a^2}}\right] + C$$

377. $$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{p}{2a}\ln|ax^2+bx+c| + \left(q-\frac{pb}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

378. $$\int \frac{(\sqrt{a}-\sqrt{x})^2}{(a^2+ax+x^2)\sqrt{x}}\,dx = \frac{2\sqrt{3}}{\sqrt{a}}\tan^{-1}\frac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}} - \frac{2}{\sqrt{3a}}\tan^{-1}\frac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}} + C$$

379. $$\int (a+x)\sqrt{a^2+x^2}\,dx = \frac{1}{6}(2x^2+3ax+2a^2)\sqrt{a^2+x^2} + \frac{1}{2}a^2\sinh^{-1}\frac{x}{a} + C$$

380. $$\int \frac{x^2+a^2}{x^4+a^2x^2+a^4}\,dx = \frac{1}{a\sqrt{3}}\tan^{-1}\frac{ax\sqrt{3}}{a^2-x^2} + C$$

381. $$\int \frac{x^2-a^2}{x^4+a^2x^2+a^4}\,dx = \frac{1}{2a^3}\ln\frac{x^2-ax+a^2}{x^2+ax+a^2} + C$$

Integrals containing sin ax

382. $$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

383. $$\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$$

384. $$\int x^2\sin ax\,dx = \frac{2}{a^2}x\sin ax + \left(\frac{2}{a^3}-\frac{x^2}{a}\right)\cos ax + C$$

385. $\displaystyle\int x^3 \sin ax\, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax + \left(\frac{6x}{a} - \frac{x^3}{a}\right)\cos ax + C$

386. $\displaystyle\int x^n \sin ax\, dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a^2}x^{n-1}\sin ax - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\, dx$

387. $\displaystyle\int \frac{\sin ax}{x}\, dx = ax - \frac{a^3x^3}{3\cdot 3!} + \frac{a^5x^5}{5\cdot 5!} - \frac{a^7x^7}{7\cdot 7!} + \cdots + \frac{(-1)^n x^{2n+1}x^{2n+1}}{(2n+1)\cdot(2n+1)!} + \cdots$

388. $\displaystyle\int \frac{\sin ax}{x^2}\, dx = -\frac{1}{a}\sin ax + a\int \frac{\cos ax}{x}\, dx$

389. $\displaystyle\int \frac{\sin ax}{x^3}\, dx = -\frac{a}{2x}\cos ax - \frac{1}{2x^2}\sin ax - \frac{a^2}{2}\int \frac{\sin ax}{x}\, dx$

390. $\displaystyle\int \frac{\sin ax}{x^n}\, dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\, dx$

391. $\displaystyle\int \frac{dx}{\sin ax} = \frac{1}{a}\ln|\csc as - \cot ax| + C$

392. $\displaystyle\int \frac{x\, dx}{\sin ax} = \frac{1}{a^2}\left[ax + \frac{a^3x^3}{18} + \frac{7a^5x^5}{1800} + \cdots + \frac{2(2^{2n-1}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$

where $\mathfrak{B}_n$ is the n*th* Bernoulli number $\mathfrak{B}_1 = 1/6, \mathfrak{B}_2 = 1/30, \ldots$ Note scaling and shifting

393. $\displaystyle\int \frac{dx}{x\sin ax} = -\frac{1}{ax} + \frac{ax}{6} + \frac{7a^3x^3}{1080} + \cdots + \frac{2(2^{2n-1}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n-1)(2n)!} + \cdots + C$

394. $\displaystyle\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

395. $\displaystyle\int x\sin^2 ax\, dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2} + C$

396. $\displaystyle\int x^2\sin^2 ax\, dx = \frac{1}{6a} - \frac{1}{4a^2}\cos 2ax + \frac{1}{24a^3}(3 - 6a^2x^2)\sin 2ax + C$

397. $\displaystyle\int \sin^3 ax\, dx = -\frac{\cos ax}{a} + \frac{\cos^2 ax}{3a} + C$

398. $\displaystyle\int x\sin^3 ax\, dx = \frac{1}{12a}x\cos 3ax - \frac{1}{36a^2}\sin 3ax - \frac{3}{4a}x\cos ax + \frac{3}{4a^2}\sin ax + C$

399. $\displaystyle\int \sin^4 ax\, dx = \frac{3}{8}x - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a} + C$

400. $\displaystyle\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax + C$

401. $\displaystyle\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\cot ax + \frac{1}{a^2}\ln|\sin ax| + C$

402. $\displaystyle\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln|\tan\frac{ax}{2}| + C$

403. $\displaystyle\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{(n-1)a\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$

404. $\displaystyle\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

405. $\displaystyle\int \frac{dx}{a-\sin ax} = \frac{2}{a\sqrt{a^2-1}}\tan^{-1}\left[\frac{a\tan(ax/2)-1}{\sqrt{a^2-1}}\right] + C, \qquad a>1$

406. $\displaystyle\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)| + C$

407. $\displaystyle\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

408. $\displaystyle\int \frac{dx}{a+\sin ax} = \frac{2}{a\sqrt{a^2-1}}\tan^{-1}\left[\frac{1+a\tan(ax/2)}{\sqrt{a^2-1}}\right] + C, \qquad a>1$

409. $\displaystyle\int \frac{x\,dx}{1+\sin ax} = \frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)| + C$

410. $\displaystyle\int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$

411. $\displaystyle\int \frac{dx}{1-\sin^2 x} = \tan x + C$

412. $\displaystyle\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

413. $\displaystyle\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

414. $$\int \frac{dx}{\alpha+\beta\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\alpha\tan\dfrac{ax}{2} + \beta\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{a\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\alpha\tan\frac{ax}{2} + \beta - \sqrt{\beta^2-\alpha^2}}{\alpha\tan\frac{ax}{2} + \beta + \sqrt{\beta^2-\alpha^2}}\right| + C, & \alpha^2 < \beta^2 \\ \dfrac{1}{a\alpha}\tan\left(\dfrac{ax}{2} \pm \dfrac{\pi}{4}\right) + C, & \beta = \pm\alpha \end{cases}$$

415. $\displaystyle\int \frac{dx}{\alpha^2+\beta^2\sin^2 ax} = \frac{1}{a\alpha\sqrt{\beta^2+\alpha^2}}\tan^{-1}\left(\frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\tan ax\right) + C$

416. $$\int \frac{dx}{\alpha^2 - \beta^2 \sin^2 ax} = \begin{cases} \frac{1}{a\alpha\sqrt{\alpha^2-\beta^2}} \tan^{-1}\left(\frac{\sqrt{\alpha^2-\beta^2}}{\alpha}\tan ax\right) + C, & \alpha^2 > \beta^2 \\ \frac{1}{2a\alpha\sqrt{\beta^2-\alpha^2}} \ln\left|\frac{\sqrt{\beta^2-\alpha^2}\tan ax + \alpha}{\sqrt{\beta^2-\alpha^2}\tan ax - \alpha}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

417. $$\int \sin^n ax\, dx = -\frac{1}{an}\sin^{n-1} ax \cos ax + \frac{n-1}{n}\int \sin^{n-2} ax\, dx$$

418. $$\int \frac{dx}{\sin^n ax} = \frac{-\cos ax}{(n-1)a\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$$

419. $$\int x^n \sin ax\, dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax\, dx$$

420. $$\int \frac{\alpha + \beta\sin ax}{1 \pm \sin ax}\, dx = \beta x + \frac{\alpha \mp \beta}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + C$$

421. $$\int \frac{\alpha + \beta\sin ax}{a + b\sin ax}\, dx = \frac{\beta}{b}x + \frac{\alpha b - a\beta}{b}\int \frac{dx}{a + b\sin ax}$$

422. $$\int \frac{dx}{\alpha + \frac{\beta}{\sin ax}} = \frac{x}{\alpha} - \frac{\beta}{\alpha}\int \frac{dx}{\beta + \alpha\sin ax}$$

Integrals containing $\cos ax$

423. $$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$$

424. $$\int x\cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

425. $$\int x^2 \cos ax\, dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax + C$$

426. $$\int x^n \cos ax\, dx = \frac{1}{a}x^n \sin ax + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\, dx$$

427. $$\int \frac{\cos ax}{x}\, dx = \ln|x| - \frac{a^2x^2}{2\cdot 2!} + \frac{a^4x^4}{4\cdot 4!} - \frac{a^6x^6}{6\cdot 6!} + \cdots + \frac{(-1)^n a^{2n}x^{2n}}{(2n)\cdot(2n)!} + \cdots + C$$

428. $$\int \frac{\cos ax\, dx}{x^n} = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}}\, dx$$

429. $$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

430. $$\int \frac{x\, dx}{\cos ax} = \frac{1}{a^2}\left[\frac{a^2x^2}{2} + \frac{a^4x^4}{4\cdot 2!} + \frac{5a^6x^6}{6\cdot 4!} + \cdots + \frac{\mathfrak{E}_n a^{2n+2}x^{2n+2}}{(2n+2)\cdot(2n)!} + \cdots\right] + C$$

431. $$\int \frac{dx}{x\cos ax} = \ln|x| + \frac{a^2x^2}{4} + \frac{5a^4x^4}{96} + \cdots + \frac{\mathfrak{E}_n a^{2n}x^{2n}}{2n(2n)!} + \cdots + C$$

where $\mathfrak{E}_n$ is the nth Euler number $\mathfrak{E}_1 = 1, \mathfrak{E}_2 = 5, \mathfrak{E}_3 = 61, \ldots$ Note scaling and shifting

432. $\displaystyle\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

433. $\displaystyle\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

434. $\displaystyle\int \sqrt{1-\cos ax}\,dx = -2\sqrt{2}\cos\frac{ax}{2} + C$

435. $\displaystyle\int \sqrt{1+\cos ax}\,dx = 2\sqrt{2}\sin\frac{ax}{2} + C$

436. $\displaystyle\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

437. $\displaystyle\int x\cos^2 ax\,dx = \frac{x^2}{4} + \frac{1}{4a}x\sin 2ax + \frac{1}{8a^2}\cos 2ax + C$

438. $\displaystyle\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a} + C$

439. $\displaystyle\int \cos^4 ax\,dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax + C$

440. $\displaystyle\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax + C$

441. $\displaystyle\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a}\tan ax + \frac{1}{a^2}\ln|\cos ax| + C$

442. $\displaystyle\int \frac{dx}{\cos^3 ax} = \frac{1}{2a}\frac{\sin ax}{\cos^2 ax} + \frac{1}{2a}\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| + C$

443. $\displaystyle\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

444. $\displaystyle\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right| + C$

445. $\displaystyle\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

446. $\displaystyle\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right| + C$

447. $\displaystyle\int \frac{dx}{1+\cos^2 ax} = -\frac{1}{\sqrt{2}a}\tan^{-1}(\sqrt{2}\cot ax) + C$

448. $\displaystyle\int \frac{dx}{1-\cos^2 ax} = -\frac{1}{a}\cot ax + C$

449. $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2} + C$$

450. $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^2\frac{ax}{2} + C$$

451. $$\int \frac{dx}{\alpha+\beta\cos ax} = \begin{cases} \dfrac{2}{a\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\sqrt{\dfrac{\alpha-\beta}{\alpha+\beta}}\tan\dfrac{ax}{2}\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{a\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\sqrt{\beta+\alpha}+\sqrt{\beta-\alpha}\tan\frac{ax}{2}}{\sqrt{\beta+\alpha}-\sqrt{\beta-\alpha}\tan\frac{ax}{2}}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

452. $$\int \frac{dx}{\alpha+\frac{\beta}{\cos ax}} = \frac{x}{\alpha} - \frac{\beta}{\alpha}\int\frac{dx}{\beta+\alpha\cos ax}$$

453. $$\int \frac{dx}{(\alpha+\beta\cos ax)^2} = \frac{\alpha\sin ax}{a(\beta^2-\alpha^2)(\alpha+\beta\cos ax)} - \frac{\alpha}{\beta^2-\alpha^2}\int\frac{dx}{\alpha+\beta\cos ax}, \qquad \alpha\neq\beta$$

454. $$\int \frac{dx}{\alpha^2+\beta^2\cos^2 ax} = \frac{1}{a\alpha\sqrt{\alpha^2+\beta^2}}\tan^{-1}\left(\frac{\alpha\tan ax}{\sqrt{\alpha^2+\beta^2}}\right) + C$$

455. $$\int \frac{dx}{\alpha^2-\beta^2\cos^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\dfrac{\alpha\tan ax}{\sqrt{\alpha^2-\beta^2}}\right) + C, & \alpha^2 > \beta^2 \\ \dfrac{1}{2a\alpha\sqrt{\beta^2-\alpha^2}}\ln\left|\dfrac{\alpha\tan ax-\sqrt{\beta^2-\alpha^2}}{\alpha\tan ax+\sqrt{\beta^2-\alpha^2}}\right| + C, & \alpha^2 < \beta^2 \end{cases}$$

456. $$\int \frac{dx}{\cos^n ax} = \frac{\sec^{(n-2)} ax\tan ax}{(n-1)a} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx + C$$

Integrals containing both sine and cosine functions

457. $$\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax + C$$

458. $$\int \frac{dx}{\sin ax\cos ax} = -\frac{1}{a}\ln|\cot ax| + C$$

459. $$\int \sin ax\cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C, \qquad a\neq b$$

460. $$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C$$

461. $$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C$$

462. $$\int \sin^n ax\cos ax\,dx = \frac{sin^{n+1}ax}{(n+1)a} + C$$

463. $$\int \cos^n ax\sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C$$

464. $$\int \frac{\sin ax\,dx}{\cos ax} = \frac{1}{a}\ln|\sec ax| + C$$

465. $$\int \frac{\cos ax\,dx}{\sin ax} = \frac{1}{a}\ln|\sin ax| + C$$

466. $$\int \frac{x\sin ax\,dx}{\cos ax} = \frac{1}{a^2}\left[\frac{a^3x^3}{3} + \frac{a^5x^5}{5} + \frac{2a^7x^7}{105} + \cdots + \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!}\right] + C$$

467. $$\int \frac{x\cos ax\,dx}{\sin ax} = \frac{1}{a^2}\left[ax - \frac{a^3x^3}{9} - \frac{a^5x^5}{225} - \cdots - \frac{2^{2n}\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} - \cdots\right] + C$$

468. $$\int \frac{\cos ax\,dx}{x\sin ax} = -\frac{1}{ax} - \frac{ax}{2} - \frac{a^3x^3}{135} - \cdots - \frac{2^{2n}\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} - \cdots + C$$

469. $$\int \frac{\sin ax}{x\cos ax}\,dx = ax + \frac{a^3x^3}{9} + \frac{2a^5x^5}{75} + \cdots + \frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$$

470. $$\int \frac{\sin^2 ax}{\cos^2 ax}\,dx = \frac{1}{a}\tan ax - x + C$$

471. $$\int \frac{\cos^2 ax}{\sin^2 ax}\,dx = -\frac{1}{a}\cot ax - x + C$$

472. $$\int \frac{x\sin^2 ax}{\cos^2 ax}\,dx = \frac{1}{a}x\tan ax + \frac{1}{a^2}\ln|\cos ax| - \frac{1}{2}x^2 + C$$

473. $$\int \frac{x\cos^2 ax}{\sin^2 ax}\,dx = -\frac{1}{a}x\cot ax + \frac{1}{a^2}\ln|\sin ax| - \frac{1}{2}x^2 + C$$

474. $$\int \frac{\cos ax}{\sin ax}\,dx = \frac{1}{a}\ln|\sin ax| + C$$

475. $$\int \frac{\sin^3 ax}{\cos^3 ax}\,dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\ln|\cos ax| + C$$

476. $$\int \frac{\cos^3 ax}{\sin^3 ax}\,dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\ln|\sin ax| + C$$

477. $$\int \sin(ax+b)\sin(ax+\beta)\,dx = \frac{x}{2}\cos(b-\beta) - \frac{1}{4a}\sin(2ax+b+\beta) + C$$

478. $$\int \sin(ax+b)\cos(ax+\beta)\,dx = \frac{x}{2}\sin(b-\beta) - \frac{1}{4a}\cos(2ax+b+\beta) + C$$

479. $$\int \cos(ax+b)\cos(ax+\beta)\,dx = \frac{x}{2}\cos(b-\beta) + \frac{1}{4a}\sin(2ax+b+\beta) + C$$

480. $$\int \sin^2 ax\cos^2 bx\,dx = \begin{cases} \dfrac{x}{4} - \dfrac{\sin 2ax}{8a} + \dfrac{\sin 2bx}{8b} - \dfrac{\sin 2(a-b)x}{16(a-b)} - \dfrac{\sin 2(a+b)x}{16(a+b)} + C, & b \neq a \\ \dfrac{x}{8} - \dfrac{\sin 4ax}{32a} + C, & b = a \end{cases}$$

481. $$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \ln|\tan ax| + C$$

482. $$\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a} \ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| - \frac{1}{a \sin ax} + C$$

483. $$\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a} \ln\left|\tan \frac{ax}{2}\right| + \frac{1}{a \cos ax} + C$$

484. $$\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2 \cos 2ax}{a} + C$$

485. $$\int \frac{\sin^2 ax}{\cos ax}\, dx = -\frac{\sin ax}{a} + \frac{1}{a} \ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right| + C$$

486. $$\int \frac{\cos^2 ax}{\sin ax}\, dx = \frac{\cos ax}{a} + \frac{1}{a} \ln\left|\tan \frac{ax}{2}\right| + C$$

487. $$\int \frac{dx}{\cos ax\,(1 + \sin ax)} = \frac{1}{2a(1 + \sin ax)} \left[-1 + (1 + \sin ax) \ln\left|\frac{\cos \frac{ax}{2} + \sin \frac{ax}{2}}{\cos \frac{ax}{2} - \sin \frac{ax}{2}}\right|\right] + C$$

488. $$\int \frac{dx}{\sin ax\,(1 + \cos ax)} = \frac{1}{4a} \sec^2 \frac{ax}{2} + \frac{1}{2a} \ln\left|\tan \frac{ax}{2}\right| + C$$

489. $$\int \frac{dx}{\sin ax\,(\alpha + \beta \sin ax)} = \frac{1}{a\alpha} \ln\left|\tan \frac{ax}{2}\right| - \frac{\beta}{\alpha} \int \frac{dx}{\alpha + \beta \sin ax}$$

490. $$\int \frac{dx}{\cos ax\,(\alpha + \beta \sin ax} = \frac{1}{\alpha^2 - \beta^2} \left[\frac{\alpha}{a} \ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| - \frac{\beta}{\alpha} \ln\left|\frac{\alpha + \beta \sin ax}{\cos ax}\right|\right] + C, \qquad \beta \neq \alpha$$

491. $$\int \frac{dx}{\sin ax\,(\alpha + \beta \cos ax)} = \frac{1}{\alpha^2 - \beta^2} \left[\frac{\alpha}{a} \ln\left|\tan \frac{ax}{2}\right| + \frac{\beta}{a} \ln\left|\frac{\alpha + \beta \cos ax}{\sin ax}\right|\right] + C, \qquad \beta \neq \alpha$$

492. $$\int \frac{dx}{\cos ax\,(\alpha + \beta \cos ax)} = \frac{1}{a\alpha} \ln\left|\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| - \frac{\beta}{\alpha} \int \frac{dx}{\alpha + \beta \cos ax}$$

493. $$\int \frac{dx}{\alpha + \beta \cos ax + \gamma \sin ax} = \begin{cases} \dfrac{2}{a\sqrt{-R}} \tan^{-1}\left(\dfrac{\gamma + (\alpha - \beta) \tan \frac{ax}{2}}{\sqrt{-R}}\right) + C, & \begin{matrix}\alpha^2 > \beta^2 + \gamma^2 \\ R = \beta^2 + \gamma^2 - \alpha^2\end{matrix} \\ \dfrac{1}{a\sqrt{R}} \ln\left|\dfrac{\gamma - \sqrt{R} + (\alpha - \beta) \tan \frac{ax}{2}}{\gamma + \sqrt{R} + (\alpha - \beta) \tan \frac{ax}{2}}\right| + C, & \alpha^2 < \beta^2 + \gamma^2 \\ \dfrac{1}{a\beta} \ln\left|\beta + \gamma \tan \dfrac{ax}{2}\right| + C, & \alpha = \beta \\ \dfrac{1}{a\beta} \ln\left|\dfrac{cos \frac{ax}{2} + \sin \frac{ax}{2}}{(\beta + \gamma) \cos \frac{ax}{2} + (\gamma - \beta) \sin \frac{ax}{2}}\right| + C, & \alpha = \gamma \\ \dfrac{1}{a\gamma} \ln\left|1 + \tan \dfrac{ax}{2}\right| + C, & \alpha = \beta = \gamma \end{cases}$$

494. $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{\sqrt{2}a} \ln\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right| + C$$

495. $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \ln|\sin ax \pm \cos ax| + C$$

496. $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln|\sin axx \pm \cos ax| + C$$

497. $$\int \frac{\sin ax\,dx}{\alpha + \beta\sin ax} = \frac{1}{a\beta}\ln|\alpha + \beta\sin ax| + C$$

498. $$\int \frac{\cos ax\,dx}{\alpha + \beta\sin ax} = \frac{1}{a\beta}\ln|\alpha + \beta\sin ax| + C$$

499. $$\int \frac{\sin ax\cos ax\,dx}{\alpha^2\cos^2 ax + \beta^2 ax} = \frac{1}{2a(\beta^2 - \alpha^2)}\ln|\alpha^2\cos^2 ax + \beta^2\sin^2 ax| + C, \quad \beta \neq \alpha$$

500. $$\int \frac{dx}{\alpha^2\sin^2 ax + \beta^2\cos^2 ax} = \frac{1}{a\alpha\beta}\tan^{-1}\left(\frac{\alpha}{\beta}\tan ax\right) + C$$

501. $$\int \frac{dx}{\alpha^2\sin^2 ax - \beta^2\cos^2 ax} = \frac{1}{2a\alpha\beta}\ln\left|\frac{\alpha\tan ax - \beta}{\alpha\tan ax + \beta}\right| + C$$

502. $$\int \frac{\sin^n ax}{\cos^{(n+2} ax}\,dx = \frac{\tan^{n+1} ax}{(n+1)a} + C$$

503. $$\int \frac{\cos^n ax}{\sin^{(n+2)} ax}\,dx = -\frac{\cot^{(n+1)} ax}{(n+1)a} + C$$

504. $$\int \frac{dx}{\alpha + \beta\frac{\sin ax}{\cos ax}} = \frac{\alpha x}{\alpha^2 + \beta^2} + \frac{\beta}{a(\alpha^2 + \beta^2)}\ln|\beta\sin ax + \alpha\cos ax| + C$$

505. $$\int \frac{dx}{\alpha + \beta\frac{\cos ax}{\sin ax}} = \frac{\alpha x}{\alpha^2 + \beta^2} - \frac{\beta}{a(\alpha^2 + \beta^2)}\ln|\alpha\sin ax + \beta\cos ax| + C$$

506. $$\int \frac{\cos^n ax}{\sin^n ax}\,dx = -\frac{\cot^{(n-1)} ax}{(n-1)a} - \int \cot^{(n-2)} ax\,dx$$

507. $$\int \frac{\sin^n ax}{\cos^n ax}\,dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \frac{\sin^{n-2} ax}{\cos^{n-2} ax}\,dx$$

508. $$\int \frac{\sin ax}{\cos^{(n+1)} ax}\,dx = \frac{1}{na}\sec^n ax + C$$

509. $$\int \frac{\alpha\sin x + \beta\cos x}{\gamma\sin x + \delta\cos x}\,dx = \frac{[(\alpha\gamma + \beta\delta)x + (\beta\gamma - \alpha\gamma)\ln|\gamma\sin x + \delta\cos x|]}{\gamma^2 + \delta^2} + C$$

510. $$\int \frac{\alpha + \beta\sin x}{a + b\cos x}\,dx = \begin{cases} \frac{2\alpha}{\sqrt{a^2 - b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2} - \frac{\beta}{b}\ln|a + b\cos x| + C, & a > b \\ \frac{2\alpha}{\sqrt{b^2 - a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\frac{x}{2} - \frac{\beta}{b}\ln|a + b\cos x| + C, & a < b \end{cases}$$

511. $$\int \frac{dx}{a^2 - b^2\cos^2 x} = \begin{cases} \dfrac{1}{a\sqrt{a^2-b^2}}\tan^{-1}\left(\dfrac{a}{\sqrt{a^2-b^2}}\tan x\right) + C, & a > b \\ \frac{-1}{a\sqrt{b^2-a^2}}\tanh^{-1}\left(\frac{a}{\sqrt{b^2-a^2}}\tan x\right) + C, & b > a \end{cases}$$

512. $$\int \frac{dx}{(a\cos x + b\sin x)^2} = \frac{1}{a^2+b^2}\tan\left(x - \tan^{-1}\frac{b}{a}\right) + C$$

513. $$\int \frac{\sin x\,dx}{\sqrt{a\cos^2 x + 2b\cos x + c}} = \begin{cases} \dfrac{-1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{\sqrt{-a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a < 0 \\ \dfrac{-1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{\sqrt{a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a > 0 \\ \dfrac{-1}{\sqrt{a}}\cosh^{-1}\left(\dfrac{\sqrt{a(a\cos^2 x + 2b\cos x + c)}}{\sqrt{ac-b^2}}\right) + C, \ b^2 < ac, \ a > 0 \end{cases}$$

514. $$\int \frac{\cos x\,dx}{\sqrt{a\sin^2 x + 2b\sin x + c}} = \begin{cases} \dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{\sqrt{-a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a < 0 \\ \dfrac{1}{\sqrt{a}}\sinh^{-1}\left(\dfrac{\sqrt{a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{b^2-ac}}\right) + C, \ b^2 > ac, \ a > 0 \\ \dfrac{1}{\sqrt{a}}\cosh^{-1}\left(\dfrac{\sqrt{a(a\sin^2 x + 2b\sin x + c)}}{\sqrt{ac-b^2}}\right) + C, \ b^2 < ac, \ a > 0 \end{cases}$$

Integrals containing $\tan ax$, $\cot ax$, $\sec ax$, $\csc ax$

Write integrals in terms of $\sin ax$ and $\cos ax$ and see previous listings.

Integrals containing inverse trigonmetric functions

515. $$\int \sin^{-1}\frac{x}{a}\,dx = x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2} + C$$

516. $$\int \cos^{-1}\frac{x}{a}\,dx = x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C$$

517. $$\int \tan^{-1}\frac{x}{a}\,dx = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln|x^2+a^2| + C$$

518. $$\int \cot^{-1}\frac{x}{a}\,dx = x\cot^{-1}\frac{x}{a} + \frac{a}{2}\ln|x^2+a^2| + C$$

519. $$\int \sec^{-1}\frac{x}{a}\,dx = \begin{cases} x\sec^{-1}\dfrac{x}{a} - a\ln|x + \sqrt{x^2-a^2}| + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ x\sec^{-1}\dfrac{x}{a} + a\ln|x + \sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$$

520. $$\int \csc^{-1}\frac{x}{a}\,dx = \begin{cases} x\csc^{-1}\dfrac{x}{a} + a\ln|x + \sqrt{x^2-a^2}| + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ x\csc^{-1}\dfrac{x}{a} - a\ln|x + \sqrt{x^2-a^2}| + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$$

521. $$\int x\sin^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\sin^{-1}\frac{x}{a} + \frac{1}{4}x\sqrt{a^2-x^2} + C$$

522. $$\int x\cos^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\cos^{-1}\frac{x}{a} - \frac{1}{4}x\sqrt{a^2-x^2} + C$$

523. $$\int x\tan^{-1}\frac{x}{a}\,dx=\frac{1}{2}(x^2+a^2)\tan^{-1}\frac{x}{a}-\frac{a}{2}\ln|x^2+a^2|+C$$

524. $$\int x\cot^{-1}\frac{x}{a}\,dx=\frac{1}{2}(x^2+a^2)\cot^{-1}\frac{x}{a}+\frac{a}{2}x+C$$

525. $$\int x\sec^{-1}\frac{x}{a}\,dx=\begin{cases}\frac{1}{2}x^2\sec^{-1}\frac{x}{a}-\frac{a}{2}\sqrt{x^2-a^2}+C, & 0<\sec^{-1}\frac{x}{a}<\pi/2\\ \frac{1}{2}x^2\sec^{-1}\frac{x}{a}+\frac{a}{2}\sqrt{x^2-a^2}+C, & \pi/2<\sec^{-1}\frac{x}{a}<\pi\end{cases}$$

526. $$\int x\csc^{-1}\frac{x}{a}\,dx=\begin{cases}\frac{1}{2}x^2\csc^{-1}\frac{x}{a}+\frac{a}{2}\sqrt{x^2-a^2}+C, & 0<\csc^{-1}\frac{x}{a}<\pi/2\\ \frac{1}{2}x^2\csc^{-1}\frac{x}{a}-\frac{a}{2}\sqrt{x^2-a^2}+C, & -\pi/2<\csc^{-1}\frac{x}{a}<0\end{cases}$$

527. $$\int x^2\sin^{-1}\frac{x}{a}\,dx=\frac{1}{3}x^3\sin^{-1}\frac{x}{a}+\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$$

528. $$\int x^2\cos^{-1}\frac{x}{a}\,dx=\frac{1}{3}x^3\cos^{-1}\frac{x}{a}-\frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}+C$$

529. $$\int x^2\tan^{-1}\frac{x}{a}\,dx=\frac{1}{3}\tan^{-1}\frac{x}{a}-\frac{a}{6}x^2+\frac{a^3}{6}\ln|x^2+a^2|+C$$

530. $$\int x^2\cot^{-1}\frac{x}{a}\,dx=\frac{1}{3}\cot^{-1}\frac{x}{a}+\frac{a}{6}x^2-\frac{a^3}{6}\ln|a^2+x^2|+C$$

531. $$\int x^2\sec^{-1}\frac{x}{a}\,dx=\begin{cases}\frac{1}{3}x^3\sec^{-1}\frac{x}{a}-\frac{a}{6}x\sqrt{x^2-a^2}-\frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}|+C, & 0<\sec^{-1}\frac{x}{a}<\pi/2\\ \frac{1}{3}x^3\sec^{-1}\frac{x}{a}+\frac{a}{6}x\sqrt{x^2-a^2}+\frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}|+c, & \pi/2<\sec^{-1}\frac{x}{a}<\pi\end{cases}$$

532. $$\int x^2\csc^{-1}\frac{x}{a}\,dx=\begin{cases}\frac{1}{3}x^3\csc^{-1}\frac{x}{a}+\frac{a}{6}x\sqrt{x^2-a^2}+\frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}|+C, & 0<\csc^{-1}\frac{x}{a}<\pi/2\\ \frac{1}{3}x^3\csc^{-1}\frac{x}{a}-\frac{a}{6}x\sqrt{x^2-a^2}-\frac{a^3}{6}\ln|x+\sqrt{x^2-a^2}|+C, & -\pi/2<\csc^{-1}\frac{x}{a}<0\end{cases}$$

533. $$\int\frac{1}{x}\sin^{-1}\frac{x}{a}\,dx=\frac{x}{a}+\frac{1}{2\cdot3\cdot3}\left(\frac{x}{a}\right)^3+\frac{1\cdot3}{2\cdot4\cdot5\cdot5}\left(\frac{x}{a}\right)^5+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot7}+\cdots+C$$

534. $$\int\frac{1}{x}\cos^{-1}\frac{x}{a}\,dx=\frac{\pi}{2}\ln|x|+-\int\frac{1}{x}\sin^{-1}\frac{x}{a}\,dx$$

535. $$\int\frac{1}{x}\tan^{-1}\frac{x}{a}\,dx=\frac{x}{a}-\frac{1}{3^2}\left(\frac{x}{a}\right)^3+\frac{1}{5^2}\left(\frac{x}{a}\right)^5-\frac{1}{7^2}\left(\frac{x}{a}\right)^7+\cdots+C$$

536. $$\int\frac{1}{x}\cot^{-1}\frac{x}{a}\,dx=\frac{\pi}{2}\ln|x|-\int\frac{1}{x}\tan^{-1}\frac{x}{a}\,dx$$

537. $$\int\frac{1}{x}\sec^{-1}\frac{x}{a}\,dx=\frac{\pi}{2}\ln|x|+\frac{a}{x}+\frac{1}{2\cdot3\cdot3}\left(\frac{x}{a}\right)^3+\frac{1\cdot3}{2\cdot4\cdot5\cdot5}\left(\frac{x}{a}\right)^5+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot7}\left(\frac{x}{a}\right)^7+\cdots+C$$

538. $\displaystyle\int \frac{1}{x}\csc^{-1}\frac{x}{a}\,dx = -\left(\frac{a}{x} + \frac{1}{2\cdot 3\cdot 3}\left(\frac{x}{a}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\left(\frac{x}{a}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\left(\frac{x}{a}\right)^7 + \cdots\right) + C$

539. $\displaystyle\int \frac{1}{x^2}\sin^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\sin^{-1}\frac{x}{a} - \frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{a}\right| + C$

540. $\displaystyle\int \frac{1}{x^2}\cos^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\cos^{-1}\frac{x}{a} + \frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{a}\right| + C$

541. $\displaystyle\int \frac{1}{x^2}\tan^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\tan^{-1}\frac{x}{a} - \frac{1}{2a}\ln\left|\frac{x^2+a^2}{a^2}\right| + C$

542. $\displaystyle\int \frac{1}{x^2}\cot^{-1}\frac{x}{a}\,dx = -\frac{1}{x}\cot^{-1}\frac{x}{a} + \frac{1}{2a}\int \frac{1}{x}\tan^{-1}\frac{x}{a}\,dx$

543. $\displaystyle\int \frac{1}{x^2}\sec^{-1}\frac{x}{a}\,dx = \begin{cases} -\frac{1}{x}\sec^{-1}\frac{x}{a} + \frac{1}{ax}\sqrt{x^2-a^2} + C, & 0 < \sec^{-1}\frac{x}{a} < \pi/2 \\ -\frac{1}{x}\sec^{-1}\frac{x}{a} - \frac{1}{ax}\sqrt{x^2-a^2} + C, & \pi/2 < \sec^{-1}\frac{x}{a} < \pi \end{cases}$

544. $\displaystyle\int \frac{1}{x^2}\csc^{-1}\frac{x}{a}\,dx = \begin{cases} -\frac{1}{x}\csc^{-1}\frac{x}{a} - \frac{1}{ax}\sqrt{x^2-a^2} + C, & 0 < \csc^{-1}\frac{x}{a} < \pi/2 \\ -\frac{1}{x}\csc^{-1}\frac{x}{a} + \frac{1}{ax}\sqrt{x^2-a^2} + C, & -\pi/2 < \csc^{-1}\frac{x}{a} < 0 \end{cases}$

545. $\displaystyle\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = (a+x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

546. $\displaystyle\int \cos^{-1}\sqrt{\frac{x}{a+x}}\,dx = (2a+x)\tan^{-1}\sqrt{\frac{x}{2a}} - \sqrt{2ax} + C$

Integrals containing the exponential function

547. $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

548. $\displaystyle\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + C$

549. $\displaystyle\int x^2e^{ax}\,dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} + C$

550. $\displaystyle\int x^ne^{ax}\,dx = \frac{1}{a}x^ne^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

551. $\displaystyle\int \frac{1}{x}e^{ax}\,dx = \ln|x| + \frac{ax}{1\cdot 1!} + \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^3}{3\cdot 3!} + \cdots + C$

552. $\displaystyle\int \frac{1}{x^n}e^{ax}\,dx = -\frac{1}{(n-1)x^{n-1}}e^{ax} + \frac{a}{n-1}\int \frac{1}{x^{n-1}}e^{ax}\,dx$

553. $\displaystyle\int \frac{e^{ax}}{\alpha+\beta e^{ax}}\,dx = \frac{1}{a\beta}\ln|\alpha+\beta e^{ax}| + C$

554. $\displaystyle\int e^{ax}\sin bx\,dx = \left(\frac{a\sin bx - b\cos bx}{a^2+b^2}\right)e^{ax} + C$

555. $\displaystyle\int e^{ax}\cos bx\,dx = \left(\frac{a\cos bx + b\sin bx}{a^2+b^2}\right)e^{ax} + C$

556. $\displaystyle\int e^{ax}\sin^n bx\,dx = \left(\frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}\right)e^{ax}\sin^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2} bx\,dx$

557. $\displaystyle\int e^{ax}\cos^n bx\,dx = \left(\frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}\right)e^{ax}\cos^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2} bx\,dx$

Another way to express the above integrals is to define

$C_n = \displaystyle\int e^{ax}\cos^n bx\,dx$ and $S_n = \displaystyle\int e^{ax}\sin^n bx\,dx$, then one can write the reduction formulas

$$\begin{aligned} C_n =& \frac{a\cos bx + nb\sin bx}{a^2+n^2b^2}e^{ax}\cos^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}C_{n-2} \\ S_n =& \frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}e^{ax}\sin^{n-1} bx + \frac{n(n-1)b^2}{a^2+n^2b^2}S_{n-2} \end{aligned}$$

558. $\displaystyle\int xe^{ax}\sin bx\,dx = \left(\frac{[2ab - b(a^2+b^2)x]\cos bx + [a(a^2+b^2)x - a^2 + b^2]\sin bx}{(a^2+b^2)^2}\right)e^{ax} + C$

559. $\displaystyle\int xe^{ax}\cos bx\,dx = \left(\frac{[a(a^2+b^2)x - a^2 + b^2]\cos bx + [b(a^2+b^2)x - 2ab]\sin bx}{(a^2+b^2)^2}\right)e^{ax} + C$

560. $\displaystyle\int e^{ax}\ln x\,dx = \frac{1}{a}e^{ax}\ln x - \frac{1}{a}\int \frac{1}{x}e^{ax}\,dx$

561. $\displaystyle\int e^{ax}\sinh bx\,dx = \left[\frac{a\sinh bx - b\cosh bx}{(a-b)(a+b)}\right]e^{ax} + C, \qquad a \neq b$

562. $\displaystyle\int e^{ax}\sinh ax\,dx = \frac{1}{4a}e^{2ax} - \frac{x}{2} + C$

563. $\displaystyle\int e^{ax}\cosh bx\,dx = \left[\frac{a\cosh bx - b\sinh bx}{(a-b)(a+b)}\right]e^{ax} + C, \qquad a \neq b$

564. $\displaystyle\int e^{ax}\cosh ax\,dx = \frac{1}{4a}e^{2ax} + \frac{x}{2} + C$

565. $\displaystyle\int \frac{dx}{\alpha + \beta e^{ax}} = \frac{x}{\alpha} - \frac{1}{a\alpha}\ln|\alpha + \beta e^{ax}| + C$

566. $\displaystyle\int \frac{dx}{(\alpha + \beta e^{ax})^2} = \frac{x}{\alpha^2} + \frac{1}{a\alpha(\alpha+\beta e^{ax})} - \frac{1}{a\alpha^2}\ln|\alpha + \beta e^{ax}| + C$

567. $\displaystyle\int \frac{dx}{\alpha e^{ax} + \beta e^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\dfrac{\alpha}{\beta}}e^{ax}\right) + C, & \alpha\beta > 0 \\ \dfrac{1}{2a\sqrt{-\alpha\beta}}\ln\left|\dfrac{e^{ax} - \sqrt{-\beta/\alpha}}{e^{ax} + \sqrt{-\beta/\alpha}}\right| + C, & \alpha\beta < 0 \end{cases}$

568. $\displaystyle\int e^{ax}\sin^2 bx\,dx = \left(\frac{a^2+4b^2-a^2\cos(2bx)-2ab\sin(2bx)}{2a(a^2+4b^2}\right)e^{ax}+C$

569. $\displaystyle\int e^{ax}\cos^2 bx\,dx = \left(\frac{a^2+4b^2+a^2\cos(2bx)+2ab\sin(2bx)}{2a(a^2+4b^2}\right)e^{ax}+C$

Integrals containing the logarithmic function

570. $\displaystyle\int \ln x\,dx = x\ln|x|+C$

571. $\displaystyle\int x\ln x\,dx = \frac{1}{2}x^2\ln|x|-\frac{1}{4}x^2+C$

572. $\displaystyle\int x^n\ln x\,dx = \frac{1}{(n+1)^2}x^{n+1}+\frac{1}{n+1}x^{n+1}\ln|x|+C, \quad n\neq -1$

573. $\displaystyle\int \frac{1}{x}\ln x\,dx = \frac{1}{2}(\ln|x|)^2+C$

574. $\displaystyle\int \frac{dx}{x\ln x} = \ln|\ln|x||+C$

575. $\displaystyle\int \frac{1}{x^2}\ln x\,dx = -\frac{1}{x}-\frac{1}{x}\ln|x|+C$

576. $\displaystyle\int (\ln|x|)^2\,dx = x(\ln|x|)^2-2x\ln|x|+2x+C$

577. $\displaystyle\int \frac{1}{x}(\ln|x|)^n\;dx = \frac{1}{n+1}(\ln|x|)^{n+1}+C, \quad n\neq -1$

578. $\displaystyle\int (\ln|x|)^n\,dx = x(\ln|x|)^n-n\int(\ln|x|)^{n-1}\,dx$

579. $\displaystyle\int \ln|x^2+a^2|\,dx = x\ln|x^2+a^2|-2x+2a\tan^{-1}\frac{x}{a}+C$

580. $\displaystyle\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2|-2x+a\ln|\frac{x+a}{x-a}|+C$

581. $\displaystyle\int (ax+b)\ln(\beta x+\gamma)\,dx = \frac{\beta^2(ax+b)^2-(b\beta-a\gamma)^2}{2a\beta^2}\ln(\beta x+\gamma)-\frac{a}{4\beta^2}(\beta x+\gamma)^2-\frac{1}{\beta}(b\beta-a\gamma)x+C$

582. $\displaystyle\int (\ln ax)^2\,dx = x(\ln ax)^2-2x\ln ax+2x+C$

Integrals containing the hyperbolic function sinh *ax*

583. $\displaystyle\int \sinh ax\,dx = \frac{1}{a}\cosh ax+C$

584. $\int x \sinh ax \, dx = \frac{1}{a} x \cosh ax - \frac{1}{a^2} \sinh ax + C$

585. $\int x^2 \sinh ax \, dx = \left(\frac{x^2}{a} + \frac{2}{a^3} \right) \cosh ax - \frac{2x}{a^2} \sinh ax + C$

586. $\int x^n \sinh ax \, dx = \frac{1}{a} x^n \cosh ax - \frac{n}{a} \int x^{n-1} \cosh ax \, dx$

587. $\int \frac{1}{x} \sinh ax \, dx = ax + \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{4 \cdot 5!} + \cdots + C$

588. $\int \frac{1}{x^2} \sinh ax \, dx = -\frac{1}{x} \sinh ax + a \int \frac{1}{x} \cosh ax \, dx$

589. $\int \frac{1}{x^n} \sinh ax \, dx = -\frac{\sinh ax}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{1}{x^{n-1}} \cosh ax \, dx$

590. $\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln |\tanh \frac{ax}{2}| + C$

591. $\int \frac{x \, dx}{\sinh ax} = \frac{1}{a^2} \left[ax - \frac{(ax)^3}{18} + frac7(ax)^5 1800 + \cdots + (-1)^n \frac{2(2^{2n}-1)\mathfrak{B}_n \, a^{2n+1} x^{2n+1}}{(2n+1)!} + \cdots \right] + C$

592. $\int \sinh^2 ax \, dx = \frac{1}{2a} x \sinh 2ax - \frac{1}{2} x + C$

593. $\int \sinh^n ax \, dx = \frac{1}{na} \sinh^{n-1} ax \cosh ax - \frac{n-1}{n} \int \sinh^{n-2} ax \, dx$

594. $\int x \sinh^2 ax \, dx = \frac{1}{4a} x \sinh 2ax - \frac{1}{8a^2} \cosh 2ax - \frac{1}{4} x^2 + C$

595. $\int \frac{dx}{\sinh^2 ax} = -\frac{1}{a} \coth ax + C$

596. $\int \frac{dx}{\sinh^3 ax} = -\frac{1}{2a} \operatorname{csch} ax \coth ax - \frac{1}{2a} \ln |\tanh \frac{ax}{2}| + C$

597. $\int \frac{x \, dx}{\sinh^2 ax} = -\frac{1}{a} x \coth ax + \frac{1}{a^2} \ln |\sinh ax| + C$

598. $\int \sinh ax \sinh bx \, dx = \frac{1}{2(a+b)} \sinh(a+b)x - \frac{1}{2(a-b)} \sinh(a-b)x + C$

599. $\int \sinh ax \sin bx \, dx = \frac{1}{a^2+b^2} [a \cosh ax \sin bx - b \sinh ax \cos bx] + C$

600. $\int \sinh ax \cos bx \, dx = \frac{1}{a^2+b^2} [a \cosh ax \cos bx + b \sinh ax \sin bx] + C$

601. $$\int \frac{dx}{\alpha + \beta \sinh ax} = \frac{1}{\sqrt{\alpha^2+\beta^2}} \ln \left| \frac{\beta e^{ax} + \alpha - \sqrt{\alpha^2+\beta^2}}{\beta e^{ax} + \alpha + \sqrt{\alpha^2+\beta^2}} \right| + C$$

602. $$\int \frac{dx}{(\alpha + \beta \sinh ax)^2} = \frac{-\beta}{a(\alpha^2+\beta^2)} \frac{\cosh ax}{\alpha + \beta \sinh ax} + \frac{\alpha}{\alpha^2+\beta^2} \int \frac{dx}{\alpha + \beta \sinh ax}$$

603. $$\int \frac{dx}{\alpha^2 + \beta^2 \sinh^2 ax} = \begin{cases} \dfrac{1}{a\alpha\sqrt{\beta^2-\alpha^2}} \tan^{-1}\left(\dfrac{\sqrt{\beta^2-\alpha^2}\tanh ax}{\alpha}\right) + C, & \beta^2 > \alpha^2 \\ \dfrac{1}{2a\alpha\sqrt{\alpha^2-\beta^2}} \ln \left| \dfrac{\alpha + \sqrt{\alpha^2-\beta^2}\tanh ax}{\alpha - \sqrt{\alpha^2-\beta^2}\tanh ax} \right| + C, & \beta^2 < \alpha^2 \end{cases}$$

604. $$\int \frac{dx}{\alpha^2 - \beta^2 \sinh^2 ax} = \frac{1}{2a\alpha\sqrt{\alpha^2+\beta^2}} \ln \left| \frac{\alpha + \sqrt{\alpha^2+\beta^2}\tanh ax}{\alpha - \sqrt{\alpha^2+\beta^2}\tanh ax} \right| + C$$

Integrals containing the hyperbolic function cosh *ax*

605. $$\int \cosh ax\, dx = \frac{1}{a} \sinh ax + C$$

606. $$\int x \cosh ax\, dx = \frac{1}{a} x \sinh ax - \frac{1}{a^2} \cosh ax + C$$

607. $$\int x^2 \cosh ax\, dx = -\frac{2}{a^2} x \cosh ax + \left(\frac{x^2}{a} + \frac{2}{a^3}\right) \sinh ax + C$$

608. $$\int x^n \cosh ax\, dx = \frac{1}{a} x^n \sinh ax - \frac{n}{a} \int x^{n-1} \sinh ax\, dx$$

609. $$\int \frac{1}{x} \cosh ax\, dx = \ln|x| + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} + \frac{(ax)^6}{6 \cdot 6!} + \cdots + C$$

610. $$\int \frac{1}{x^2} \cosh ax\, dx = -\frac{1}{x} \cosh ax + a \int \frac{1}{x} \sinh ax\, dx$$

611. $$\int \frac{1}{x^n} \cosh ax\, dx = -\frac{1}{n-1} \frac{\cosh ax}{x^{n-1}} + \frac{a}{n-1} \int \frac{\sinh ax}{x^{n-1}}\, dx, \qquad n > 1$$

612. $$\int \frac{dx}{\cosh ax} = \frac{2}{a} \tan^{-1} e^{ax} + C$$

613. $$\int \frac{x\, dx}{\cosh ax} = \frac{1}{a^2} \left[\frac{a^2x^2}{2} - \frac{a^4x^4}{8} + \frac{5a^6x^6}{144} + \cdots + (-1)^n \frac{\mathfrak{E}_n a^{2n+2} x^{2n+2}}{(2n+2)\cdot(2n)!} + \cdots \right] + C$$

614. $$\int \cosh^2 ax\, dx = \frac{1}{2} x + \frac{1}{2} \sinh ax \cosh ax + C$$

615. $$\int \cosh^n ax\, dx = \frac{1}{na} \cosh^{n-1} ax \sinh ax + \frac{n-1}{n} \int \cosh^{n-2} ax\, dx$$

616. $$\int x \cosh^2 ax\, dx = \frac{1}{4} x^2 + \frac{1}{4a} x \sinh 2ax - \frac{1}{8a^2} \cosh 2ax + C$$

617. $\displaystyle\int \frac{dx}{\cosh^2 ax} = \frac{1}{a}\tanh ax + C$

618. $\displaystyle\int \frac{x\,dx}{\cosh^2 ax} = \frac{1}{a}x\tanh ax - \frac{1}{a^2}\ln|\cosh ax| + C$

619. $\displaystyle\int \frac{dx}{\cosh^n ax} = \frac{1}{(n-1)a}\frac{x\sinh ax}{\cosh^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} ax, \quad n>1}$

620. $\displaystyle\int \cosh ax\cosh bx\,dx = \frac{1}{2(a-b)}\sinh(a-b)x + \frac{1}{2(a+b)}\sinh(a+b)x + C$

621. $\displaystyle\int \cosh ax\sin bx\,dx = \frac{1}{a^2+b^2}\left[a\sinh ax\sin bx - b\cosh ax\cos bx\right] + C$

622. $\displaystyle\int \cosh ax\cos bx\,dx = \frac{1}{a^2+b^2}\left[a\sinh ax\cos bx + b\cosh ax\sin bx\right] + C$

623. $$\int \frac{dx}{\alpha+\beta\cosh ax} = \begin{cases} \dfrac{2}{\sqrt{\beta^2-\alpha^2}}\tan^{-1}\dfrac{\beta e^{ax}+\alpha}{\sqrt{\beta^2-\alpha^2}} + C, & \beta^2>\alpha^2 \\[2ex] \dfrac{1}{a\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\beta e^{ax}+\alpha-\sqrt{\alpha^2-\beta^2}}{\beta e^{ax}+\alpha+\sqrt{\alpha^2-\beta^2}}\right| + C, & \beta^2<\alpha^2 \end{cases}$$

624. $\displaystyle\int \frac{dx}{1+\cosh ax} = \frac{1}{a}\tanh ax + C$

625. $\displaystyle\int \frac{x\,dx}{1+\cosh ax} = \frac{x}{a}\tanh\frac{ax}{2} - \frac{2}{a^2}\ln|\cosh\frac{ax}{2}| + C$

626. $\displaystyle\int \frac{dx}{-1+\cosh ax} = -\frac{1}{a}\coth\frac{ax}{2} + C$

627. $\displaystyle\int \frac{dx}{(\alpha+\beta\cosh ax)^2} = \frac{\beta\sinh ax}{a(\beta^2-\alpha^2)(\alpha+\beta\cosh ax)} - \frac{\alpha}{\beta^2-\alpha^2}\int \frac{dx}{\alpha+\beta\cosh ax}$

628. $$\int \frac{dx}{\alpha^2-\beta^2\cosh^2 ax} = \begin{cases} \dfrac{1}{2a\alpha\sqrt{\alpha^2-\beta^2}}\ln\left|\dfrac{\alpha\tanh ax+\sqrt{\alpha^2-\beta^2}}{\alpha\tanh ax-\sqrt{\alpha^2-\beta^2}}\right| + C, & \alpha^2>\beta^2 \\[2ex] \dfrac{-1}{a\alpha\sqrt{\beta^2-\alpha^2}}\tan^{-1}\dfrac{\alpha\tanh ax}{\sqrt{\beta^2-\alpha^2}} + C, & \alpha^2<\beta^2 \end{cases}$$

629. $\displaystyle\int \frac{dx}{\alpha^2+\beta^2\cosh^2 ax} = \frac{1}{a\alpha\sqrt{\alpha^2+\beta^2}}\tanh^{-1}\left(\frac{\alpha\tanh ax}{\sqrt{\alpha^2+\beta^2}}\right) + C$

Integrals containing the hyperbolic functions $\sinh ax$ and $\cosh ax$

630. $\displaystyle\int \sinh ax\cosh ax\,dx = \frac{1}{2a}\sinh^2 ax + C$

631. $\displaystyle\int \sinh ax\cosh bx\,dx = \frac{1}{2(a+b)}\cosh(a+b)x + \frac{1}{2(a-b)}\cosh(a-b)x + C$

632. $\displaystyle\int \sinh^2 ax \cosh^2 ax\,dx = \frac{1}{32a}\sinh 4ax - \frac{1}{8}x + C$

633. $\displaystyle\int \sinh^n ax \cosh ax\,dx = \frac{1}{(n+1)a}\sinh^{n+1} ax + C, \quad n \neq -1$

634. $\displaystyle\int \cosh^n ax \sinh ax\,dx = \frac{1}{(n+1)a}\cosh^{n+1} ax + C, \quad n \neq -1$

635. $\displaystyle\int \frac{\sinh ax}{\cosh ax}\,dx = \frac{1}{a}\ln|\cosh ax| + C$

636. $\displaystyle\int \frac{\cosh ax}{\sinh ax}\,dx = \frac{1}{a}\ln|\sinh ax| + C$

637. $\displaystyle\int \frac{dx}{\sinh ax \cosh ax} = \frac{1}{a}\ln|\tanh ax| + C$

638. $\displaystyle\int \frac{x\sinh ax}{\cosh ax}\,dx = \frac{1}{a^2}\left[\frac{a^3x^3}{3} - \frac{a^5x^5}{15} + \cdots + (-1)^n\frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$

639. $\displaystyle\int \frac{x\cosh ax}{\sinh ax}\,dx = \frac{1}{a^2}\left[ax + \frac{a^3x^3}{9} - \frac{a^5x^5}{225} + \cdots + (-1)^{n-1}\frac{2^{2n}\mathfrak{B}_n a^{2n+1}x^{2n+1}}{(2n+1)!} + \cdots\right] + C$

640. $\displaystyle\int \frac{\sinh^2 ax}{\cosh^2 ax}\,dx = x - \frac{1}{a}\tanh ax + C$

641. $\displaystyle\int \frac{\cosh^2 ax}{\sinh^2 ax}\,dx = x - \frac{1}{a}\coth ax + C$

642. $\displaystyle\int \frac{x\sinh^2 ax}{\cosh^2 ax}\,dx = \frac{1}{2}x^2 - \frac{1}{a}x\tanh ax + \frac{1}{a^2}\ln|\cosh ax| + C$

643. $\displaystyle\int \frac{x\cosh^2 ax}{\sinh^2 ax}\,dx = \frac{1}{2}x^2 - \frac{1}{a}x\coth ax + \frac{1}{a^2}\ln|\sinh ax| + C$

644. $\displaystyle\int \frac{\sinh ax}{x\cosh ax}\,dx = ax - \frac{a^3x^3}{9} + \cdots + (-1)^{n-1}\frac{2^{2n}(2^{2n}-1)\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$

645. $\displaystyle\int \frac{\cosh ax}{x\sinh ax}\,dx = -\frac{1}{ax} + \frac{ax}{3} - \frac{a^3x^3}{135} + \cdots + (-1)^n\frac{2^{2n}\mathfrak{B}_n a^{2n-1}x^{2n-1}}{(2n-1)(2n)!} + \cdots + C$

646. $\displaystyle\int \frac{\sinh^3 ax}{\cosh^3 ax}\,dx = \frac{1}{a}\ln|\cosh ax| - \frac{1}{2a}\tanh^2 ax + C$

647. $\displaystyle\int \frac{\cosh^3 ax}{\sinh^3 ax}\,dx = \frac{1}{a}\ln|\sinh ax| - \frac{1}{2a}\coth^2 ax + C$

648. $\displaystyle\int \frac{dx}{\sinh ax \cosh^2 ax} = \frac{1}{a}\operatorname{sech} ax + \frac{1}{a}\ln\tanh\frac{ax}{2}| + C$

649. $$\int \frac{dx}{\sinh^2 ax \cosh ax} = -\frac{1}{a}\tan^{-1}(\sinh ax) - \frac{1}{a}\operatorname{csch} ax + C$$

650. $$\int \frac{dx}{\sinh^2 ax \cosh^2 ax} = -\frac{2}{a}\coth ax + C$$

651. $$\int \frac{\sinh^2 ax}{\cosh ax}\,dx = \frac{1}{a}\sinh ax - \frac{1}{a}\tan^{-1}(\sinh ax) + C$$

652. $$\int \frac{\cosh^2 ax}{\sinh ax}\,dx = \frac{1}{a}\cosh ax + \frac{1}{a}\ln|\tanh\frac{ax}{2}| + C$$

653. $$\int \frac{dx}{\cosh ax\,(1+\sinh ax)} = \frac{1}{2a}\ln\left|\frac{1+\sinh ax}{\cosh ax}\right| + \frac{1}{a}\tan^{-1} e^{ax} + C$$

654. $$\int \frac{dx}{\sinh ax\,(\cosh ax + 1)} = \frac{1}{2a}\ln|\tanh\frac{ax}{2}| + \frac{1}{2a(\cosh ax + 1)} + C$$

655. $$\int \frac{dx}{\sinh ax\,(\cosh ax - 1)} = -\frac{1}{2a}\ln|\tanh\frac{ax}{2}| - \frac{1}{2a(\cosh ax - 1)} + C$$

656. $$\int \frac{dx}{\alpha + \beta\frac{\sinh ax}{\cosh ax}} = \frac{\alpha x}{\alpha^2-\beta^2} - \frac{\beta}{a(\alpha^2-\beta^2)}\ln|\beta\sinh ax + \alpha\cosh ax| + C$$

657. $$\int \frac{dx}{\alpha + \beta\frac{\cosh ax}{\sinh ax}} = \frac{\alpha x}{\alpha^2-\beta^2} + \frac{\beta}{a(\alpha^2-\beta^2)}\ln|\alpha\sinh ax + \beta\cosh ax| + C$$

658. $$\int \frac{dx}{b\cosh ax + c\sinh ax} = \begin{cases} \frac{1}{a\sqrt{b^2-c^2}}\sec^{-1}\left[\frac{b\cosh ax + c\sinh ax}{\sqrt{b^2-c^2}}\right] + C, & b^2 > c^2 \\ \frac{-1}{a\sqrt{c^2-b^2}}\operatorname{csch}^{-1}\left[\frac{b\cosh ax + c\sinh ax}{\sqrt{c^2-b^2}}\right] + C, & b^2 < c^2 \end{cases}$$

Integrals containing the hyperbolic functions tanh ax, coth ax, sech ax, csch ax

Express integrals in terms of $\sinh ax$ and $\cosh ax$ and see previous listings.

Integrals containing inverse hyperbolic functions

659. $$\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2} + C$$

660. $$\int \cosh^{-1}\frac{x}{a}\,dx = \begin{cases} x\cosh^{-1}(x/a) - \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) > 0 \\ x\cosh^{-1}(x/a) + \sqrt{x^2-a^2}, & \cosh^{-1}(x/a) < 0 \end{cases}$$

661. $$\int \tanh^{-1}\frac{x}{a}\,dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2| + C$$

662. $$\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\ln|x^2-a^2| + C$$

663. $$\int \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} x\operatorname{sech}^{-1}\frac{x}{a} + a\sin^{-1}\frac{x}{a} + C, & \operatorname{sech}^{-1}(x/a) > 0 \\ x\operatorname{sech}^{-1}\frac{x}{a} - a\sin^{-1}\frac{x}{a} + C, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$$

664. $\displaystyle\int \operatorname{csch}^{-1}\frac{x}{a}\,dx = x\operatorname{csch}^{-1}\frac{x}{a} \pm a\sinh^{-1}\frac{x}{a}, \quad + \text{ for } x>0 \text{ and } - \text{ for } x<0$

665. $\displaystyle\int x\sinh^{-1}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{1}{4}xx\sqrt{x^2+a^2}+C$

666. $\displaystyle\int x\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{4}(2x^2-a^2)\cosh^{-1}\frac{x}{a} - \frac{1}{4}x\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)>0 \\ \frac{1}{4}(2x^2-a^2)\cosh^{-1}\frac{x}{a} + \frac{1}{4}x\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)<0 \end{cases}$

667. $\displaystyle\int x\tanh^{-1}\frac{x}{a}\,dx = \frac{ax}{2}+\frac{1}{2}(x^2-a^2)\tanh^{-1}\frac{x}{a}+C$

668. $\displaystyle\int x\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2}+\frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a}+C$

669. $\displaystyle\int x\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}x^2\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{2}a\sqrt{a^2-x^2}, & \operatorname{sech}^{-1}(x/a)>0 \\ \frac{1}{2}x\operatorname{sech}^{-1}\frac{x}{a} + \frac{1}{2}a\sqrt{a^2-x^2}+C, & \operatorname{sech}^{-1}(x/a)<0 \end{cases}$

670. $\displaystyle\int x\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{2}x^2\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{2}\sqrt{x^2+a^2}+C, \quad + \text{ for } x>0 \text{ and } - \text{ for } x<0$

671. $\displaystyle\int x^2\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\sinh^{-1}\frac{x}{a} + \frac{1}{9}(2a^2-x^2)\sqrt{x^2+a^2}+C$

672. $\displaystyle\int x^2\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{3}x^3\cosh^{-1}\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)>0 \\ \frac{1}{3}x^3\cosh^{-1}\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{x^2-a^2}+C, & \cosh^{-1}(x/a)<0 \end{cases}$

673. $\displaystyle\int x^2\tanh^{-1}\frac{x}{a}\,dx = \frac{a}{6}x^2+\frac{1}{3}x^3\tanh^{-1}\frac{x}{a}+\frac{1}{6}a^3\ln|a^2-x^2|+C$

674. $\displaystyle\int x^2\coth^{-1}\frac{x}{a}\,dx = \frac{a}{6}x^2+\frac{1}{3}x^3\coth^{-1}\frac{x}{a}+\frac{1}{6}a^3\ln|x^2-a^2|+C$

675. $\displaystyle\int x^2\operatorname{sech}^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\operatorname{sech}^{-1}\frac{x}{a} - \frac{1}{3}\int\frac{x^3\,dx}{\sqrt{x^2+a^2}}$

676. $\displaystyle\int x^2\operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{3}x^3\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{3}\int\frac{x^2\,dx}{\sqrt{x^2+a^2}}$

677. $\displaystyle\int x^n\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\sinh^{-1}\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{x^2-a^2}}$

678. $\displaystyle\int x^n\cosh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{n+1}x^{n+1}\cosh^{-1}\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}}{\sqrt{x^2-a^2}}, & \cosh^{-1}(x/a)>0 \\ \frac{1}{n+1}x^{n+1}\cosh^{-1}\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{x^2-a^2}}, & \cosh^{-1}(x/a)<0 \end{cases}$

679. $\displaystyle\int x^n\tanh^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\tanh^{-1}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2-x^2}$

680. $\int x^n \coth^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\coth^{-1}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2-x^2}$

681. $\int x^n \operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{n+1}x^{n+1}\operatorname{sech}^{-1}\frac{x}{a} + \frac{a}{n+1}\int\frac{x^n\,dx}{\sqrt{a^2-x^2}}, & \operatorname{sech}^{-1}(x/a) > 0 \\ \frac{1}{n+1}x^{n+1}\operatorname{sech}^{-1}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^n\,dx}{\sqrt{a^2-x^2}}, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

682. $\int x^n \operatorname{csch}^{-1}\frac{x}{a}\,dx = \frac{1}{n+1}x^{n+1}\operatorname{csch}^{-1}\frac{x}{a} \pm \frac{a}{n+1}\int\frac{x^n\,dx}{\sqrt{x^2+a^2}},\quad + \text{ for } x>0,\ - \text{ for } x<0$

683. $\int \frac{1}{x}\sinh^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{x}{a} - \frac{(x/a)^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3(x/a)^5}{2\cdot 4\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5(x/a)^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots + C, & |x| > a \\ \frac{1}{2}\left(\ln|\frac{2x}{a}|\right)^2 - \frac{(a/x)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} - \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots + C, & x > a \\ -\frac{1}{2}\left(\ln|\frac{-2x}{a}|\right)^2 + \frac{(a/x)^2}{2\cdot 2\cdot 2} - \frac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots + C, & x < -a \end{cases}$

684. $\int \frac{1}{x}\cosh^{-1}\frac{x}{a}\,dx = \pm\left[\frac{1}{2}\left(\ln|\frac{2x}{a}|\right)^2 + \frac{(a/x)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(a/x)^4}{2\cdot 4\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5(a/x)^6}{2\cdot 4\cdot 6\cdot 6\cdot 6} + \cdots\right] + C$

$+ \text{ for } \cosh^{-1}(x/a) > 0,\ - \text{ for } \cosh^{-1}(x/a) < 0$

685. $\int \frac{1}{x}\tanh^{-1}\frac{x}{a}\,dx = \frac{x}{a} + \frac{(x/a)^3}{3^2} + \frac{(x/a)^5}{5^2} + \cdots + C$

686. $\int \frac{1}{x}\coth^{-1}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{1}{2}(x^2-a^2)\coth^{-1}\frac{x}{a} + C$

687. $\int \frac{1}{x}\operatorname{sech}^{-1}\frac{x}{a}\,dx = \begin{cases} -\frac{1}{2}\ln|\frac{a}{x}|\ \ln|\frac{4a}{x}| - \frac{(x/a)^2}{2\cdot 2\cdot 2} - \frac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots + C, & \operatorname{sech}^{-1}(x/a) > 0 \\ \frac{1}{2}\ln|\frac{a}{x}|\ \ln|\frac{4a}{x}| + \frac{(x/a)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots, & \operatorname{sech}^{-1}(x/a) < 0 \end{cases}$

688. $\int \frac{1}{x}\operatorname{csch}^{-1}\frac{x}{a}\,dx = \begin{cases} \frac{1}{2}\ln|\frac{x}{a}|\ \ln|\frac{4a}{x}| + \frac{(x/a)^2}{2\cdot 2\cdot 2} - \frac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} + \cdots + C, & 0 < x < a \\ \frac{1}{2}\ln|\frac{-x}{a}|\ \ln|\frac{-x}{4a}| - \frac{(x/a)^2}{2\cdot 2\cdot 2} + \frac{1\cdot 3(x/a)^4}{2\cdot 4\cdot 4\cdot 4} - \cdots, & -a < x < 0 \\ -\frac{a}{x} + \frac{(a/x)^3}{2\cdot 3\cdot 3} - \frac{1\cdot 3(a/x)^5}{2\cdot 4\cdot 5\cdot 5} + \cdots + C, & |x| > a \end{cases}$

Integrals evaluated by reduction formula

689. If $S_n = \int \sin^n x\,dx$, then $S_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}S_{n-2}$

690. If $C_n = \int \cos^n x\,dx$, then $C_n = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}C_{n-2}$

691. If $I_n = \int \frac{\sin^n ax}{\cos ax}\,dx$, then $I_n = \frac{-1}{(n-1)\,a}\sin^{n-1}ax + I_{n-2}$

692. If $I_n = \int \frac{\cos^n ax}{\sin ax}\,dx$, then $I_n = \frac{1}{(n-1)\,a}\cos^{n-1}ax + I_{n-2}$

693. If $S_m = \int x^m \sin nx\,dx$ and $C_m = \int x^m \cos nx\,dx$, then

$$S_m = \frac{-1}{n}x^m \cos nx + \frac{m}{n}C_{m-1} \qquad \text{and} \qquad C_m = \frac{1}{n}x^m \sin nx - \frac{m}{n}S_{m-1}$$

694. If $I_1 = \int \tan x\,dx$, and $I_n = \int \tan^n x\,dx$, then $I_n = \dfrac{1}{n-1}\tan^{n-1} x - I_{n-2}, \quad n = 2, 3, 4, \ldots$

695. If $I_n = \int \dfrac{\sin^n ax}{\cos ax}\,dx$, then $I_n = -\dfrac{\sin^{n-1} ax}{(n-1)a} + I_{n-2}$

696. If $I_n = \int \dfrac{\cos^n ax}{\sin ax}\,dx$, then $I_n = \dfrac{\cos^{n-1} ax}{(n-1)a} + I_{n-2}$

697. If $I_{n,m} = \int \sin^n x \cos^m x\,dx$, then

$$\begin{aligned}
I_{n,m} &= \frac{-1}{n+m}\sin^{n-1} x \cos^{m+1} x + \frac{n-1}{n+m}I_{n-2,m} \\
I_{n,m} &= \frac{1}{n+1}\sin^{n+1} x \cos^{m+1} x + \frac{n+m+2}{n+1}I_{n+2,m} \\
I_{n,m} &= \frac{1}{n+m}\sin^{n+1} x \cos^{m-1} x + \frac{m-1}{n+m}I_{n,m+2} \\
I_{n,m} &= \frac{-1}{m+1}\sin^{n+1} x \cos^{m+1} x + \frac{n+m+2}{m+1}I_{n,m+2} \\
I_{n,m} &= \frac{-1}{m+1}\sin^{n-1} x \cos^{m+1} x + \frac{n-1}{m+1}I_{n-2,m+2} \\
I_{n,m} &= \frac{1}{n+1}\sin^{n+1} x \cos^{m-1} x + \frac{m-1}{n+1}I_{n+2,m-2}
\end{aligned}$$

698. If $S_n = \int e^{ax} \sin^n bx\,dx$ and $C_n = \int e^{ax} \cos^n bx\,dx$, then

$$\begin{aligned}
C_n &= e^{ax} \cos^{n-1} bx \left[\frac{a \cos bx + nb \sin bx}{a^2 + n^2b^2}\right] + \frac{n(n-1)\,b^2}{a^2 + n^2b^2}C_{n-2} \\
S_n &= e^{ax} \sin^{n-1} ax \left[\frac{a \sin bx - nb \cos nx}{a^2 + n^2b^2}\right] + \frac{n(n-1)\,b^2}{a^2 + n^2b^2}S_{n-2}
\end{aligned}$$

699. If $I_n = \int x^m (\ln x)^n\,dx$, then $I_n = \dfrac{1}{m+1}x^{m+1}(\ln x)^n - \dfrac{n}{m+1}I_{n-1}$

Integrals involving Bessel functions

700. $\int J_1(x)\,dx = -J_0(x) + C$

701. $\int xJ_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx$

702. $\int x^n J_1(x)\,dx = -x^n J_0(x) + n\int x^{n-1} J_0(x)\,dx$

703. $\int \frac{J_1(x)}{x}\,dx = -J_1(x) + \int J_0(x)\,dx$

704. $\int x^{\nu} J_{\nu-1}(x)\,dx = x^{\nu} J_{\nu}(x) + C$

705. $\int x^{-\nu} J_{\nu+1}(x)\,dx = x^{-\nu} J_{\nu}(x) + C$

706. $\int \frac{J_1(x)}{x^n}\,dx = \frac{-1}{n}\frac{J_1(x)}{x^{n-1}} + \frac{1}{n}\int \frac{J_0(x)}{x^{n-1}}\,dx$

707. $\int xJ_0(x)\,dx = xJ_1(x) + C$

708. $\int x^2 J_0(x)\,dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx$

709. $\int x^n J_0(x)\,dx = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int x^{n-2} J_0(x)\,dx$

710. $\int \frac{J_0(x)}{x^n}\,dx = \frac{J_1(x)}{(n-1)^2 x^{n-2}} - \frac{J_0(x)}{(n-1)x^{n-1}} - \frac{1}{(n-1)^2}\int \frac{J_0(x)}{x^{n-2}}\,dx$

711. $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x)$

712. $\int xJ_n(\alpha x)J_n(\beta x)\,dx = \frac{x}{\beta^2 - \alpha^2}\left[\alpha J_n'(\alpha x)J_n(\beta x) - \beta J_n'(\beta x)J_n(\alpha x)\right] + C$

713. If $I_{m,n} = \int x^m J_n(x)\,dx, \quad m \geq -n$, then

$$I_{m,n} = -x^m J_{n-1}(x) + (m+n-1)\,I_{m-1,n-1}$$

714. If $I_{n,0} = \int x^n J_0(x)\,dx$, then $I_{n,0} = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 I_{n-2,0}$ Note that $I_{1,0} = \int xJ_0(x)\,dx = xJ_1(x) + C$ and $I_{0,1} = \int J_1(x)\,dx = -J_0(x) + C$ Note also that the integral $I_{0,0} = \int J_0(x)\,dx$ cannot be given in closed form.

Definite integrals

General integration properties

1. If $\dfrac{dF(x)}{dx} = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$

2.

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty}\int_0^b f(x)\,dx, \qquad \int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{b\to\infty\\ a\to-\infty}}\int_a^b f(x)\,dx$$

3. If $f(x)$ has a singular point at $x=b$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)\,dx$

4. If $f(x)$ has a singular point at $x=a$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^{b} f(x)\,dx$

5. If $f(x)$ has a singular point at $x=c$, $a<c<b$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx$

6.

$$\begin{aligned}\int_a^b cf(x)\,dx &= c\int_a^b f(x)\,dx, \quad c \text{ constant}\\ \int_a^a f(x)\,dx &= 0,\\ \int_0^b f(x)\,dx &= \int_0^b f(b-x)\,dx\end{aligned}$$

$$\begin{aligned}\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx,\\ \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\end{aligned}$$

7. Mean value theorems

$$\int_a^b f(x)\,dx = f(c)(b-a), \quad a\le c\le b$$

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx, \quad g(x)\ge 0,\ a\le c\le b$$

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx, \quad a<\xi<b \qquad \int_a^b f(x)g(x)\,dx = f(b)\int_\eta^b g(x)\,dx, \quad a<\eta<b$$

The last mean value theorem requires that $f(x)$ be monotone increasing and nonnegative throughout the interval (a,b)

8. Numerical integration

Divide the interval (a,b) into n equal parts by defining a step size $h = \frac{b-a}{n}$.

Two numerical integration schemes are

(a) Trapezoidal rule with global error $-\frac{(b-a)}{12}h^2 f''(\xi)$ for $a<\xi<b$.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)\right]$$

(b) Simpson's 1/3 rule with global error $-\frac{(b-a)}{90}h^4 f^{(iv)}(\xi)$ for $a<\xi<b$.

$$\int_a^b f(x)\,dx = \frac{2h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

9. If $f(x)$ is periodic with period L, then $f(x+L) = f(x)$ for all x and $\int_0^{nL} f(x)\,dx = n\int_0^L f(x)\,dx$, for integer values of n.

10.
$$\underbrace{\int_0^x dx \int_0^x dx \cdots \int_0^x}_{n\ integration\ signs} dx\, f(x) = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)\,du$$

Integrals containing algebraic terms

11. $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad m>0, n>0$

12. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left[\Gamma(\frac{1}{4})\right]^2$

13. $\displaystyle\int_0^1 \frac{dx}{(1-x^{2n})^{n/2}} = \frac{\pi}{2n\sin\frac{\pi}{2n}}$

14. $\displaystyle\int_0^1 \frac{1}{\beta-\alpha x}\frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{\beta(\beta-\alpha)}}$

15. $\displaystyle\int_0^1 \frac{x^p-x^{-p}}{x^q-x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\tan\frac{p\pi}{2q}, \qquad |p|<q$

16. $\displaystyle\int_0^1 \frac{x^p+x^{-p}}{x^q+x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\sec\frac{p\pi}{2q}, \qquad |p|<q$

17. $\displaystyle\int_0^1 \frac{x^{p-1}-x^{1-p}}{1-x^2}\,dx = \frac{\pi}{2}\cot\frac{p\pi}{2}, \qquad 0<p<2$

18. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}$

19. $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx = \frac{\pi}{4}a^2$

20. $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$

21. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}, \qquad 0<\alpha<1$

22. $\displaystyle\int_0^1 \frac{x^{\alpha-1}+x^{-\alpha}}{1+x}\,dx = \frac{\pi}{\sin\alpha\pi}, \qquad 0<\alpha<1$

23. $\displaystyle\int_0^\infty \frac{x^m\,dx}{1+x^2} = \frac{\pi}{2}\sec\frac{m\pi}{2}$

24. $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1-x^2}\,dx = \frac{\pi}{2}\cot\frac{\alpha\pi}{2}$

25. $\displaystyle\int_0^\infty \frac{dx}{1-x^n} = \frac{\pi}{n}\cot\frac{\pi}{n}$

26. $\displaystyle\int_0^\infty \frac{dx}{(a^2x^2+c^2)(x^2+b^2)} = \frac{\pi}{2bc}\frac{1}{c+ab}$

27. $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2}\frac{1}{ab\,(a+b)}$

28. $\displaystyle\int_0^\infty \frac{dx}{(a^2-x^2)(x^2+p^2)} = \frac{\pi}{2p}\frac{1}{a^2+p^2}$

29. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(a^2-x^2)(x^2+p^2)} = \frac{\pi}{2}\frac{p}{a^2+p^2}$

30. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$

31. $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}}$

32. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$

Integrals containing trigonometric terms

33. $\displaystyle\int_0^1 \frac{\sin^{-1}x}{x}\,dx = \frac{\pi}{2}\ln 2$

34. $\displaystyle\int_0^{\pi/2} \frac{\tan^{-1}(\frac{b}{a}\tan\theta)\,d\theta}{\tan\theta} = \frac{\pi}{2}\ln|1+\frac{b}{a}|$

35. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$

36. $\displaystyle\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

37. $\displaystyle\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{\cos^{-1}(b/a)}{\sqrt{a^2-b^2}}$

38. $\displaystyle\int_0^{\pi/2} \sin^{2m-1}x\cos^{2n-1}x\,dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad m>0, n>0$

39. $\displaystyle\int_0^{\pi/2} \sin^p x\cos^q x\,dx = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q}{2}+1)}$

40. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\tan^m x} = \frac{\pi}{4}$

41. $\displaystyle\int_0^{\pi} \cos p\theta\cos q\theta\,d\theta = \begin{cases} 0, & p\neq q \\ \frac{\pi}{2}, & p=q \end{cases}$

42. $$\int_0^\pi \sin p\theta \sin q\theta \, d\theta = \begin{cases} 0, & p \neq q \\ \dfrac{\pi}{2}, & p = q \end{cases}$$

43. $$\int_0^\pi \sin p\theta \cos q\theta \, d\theta = \begin{cases} 0, & p+q \text{ even} \\ \dfrac{2p}{p^2 - q^2}, & p+q \text{ odd} \end{cases}$$

44. $$\int_0^\pi \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}}$$

45. $$\int_0^\pi \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

46. $$\int_0^\pi \frac{\sin\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \frac{2}{a}\tanh^{-1} a$$

47. $$\int_0^\pi \frac{\sin 2\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \frac{2}{a^2}(1+a^2)\tanh^{-1} a - \frac{2}{a}$$

48. $$\int_0^\pi \frac{x\sin x\,dx}{1 - 2a\cos x + a^2} = \begin{cases} \dfrac{\pi}{a}\ln(1+a), & |a| < 1 \\ \pi\ln\left(1 + \dfrac{1}{a}\right), & |a| > 1 \end{cases}$$

49. $$\int_0^\pi \frac{\cos p\theta\,d\theta}{1 - 2a\cos\theta + a^2} = \begin{cases} \dfrac{\pi a^p}{1-a^2}, & a^2 < 1 \\ \dfrac{\pi a^{-p}}{a^2-1}, & a^2 > 1 \end{cases}$$

50. $$\int_0^\pi \frac{\cos p\theta\,d\theta}{(1 - 2a\cos\theta + a^2)^2} = \begin{cases} \dfrac{\pi a^p}{(1-a^2)^3}[(p+1) - (p-1)a^2], & a^2 < 1 \\ \dfrac{\pi a^{-p}}{(a^2-1)^3}[(1-p) + (1+p)a^2], & a^2 > 1 \end{cases}$$

51. $$\int_0^\pi \frac{\cos p\theta\,d\theta}{(1 - 2a\cos\theta + a^2)^3} = \begin{cases} \dfrac{\pi a^p}{2(1-a^2)^5}\left[(p+2)(p+1) + 2(p+2)(p-2)a^2 + (p-2)(p-1)a^4\right], & a^2 < 1 \\ \dfrac{\pi a^{-p}}{2(a^2-1)^5}\left[(1-p)(2-p) + 2(2-p)(2+p)a^2 + (2+p)(1+p)a^4\right], & a^2 > 1 \end{cases}$$

52. $$\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

53. $$\int_0^{2\pi} \frac{dx}{a + b\sin x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

54. $$\int_0^{2\pi} \frac{dx}{a + b\cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

55. $$\int_0^{2\pi} \frac{dx}{(a + b\sin x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

56. $$\int_0^{2\pi} \frac{dx}{(a + b\cos x)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

57. $$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \quad m, n \text{ integers} \\ \frac{L}{2}, & m = n \end{cases}$$

58. $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$ for all integer m, n values

59. $$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \neq 0 \\ L, & m = n = 0 \end{cases}$$

60. $$\int_0^\infty \frac{x^m\,dx}{1+2x\cos\beta + x^2} = \frac{\pi}{\sin m\pi}\frac{\sin m\beta}{\sin\beta}$$

61. $$\int_0^\infty \frac{\sin\alpha x}{x}\,dx = \begin{cases} \pi/2, & \alpha > 0 \\ 0, & \alpha = 0 \\ -\pi/2, & \alpha < 0 \end{cases}$$

62. $$\int_0^\infty \frac{\sin\alpha x\sin\beta x}{x}\,dx = \begin{cases} 0, & \alpha > \beta > 0 \\ \pi/2, & 0 < \alpha < \beta \\ \pi/4, & \alpha = \beta > 0 \end{cases}$$

63. $$\int_0^\infty \frac{\sin\alpha x\sin\beta x}{x^2}\,dx = \begin{cases} \frac{\pi\alpha}{2}, & 0 < \alpha \leq \beta \\ \frac{\pi\beta}{2}, & \alpha \geq \beta > 0 \end{cases}$$

64. $$\int_0^\infty \frac{\sin^2\alpha x}{x^2}\,dx = \frac{\pi\alpha}{2}$$

65. $$\int_0^\infty \frac{1-\cos\alpha x}{x^2}\,dx = \frac{\pi\alpha}{2}$$

66. $$\int_0^\infty \frac{\cos\alpha x}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-\alpha a}$$

67. $$\int_0^\infty \frac{x\sin\alpha x}{x(x^2+a^2)}\,dx = \frac{\pi}{2}e^{-\alpha a}$$

68. $$\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}$$

69. $$\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}$$

70. $$\int_0^\infty \frac{\tan x}{x}\,dx = \frac{\pi}{2}$$

71. $$\int_0^\infty \frac{\sin\alpha x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-\alpha a})$$

72. $$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

73. $$\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

74. $$\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

75. $\int_0^\infty \sin ax^2 \cos 2bx \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right)$

76. $\int_0^\infty \cos ax^2 \cos 2bx \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left(\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right)$

77. $\int_0^\infty \frac{dx}{x^4 + 2a^2x^2\cos 2\beta + a^4} = \frac{\pi}{4a^3\cos\beta}$

78. $\int_0^\infty \cos\left(x^2 + \frac{a^2}{x^2}\right) dx = \frac{\sqrt{\pi}}{2}\cos(\frac{\pi}{4} + 2a)$

79. $\int_0^\infty \sin\left(x^2 + \frac{a^2}{x^2}\right) dx = \frac{\sqrt{\pi}}{2}\sin(\frac{\pi}{4} + 2a)$

80. $\int_0^\infty \frac{\tan bx \, dx}{x(p^2 + x^2)} = \frac{\pi}{2p^2} \tanh bp$

81. $\int_0^\infty \frac{x \tan bx \, dx}{p^2 + x^2} = \frac{\pi}{2} - \frac{\pi}{2}\tanh bp$

82. $\int_0^\infty \frac{x \cot bx \, dx}{p^2 + x^2} = \frac{\pi}{2}\coth bp$

83. $\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{(p^2 + x^2)} = \frac{\pi}{2p}\frac{\sinh ap}{\sinh bp}, \quad a < b$

84. $\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{(p^2 + x^2)} = \frac{\pi}{2p}\frac{\cosh ap}{\cosh bp}, \quad a < b$

85. $\int_0^\infty \frac{\sin ax}{\cos bx}\frac{dx}{(p^2 + x^2)} = \frac{\pi}{2p^2}\frac{\sinh ap}{\cosh bp}, \quad a < b$

86. $\int_0^\infty \frac{\sin ax}{\cos bx}\frac{x \, dx}{(x^2 + p^2)} = -\frac{\pi}{2}\frac{\sinh ap}{\cosh bp}, \quad a < b$

87. $\int_0^\infty \frac{\cos ax}{\sin bx}\frac{x \, dx}{(p^2 + x^2)} = \frac{\pi}{2}\frac{\cosh ap}{\sinh bp}, \quad a < b$

Integrals containing exponential and logarithmic terms

88. $\int_0^1 \frac{\ln\frac{1}{x}}{1 + x} dx = \frac{\pi^2}{12}$

89. $\int_0^1 \frac{\ln\frac{1}{x}}{(1 - x)} dx = \frac{\pi^2}{6}$

90. $\int_0^1 \frac{\left(\ln\frac{1}{x}\right)^3}{1 - x} dx = \frac{\pi^4}{15}$

91. $\int_0^1 \frac{\ln(1 + x)}{x} dx = \frac{\pi^2}{12}$

92. $\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$

93. $\int_0^1 (ax^2+bx+c)\frac{\ln\frac{1}{x}}{1-x}\,dx = (a+b+c)\frac{\pi^2}{6} - (a+b) - \frac{a}{4}$

94. $\int_0^1 \frac{\ln\frac{1}{x}}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln 2$

95. $\int_0^1 \frac{1-x^{p-1}}{(1-x)(1-x^p)}(\ln\frac{1}{x})^{2n-1}\,dx = \frac{1}{4n}(1-\frac{1}{p^{2n}})(2\pi)^{2n}\mathfrak{B}_{2n-1}$

96. $\int_0^1 \frac{x^m-x^n}{\ln x}\,dx = \ln\left|\frac{1+m}{1+n}\right|$

97. $\int_0^1 x^p(\ln x)^n\,dx = \begin{cases} (-1)^n\frac{n!}{(p+1)^{n+1}}, & \text{n an integer} \\ (-1)^n\frac{\Gamma(n+1)}{(p+1)^{n+1}}, & \text{n noninteger} \end{cases}$

98. $\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$

99. $\int_0^{\pi/2} \ln\sin\theta\,d\theta = \frac{\pi}{2}\ln(\frac{1}{2})$

100. $\int_0^{\pi} \ln(a+b\cos x)\,dx = \pi\ln\left|\frac{a+\sqrt{a^2+b^2}}{2}\right|$

101. $\int_0^{2\pi} \ln(a+b\cos x)\,dx = 2\pi\ln|a+\sqrt{a^2-b^2}|$

102. $\int_0^{2\pi} \ln(a+b\sin x)\,dx = 2i\ln|a+\sqrt{a^2-b^2}|$

103. $\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}$

104. $\int_0^{\infty} x^n e^{-ax}\,dx = \frac{\Gamma(n+1}{a^{n+1}}$

105. $\int_0^{\infty} e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma(\frac{1}{2})$

106. $\int_0^{\infty} x^n e^{-a^2x^2}\,dx = \frac{\Gamma(\frac{m+1}{2})}{2a^{m+1}}$

107. $\int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

108. $\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

109. $\int_0^\infty e^{-ax}\frac{\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$

110. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}$

111. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/4a^2}$

112. $\int_0^\infty e^{-(ax^2+b/x^2)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$

113. $\int_0^\infty x^{2n}e^{-\beta x^2}\,dx = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^{n+1}\beta^n}\sqrt{\frac{\pi}{\beta}}$

114. $\int_0^\infty e^{-k\left(\frac{x^2}{a^2}+\frac{b^2}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}\frac{a}{\sqrt{k}}e^{-2kb/a}$

115. $\int_0^\infty \frac{\sin rx\,dx}{x(x^4+2a^2x^2\cos 2\beta+a^4)} = \frac{\pi}{2a^4}\left[1-\frac{\sin(ar\sin\beta+2\beta)}{\sin 2\beta}\,e^{-\beta r\cos\beta}\right]$

116. $\int_0^\infty \frac{\cos rx\,dx}{x^4+2a^2x^2\cos 2\beta+a^4} = \frac{\pi}{2a^3}\frac{\sin(\beta+ar\sin\beta)}{\sin 2\beta}\,e^{-ar\cos\beta}$

117. $\int_0^\infty \frac{\sin rx\,dx}{x(x^6+a^6)} = \frac{\pi}{6a^6}\left[3-e^{-ar}-2e^{-ar/2}\cos\frac{ar\sqrt{3}}{2}\right]$

118. $\int_0^\infty \frac{\cos rx\,dx}{x^6+a^6} = \frac{\pi}{6a^5}\left[e^{-ar}-2e^{-ar/2}\cos(\frac{ar\sqrt{3}}{2}+\frac{2\pi}{3})\right]$

119. $\int_0^\infty \frac{\sin \pi x\,dx}{x(1-x^2)} = \pi$

120. $\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\cos bx\,dx = \frac{1}{2}\ln\left|\frac{p^2+b^2}{q^2+b^2}\right|$

121. $\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\sin bx\,dx = \tan^{-1}\frac{p}{b}-\tan^{-1}\frac{q}{b}$

122. $\int_0^\infty e^{-ax}\frac{\sin px-\sin qx}{x}\,dx = \tan^{-1}\frac{p}{a}-\tan^{-1}\frac{q}{b}$

123. $\int_0^\infty e^{-ax}\frac{\cos px-\cos qx}{x}\,dx = \frac{1}{2}\ln\left|\frac{a^2+a^2}{a^2+p^2}\right|$

124. $\int_0^\infty xe^{-x^2}\sin ax\,dx = \frac{a\sqrt{\pi}}{4}e^{-a^2/4}$

125. $\int_0^\infty x^2e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{4}\left(1-\frac{a^2}{2}\right)\,e^{-a^2/4}$

126. $\int_0^\infty x^3e^{-x^2}\sin ax\,dx = \frac{\sqrt{\pi}}{8}\left(3a-\frac{a^3}{2}\right)\,e^{-a^2/4}$

127. $$\int_0^\infty x^4 e^{-x^2} \cos ax\, dx = \frac{\sqrt{\pi}}{8}\left(3 - 3a^2 + \frac{a^4}{4}\right) e^{-a^2/4}$$

128. $$\int_0^\infty \left(\frac{\ln x}{x-1}\right)^3 dx = \pi^2$$

129. $$\int_{-\infty}^\infty \frac{x \sin rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a}(a \cos br + b \sin br)\, e^{-ar}$$

130. $$\int_{-\infty}^\infty \frac{\sin rx\, dx}{x[(x-b)^2 + a^2]} = \frac{\pi}{a(a^2+b^2)}\left[a - (\cos br - b \sin br)\, e^{-ar}\right]$$

131. $$\int_{-\infty}^\infty \frac{\cos rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a} e^{-ar} \cos br$$

132. $$\int_{-\infty}^\infty \frac{\sin rx\, dx}{(x-b)^2 + a^2} = \frac{\pi}{a} e^{-ar} \sin br$$

133. $$\int_{-\infty}^\infty e^{-x^2} \cos 2nx\, dx = \sqrt{\pi}\, e^{-n^2}$$

134. $$\int_0^\infty \frac{x^{p-1} \ln x}{1+x}\, dx = \frac{-\pi^2}{\sin p\pi} \cot p\pi, \qquad 0 < p < 1$$

135. $$\int_0^\infty e^{-x} \ln x\, dx = -\gamma$$

136. $$\int_0^\infty e^{-x^2} \ln x\, dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2\ln 2)$$

137. $$\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}$$

138. $$\int_0^\infty \frac{x\, dx}{e^x - 1} = \frac{\pi^2}{6}$$

139. $$\int_0^\infty \frac{x\, dx}{e^x + 1} = \frac{\pi^2}{12}$$

Integrals containing hyperbolic terms

140. $$\int_0^1 \frac{\sinh(m \ln x)}{\sinh(\ln x)}\, dx = \frac{\pi}{2} \tan \frac{m\pi}{2}, \qquad |m| < 1$$

141. $$\int_0^\infty \frac{\sin ax}{\sinh bx}\, dx = \frac{\pi}{2b} \tanh(\frac{\pi a}{2b})$$

142. $$\int_0^\infty \frac{\cos ax}{\cosh bx}\, dx = \frac{\pi}{2b} \text{sech}\,(\frac{\pi a}{2b})$$

143. $$\int_0^\infty \frac{x\, dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

144. $$\int_0^\infty \frac{\sinh px}{\sinh qx}\,dx = \frac{\pi}{2q}\tan(\frac{\pi p}{2q}), \quad |p| < q$$

145. $$\int_0^\infty \frac{\cosh ax - \cosh bx}{\sinh \pi x}\,dx = \ln\left|\frac{\cos\frac{b}{2}}{\cos\frac{a}{2}}\right|, \quad -\pi < b < a < \pi$$

146. $$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos mx\,dx = \frac{\pi}{2q}\,\frac{\sin\frac{\pi p}{q}}{\cos\frac{\pi p}{q} + \cosh\frac{\pi m}{q}}, \quad q > 0, p^2 < q^2$$

147. $$\int_0^\infty \frac{\sinh px}{\cosh qx}\sin mx\,dx = \frac{\pi}{q}\,\frac{\sin\frac{p\pi}{2q}\sinh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q} + \cosh\frac{m\pi}{q}}$$

148. $$\int_0^\infty \frac{\cosh px}{\cosh qx}\cos mx\,dx = \frac{\pi}{q}\,\frac{\cos\frac{p\pi}{2q}\cosh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q} + \cosh\frac{m\pi}{q}}$$

Miscellaneous Integrals

149. $$\int_0^x \xi^{\lambda-1}[1-\xi^\mu]^\nu\,d\xi = \frac{x^\lambda}{\lambda}F(-\nu, \frac{\lambda}{\mu}; \frac{\lambda}{\mu}+1; x^\mu)$$ See hypergeometric function

150. $$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi = \pi\,J_n(x)$$

151. $$\int_{-a}^a (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

152. If $f'(x)$ is continuous and $\int_1^\infty \frac{f(x)-f(\infty)}{x}\,dx$ converges, then

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = [f(0)-f(\infty)]\,\ln\frac{b}{a}$$

153. If $f(x) = f(-x)$ so that $f(x)$ is an even function, then

$$\int_0^\infty f\left(x - \frac{1}{x}\right)\,dx = \int_0^\infty f(x)\,dx$$

154. Elliptic integral of the first kind

$$\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\theta, k), \quad 0 < k < 1$$

155. Elliptic integral of the second kind

$$\int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta = E(\theta, k)$$

156. Elliptic integral of the third kind

$$\int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \Pi(\theta, k, n)$$

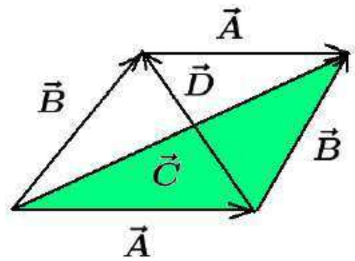
Appendix D
Solutions to Selected Problems

Chapter 6

► **6-1.** $(a)\ \vec{A}+\vec{B} = 9\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 3\,\hat{\mathbf{e}}_3 \quad (b)\ 6\vec{A}-3\vec{B} = 15\,\hat{\mathbf{e}}_2 \quad (c)\ \vec{A}+2\vec{B} = 15\,\hat{\mathbf{e}}_1 + 5\,\hat{\mathbf{e}}_3$

► **6-2.**

$$\vec{A}+\vec{B} = \vec{C}$$
$$\vec{A}+\vec{D} = \vec{B}$$



Since the vectors are coplaner there exists scalar constants α and β such that $\vec{A}+\alpha\vec{D} = \beta\vec{C}$ This implies

$$\vec{A}+\alpha(\vec{B}-\vec{A}) = \beta(\vec{A}+\vec{B}) \quad \text{or} \quad \vec{A}(1-\alpha-\beta)+\vec{B}(\alpha-\beta) = \vec{0}$$

Since $\vec{A}$ and $\vec{B}$ are linearly independent and noncolinear one can state that

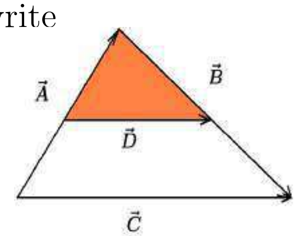
$$1 = \alpha+\beta \quad \text{and} \quad 0 = \alpha-\beta$$

Solving these simultaneous equations gives $\alpha = 1/2$ and $\beta = 1/2$ which demonstrates that the diagonals bisect one another.

► **6-3.**

Let $\vec{A}+\vec{B} = \vec{C}$ and then by construction write

$$\frac{1}{2}\vec{A}+\vec{D}+\frac{1}{2}\vec{B} = \vec{C} = \vec{A}+\vec{B}$$

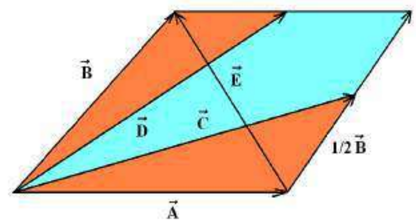


so that

$$\vec{D} = \frac{1}{2}\vec{A}+\frac{1}{2}\vec{B} = \frac{1}{2}\vec{C}$$

► 6-4. By construction

$$\vec{A} + \frac{1}{2}\vec{B} = \vec{C}, \qquad \vec{B} + \frac{1}{2}\vec{A} = \vec{D}, \qquad \vec{A} + \vec{E} = \vec{B}$$



All these vectors are coplaner so that there exists scala constants $\alpha, \beta, \gamma, \delta$ such that

$$\vec{A} + \alpha\vec{E} = \beta\vec{C} \qquad \text{and} \qquad \vec{A} + \gamma\vec{E} = \delta\vec{D}$$

or

$$\vec{A} + \alpha(\vec{B} - \vec{A}) = \beta(\vec{A} + \frac{1}{2}\vec{B}) \qquad \text{and} \qquad \vec{A} + \gamma(\vec{B} - \vec{A}) = \delta(\vec{B} + \frac{1}{2}\vec{A})$$

This implies that

$$\vec{A}(1 - \alpha - \beta) + \vec{B}(\alpha - \frac{1}{2}\beta) = \vec{0} \qquad \text{and} \qquad \vec{A}(1 - \gamma - \frac{1}{2}\delta) + \vec{B}(\gamma - \delta) = \vec{0}$$

This produces the simultaneous equations

$$\alpha + \beta = 1, \quad \alpha - \frac{1}{2}\beta = 0, \quad \gamma + \frac{1}{2}\delta = 1, \quad \gamma - \delta = 0$$

Solving for $\alpha, \beta, \gamma, \delta$ one finds

$$\alpha = 1/3, \qquad \beta = 2/3, \qquad \gamma = 2/3, \qquad \delta = 2/3$$

► 6-5. (a)$c_1\vec{A} + c_2\vec{B} + c_3\vec{C} = \vec{0}$ gives the system of equations

$$\begin{aligned} c_1 - 4c_2 + 7c_3 &= 0 \\ c_1 - 3c_2 + 6c_3 &= 0 \\ -2c_1 \qquad - 6c_3 &= 0 \end{aligned} \qquad \Longrightarrow \qquad c_1 = -3c_2, \qquad c_3 = c_2$$

Since $c_2 \neq 0$, select $c_2 = 1$ for convenience, then $c_1 = -3,\ c_2 = 1,\ c_3 = 1$, so vectors are linearly dependent.

(b) Linearly independent

(c) Linearly independent

► 6-6. The vectors $\vec{A}, \vec{B}, \vec{C}$ are linearly dependent if and only if $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$
If $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ and $\vec{A} \neq 0$, then $\vec{B} \times \vec{C} = \vec{0}$ which implies the vectors $\vec{B}$ and $\vec{C}$ are colinear. If the determinant $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = 0$, then two rows of the determinant are proportional which implies two of the vectors are colinear.

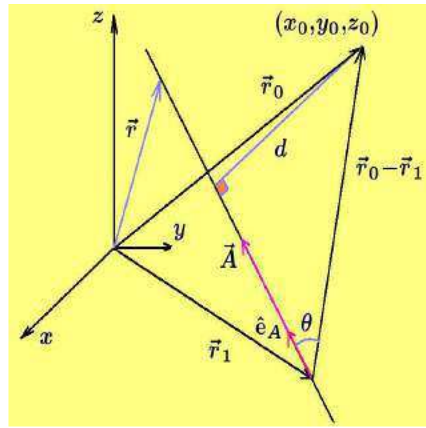
▶ **6-7.** (a) $\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos 0 = |\vec{A}|^2 = C^2$

(b) $\frac{d}{dt}(\vec{A}\cdot\vec{A}) = \frac{d}{dt}C^2 \implies \vec{A}\cdot\frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{A} = 0 \implies \vec{A}\cdot\frac{d\vec{A}}{dt} = 0$ which shows $\vec{A}$ is perpendicular to $\frac{d\vec{A}}{dt}$

▶ **6-8.**
Equation of line is $\vec{r} = \vec{r}_1 + t\vec{A}$ where t is a parameter. Use the property of right triangles and write

$$d = |\vec{r}_0 - \vec{r}_1|\sin\theta = |(\vec{r}_0 - \vec{r}_1)\times\hat{\mathbf{e}}_A|$$

where the absolute value sign insures that d is positive and $\hat{\mathbf{e}}_A$ is a unit vector in the direction of $\vec{A}$.



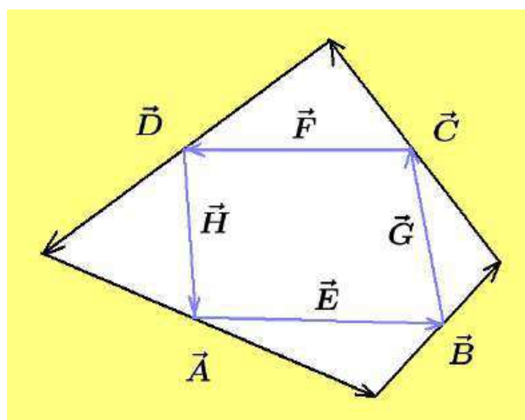
▶ **6-9.** Area is 54 square units

▶ **6-10.**

$$\vec{A}+\vec{B}+\vec{C}+\vec{D} = \vec{0}$$

By construction

$$\vec{E} = \frac{1}{2}\vec{A} + \frac{1}{2}\vec{B}$$
$$\vec{F} = \frac{1}{2}\vec{C} + \frac{1}{2}\vec{D}$$
$$\vec{G} = \frac{1}{2}\vec{B} + \frac{1}{2}\vec{C}$$
$$\vec{H} = \frac{1}{2}\vec{D} + \frac{1}{2}\vec{A}$$



To show $\vec{E}$ is parallel to $\vec{F}$, show $\vec{E}\times\vec{F} = \vec{0}$ and to show $\vec{G}$ is parallel to $\vec{H}$, show $\vec{G}\times\vec{H} = \vec{0}$

$$\vec{E}\times\vec{F} = (\frac{1}{2}\vec{A}+\frac{1}{2}\vec{B})\times(\frac{1}{2}\vec{C}+\frac{1}{2}\vec{D}) = \frac{1}{2}(\vec{A}+\vec{B})\times(-\frac{1}{2})(\vec{A}+\vec{B}) = \vec{0}$$
$$\vec{G}\times\vec{H} = (\frac{1}{2}\vec{B}+\frac{1}{2}\vec{C})\times(\frac{1}{2}\vec{D}+\frac{1}{2}\vec{A}) = \frac{1}{2}(\vec{B}+\vec{C})\times(-\frac{1}{2})(\vec{B}+\vec{C}) = \vec{0}$$

▶ **6-11.**

$$\vec{r}-\vec{r}_0 = (x-x_0)\,\hat{\mathbf{e}}_1 + (y-y_0)\,\hat{\mathbf{e}}_2 + (z-z_0)\,\hat{\mathbf{e}}_3$$
$$(\vec{r}-\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) = \rho^2$$
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \rho^2$$

▶ **6-12.** (c) 23/9 (d) 23/3

▶ **6-13.** (a) $\hat{\mathbf{e}}_C = \frac{\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3}{\sqrt{2}}$ also $-\hat{\mathbf{e}}_C$ (b) $-1/\sqrt{3}$

▶ **6-14.** (b) $\vec{A} \cdot \hat{\mathbf{e}}_\alpha = -\cos\alpha + \sqrt{3}\sin\alpha$

(c) $\alpha = \pi/6$ or $7\pi/6$

(d) $y(\alpha) = \sqrt{3}\sin\alpha - \cos\alpha$ and $\frac{dy}{d\alpha} = 0$ when $\alpha = -\pi/3$ or $2\pi/3$

$y''(\alpha) = -\sqrt{3}\sin\alpha + \cos\alpha$, $y''(-\pi/3) > 0$ and $y''(2\pi/3) < 0$, Maximum +2, Minimum -2

▶ **6-15.**

$$\begin{aligned}
\vec{A}(t) =& \vec{A}_0 + \vec{A}_1(t-t_0) + \vec{A}_2\frac{(t-t_0)^2}{2!} + \vec{A}_3\frac{(t-t_0)^3}{3!} + \cdots + \vec{A}_n\frac{(t-t_0)^n}{n!} + \cdots \\
\vec{A}'(t) =& \vec{A}_1 + \vec{A}_2(t-t_0) + \cdots + \vec{A}_n\frac{(t-t_0)^{n-1}}{(n-1)!} + \cdots \\
\vec{A}''(t) =& \vec{A}_2 + \vec{A}_3(t-t_0) + \cdots + \vec{A}_n\frac{(t-t_0)^{n-2}}{(n-2)!} + \cdots \\
\vdots \quad & \quad \vdots
\end{aligned}$$

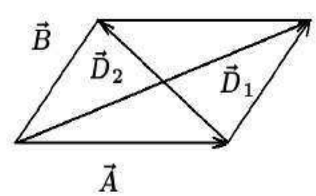
Evaluate the derivatives at $t = t_0$ and show

$$\vec{A}(t_0) = \vec{A}_0, \quad \vec{A}'(t_0) = \vec{A}_1, \quad \vec{A}''(t_0) = \vec{A}_2, \quad \ldots \quad , \quad \vec{A}^{(n)}(t_0) = \vec{A}_n, \quad \ldots$$

▶ **6-16.** (a) $\vec{A} \times \vec{B} = -16\,\hat{\mathbf{e}}_1 + 8\,\hat{\mathbf{e}}_3$ (b) $16\,\hat{\mathbf{e}}_1 - 8\,\hat{\mathbf{e}}_3$ (e) $\theta = \cos^{-1}(11/21) \approx 58.41°$

▶ **6-17.**

$$\begin{aligned}
\vec{D}_1 =& \vec{A} + \vec{B} = 3\,\hat{\mathbf{e}}_1 + 11\,\hat{\mathbf{e}}_2 + 4\,\hat{\mathbf{e}}_3 \\
\vec{D}_2 =& \vec{B} - \vec{A} = \hat{\mathbf{e}}_1 + 7\,\hat{\mathbf{e}}_2 \\
\text{Area} =& |\vec{A} \times \vec{B}| = 15
\end{aligned}$$



▶ **6-18.**

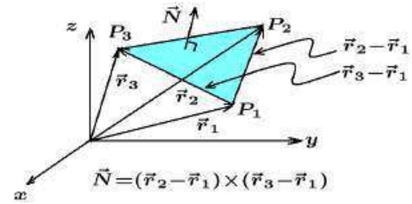
$$\begin{aligned}
\cos\alpha =& \sqrt{2}/2 \\
\cos\beta =& 1/2 \\
\cos\gamma =& -1/2
\end{aligned}$$

$$\vec{e} = \frac{\vec{\rho}}{|\vec{\rho}|} = \cos\alpha\,\hat{\mathbf{e}}_1 + \cos\beta\,\hat{\mathbf{e}}_2 + \cos\gamma\,\hat{\mathbf{e}}_3$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2/4 + 1/4 + 1/4 = 1$$

▶ **6-19.** If $\vec{A} \times \vec{B} = \vec{0}$, then $\vec{A}$ is parallel to $\vec{B}$ or $\vec{A} = c\vec{B}$ for some constant c.

► **6-20.**

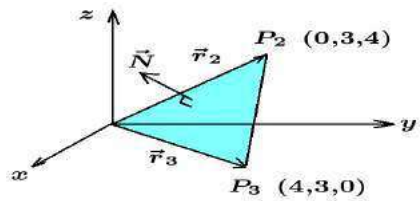


(d) $(\vec{r}-\vec{r}_1)\cdot\vec{N}=0$

is equation of plane

► **6-21.** $\vec{T}=\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2$ is tangent to line. $x=3+\lambda,\ y=4-\lambda,\ z=2$

► **6-22.**



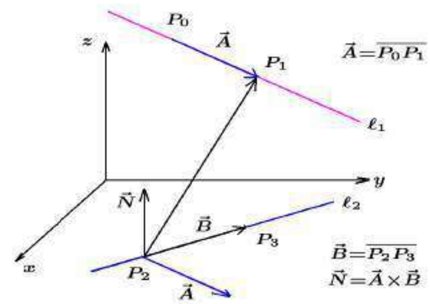
$\vec{N}=12\,\hat{\mathbf{e}}_1-16\,\hat{\mathbf{e}}_2+12\,\hat{\mathbf{e}}_3$

Equation of plane is

$(\vec{r}-r_1)\cdot\vec{N}=0$

or $3(x-4)-4(y-3)+z=0$

► **6-23.**



Plane through $\overline{P_0P_1}$ and perpendicular to $\vec{N}$

and plane through $\overline{P_2P_3}$ also perpendicular to $\vec{N}$

are parallel planes.

The vector $\overline{P_2P_1}$ is vector from one plane

to the other and its projection onto $\vec{N}$ is distance between planes

and also equal to the minimum distance between the skew lines.

► **6-24.** $x=1+2t,\ y=5t,\ z=1+2t$ is parametric equation of line. The point $(6,13,12)$ is not on the line.

► **6-25.** If $\vec{r}-\vec{r}_1$ is colinear with $(\vec{r}_2-\vec{r}_1)$, then $(\vec{r}-\vec{r}_1)\times(\vec{r}_2-\vec{r}_1)=\vec{0}$ By vector addition $\vec{r}=\vec{r}_1+\lambda(\vec{r}_2-\vec{r}_1)$ where λ is a parameter.

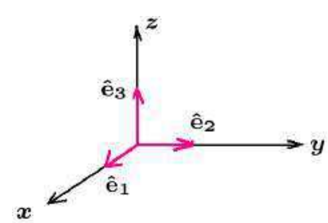
► **6-27.**

even permutations

$i=1,\ j=2,\ k=3,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2=\hat{\mathbf{e}}_3$

$i=2,\ j=3,\ k=1,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3=\hat{\mathbf{e}}_1$

$i=3,\ j=1,\ k=2,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_1=\hat{\mathbf{e}}_2$



odd permutations

$i=3,\ j=2,\ k=1,\quad \hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_2=-\,\hat{\mathbf{e}}_1$

$i=2,\ j=1,\ k=3,\quad \hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_1=-\,\hat{\mathbf{e}}_3$

$i=1,\ j=3,\ k=2,\quad \hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3=-\,\hat{\mathbf{e}}_2$

► **6-28.** (a) If $\hat{\mathbf{e}}_{\ell_1} = \cos\alpha_1\,\hat{\mathbf{e}}_1 + \cos\beta_1\,\hat{\mathbf{e}}_2 + \cos\gamma_1\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_{\ell_2} = \cos\alpha_2\,\hat{\mathbf{e}}_1 + \cos\beta_2\,\hat{\mathbf{e}}_2 + \cos\gamma_2\,\hat{\mathbf{e}}_3$, then

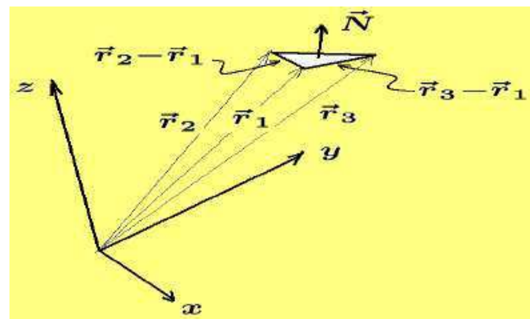
$$\hat{\mathbf{e}}_{\ell_1}\cdot\hat{\mathbf{e}}_{\ell_2} = \cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2$$

(b) $\theta = \cos^{-1}(8/9) = .475882$ radians $\approx 27.266°$

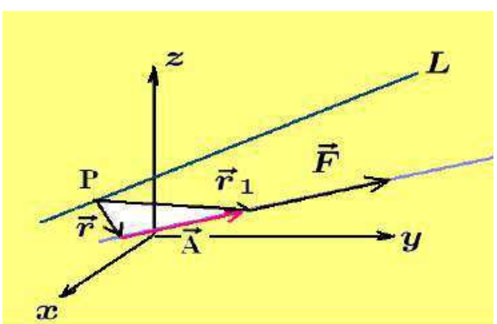
► **6-29.** Shortest distance is 9 units.

► **6-31.** If $\vec{A}\times\vec{B} = \vec{0}$, then $\vec{A}$ is colinear with $\vec{B}$ and if $\vec{B}\times\vec{C} = \vec{0}$, then $\vec{B}$ is colinear with $\vec{C}$. Therefore, $\vec{A}$ is colinear with $\vec{C}$ so that $\vec{A}\times\vec{C} = \vec{0}$.

► **6-32.** Normal to plane is $\vec{N} = (\vec{r}_3 - \vec{r}_1)\times(\vec{r}_2 - \vec{r}_1)$
Equation of plane is $(\vec{r} - \vec{r}_1)\cdot\vec{N} = 0$ or
$(x-3) - 2(y-10) + 2(z-13) = 0$ The distance from given point to plane is projection of $(\vec{r}_0 - \vec{r}_1)$ onto $\hat{\mathbf{e}}_N$ giving 9 units for the distance.
Here $\vec{r}_0 = 6\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 + 18\,\hat{\mathbf{e}}_3$



► **6-33.**



$$\vec{M}_P = \vec{r}_1\times\vec{F}$$
$$\vec{r}_1 = \vec{r} + \vec{A}$$
$$\vec{M}_P = (\vec{r} + \vec{A})\times\vec{F} = \vec{r}\times\vec{F} + \vec{A}\times\vec{F} = \vec{r}\times\vec{F}$$

because $\vec{A}\times\vec{F} = \vec{0}$, $\vec{A}$ and $\vec{F}$ are colinear

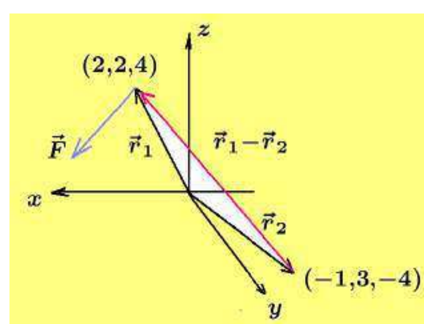
$\vec{M}_P\cdot\hat{\mathbf{e}}_L =$ projection of $\vec{M}_P$ on the line L

► **6-34.** (a) $\vec{M}_0 = \vec{r}_1\times\vec{F} = -1200\,\hat{\mathbf{e}}_1 + 800\,\hat{\mathbf{e}}_2 + 200\,\hat{\mathbf{e}}_3$

(b) $\vec{M}_{P_2} = (\vec{r}_1 - \vec{r}_2)\times\vec{F} = -1400\,\hat{\mathbf{e}}_1 + 1400\,\hat{\mathbf{e}}_2 + 700\,\hat{\mathbf{e}}_3$

(c) $\hat{\mathbf{e}}_L = \dfrac{-\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2 - 4\,\hat{\mathbf{e}}_3}{\sqrt{26}}$

$$M_L = \vec{M}_{P_2}\cdot\hat{\mathbf{e}}_L = \frac{2800}{\sqrt{26}}$$



▶ **6-35.** (a) $\frac{t^2}{2}\,\hat{\mathbf{e}}_1 + t\,\hat{\mathbf{e}}_2 - \frac{t^3}{3}\,\hat{\mathbf{e}}_3 + \vec{c}$

▶ **6-36.**

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} = \cos t\,\hat{\mathbf{e}}_1 + \sin t\,\hat{\mathbf{e}}_2 \\
\vec{v} &= \vec{v}(t) = \sin t\,\hat{\mathbf{e}}_1 - \cos t\,\hat{\mathbf{e}}_2 + \vec{c} \\
\vec{v}(0) &= -\,\hat{\mathbf{e}}_2 + \vec{c} = 2\,\hat{\mathbf{e}}_3 \implies \vec{c} = 2\,\hat{\mathbf{e}}_3 + \,\hat{\mathbf{e}}_2 \\
\vec{v} &= \frac{d\vec{r}}{dt} = \sin t\,\hat{\mathbf{e}}_1 + (1-\cos t)\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 \\
\vec{r} &= \vec{r}(t) = -\cos t\,\hat{\mathbf{e}}_1 + (t - \sin t)\,\hat{\mathbf{e}}_2 + 2t\,\hat{\mathbf{e}}_3
\end{aligned}$$

▶ **6-39.** (a) $\omega = 5\,\hat{\mathbf{e}}_3$

(b) $\vec{v} = \omega \times \vec{r} = -5\sin 5t\;\hat{\mathbf{e}}_1 + 5\cos 5t\;\hat{\mathbf{e}}_2$

▶ **6-40.** $\vec{r} = e^t\;\hat{\mathbf{e}}_1 + \cos t\;\hat{\mathbf{e}}_2 + \sin t\;\hat{\mathbf{e}}_3$ with $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

▶ **6-41.**

$$\vec{C} = \vec{C}(x) = \vec{r}(x) + \alpha\vec{N}(x) = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + \frac{(1+(y')^2)}{y''}\,[-y'\,\hat{\mathbf{e}}_1 + \,\hat{\mathbf{e}}_2]$$

If $x = x(t)$ and $y = y(t)$, then

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} \\
y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)}{\frac{dx}{dt}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}
\end{aligned}$$

Substitute these derivatives into $\vec{C} = \vec{C}(x)$ and simplify.

▶ **6-42.** (b) $\vec{C} = \vec{C}(x) = x\,\hat{\mathbf{e}}_1 + e^x\,\hat{\mathbf{e}}_2 + \frac{(1+e^{2x})}{e^x}\,[-e^x\,\hat{\mathbf{e}}_1 + \,\hat{\mathbf{e}}_2]$

▶ **6-43.** When $\hat{\mathbf{e}} = \hat{\mathbf{e}}_A = \frac{1}{|\vec{A}|}\vec{A}$, then $\hat{\mathbf{e}}_A \cdot \vec{A} = |\vec{A}|$

▶ **6-47.** $\frac{\partial U}{\partial x} = (4xy + y^2)\,\hat{\mathbf{e}}_1 + (y + 6xy)\,\hat{\mathbf{e}}_2$ and $\frac{\partial^2 U}{\partial x^2} = 4y\,\hat{\mathbf{e}}_1 + 6y\,\hat{\mathbf{e}}_2$

▶ **6-48.** If $\vec{v} = \frac{d\vec{r}}{dt} = \omega \times \vec{r}$, then

$$\frac{dx}{dt}\,\hat{\mathbf{e}}_1 + \frac{dy}{dt}\,\hat{\mathbf{e}}_2 + \frac{dz}{dt}\,\hat{\mathbf{e}}_3 = (z\omega_2 - y\omega_3)\,\hat{\mathbf{e}}_1 + (x\omega_3 - z\omega_1)\,\hat{\mathbf{e}}_2 + (y\omega_1 - x\omega_2)\,\hat{\mathbf{e}}_3$$

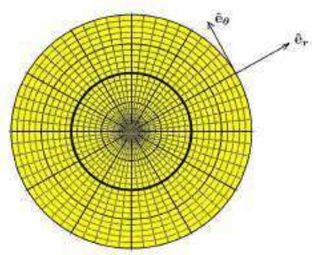
Equate like components to show result.

► **6-50.** If $\frac{d}{dt}(\vec{B}\times\vec{C}) = \vec{B}\times\frac{d\vec{C}}{dt} + \frac{d\vec{B}}{dt}\times\vec{C}$, then

$$\begin{aligned}\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right] =& \vec{A}\times\frac{d}{dt}(\vec{B}\times\vec{C}) + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})\\ =& \vec{A}\times\left[\vec{B}\times\frac{d\vec{C}}{dt} + \frac{d\vec{B}}{dt}\times\vec{C}\right] + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})\\ =& \vec{A}\times(\vec{B}\times\frac{d\vec{C}}{dt}) + \vec{A}\times(\frac{d\vec{B}}{dt}\times\vec{C}) + \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})\end{aligned}$$

► **6-51.** The curves $\vec{r} = \vec{r}(r_0,\theta) = r_0\cos\theta\,\hat{\mathbf{e}}_1 + r_0\sin\theta\,\hat{\mathbf{e}}_2$ are coordinate curves which are circles of radius r_0. The curves $\vec{r} = \vec{r}(r,\theta_0) = r\cos\theta_0\,\hat{\mathbf{e}}_1 + r\sin\theta_0\,\hat{\mathbf{e}}_2$ are coordinate curves which are the rays $\theta = \theta_0 = a\ constant$

$$\begin{aligned}\frac{\partial\vec{r}}{\partial r} =& \cos\theta + \sin\theta\,\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_r\\ \frac{\partial\vec{r}}{\partial\theta} =& -r\sin\theta\,\hat{\mathbf{e}}_1 + r\cos\theta\,\hat{\mathbf{e}}_2 = r\,\hat{\mathbf{e}}_\theta\end{aligned}$$



► **6-52.** $\displaystyle\int_C \vec{F}\times d\vec{r} = \int_{(1,3)}^{(2,6)} \hat{\mathbf{e}}_1(y-x)\,dz - \hat{\mathbf{e}}_2\,xy\,dz + \hat{\mathbf{e}}_3(xy\,dy - (y-x)\,dx)$

On the line $y = 3x,\ z = 0,\ dz = 0,\ dy = 3dx$

so that $\displaystyle\int_C \vec{F}\times d\vec{r} = \int_1^2 [x(3x)\,3dx - (3x-x)\,dx]\,\hat{\mathbf{e}}_3 = 18\,\hat{\mathbf{e}}_3$

► **6-53.** $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C[(xy+1)dx + (x+z+1)dy + (z+1)dz]$

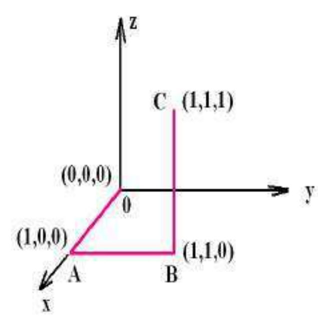
$\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{0A}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3$

On 0A, $y = 0, dy = 0, z = 0, dz = 0$ and $I_1 = \int_0^1 dx = 1$

On AB, $x = 1, dx = 0, z = 0, dz = 0$ and $I_2 = \displaystyle\int_0^1 2dy = 2$

On BC, $x = 1, dx = 0, y = 1, dy = 0$ and $I_3 = \displaystyle\int_0^1 (z+1)dz = 3/2$

Therefore $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 9/2$



► **6-55.** $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{0A}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} = I_1 + I_2$

On 0A $y = x, dy = dx, z = 0, dz = 0, 0 \le x \le 1,\ I_1 = \displaystyle\int_0^1 (x + 2x^2)\,dx = 7/6$

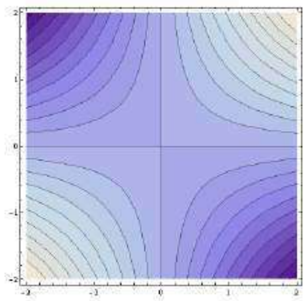
On AB $x = 1, y = 1, dx = dy = 0, 0 \le z \le 2 \quad I_2 = \int_0^2 dz = 2$

Therefore, $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 19/6$

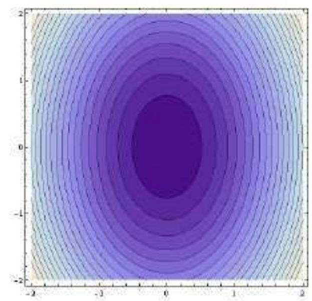
▶ **6-57.** On circle $x = \cos\theta,\ y = \sin\theta,\ dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C yz\, dx + 2x\, dy + y\, dz$$
$$= \int_0^{2\pi} -2\sin^2\theta\, d\theta + 2\cos^2\theta\, d\theta = 0$$

▶ **6-60.** (b)

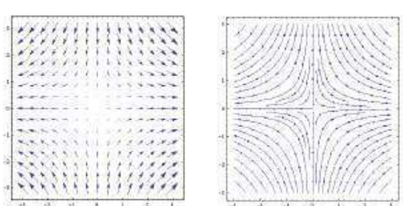


(d)

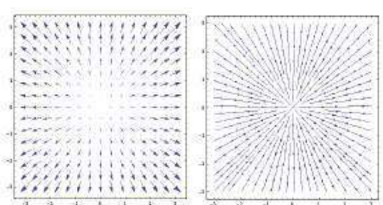


▶ **6-61.**

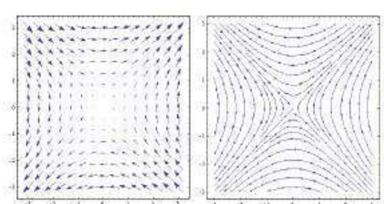
(a) $\dfrac{dx}{x} = \dfrac{dy}{-y} \implies xy = c$



(b) $\dfrac{dx}{2x} = \dfrac{dy}{2y} \implies y = cx$



(c) $\dfrac{dx}{2y} = \dfrac{dy}{2x} \implies x^2 - y^2 = c$



▶ **6-62.** (c) Vectors $\vec{A}$ and $\vec{B}$ form a plane. $\vec{N} = \vec{A} \times \vec{B}$ is normal to plane and has the same direction as $\vec{r} - \vec{r}_0$.

(d) $(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1) = \vec{N}$ is normal to plane and $(\vec{r} - \vec{r}_1) \times \vec{N} = \vec{0}$ is the equation of the line.

▶ **6-63.** (b)

$$\vec{r} = \cos 2t\ \hat{\mathbf{e}}_1 + \sin 2t\ \hat{\mathbf{e}}_2$$
$$\vec{v} = \frac{d\vec{r}}{dt} = -2\sin 2t\ \hat{\mathbf{e}}_1 + 2\cos 2t\ \hat{\mathbf{e}}_2$$
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = -4\cos 2t\ \hat{\mathbf{e}}_1 - 4\sin 2t\ \hat{\mathbf{e}}_2$$

▶ **6-65.**

$$(a) \quad \frac{\partial \vec{F}}{\partial x} = 2x\, e1 + yz\ \hat{\mathbf{e}}_2 + 2xy^2z^2\ \hat{\mathbf{e}}_3$$
$$(d) \quad \frac{\partial^2 \vec{F}}{\partial x^2} = 2\,\hat{\mathbf{e}}_1 + 2y^2z^2$$

► **6-68.** If $\vec{r}_0$ is center of sphere , $\vec{r}_1$ is point on sphere where tangent plane is constructed and $\vec{r}$ is a general point on the tangent plane, then the vector $(\vec{r}-\vec{r}_1)$ must be perpendicular to the vector $\vec{r}_1-\vec{r}_0$.

► **6-69.**

$$\begin{aligned}\frac{\partial F}{\partial x} =& \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}\\ \frac{\partial^2 F}{\partial x^2} =& \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[\frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 F}{\partial u\,\partial v}\frac{\partial v}{\partial x}\right]\\ &+\frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\left[\frac{\partial^2 F}{\partial v\,\partial u}\frac{\partial u}{\partial x}+\frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial x}\right]\end{aligned}$$

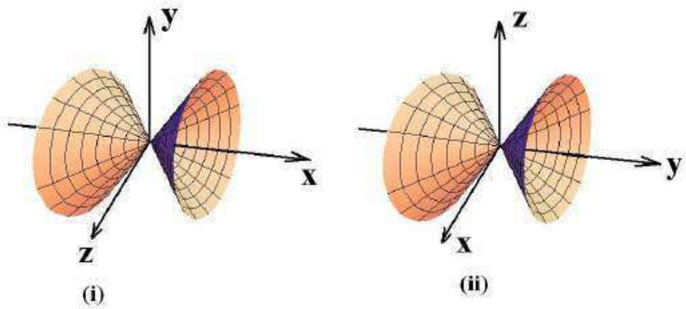
► **6-70.** (a) Use area of parallelogram $\vec{A}\times\vec{B}$ so that the area of 1/2 of parallelogram is $\frac{1}{2}\vec{A}\times\vec{B}$.

► **6-72.** (b) 58

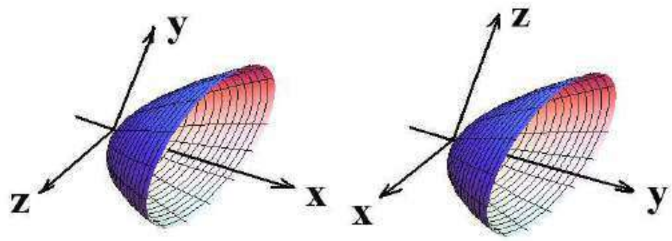
► **6-74.** (b) -12

Chapter 7

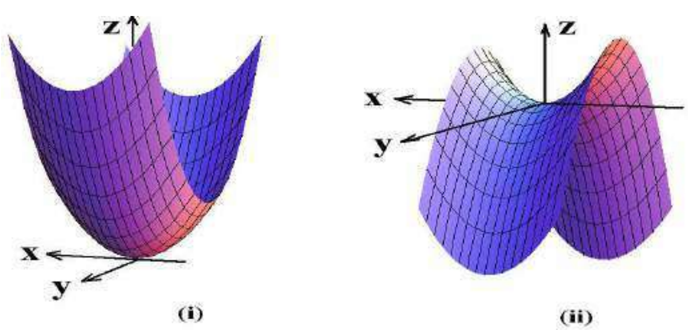
► **7-1.**



► **7-2.**



► **7-3.**



► **7-4.**

$$\vec{r} = \alpha\cos\omega t\ \hat{\mathbf{e}}_1 + \alpha\sin\omega t\ \hat{\mathbf{e}}_2 + \beta t\ \hat{\mathbf{e}}_3$$

$$\frac{d\vec{r}}{dt} = -\ \alpha\omega\sin\omega t\ \hat{\mathbf{e}}_1 + \alpha\omega\cos\omega t\ \hat{\mathbf{e}}_2 + \beta\ \hat{\mathbf{e}}_3$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\alpha^2\omega^2 + \beta^2}$$

unit normal $\quad \hat{\mathbf{e}}_t = \dfrac{-\alpha\omega\sin\omega t\ \hat{\mathbf{e}}_1 + \alpha\omega\cos\omega t\ \hat{\mathbf{e}}_2 + \beta\ \hat{\mathbf{e}}_3}{\sqrt{\alpha^2\omega^2+\beta^2}} = \dfrac{d\vec{r}}{ds} = \dfrac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}$

$$\frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{-\alpha\omega^2\cos\omega t\ \hat{\mathbf{e}}_1 - \alpha\omega^2\sin\omega t\ \hat{\mathbf{e}}_2}{\sqrt{\alpha^2\omega^2+\beta^2}}$$

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{\frac{d\,\hat{\mathbf{e}}_t}{dt}}{\frac{ds}{dt}} = \frac{-\alpha\omega^2\cos\omega t\ \hat{\mathbf{e}}_1 - \alpha\omega^2\sin\omega t\ \hat{\mathbf{e}}_2}{\alpha^2\omega^2+\beta^2} = \kappa\ \hat{\mathbf{e}}_n, \qquad \hat{\mathbf{e}}_n \text{ is unit normal}$$

▶7-4 (Continued)

$$\kappa^2 = \frac{d\,\hat{\mathbf{e}}_t}{ds}\cdot\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{\alpha^2\omega^4}{(\alpha^2\omega^2+\beta^2)^2}$$

$$\text{Curvature} \qquad \kappa = \frac{\alpha\omega^2}{\alpha^2\omega^2+\beta^2}$$

$$\text{radius of curvature} \quad \rho = \frac{1}{\kappa} = \frac{\alpha^2\omega^2+\beta^2}{\alpha\omega^2}$$

$$\text{unit normal} \quad \hat{\mathbf{e}}_n = \frac{1}{\kappa}\frac{d\,\hat{\mathbf{e}}_t}{ds} = -\cos\omega t\ \hat{\mathbf{e}}_1 - \sin\omega t\ \hat{\mathbf{e}}_2$$

$$\text{unit binormal} \quad \hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n = \beta\sin\omega t\ \hat{\mathbf{e}}_1 - \beta\cos\omega t\ \hat{\mathbf{e}}_2 + \frac{\alpha\omega}{\sqrt{\alpha^2\omega^2+\beta^2}}\ \hat{\mathbf{e}}_3$$

$$\frac{d\,\hat{\mathbf{e}}_b}{ds} = \frac{\frac{d\,\hat{\mathbf{e}}_b}{dt}}{\frac{ds}{dt}} = \frac{\beta\omega\cos\omega t\ \hat{\mathbf{e}}_1 + \beta\omega\sin\omega t\ \hat{\mathbf{e}}_2}{\alpha^2\omega^2+\beta^2} = -\tau\ \hat{\mathbf{e}}_n$$

$$\tau^2 = \frac{d\,\hat{\mathbf{e}}_b}{ds}\cdot\frac{d\,\hat{\mathbf{e}}_b}{ds} = \frac{\beta^2\omega^2}{(\alpha^2\omega^2+\beta^2)^2}$$

$$\tau = \frac{\beta\omega}{\alpha^2\omega^2+\beta^2}, \qquad \sigma = \frac{1}{\tau} = \frac{\alpha^2\omega^2+\beta^2}{\beta\omega}$$

▶ 7-5.

$$\vec{r}\,' = \frac{d\vec{r}}{ds} \qquad \left(\frac{ds}{dt}\right)^2 = \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = \vec{r}\,'\cdot\vec{r}\,', \qquad \frac{ds}{dt} = (\vec{r}\,'\cdot\vec{r}\,')^{1/2}$$

$$\hat{\mathbf{e}}_t = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}, \qquad \frac{d\,\hat{\mathbf{e}}_t}{dt} = \frac{\frac{ds}{dt}\frac{d^2\vec{r}}{dt^2} - \frac{d\vec{r}}{dt}\frac{d^2 s}{dt^2}}{(\frac{ds}{dt})^2}$$

$$\frac{d\,\hat{\mathbf{e}}_t}{ds} = \frac{\frac{d\,\hat{\mathbf{e}}_t}{dt}}{\frac{ds}{dt}} = \frac{\vec{r}\,''}{\vec{r}\,'\cdot\vec{r}\,'} - \frac{\vec{r}\,'\frac{d^2 s}{dt^2}}{\vec{r}\,'\cdot\vec{r}\,'\,\frac{ds}{dt}} = \kappa\ \hat{\mathbf{e}}_n$$

$$= \frac{\vec{r}\,''}{\vec{r}\,'\cdot\vec{r}\,'} - \frac{\frac{\vec{r}\,'(\vec{r}\,'\cdot\vec{r}\,'')}{(\vec{r}\,'\cdot\vec{r}\,')^{1/2}}}{\vec{r}\,'\cdot\vec{r}\,'\,(\vec{r}\,'\cdot\vec{r}\,')^{1/2}} = \frac{\vec{r}\,''}{\vec{r}\,'\cdot\vec{r}\,'} - \frac{\vec{r}\,'\,(\vec{r}\,'\cdot\vec{r}\,'')}{(\vec{r}\,'\cdot\vec{r}\,')^2} = \kappa\ \hat{\mathbf{e}}_n$$

$$\kappa^2 = (\kappa\ \hat{\mathbf{e}}_n)\cdot(\kappa\ \hat{\mathbf{e}}_n) = \frac{(\vec{r}\,'\cdot\vec{r}\,')(\vec{r}\,''\cdot\vec{r}\,'') - (\vec{r}\,'\cdot\vec{r}\,'')^2}{(\vec{r}\,'\cdot\vec{r}\,')^3}$$

▶ 7-6. Special case of previous problem

$$\vec{r} = x\,\hat{\mathbf{e}}_1 + y(x)\,\hat{\mathbf{e}}_2, \quad \vec{r}\,' = \hat{\mathbf{e}}_1 + y'\ \hat{\mathbf{e}}_2, \quad \vec{r}\,'' = y''\ \hat{\mathbf{e}}_2$$

and

$$\vec{r}\,'\cdot\vec{r}\,' = 1+(y')^2, \qquad \vec{r}\,'\cdot\vec{r}\,'' = y'y'', \qquad \vec{r}\,''\cdot\vec{r}\,'' = (y'')^2$$

giving

$$\kappa = \frac{\sqrt{(1+(y')^2)(y'')^2 - (y')^2(y'')^2}}{(1+(y')^2)^{3/2}} \qquad \Longrightarrow \qquad \kappa = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

► **7-7.**

$$\frac{d\vec{r}}{dt} = \vec{r}\,', \qquad \frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{d\vec{r}}{dt}, \qquad \frac{ds}{dt} = s' = (\vec{r}\,'\cdot\vec{r}\,')^{1/2}$$

$$\frac{d\vec{r}}{ds} = \hat{\mathbf{e}}_t = \frac{\vec{r}\,'}{s'}, \qquad \frac{d^2s}{dt^2} = s'' = \frac{\vec{r}\,'\cdot\vec{r}\,''}{(\vec{r}\,'\cdot\vec{r}\,')^{1/2}}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\,\hat{\mathbf{e}}_t}{ds} = \kappa\,\hat{\mathbf{e}}_n = \frac{s'\vec{r}\,'' - \vec{r}\,'s''}{(s')^3}, \qquad \frac{d\vec{r}}{ds}\times\frac{d^2\vec{r}}{ds^2} = \kappa\,\hat{\mathbf{e}}_t\times\hat{\mathbf{e}}_n = \kappa\,\hat{\mathbf{e}}_b$$

► **7-8.** See derivatives from previous problem.

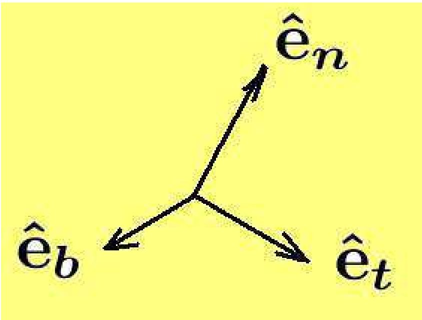
$$\frac{d^3\vec{r}}{ds^3} = \kappa\frac{d\,\hat{\mathbf{e}}_n}{ds} + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n = \kappa(\tau\,\hat{\mathbf{e}}_b - \kappa\,\hat{\mathbf{e}}_t) + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n$$

$$\frac{d^3\vec{r}}{ds^3} = \kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n$$

$$\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3} = \kappa\,\hat{\mathbf{e}}_n\times(\kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n) = \kappa^2\tau\,\hat{\mathbf{e}}_t + \kappa^3\,\hat{\mathbf{e}}_b$$

$$\frac{d\vec{r}}{ds}\cdot(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}) = \hat{\mathbf{e}}_t\cdot(\kappa^2\tau\,\hat{\mathbf{e}}_t + \kappa^3\,\hat{\mathbf{e}}_b) = \kappa^2\tau$$

$$\tau = \frac{1}{\kappa^2}\frac{d\vec{r}}{ds}\cdot(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3})$$



In terms of a parameter t one can write

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}, \qquad \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{r}}{ds}\frac{d^2s}{dt^2} + \frac{d^2\vec{r}}{ds^2}(\frac{ds}{dt})^2$$

$$\frac{d^3\vec{r}}{dt^3} = \frac{d\vec{r}}{ds}\frac{d^3s}{dt^3} + \frac{d^2\vec{r}}{ds^2}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{d^2\vec{r}}{ds^2}2(\frac{ds}{dt})\frac{d^2s}{dt^2} + \frac{d^3\vec{r}}{ds^3}(\frac{ds}{dt})^3$$

which can be express

$$\vec{r}\,'' = \hat{\mathbf{e}}_t\frac{d^2s}{dt^2} + \kappa\,\hat{\mathbf{e}}_n(\frac{ds}{dt})^2$$

$$\vec{r}\,''' = \hat{\mathbf{e}}_t\frac{d^3s}{dt^3} + \kappa\,\hat{\mathbf{e}}_n\frac{ds}{dt}\frac{d^2s}{dt^2} + \kappa\,\hat{\mathbf{e}}_n 2\frac{ds}{dt}\frac{d^2s}{dt^2} + (\frac{ds}{dt})^3[\kappa\tau\,\hat{\mathbf{e}}_b - \kappa^2\,\hat{\mathbf{e}}_t + \frac{d\kappa}{ds}\,\hat{\mathbf{e}}_n]$$

$$\vec{r}\,''\times\vec{r}\,''' = (stuff_1)\,\hat{\mathbf{e}}_b + (stuff_2)\,\hat{\mathbf{e}}_n + \kappa^2\tau(\frac{ds}{dt})^5\,\hat{\mathbf{e}}_t$$

$$\vec{r}\,'\cdot(\vec{r}\,''\times\vec{r}\,''') = \kappa^2\tau(\frac{ds}{dt})^6, \qquad \text{but} \quad (\frac{ds}{dt})^6 = (\vec{r}\,'\cdot\vec{r}\,')^3$$

and κ^2 can be obtained from problem 7-5 to obtain

$$\tau = \frac{\vec{r}\,'\cdot(\vec{r}\,''\times\vec{r}\,''')}{(\vec{r}\,'\cdot\vec{r}\,')(\vec{r}\,''\cdot\vec{r}\,'') - (\vec{r}\,'\cdot\vec{r}\,'')^2}$$

► **7-9.** (a) zero (b) zero

▶ **7-10.**

$$\frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{d\vec{r}}{dt} = \vec{r}\,' \quad \text{or} \quad \hat{\mathbf{e}}_t\frac{ds}{dt} = \vec{r}\,', \qquad |\vec{r}\,'| = \frac{ds}{dt}$$

$$\frac{d}{dt}\left(\hat{\mathbf{e}}_t\frac{ds}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \vec{r}\,''$$

$$\hat{\mathbf{e}}_t\frac{d^2s}{dt^2} + \frac{ds}{dt}\frac{d\,\hat{\mathbf{e}}_t}{ds}\frac{ds}{dt} = \vec{r}\,''$$

$$\hat{\mathbf{e}}_t\frac{d^2s}{dt^2} + (\frac{ds}{dt})^2\kappa\,\hat{\mathbf{e}}_n = \vec{r}\,''$$

$$\vec{r}\,' \times \vec{r}\,'' = \hat{\mathbf{e}}_t\frac{ds}{dt} \times (\,\hat{\mathbf{e}}_t\frac{d^2s}{dt^2} + (\frac{ds}{dt})^2\kappa\,\hat{\mathbf{e}}_n) = (\frac{ds}{dt})^3\kappa\,\hat{\mathbf{e}}_b = |\vec{r}\,'|^3\kappa\,\hat{\mathbf{e}}_b$$

$$|\vec{r}\,' \times \vec{r}\,''| = \kappa|\vec{r}\,'|^3$$

▶ **7-11.** (i)

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot \hat{\mathbf{e}}\,\Big|_{(1,1,1)}$$

$$= [(2xy^2z)\,\hat{\mathbf{e}}_1 + 2yx^2z\,\hat{\mathbf{e}}_2 + (x^2y^2 + x^3)\,\hat{\mathbf{e}}_3] \cdot \left(\frac{3\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + 6\,\hat{\mathbf{e}}_3}{7}\right)\Big|_{(1,1,1)} = \frac{23}{7}$$

▶ **7-12.**

$$(i) \qquad \text{grad}\,\phi = 2xy\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2$$

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot\,\hat{\mathbf{e}}_\alpha = (2xy\,\hat{\mathbf{e}}_1 + x^2\,\hat{\mathbf{e}}_2)\cdot(\cos\alpha\,\hat{\mathbf{e}}_1 + \sin\alpha\,\hat{\mathbf{e}}_2)\Big|_{(2,\sqrt{3})}$$

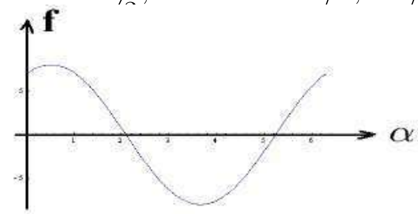
$$\frac{d\phi}{ds} = 4\sqrt{3}\cos\alpha + 4\sin\alpha = f(\alpha)$$

$$(ii) \qquad \frac{df}{d\alpha} = -4\sqrt{3}\sin\alpha + 4\cos\alpha = 0, \implies \tan\alpha = \frac{1}{\sqrt{3}}, \implies \alpha = \pi/6,\ 7\pi/6$$

$$f''(\alpha) = -4\sqrt{3}\cos\alpha - 4\sin\alpha$$

$f''(\pi/6) < 0$ maximum at $\pi/6$

$f''(7\pi/6) > 0$ minimum at $7\pi/6$



▶ **7-13.** Show derivative equals

$$\vec{r}\cdot(\vec{r}\,' \times \vec{r}\,''') + \vec{r}\cdot(\vec{r}\,'' \times \vec{r}\,'') + \vec{r}\,'\cdot(\vec{r}\,' \times \vec{r}\,'')$$

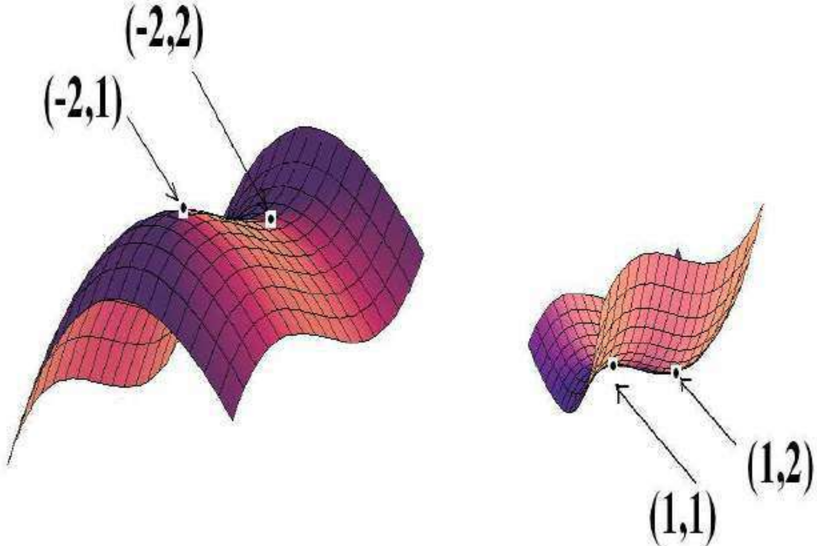
where last two terms are zero. Use triple scalar product relation on last term.

▶ **7-14.** Use the vector identity $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$ and show

$$\begin{aligned}\vec{A}\times(\vec{B}\times\vec{C}) =&\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})\\ \vec{B}\times(\vec{C}\times\vec{A}) =&\vec{C}(\vec{B}\cdot\vec{A})-\vec{A}(\vec{B}\cdot\vec{C})\\ \vec{C}\times(\vec{A}\times\vec{B}) =&\vec{A}(\vec{C}\cdot\vec{B})-\vec{B}(\vec{C}\cdot\vec{A})\end{aligned}$$

Add both sides to obtain desired result.

▶ **7-15.**



$$\frac{\partial z}{\partial x} =x^2+x+2, \qquad \frac{\partial z}{\partial y} = y^2-3y+2$$

critical points are where $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$ simultaneously

critical points $(-2,2),\quad (-2,1),\quad (1,2),\ (1,1)$

$A=\dfrac{\partial^2 z}{\partial x^2}=2x+1=A,\qquad \dfrac{\partial^2 z}{\partial x\,\partial y}=0=B,\qquad \dfrac{\partial^2 z}{\partial y^2}=2y-3=C,\qquad \Delta=AC-B^2$

At $(2,2),\ \Delta=-3<0,$ saddle point

At $(-2,1),\ \Delta=3>0,\ A<0,$ relative minimum

At $(1,2),\ \Delta=3>0,\ A>0,$ relative minimum

At $(1,1),\ \Delta=-3<0,$ saddle point

▶ **7-16.** $\vec{\omega}=\alpha\,\hat{\mathbf{e}}_t+\beta\,\hat{\mathbf{e}}_n+\gamma\,\hat{\mathbf{e}}_b$ and if

$$\omega\times\ \hat{\mathbf{e}}_t=\kappa\,\hat{\mathbf{e}}_n,\quad \omega\times\ \hat{\mathbf{e}}_b=-\tau\,\hat{\mathbf{e}}_n,\quad \omega\times\ \hat{\mathbf{e}}_n=\tau\,\hat{\mathbf{e}}_b-\kappa\,\hat{\mathbf{e}}_t$$

show that $\alpha=\tau,\beta=0,\gamma=\kappa$ so that $\omega=\tau\,\hat{\mathbf{e}}_t+\kappa\,\hat{\mathbf{e}}_b$

▶ **7-17.** Vector equation of plane is $(\vec{r}-\vec{r}_0)\cdot\vec{N}=0$ where $\vec{r}=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3$ is variable point in plane, $\vec{r}_0$ is fixed point in plane and $\vec{N}$ is normal to plane. Note that

$$\begin{aligned}\frac{d\vec{r}}{ds} =&\hat{\mathbf{e}}_t \text{ is normal to normal plane}\\ \frac{d\vec{r}}{ds}\times\frac{d^2\vec{r}}{ds^2} =&\hat{\mathbf{e}}_t\times\kappa\,\hat{\mathbf{e}}_n=\kappa\,\hat{\mathbf{e}}_b \text{ is normal to osculating plane}\\ \frac{d^2\vec{r}}{ds^2} =&\kappa\,\hat{\mathbf{e}}_n \text{ is normal to rectifying plane}\end{aligned}$$

▶ **7-18.** If $\vec{r}(u,v)=x(u,v)\,\hat{\mathbf{e}}_1+y(u,v)\,\hat{\mathbf{e}}_2+z(u,v)\,\hat{\mathbf{e}}_3$, then

$$\begin{aligned}&\frac{\partial\vec{r}}{\partial u}=\frac{\partial x}{\partial u}\,\hat{\mathbf{e}}_1+\frac{\partial y}{\partial u}\,\hat{\mathbf{e}}_2+\frac{\partial z}{\partial u}\,\hat{\mathbf{e}}_3 \text{ is tangent to coordinate curve } \vec{r}(u,v_0)\\ &\frac{\partial\vec{r}}{\partial v}=\frac{\partial x}{\partial v}\,\hat{\mathbf{e}}_1+\frac{\partial y}{\partial v}\,\hat{\mathbf{e}}_2+\frac{\partial z}{\partial v}\,\hat{\mathbf{e}}_3 \text{ is tangent to coordinate curve } \vec{r}(u_0,v)\\ &\vec{N}=\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v} \text{ is normal to surface and}\\ &\hat{\mathbf{e}}_n=\frac{\vec{N}}{|\vec{N}|}=\ell_1\,\hat{\mathbf{e}}_1+\ell_2\,\hat{\mathbf{e}}_2+\ell_3\,\hat{\mathbf{e}}_3 \text{ is unit normal to surface where}\\ &|\vec{N}|=\sqrt{(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v})\cdot(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v})}=\sqrt{EG-F^2}\end{aligned}$$

▶ **7-19.** $\vec{N}=\operatorname{grad}F=\frac{\partial F}{\partial x}\,\hat{\mathbf{e}}_1+\frac{\partial F}{\partial y}\,\hat{\mathbf{e}}_2+\frac{\partial F}{\partial z}\,\hat{\mathbf{e}}_3$ and $\hat{\mathbf{e}}_n=\frac{\vec{N}}{|\vec{N}|}$ where $|\vec{N}|=H$

▶ **7-20.** If surface is $\vec{r}=\vec{r}(x,y)=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z(x,y)\,\hat{\mathbf{e}}_3$, then

$$\frac{\partial\vec{r}}{\partial x}=\hat{\mathbf{e}}_1+\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_3 \qquad \frac{\partial\vec{r}}{\partial y}=\hat{\mathbf{e}}_2+\frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_3$$

and $\vec{N}=\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y}=-\frac{\partial z}{\partial x}\,\hat{\mathbf{e}}_1-\frac{\partial z}{\partial y}\,\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3$ with $\hat{\mathbf{e}}_n=\frac{\vec{N}}{|\vec{N}|}$ and $|\vec{N}|=\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}$

▶ **7-21.** $\hat{\mathbf{e}}_n=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2$ or $\hat{\mathbf{e}}_n=\cos\theta\,\hat{\mathbf{e}}_1+\sin\theta\,\hat{\mathbf{e}}_2$

▶ **7-22.** $\hat{\mathbf{e}}_n=x\,\hat{\mathbf{e}}_1+y\,\hat{\mathbf{e}}_2+z\,\hat{\mathbf{e}}_3$ or $\hat{\mathbf{e}}_n=\sin\theta\cos\phi\,\hat{\mathbf{e}}_1+\sin\theta\sin\phi\,\hat{\mathbf{e}}_2+\cos\theta\,\hat{\mathbf{e}}_3$

▶ **7-23.** $\phi=ax+by+cz-d=0$ and $\vec{N}=\operatorname{grad}\phi=a\,\hat{\mathbf{e}}_1+b\,\hat{\mathbf{e}}_2+c\,\hat{\mathbf{e}}_3$ so that

$$\hat{\mathbf{e}}_n=\frac{a\,\hat{\mathbf{e}}_1+b\,\hat{\mathbf{e}}_2+c\,\hat{\mathbf{e}}_3}{\sqrt{a^2+b^2+c^2}}$$

▶ **7-24.** $\frac{1}{2}\,\hat{\mathbf{e}}_1+\frac{\pi}{8}\,\hat{\mathbf{e}}_2$

▶ **7-25.** $\frac{1}{15}\,\hat{\mathbf{e}}_1+\frac{1}{15}\,\hat{\mathbf{e}}_2+\frac{1}{15}\,\hat{\mathbf{e}}_3$

▶ **7-26.** 2π

▶ **7-27.** $\dfrac{19}{24}$

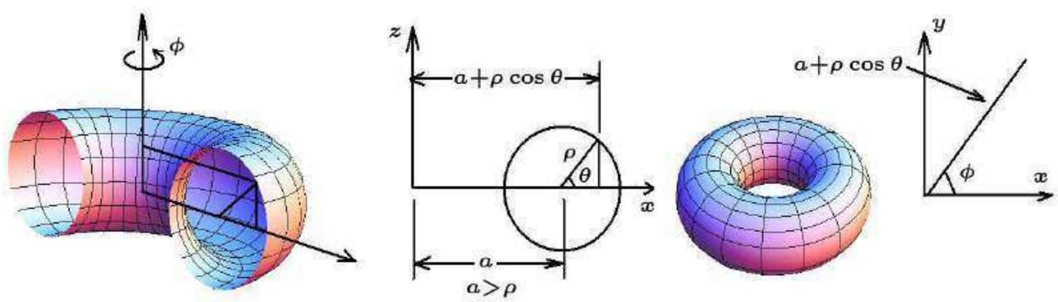
▶ **7-28.** π

▶ **7-30.** If $\vec{r} = \vec{r}(u,v)$, then $d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv$ and

$$ds^2 = d\vec{r}\cdot d\vec{r} = \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial u}\,(du)^2 + 2\frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v}\,du\,dv + \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial v}\,(dv)^2$$

▶ **7-31.** (b) $S = \pi R\sqrt{R^2+H^2}$

▶ **7-32.**



$dS = \rho(a+\rho\cos\theta)\,d\theta\,d\phi$ and $dV = \rho(a+\rho\cos\theta)\,d\rho\,d\theta\,d\phi$ Volume is $V = 2\pi^2 a\rho^2$ and surface area is $S = 4\pi^2 a\rho$

What does the Pappus theorem tell you about the volume?

▶ **7-33.** (d) $ds = \sqrt{1+(y')^2}\,dx$ where $y = x^2$, $\quad y' = 2x$ so that

$$S = \int_0^2 \sqrt{1+(2x)^2}\,dx$$

To evaluate this integral make the substitution $2x = \sinh u$ with $2dx = \cosh u\,du$ and show

$$S = \int_0^{\sinh^{-1}(4)} \frac{1}{2}\cosh^2 u\,du = \frac{1}{2}\int_0^{\sinh^{-1}(4)} [\frac{1}{2}\cosh 2u + \frac{1}{2}]\,du$$

Show that

$$S = \sqrt{17} + \frac{1}{4}\sinh^{-1}(4)$$

▶ **7-34.** (a) x, y-plane, coordinate curves $\vec{r}(u_0, v)$ vertical lines, $\vec{r}(u, v_0)$ horizontal lines
(b) x, y-plane polar coordinates, $\vec{r}(u_0, v)$ circles of radius u_0, $\vec{r}(u, v_0)$ rays at angle v_0.

▶ **7-35.** $I = 4\int_0^1 \int_0^{1-x} dydx = 2$

▶ **7-36.**

$$I = \iint_S \vec{F}\cdot \hat{\mathbf{e}}_n\, dS = \left[\iint_{OCDG} + \iint_{GFA0} + \iint_{FABE} + \iint_{BEDC} + \iint_{EDGF} + \iint_{ABC0}\right] \vec{F}\cdot \hat{\mathbf{e}}_n\, dS$$

On face 0CDG $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_1,\ \vec{F}\cdot\hat{\mathbf{e}}_n = -x^2\Big|_{x=0} = 0,\ dS = dydz$

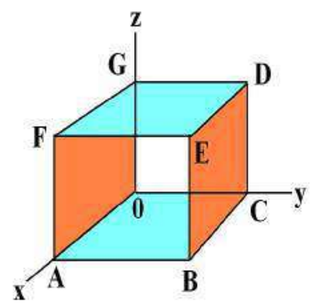
On face GFA0 $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_2,\ \vec{F}\cdot\hat{\mathbf{e}}_n = -y^2\Big|_{y=0} = 0,\ dS = dxdz$

On face FABE $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_1,\ \vec{F}\cdot\hat{\mathbf{e}}_n = x^2\Big|_{x=1} = 1,\ dS = dydz$

On face BEDC, $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_2,\ \vec{F}\cdot\hat{\mathbf{e}}_n = y^2\Big|_{y=1} = 1,\ dS = dxdz$

On face EDGF, $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_3,\ \vec{F}\cdot\hat{\mathbf{e}}_n = z^2\Big|_{z=1} = 1,\ dS = dxdy$

On face ABC0, $\hat{\mathbf{e}}_n = -\hat{\mathbf{e}}_3,\ \vec{F}\cdot\hat{\mathbf{e}}_n = -z^2\Big|_{z=0} = 0,\ dS = dxdy$



Add the above surface integrals over each face and show $I = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\, dS = 1+1+1 = 3$

▶ **7-37.**

$$\phi = x^2 + y^2 - 1 = 0, \quad 0 \le z \le 3, \quad \vec{N} = \text{grad}\,\phi = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2$$

$$\hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 \quad \text{since } x^2+y^2 = 1, \qquad dS = \frac{dxdz}{|\,\hat{\mathbf{e}}_n\cdot\hat{\mathbf{e}}_2|} = \frac{dxdz}{y}$$

$$I = \iint_S f(x,y,z)\,dS = \int_0^3 \int_0^1 2(x+1)y\,\frac{dxdz}{y} = 9$$

▶ **7-38.** Method I Use cartesian coordinates

$$\hat{\mathbf{e}}_n = \frac{x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3}{3} \qquad \frac{dS}{=}\frac{dxdy}{|\,\hat{\mathbf{e}}_n\cdot\hat{\mathbf{e}}_3|} = 3\frac{dxdy}{z}$$

$$\vec{F}\cdot\hat{\mathbf{e}}_n = \frac{x(x+z) + y(y+z) - (x+y)z}{3} = \frac{x^2+y^2}{3}$$

$$I = \iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = \int_{-3}^{3}\int_{y=-\sqrt{9-x^2}}^{+\sqrt{9-x^2}} \frac{x^2+y^2}{\sqrt{9-x^2-y^2}}\,dydx = 36\pi$$

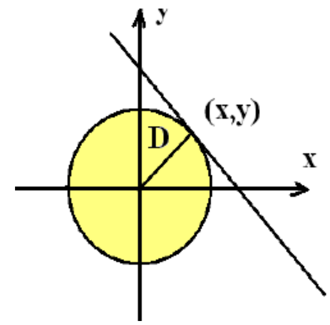
▶ **7-38.** (continued)

Method II Use spherical coordinates $x = 3\sin\theta\cos\phi,\ y = 3\sin\theta\sin\phi,\ z = 3\cos\theta$

$$dS = 3d\theta\, 3\sin\theta\, d\phi, \qquad \vec{F}\cdot\hat{\mathbf{e}}_n = \frac{x^2+y^2}{3}$$

$$I = \iint_S (3)(x^2+y^2)\sin\theta\, d\theta d\phi = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(3)(9\sin^2\theta\cos^2\phi + 9\sin^2\theta\sin^2\phi)\,\sin\theta\, d\theta d\phi$$

$$I = 27\int_0^{\pi/2}\int_0^{2\pi}\sin^3\theta\, d\theta d\phi = 36\pi$$



▶ **7-40.** $G = x+y-2 = 0$ and $D^2 = F(x,y) = x^2+y^2$ one finds $\text{grad}\, F = \nabla F = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2$ and $\text{grad}\, G = \nabla G = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$ Let $\vec{r} = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2$ denote position vector to point on circle, then $\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 + \frac{dy}{dx}\hat{\mathbf{e}}_2$ is tangent to circle. Differentiate equation of circle to show $2x + 2y\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{x}{y}$, then one can write $\frac{d\vec{r}}{dx} = \hat{\mathbf{e}}_1 - \frac{x}{y}\hat{\mathbf{e}}_2$ is tangent to circle. Observe that $\frac{d\vec{r}}{dx}\cdot\nabla F = 2x + 2y\frac{-x}{y} = 0$ which shows ∇F is perpendicular to $\frac{d\vec{r}}{dx}$ or ∇F is perpendicular to line. The slope of the line is -1 and the vectors $\vec{r}_1 = 2\,\hat{\mathbf{e}}_1,\quad \vec{r}_2 = 2\,\hat{\mathbf{e}}_2$ point to points on the line so that $\vec{\alpha} = \vec{r}_1 - \vec{r}_2$ is a direction vector of the line. Note that $\vec{\alpha}\cdot\nabla G = (2\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2)\cdot(\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2) = 0$ so that ∇G is perpendicular to line.

The quantity D^2 is a minimum when the circle just touches the line and at this point of contact the normal to the circle is ∇G and this vector is also perpendicular to the line and has the same direction as ∇F. Consequently, there exists a scalar λ such that $\nabla F + \lambda\nabla G = 0$ at this point of contact.

Here $H = F + \lambda G = x^2 + y^2 + \lambda(x+y-2)$ with

$$\begin{aligned}\frac{\partial H}{\partial \lambda} &= x + y - 2 = 0 \quad \text{constraint equation}\\ \frac{\partial H}{\partial x} &= 2x + \lambda = 0\\ \frac{\partial H}{\partial y} &= 2y + \lambda = 0\end{aligned}$$

where the last two equations are a necessary condition for H to have an extreme value. Solve the above system of equations and show $x = 1,\ y = 1,\ \lambda = -2$ so that the minimum distance from the origin to the line is $\sqrt{2}$.

▶ **7-41.**

$$\begin{aligned} F =&\omega + \lambda_1 g + \lambda_2 h = x^2 + y^2 + z^2 + \lambda_1 g + \lambda_2 h \\ \frac{\partial F}{\partial \lambda_1} =&g = x + y + z - 6 = 0 \\ \frac{\partial F}{\partial \lambda_2} =&h = 3x + 5y + 7z - 34 = 0 \\ \frac{\partial F}{\partial x} =&2x + \lambda_1 + 3\lambda_2 = 0 \\ \frac{\partial F}{\partial y} =&2y + \lambda_1 + 5\lambda_2 = 0 \\ \frac{\partial F}{\partial z} =&2z + \lambda_1 + 7\lambda_2 = 0 \end{aligned}$$

Solve this system of equations and show

$$x = 1, \quad y = 2, \quad z = 3, \quad \lambda_1 = 1, \quad \lambda_2 = -1$$

▶ **7-42.**

$$\nabla F = -2z\,\hat{\mathbf{e}}_2 + (3y-1)\,\hat{\mathbf{e}}_3 \quad \text{and} \quad \hat{\mathbf{e}}_n = x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3, \quad \text{with } dS = \frac{dxdy}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_3|} = \frac{dxdy}{z}$$

$$I = \int_1^1 \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} (y-1)dydx = -\pi$$

Problem can also be solved using spherical coordinates $x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta$

▶ **7-44.** (b) $\dfrac{\partial \vec{r}}{\partial x} = \hat{\mathbf{e}}_1 + \dfrac{\partial y}{\partial x}\,\hat{\mathbf{e}}_2, \quad \dfrac{\partial \vec{r}}{\partial z} = \dfrac{\partial y}{\partial z}\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ giving

$$\begin{aligned} E =&\frac{\partial \vec{r}}{\partial x} \cdot \frac{\partial \vec{r}}{\partial x} = 1 + (\frac{\partial y}{\partial x})^2 \\ F =&\frac{\partial \vec{r}}{\partial x} \cdot \frac{\partial \vec{r}}{\partial z} = \frac{\partial y}{\partial x}\frac{\partial y}{\partial z} \\ G =&\frac{\partial \vec{r}}{\partial z} \cdot \frac{\partial \vec{r}}{\partial z} = (\frac{\partial y}{\partial z})^2 + 1 \end{aligned}$$

Show

$$dS = \sqrt{EG - F^2} = \sqrt{1 + (\frac{\partial y}{\partial x})^2 + (\frac{\partial y}{\partial z})^2}$$

Normal to surface is $\vec{N} = \dfrac{\partial \vec{r}}{\partial z} \times \dfrac{\partial \vec{r}}{\partial x} = -\dfrac{\partial y}{\partial x}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \dfrac{\partial y}{\partial z}\,\hat{\mathbf{e}}_3$ Note that if you use $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial z}$ you get $-\vec{N}$

The vector element of area is $d\vec{S} = \hat{\mathbf{e}}_n\,dS = (-\dfrac{\partial y}{\partial x}\,\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \dfrac{\partial y}{\partial z}\,\hat{\mathbf{e}}_3)\,dxdz$ Take the dot product of both sides of this equation with $\hat{\mathbf{e}}_2$ and show

$$|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|\,dS = dxdz, \qquad \text{or} \qquad dS = \frac{dxdz}{|\,\hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2|}$$

Here the absolute value sign is used to insure that the element of surface area dS is positive.

► **7-45.** $\vec{r}_c = \dfrac{\sum_{j=1}^{n} m_j \vec{r}_j}{\sum_{j=1}^{n} m_j}$ Note that this is a weighted sum of the vectors $\vec{r}_i$ where the weight factors are $m_1, m_2, \dots, m_n$.

► **7-46.** We have verified the triple scalar product $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ Change symbols and write $\vec{X}\cdot(\vec{C}\times\vec{D}) = \vec{D}\cdot(\vec{X}\times\vec{C})$ Substitute $\vec{X} = \vec{A}\times\vec{B}$ to show $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \vec{D}\left\{(\vec{A}\times\vec{B})\times\vec{C}\right\}$ Next one can employ the triple vector product relation

$$(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C})$$

to obtain

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

► **7-48.** (a) At extremum value for $E(\alpha,\beta) = \sum_{i=1}^{N}(\alpha x_i - \beta - y_i)^2$ require that

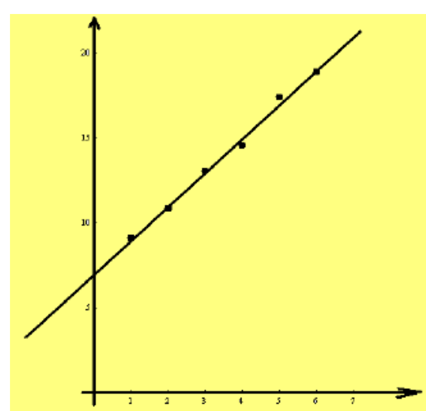
$$\frac{\partial E}{\partial \alpha} = \sum_{i=1}^{N} 2(\alpha x_i - \beta - y_i)x_i = 0$$
$$\frac{\partial E}{\partial \beta} = \sum_{i=1}^{N} 2(\alpha x_i - \beta - y_i)(-1) = 0$$

The above equations can be expressed in the form

$$\alpha \sum_{i=1}^{N} x_i^2 - \beta \sum_{i=1}^{n} x_i = \sum_{i=1}^{N} x_i y_i$$
$$\alpha \sum_{i=1}^{N} x_i - \beta \sum_{i=1}^{N} 1 = \sum_{i=1}^{N} y_i \qquad \text{where} \qquad \sum_{i=1}^{N} 1 = N$$

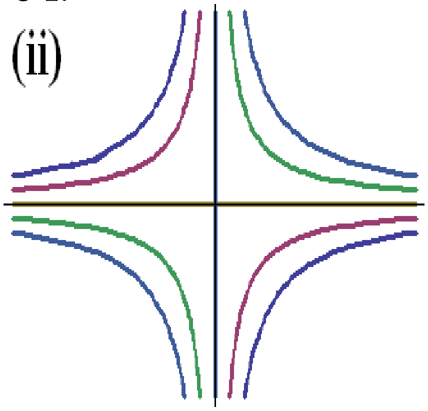
Solve this system of equations to obtain desired result.

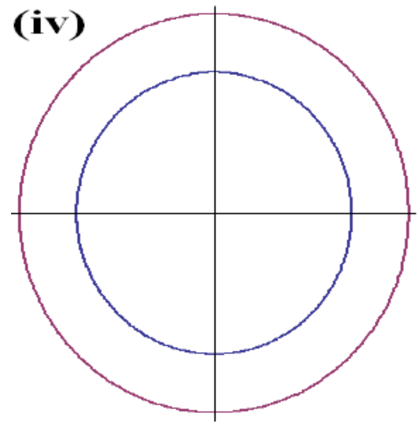
(b) $y = 7 + 2x$



Chapter 8

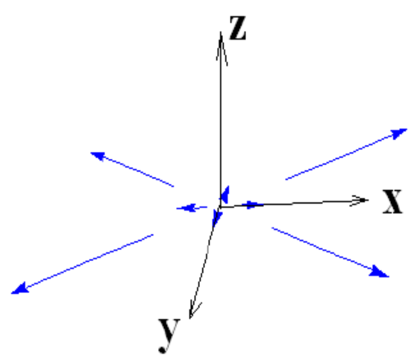
► **8-1.**





► **8-2.** (i) $\text{grad}\,\phi = 4\,\hat{\mathbf{e}}_1 - 3\,\hat{\mathbf{e}}_2$

(iii) $\text{grad}\,\phi = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2$ (ii)



► **8-3.** (iii) $z = x^2 + y^2$ paraboloid $\vec{N} = -2x\,\hat{\mathbf{e}}_1 - 2y\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\Big|_{3,4,25} = -6\,\hat{\mathbf{e}}_1 - 8\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

(iv) $z - xy = 0$ hyperbolic paraboloid $\vec{N} = -y\,\hat{\mathbf{e}}_1 - x\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\Big|_{2,3,6} = -3\,\hat{\mathbf{e}}_1 - 2\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

► **8-4.** $\dfrac{\partial z}{\partial x} = y = 0$ and $\frac{\partial z}{\partial y} = x = 0$ simultaneously at the origin.

► **8-5.** Normal to sphere $\vec{N}_1 = 2x\,\hat{\mathbf{e}}_1 + 2y\,\hat{\mathbf{e}}_2 + 2z\,\hat{\mathbf{e}}_3$ and normal to plane is $\vec{N}_2 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

At the point $(3,2,6)$ one finds $\vec{N}_1 = 6\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + 12\,\hat{\mathbf{e}}_3$ and $\vec{N}_2 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

A tangent vector to the curve of intersection is

$$\vec{T} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 6 & 4 & 12 \\ 1 & 1 & 1 \end{vmatrix} = -8\,\hat{\mathbf{e}}_1 + 6\,\hat{\mathbf{e}}_2 + 2\,\hat{\mathbf{e}}_3 \qquad \text{or} \qquad -\vec{T}$$

► **8-6.** (i) $r = \sqrt{x^2+y^2+z^2}$ and

$$\begin{aligned}\nabla r^n =&\frac{\partial r^n}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r^n}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r^n}{\partial z}\,\hat{\mathbf{e}}_3\\ =&nr^{n-1}\left[\frac{\partial r}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial r}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial r}{\partial z}\,\hat{\mathbf{e}}_3\right] = nr^{n-1}\left[\frac{x}{r}\,\hat{\mathbf{e}}_1 + \frac{y}{r}\,\hat{\mathbf{e}}_2 + \frac{z}{r}\,\hat{\mathbf{e}}_3\right]\\ =&nr^{n-2}\vec{r}\end{aligned}$$

(iii)

$$\begin{aligned}\nabla f(r) =&\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\,\hat{\mathbf{e}}_3\\ =&f'(t)\left[\frac{x}{r}\,\hat{\mathbf{e}}_1 + \frac{y}{r}\,\hat{\mathbf{e}}_2 + \frac{z}{r}\,\hat{\mathbf{e}}_3\right] = f'(r)\frac{\vec{r}}{|\vec{r}|} = f'(r)\frac{\vec{r}}{r}\end{aligned}$$

► **8-7.** Let $\vec{r}_1(\tau) = (\tau-1)\,\hat{\mathbf{e}}_1 + (16-\tau)\,\hat{\mathbf{e}}_2 + (2\tau-2)\,\hat{\mathbf{e}}_3$ and $\vec{r}_2(t) = -t\,\hat{\mathbf{e}}_1 + 2t\,\hat{\mathbf{e}}_2 + 3t\,\hat{\mathbf{e}}_3$ and define vector from line 1 to line 2 as $\vec{r}_2 - \vec{r}_1$. Minimize the distance squared

$$f(t,\tau) = |\vec{r}_2 - \vec{r}_1|^2 = (-t-\tau+1)^2 + (2t+\tau-16)^2 + (3t-2\tau+2)^2$$

by examining the critical points where $\dfrac{\partial f}{\partial \tau} = 0$ and $\dfrac{\partial f}{\partial t} = 0$ and show minimum occurs where $t=3$ and $\tau=5$ giving minimum distance $\sqrt{75}$

► **8-8.** Method I: Normal to plane is $\vec{N} = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$ and unit normal to plane is $\hat{\mathbf{e}}_N = \frac{\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_3}{\sqrt{3}}$. Consider vector $\vec{r}_1 = \hat{\mathbf{e}}_1$ projected onto $\hat{\mathbf{e}}_N$ to obtain distance $\frac{1}{\sqrt{3}}$

Method II: $f = d^2 = x^2+y^2+z^2$ is distance from origin to (x,y,z) squared. Here $z = 1-x-y$ so that $f = d^2 = x^2+y^2+(1-x-y)^2$ Show f has critical points at $x = 1/3$, $y = 1/3$ and $z = 1/3$ giving $d^2 = 1/3$ or $d = \frac{1}{\sqrt{3}}$

► **8-9.** (iii) $\dfrac{d\phi}{dn} = [(3x^2y+y^2)\,\hat{\mathbf{e}}_1 + (x^3+2xy)\,\hat{\mathbf{e}}_2]\cdot\hat{\mathbf{e}}_n$ where

on bottom of square $\hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_2,\ y=0 \implies \dfrac{d\phi}{dn} = -x^3$

on right side of square $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_1,\ x=1 \implies \dfrac{d\phi}{dn} = 3y+y^2$

on top of square $\hat{\mathbf{e}}_n = \hat{\mathbf{e}}_2,\ y=1 \implies \dfrac{d\phi}{dn} = x^3+2x$

on left side of square $\hat{\mathbf{e}}_n = -\,\hat{\mathbf{e}}_2,\ x=0 \implies \dfrac{d\phi}{dn} = 0$

► **8-10.** (ii) $z = (x-2)^2-(y-3)^2$ has critical points at $x=2$ and $y=3$. Here $A = \frac{\partial^2 z}{\partial x^2} = 2$, $C = \frac{\partial^2 z}{\partial y^2} = -2$ and $B = \frac{\partial^2 z}{\partial x\,\partial y} = 0$ gives $AC - B^2 = -4 < 0$ so there is a saddle point at the critical value.

▶ **8-13.** $V = \frac{1}{2}kx^2$ with $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$. Use $\vec{F} = \operatorname{grad} V = -kx\,\hat{\mathbf{e}}_1$. That is, if the spring is stretched in the positive direction a distance x, then the restoring force is in the negative direction and proportional to the displacement. This gives the equation of motion for the spring mass system as

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \omega^2 x = 0, \qquad \omega^2 = \frac{k}{m}$$

▶ **8-14.**

$$\begin{aligned}
\vec{F}(x+\Delta x, y, z) =& \vec{F}(x,y,z) + \frac{\partial \vec{F}}{\partial x}\,\Delta x + h.o.t.\\
\vec{F}(x, y+\Delta y, z) =& F(x,y,z) + \frac{\partial \vec{F}}{\partial y}\,\Delta y + h.o.t.\\
\vec{F}(x, y, z+\Delta z) =& \vec{F}(x,y,z) + \frac{\partial \vec{F}}{\partial z}\,\Delta z + h.o.t.
\end{aligned}$$

where $h.o.t.$ denotes "higher order terms" which are neglected.

The flux in the x-direction on face CGBF is $\vec{F}\cdot(-\hat{\mathbf{e}}_1)\,\Delta S = -F_1\,\Delta y\Delta z$ and the flux in the x-direction of the face DHAE is

$$(\vec{F} + \frac{\partial \vec{F}}{\partial x}\,\Delta x)\cdot(\,\hat{\mathbf{e}}_1)\,\Delta S = (F_1 + \frac{\partial F_1}{\partial x}\,\Delta x)\,\Delta y\Delta z$$

The flux in the y-direction on face DCBA is $\vec{F}\cdot(-\hat{\mathbf{e}}_2)\,\Delta S = -F_2\Delta x\Delta z$ and the flux in the y-direction on face HGEF is

$$(\vec{F} + \frac{\partial \vec{F}}{\partial y}\,\Delta y)\cdot(\,\hat{\mathbf{e}}_2)\,\Delta S = (F_2 + \frac{\partial F_2}{\partial y}\,\Delta y)\,\Delta x\Delta z$$

The flux in the z-direction on face AEFB is $\vec{F}\cdot(-\hat{\mathbf{e}}_3)\,\Delta S = F_3\,\Delta x\Delta y$ and the flux in the z-direction on face HGCD is

$$(\vec{F} + \frac{\partial \vec{F}}{\partial z}\,\Delta z)\cdot\,\hat{\mathbf{e}}_3\,\Delta S = (F_3 + \frac{\partial F_3}{\partial z}\,\Delta z)\,\Delta x\Delta y$$

Add the flux over each surface and show

$$Total\ \ Flux = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\,\Delta x\Delta y\Delta z$$

so that

$$\frac{Flux}{Volume} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \operatorname{div}\vec{F}$$

▶ **8-15.** (i) $\operatorname{div}\vec{F} = 2yz - 2x, \qquad \operatorname{curl}\vec{F} = \vec{0}$

(iii) $\operatorname{div}\vec{F} = 2y - 2z \qquad \operatorname{curl}\vec{F} = \vec{0}$

▶ **8-19.** If $\vec{V} = \nabla\phi$, then $\operatorname{div}\vec{V} = \nabla^2\phi = 0$ and $\operatorname{curl}\vec{V} = \operatorname{curl}(\nabla\phi) = \vec{0}$

▶ **8-21.** Only flux is from top surface and $\iint\limits_S \vec{F}\cdot d\vec{S} = 16\pi$ and from divergence theorem $\iiint\limits_V \operatorname{div}\vec{F}\,dV = 16\pi$

▶ **8-22.** (i) Evaluating the left and right-hand sides of the Green's theorem one finds the value -16/3. (ii) See page 192

▶ **8-23.** Both sides of the Stokes theorem yield the value $\pi/4$

▶ **8-24.** (i) $\phi = x^2y + xy^2 = C$ is family of solution curves.

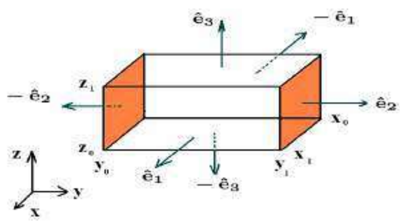
▶ **8-25.** Area$=\pi$

▶ **8-26.** $\iint\limits_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = \iiint\limits_V \operatorname{div}\vec{F}\,dV = 4\pi a^3$

▶ **8-27.** On sphere with radius $r=1$, $\iint\limits_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = -4\pi/3$ and on sphere with radius $r=2$ one finds $\iint_S \vec{F}\cdot\hat{\mathbf{e}}_n\,dS = 32\pi/3$ Total flux $= 32\pi/3 - 4\pi/3 = 28\pi/3$

▶ **8-28.** On inner surface flux is -2π and on outer surface flux is 8π. Zero flux across top and bottom surfaces. Total flux $= 8\pi - 2\pi = 6\pi$

▶ **8-29.** The divergence of $\vec{F}$ is zero and so the sum of the fluxes associated with the $\pm\hat{\mathbf{e}}_n$ faces must sum to zero. For example, $\int_{z_0}^{z_1}\int_{y_0}^{y_1} y\,dydz - \int_{z_0}^{z_1}\int_{y_0}^{y_1} y\,dydz = 0$



▶ **8-31.** $3V$

▶ **8-32.** If origin outside of S, then use the divergence theorem and show

$$\iint_S \frac{\hat{\mathbf{e}}_n \cdot \vec{r}}{r^3}\, dS = \iiint_V \nabla \cdot \left(\frac{\vec{r}}{r^3}\right)\, dV$$

and then show $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$

If origin is inside of S, then place a sphere of radius ϵ about the origin and show

$$\iint\limits_S \hat{\mathbf{e}}_n \cdot \frac{\vec{r}}{r^3}\, dS + \iint\limits_{S_\epsilon} \hat{\mathbf{e}}_n \cdot \frac{\vec{r}}{r^3}\, dS = \iiint_V \nabla \cdot \frac{\vec{r}}{r^3}\, dV = 0$$

On sphere of radius ϵ show that $\iint_{S_\epsilon} \hat{\mathbf{e}}_n \cdot \frac{\vec{r}}{r^3}\, dS = -4\pi$

▶ **8-33.** $5x^2yz^3 + 3xy^2z^2$

▶ **8-35.** Area $= \frac{1}{2}$ (base) (height)

▶ **8-36.** -162π

▶ **8-37.** $I = 216$

▶ **8-38.** 4

Problems 8-40 to 8-49 See equations (8.74) to (8.82)

▶ **8-50.** (c) If $\vec{r} = \vec{r}(u, v, w)$, then $d\vec{r} = \frac{\partial \vec{r}}{\partial u}\, du + \frac{\partial \vec{r}}{\partial v}\, dv + \frac{\partial \vec{r}}{\partial w}\, dw$ is the diagonal of a volume element in the shape of a parallelepiped having sides $\vec{A} = \frac{\partial \vec{r}}{\partial u}\, du$, $\vec{B} = \frac{\partial \vec{r}}{\partial v}\, dv$ and $\vec{C} = \frac{\partial \vec{r}}{\partial w}\, dw$. The volume of this elemental parallelepiped is

$$dV = |\vec{A} \cdot (\vec{B} \times \vec{C})| = |\frac{\partial \vec{r}}{\partial u} \cdot (\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w})|\, du dv dw$$

Use vector identity and orthogonality property $\frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial w} = 0$ to show

$$|\frac{\partial \vec{r}}{\partial u} \cdot (\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w})| = h_u h_v h_w$$

▶ **8-51.** Calculate $\text{grad}\, v \times \text{grad}\, w$ and then make use of the fact that mixed partial derivatives are equal to show $\text{div}\,(\text{grad}\, v \times \text{grad}\, w) = 0$

► **8-52.** If $\vec{A} = \alpha\vec{E}_1 + \beta\vec{E}_2 + \gamma\vec{E}_3$ and $\vec{E}_1 \cdot \vec{E}_2 = 0$, $\vec{E}_1 \cdot \vec{E}_3 = 0$, $\vec{E}_2 \cdot \vec{E}_3 = 0$, then show

$$\begin{aligned}
\vec{A} \cdot \vec{E}_1 =& \alpha\vec{E}_1 \cdot \vec{E}_1 \quad \text{or} \quad \alpha = \frac{\vec{A} \cdot \vec{E}_1}{\vec{E}_1 \cdot \vec{E}_1} \\
\vec{A} \cdot \vec{E}_2 =& \beta\vec{E}_2 \cdot \vec{E}_2 \quad \text{or} \quad \beta = \frac{\vec{A} \cdot \vec{E}_2}{\vec{E}_2 \cdot \vec{E}_2} \\
\vec{A} \cdot \vec{E}_3 =& \gamma\vec{E}_3 \cdot \vec{E}_3 \quad \text{or} \quad \gamma = \frac{\vec{A} \cdot \vec{E}_3}{\vec{E}_3 \cdot \vec{E}_3}
\end{aligned}$$

► **8-53.** If $\vec{E}^1$ had the same direction as $\vec{E}_2 \times \vec{E}_3$, then $\vec{E}^1 = \alpha\vec{E}_2 \times \vec{E}_3$. If $\vec{E}^1 \cdot \vec{E}_1 = 1$, then $1 = \alpha\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$ or $\alpha = \dfrac{1}{V}$ for $V = \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$

Do the same type of arguments for $\vec{E}^2$ and $\vec{E}^3$.

Reverse the roles of $\vec{E}_1, \vec{E}_2, \vec{E}_3$ with $\vec{E}^1, \vec{E}^2, \vec{E}^3$ to show

$$\vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3) = \frac{1}{V}(\vec{E}_2 \times \vec{E}_3) \cdot \left[\frac{1}{V}(\vec{E}_3 \times \vec{E}_1) \times \frac{1}{V}(\vec{E}_1 \times \vec{E}_2)\right]$$

then use the triple scalar product relation to show

$$\vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3) = \frac{1}{V^3}(\vec{E}_1 \times \vec{E}_2)\left[(\vec{E}_2 \times \vec{E}_3) \times (\vec{E}_3 \times \vec{E}_1)\right]$$

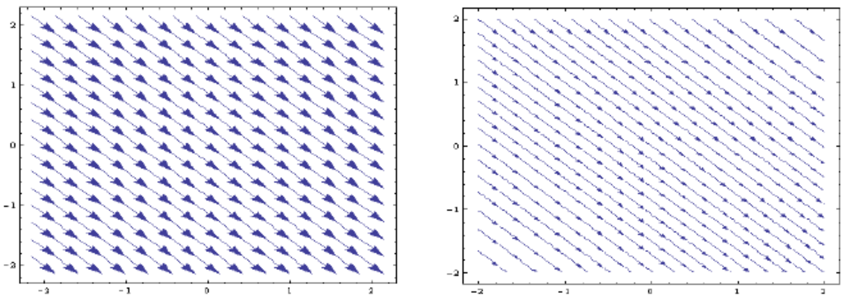
Use another vector identity to show

$$\vec{E}^1 \cdot (\vec{E}^2 \times \vec{E}^3) = \frac{1}{V^3}[\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)]^2 = \frac{1}{V^3}V^2 = \frac{1}{V}$$

Chapter 9

► **9-1.** $U = U(x) = c_1 x + c_2, \quad U = U(r) = c_1 \ln r + c_2, \quad U = U(\rho) = -c_1/\rho + c_2$

► **9-2.**

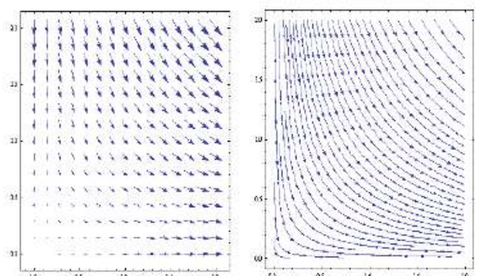


Streamlines $\sin \alpha x + \cos \alpha y = c$ and velocity field is derivable from potential function $\phi = V_0 \cos \alpha\, x - V_0 \sin \alpha\, y$

► **9-3.**

Streamlines $xy = c$

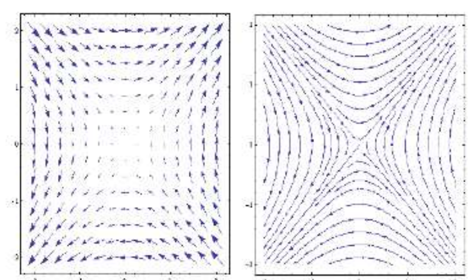
Potential $\phi = x^2 - y^2$



► **9-4.**

Streamlines $y^2 - x^2 = c$

Potential $\phi = 2xy$

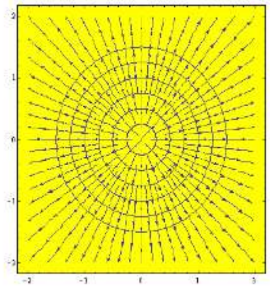


► **9-5.** $\phi = \phi(r) = \ln(r_0/r)$

► **9-6.** True

► **9-7.** Write $\nabla^2 \phi = \nabla \cdot (\nabla \phi) = 0$ and $\nabla \times (\nabla \phi) = 0$

► **9-8.** (c)



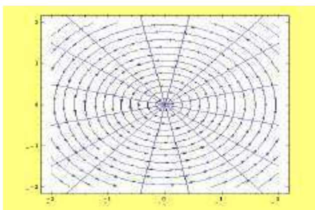
► **9-9.** Integrate the equations $\frac{\partial \phi}{\partial x} = \frac{-ky}{x^2+y^2}$ and $\frac{\partial \phi}{\partial y} = \frac{kx}{x^2+y^2}$ to obtain

$$\phi = -k \tan^{-1}\left(\frac{x}{y}\right) + C_1 \qquad \text{and} \qquad \phi = k \tan^{-1}\left(\frac{y}{x}\right) + C_2$$

Show that $\tan^{-1} h + \tan^{-1}(\frac{1}{h}) = \frac{\pi}{2}$ for $h > 0$ and then let $c_1 = \frac{\pi}{2}k,\ c_2 = 0$ to obtain the potential

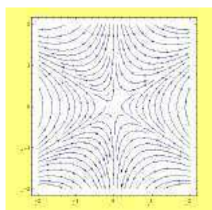
$$\phi = k\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{x}{y}\right)\right] = k \tan^{-1}\left(\frac{y}{x}\right)$$

Streamlines are circles $\psi = x^2 + y^2 = c$

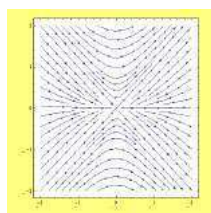


► **9-10.**

(d) $2xy \quad x^2 - y^2$



(e) $x^2 + y^2 \quad 2xy$



► **9-11.** Show that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \implies \left(\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\cos\theta = \left(\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)\sin\theta$$

and

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \implies \left(\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\sin\theta = -\left(\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\cos\theta$$

If the above equations are to hold for all values of θ, then the coefficient of the $\sin\theta$ and $\cos\theta$ terms must equal zero.

► **9-12.**

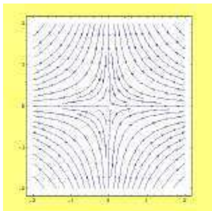
On upper semi-circle $\int_{C_1} \vec{F} \cdot d\vec{r} = -\pi/4$

On the lower semi-circle $\int_{C_2} \vec{F} \cdot d\vec{r} = \pi/4$

▶ **9-13.** (b) $\vec{V} = [-(x-y)z - yz_0 + x_0 z_0]\,\hat{\mathbf{e}}_2$

▶ **9-14.** $\phi = -mgz, \quad W = \int_{h_1}^{h_2} \vec{F}\cdot d\vec{r} = \int_{h_1}^{h_2} -mg\,dz = -mg(h_2 - h_1) = \phi(h_2) - \phi(h_1)$

▶ **9-16.** If $\phi = x^2 - y^2$, then $\operatorname{grad}\phi = \vec{F} = 2x\,\hat{\mathbf{e}}_1 - 2y\,\hat{\mathbf{e}}_2$ with field lines $xy = c$



▶ **9-18.** Show

$$\phi = y^2 \sin x + xz^3 + f_1(y,z)$$
$$\phi = y^2 \sin x - 4y + f_2(x,z)$$
$$\phi = zx^3 + f_3(x,y)$$

Select f_1, f_2, f_3 such that all three integrations are the same to obtain $\phi = y^2 \sin x + xz^3 - 4y$

▶ **9-19.** $\phi = x^2yz + xy$

▶ **9-20.** $\pi(2-\sqrt{3})$

▶ **9-21.** $3V$

▶ **9-22.** Use the results $\nabla\cdot(\phi\vec{C}) = \nabla\phi\cdot\vec{C} = \vec{C}\cdot\nabla\phi$ and $\phi\vec{C}\cdot\hat{\mathbf{e}}_n = \vec{C}\cdot(\phi\,\hat{\mathbf{e}}_n)$ to show

$$\vec{C}\cdot\left[\iiint_V \nabla\phi\,dV - \iint_S \phi\,\hat{\mathbf{e}}_n\,dS\right] = 0$$

and since $\vec{C}$ is arbitrary the term in the brackets must equal zero.

▶ **9-23.** (a) $\vec{F} = \nabla\phi$ so that $\operatorname{div}\vec{F} == \nabla\cdot(\nabla\phi) = \nabla^2\phi = 0$

(b) Use the Gauss divergence theorem to show

$$\iiint_V \operatorname{div}\vec{F}\,dV = 0 = \iint_S \nabla\phi\cdot\hat{\mathbf{e}}_n\,dS = \iint_S \frac{\partial\phi}{\partial n}\,dS$$

▶ **9-24.** $|\vec{r}| = (x^2+y^2+z^2)^{1/2} = (\vec{r}\cdot\vec{r})^{1/2}$ so that $|\vec{r}|^\nu = (x^2+y^2+z^2)^{\nu/2} = \phi$

$$\frac{\partial\phi}{\partial x} = \frac{\nu}{2}(x^2+y^2+z^2)^{\nu/2-1}(2x) \quad \frac{\partial\phi}{\partial y} = \frac{\nu}{2}(x^2+y^2+z^2)^{\nu/2-1}(2y \quad \frac{\partial\phi}{\partial z} = \frac{\nu}{2}(x^2+y^2+z^2)^{\nu/2-1}(2z)$$

so that

$$\begin{aligned}\nabla|\vec{r}|^\nu =& \frac{\partial\phi}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{e}}_3 = \nu|\vec{r}|^{\nu-2}(x\,\hat{\mathbf{e}}_1 + y\,\hat{\mathbf{e}}_2 + z\,\hat{\mathbf{e}}_3)\\ =&\nu|\vec{r}|^{\nu-2}\vec{r} = \nu|\vec{r}|^{\nu-2}|\vec{r}|\frac{\vec{r}}{|\vec{r}|} = \nu|\vec{r}|^{\nu-1}\,\hat{\mathbf{e}}_{\vec{r}}\end{aligned}$$

▶ **9-26.** In problem 9-24 replace $\vec{r}$ by $\vec{r}-\vec{r}_0$ and then use the results from problem 9-25 to obtain the result for part (b).

▶ **9-27.** Let $M = \frac{\partial U}{\partial p}$ and $N = \frac{\partial U}{\partial V} + P$ and show $\frac{\partial M}{\partial V} = \frac{\partial^2 U}{\partial p\,\partial v}$ and $\frac{\partial N}{\partial p} = \frac{\partial^2 U}{\partial v\,\partial p} + 1$
Here $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial p}$ so that the line integral is path dependent.

▶ **9-29.** $I = -\frac{3}{4}(5+\pi)$

▶ **9-30.** Potential energy $\frac{1}{2}Kx^2$
Kinetic +potential energy is constant or $\frac{1}{2}mv^2 + \frac{1}{2}Kx^2 = constant$

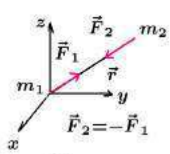
▶ **9-32.** (a) $T = T(x) = c_1x + c_2,$
$T(0) = c_2 = T_0$ and $T(L) = c_1L + c_2 = T_1 \implies T = T(x) = (\frac{T_1-T_0}{L})x + T_0$

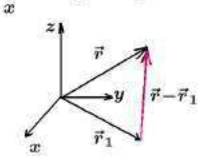
▶ **9-33.** $\phi = 3x^2z + 4y^2$

▶ **9-34.**

(a) $\vec{F}_1 = \dfrac{Gm_1m_2}{|\vec{r}|^3}\,\vec{r}$ where $|\vec{r}|^2 = x^2+y^2+z^2$

(b) $\vec{F}_1 = \dfrac{Gm_1m_2}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1)$
where $|\vec{r}-\vec{r}_1|^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$





▶ **9-39.** $\vec{y} = \vec{y}_c + \vec{y}_p = \vec{C}_1(1) + \vec{C}_2(t) - \sin t\,\hat{\mathbf{e}}_1 + \cos t\,\hat{\mathbf{e}}_2$

▶ **9-40.** $\vec{y}_1 = \vec{C}_1\,\sin t - \vec{C}_2\,\cos t, \qquad \vec{y}_2 = \vec{C}_1\,\cos t + \vec{C}_2\,\sin t$

▶ **9-41.** $\dfrac{d\vec{r}}{dt} = \underbrace{r_0e^{\theta\cot\alpha}\omega}_{r\omega}\;\hat{\mathbf{e}}_\theta + \underbrace{r_0e^{\theta\cot\alpha}\frac{d\theta}{dt}\cot\alpha}_{r\omega\cot\alpha}\;\hat{\mathbf{e}}_r$

Chapter 10

▶ **10-1.** (c) $AB = \begin{bmatrix} 36 & 59 \\ 11 & 18 \end{bmatrix}$ (d) $B^{-1}A^{-1} = \begin{bmatrix} -18 & 59 \\ 11 & -36 \end{bmatrix}$

▶ **10-2.**

$$\begin{aligned} AA^{-1} =& I \\ (AA^{-1})^T =& I^T = I \\ (A^{-1})^T A^T =& I \quad \text{multiply by } (A^T)^{-1} \\ (A^{-1})^T =& (A^T)^{-1} \end{aligned}$$

▶ **10-4.** (a)

$$\begin{aligned} \text{If} \qquad AB =& BA, \text{ left multipley by } A^{-1} \\ A^{-1}AB =& A^{-1}BA \\ B =& A^{-1}BA \text{ right multiply by } A^{-1} \\ BA^{-1} =& A^{-1}BAA^{-1} \\ BA^{-1} =& A^{-1}B \end{aligned}$$

▶ **10-5.** If A is symmetric, then $A^T = A$ so that if $(A^{-1})TA^T = I$ one can write $(A^{-1})^T A = I$. Multiply this last equation on the right by A^{-1} to obtain $(A^{-1})^T = A^{-1}$ which shows A^{-1} is symmetric.

▶ **10-6.** Left multiply both sides of equation by A^{-1}

▶ **10-7.** If $AB = A$ and $BA = B$, then one can write

$$\begin{aligned} AB =& A \text{ right multiply by A} \\ (AB)A =& A^2 \text{ associative property} \\ A(BA) =& A^2 \text{ properties } BA = B \text{ and } AB = A \\ AB =& A^2 \\ A =& A^2 \end{aligned}$$

$$\begin{aligned} BA =& B \text{ right multiply by B} \\ (BA)B =& B^2 \text{ associative property} \\ B(AB) =& B^2 \text{ properties } AB = A \text{ and } BA = B \\ BA =& B^2 \\ B =& B^2 \end{aligned}$$

▶ **10-8.** (a) Let $Y = A^2 = AA$ with $Y^{-1} = (A^2)^{-1}$, then one can write

$$\begin{aligned} I =& YY^{-1} = (A^2)(A^2)^{-1} \text{ left multiply by } A^{-1} \\ A^{-1} =& A^{-1}AA(A^2)^{-1} \text{ left mulitply by } A^{-1} \\ (A^{-1})^2 =& A^{-1}A(A^2)^{-1} = (A^2)^{-1} \end{aligned}$$

▶ **10-9.** (a) Let $B = AA^T$, then $B^T = (AA^T)^T = AA^T = B$

(d) $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

► **10-10.** (a) If $A^T = A$ and $B^T = B$, then $(AB)^T = B^T A^T = BA = AB$
(b) If $((AB)^T = (AB)$, then $B^T A^T = AB$ which implies $BA = AB$

► **10-12.** If $AB = BA$, then one must have $c = 0$ and $d = a$.

► **10-13.** Show $A^2 = A$ and $A^3 = A$, then show $B^2 = I$ and $B^3 = B$ so A is idempotnt and $B$is involutory.

► **10-14.** Show $A^2 = I$ and $A^3 = A$ so that A is involutory.

► **10-15.** Show $B = A^2$

► **10-16.** $X = A^{-1}B$

► **10-18.** (a) -11 (b) 6 (c) -2

► **10-19.** (a) 8 (b) 5 (c) 4

► **10-20.** (a) $(m_{ij}) = \begin{pmatrix} 14 & -5 & -8 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$

(b) $(c_{ij}) = \begin{pmatrix} 14 & -5 & -8 \\ -2 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix}$

(c) $AC^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

► **10-21.** $6xyz$

► **10-22.** (a) 0 (b) 0 (c) 36
(d) $a_1a_2a_3a_4$ (e) $a_1a_2a_3a_4$ (f) $45,000 = (36)(25)(5)(2)(5)$

► **10-23.** (a) $Z^{-1} = \dfrac{1}{(z_{11}z_{22} - z_{12}z_{21})} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix}$

► **10-24.** $|A| = 3, \quad |B| = 1, \quad |A| \cdot |B| = 3, \qquad |AB| = 3$

► **10-25.** $(x_2 - x_1)y - (y_2 - y_1)x = x_2y_1 - x_1y_2$

▶ **10-28.** $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 11 & -5 & 1 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 1 & 2 & -7 & -8 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C^{-1} = \begin{bmatrix} 8 & -6 & 4 & -2 \\ -6 & 12 & -8 & -4 \\ 4 & -8 & 12 & -6 \\ -2 & 4 & -6 & 8 \end{bmatrix}$

▶ **10-29.** If $AA^T = \begin{bmatrix} \alpha_2^2 + 1/4 & \frac{\alpha_1+\alpha_2}{4} & 0 \\ \frac{\alpha_1+\alpha_2}{2} & \alpha_1^2 + 1/4 & 0 \\ 0 & 0 & \alpha_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then one must select $\alpha_1^2 = \alpha_2^2 = 3/4$, $\alpha_1 = -\alpha_2$ and $\alpha_3^2 = 1$

▶ **10-30.** (b) $\dfrac{d|A|}{dt} = -4t^3 + 6t^2 - 6t$

▶ **10-32.** $|A| = 1, \quad |B| = 1, \quad |A + B| = 3, \quad |A| + |B| = 2$

▶ **10-36.** (a)

$$\begin{aligned} e^{At} &= I + At + A^2 \frac{t^2}{2!} + \cdots \\ \int_0^t e^{At}\,dt &= It + A\frac{t^2}{2!} + A^2\frac{t^3}{3!} + \cdots \\ A\int_0^t e^{At}\,dt &= At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots \\ A\int_0^t e^{At}\,dt + I &= e^{At} \end{aligned}$$

(b)

$$\begin{aligned} \int_0^t e^{At}\,dt &= It + A\frac{t^2}{2!} + A^2\frac{t^3}{3!} + \cdots \\ \int_0^t e^{At}\,dt &= A^{-1}\left[At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots\right] = [At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots]A^{-1} \\ \int_0^t e^{At}\,dt &= A^{-1}\left[e^{At} - I\right] = [e^{At} - I]A^{-1} \end{aligned}$$

▶ **10-42.** (b) $y_{n+2} - 6y_{n+1} + 9y_n = 0$ Assume a solution $y_n = A^n$ with $y_{n+1} = A^{n+1}$ and $y_{n+2} = A^{n+2}$, then substitute the assumed solution into the difference equation and obtain the characteristic equation $(A-3)^2 = 0$ with repeated roots $A = 3, 3$. The fundamental set of solutions is $\{3^n, n3^n\}$ and the general solution is $y_n = c_0(3)^n + c_1 n(3)^n$

Chapter 11

▶ **11-1.** (a) 10/15 (b) 3/15 (c) 2/15.

▶ **11-2.** (a) (10/15)(3/14) (b) (10/15)(9/14) (c) (2/15)(3/14)

▶ **11-3.**

Method I Label the fruit A1,A2,A3,O1,O2,O3,O4,O5,P1,P2

Next collect all possible pairs — Note the pair (A1,A2) is the same as the pair (A2,A1) These are all possible event pairs

(A1,A2)	(A1,A3)	(A1,O1)	(A1,O2)	(A1,O3)	(A1,O4)	(A1,O5)	(A1,P1)	(A1,P2)	9
(A2,A3)	(A2,O1)	(A2,O2)	(A2,O3)	(A2,04)	(A2,05)	(A2,P1)	(A2,P2)		8
(A3,01)	(A3,02)	(A3,03)	(A3,04)	(A3,05)	(A3,P1)	(A3,P2)			7
(O1,02)	(O1,03)	(O1,04)	(O1,05)	(O1,P1)	(O1,P2)				6
(O2,03)	(02,04)	(O2,05)	(O2,P1)	(O2,P2)					5
(03,04)	(03,05)	(O3,P1)	(O3,P2)						4
(04,05)	(O4,P1)	(O4,P2)							3
(O5,P1)	(O5,P2)								2
(P1,P2)									1

Total of 45 possible two event pairs. Assign an equal probability to each event of 1/45

Since the event (P1,P2) can only happen once —its probability is 1/45
The probability of getting two oranges is 10/45 since there are 10 (0,0) pairs above
The probability of getting two apples is 3/45 since there are 3 (A,A) pairs above.

Method 2 From AAA, OOOOO, PP assign a probability of 1/10 to each single event of selecting one fruit. (All have same probability)

Probability of getting a pear is $1/10 + 1/10 = 2/10 = 1/5$

If a pear is selected, then we are left with AAA,OOOOO,P Now each single event has probability of 1/9

Probability of getting a pear on second selection is 1/9.

The product $(1/5)(1/9) = 1/45$ is probability of getting a pear plus another pear.

Similarly, the probability of getting an orange is $5/10 = 1/2$ the first time and 4/9 the second time, so that $(1/2)(4/9) = 2/9$ is probability of getting two oranges.

The probability of getting two apples is $(3/10)(2/9) = 6/90 = 1/15$

▶ **11-4.** (a) mean 80, variance 79.6, standard deviation 8.922

▶ **11-5.** $(x_j - \bar{x})^2 = x_j^2 - 2x_j\bar{x} + \bar{x}^2$ so that

$$s^2 = \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - 2\bar{x}\sum_{j=1}^{m} x_j n f_j + \bar{x}^2 n \sum_{j=1}^{m} f_j\right\}$$

But $\sum_{j=1}^{m} f_j = 1$ and $n\sum_{j=1}^{m} x_j f_j = \bar{x}n$ so that

$$\begin{aligned} s^2 =& \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - 2n\bar{x}^2 + n\bar{x}^2\right\} \\ =& \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - n\bar{x}^2\right\} \\ =& \frac{1}{n-1}\left\{\sum_{j=1}^{m} x_j^2 n f_j - n\left(\frac{1}{n^2}\right)\left(\sum_{j=1}^{m} x_j f_j n\right)^2\right\} \end{aligned}$$

▶ **11-6.** $P(2) = P(12) = 1/36$, $P(3) = P(11) = 2/16$, $P(4) = P(10) = 3/36$, $P(5) = P(9) = 4/36$, $P(6) = P(8) = 5/36$, $P(7) = 6/36$

▶ **11-8.** Binomial distribution $(p+q)^n = p^n + np^{n-1}q + \cdots$ where $p = 1/2$, $q = 1/2$ and $n = 10$ gives $(1/2)^{10} = 0.00097956$

▶ **11-9.** (a)

x_j	$\tilde{f}_j$	f_j	$F(x)$	$x_j\tilde{f}_j$	$x_j^2\tilde{f}_j$
0.725	2	$0.0333 = 2/60$	$0.0333 = 2/60$	1.450	1.0513
0.728	7	$0.1167 = 7/60$	$0.1500 = 9/60$	5.096	3.7099
0.731	11	$0.1833 = 11/60$	$0.3333 = 20/60$	8.041	5.878
0.734	14	$0.2333 = 14/60$	$0.5666 = 34/60$	10.276	7.5426
0.737	13	$0.2167 = 13/60$	$0.7833 = 47/60$	9.581	7.0612
0.740	7	$0.1167 = 7/60$	$0.9000 = 54/60$	5.180	3.8332
0.743	3	$0.05 = 3/60$	$0.95 = 57/60$	2.229	1.6561
0.746	3	$0.05 = 3/60$	$1.00 = 60/60$	2.238	1.6695

(c) *mean* $= 0.73488$, *variance* $= 0.00002461$, *Standard deviation* $= 0.00496$

(e) (*i*) $P(X \le 0.737) = 0.783$,

(*ii*) $P(0.728 < X < 0.734) = P(X \le 0.734) - P(X \le 0.728) = 0.42$,

(*iii*) $P(X > 0.734) = 1 - P(X \le 0.734) = 0.4334$

▶ **11-11.** (a)

$$
\begin{aligned}
(p+q)^n =& \binom{n}{0}p^n + \binom{n}{1}p^{n-1}q + \cdots + \binom{n}{n-x}p^x q^{n-x} + \cdots \binom{n}{n}q^n \\
\binom{n}{k} =& \binom{n}{n-k}
\end{aligned}
$$

so that $(p+q)^n = \sum_{x=0}^{n} \binom{n}{n-x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = 1 = \sum_{x=0}^{n} f(x)$

(b)

$$
(p+q)^{n-1} = \sum_{x=0}^{n-1} \binom{n-1}{n-1-x} p^x q^{n-1-x}
$$

shift summation index by letting $x = X - 1$ so that

$$
(p+q)^{n-1} = \sum_{X=1}^{n} \binom{n-1}{n-1-X+1} p^{X-1} q^{n-X} = \sum_{X=1}^{n} \binom{n-1}{X-1} p^{X-1} q^{n-X} = 1
$$

since $\binom{n-1}{n-1-X+1} = \binom{n-1}{X-1}$

(c) $x\binom{n}{x} = x\dfrac{n!}{x!(n-x)!} = \dfrac{n(n-1)!}{(x-1)!(n-1-(x-1))!} = n\binom{n-1}{x-1}$

(d)

$$
\mu = \sum_{x=0}^{n} x f(x) = \sum_{x=1}^{n} x\binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} n\binom{n-1}{x-1} p^x q^{n-x} = np
$$

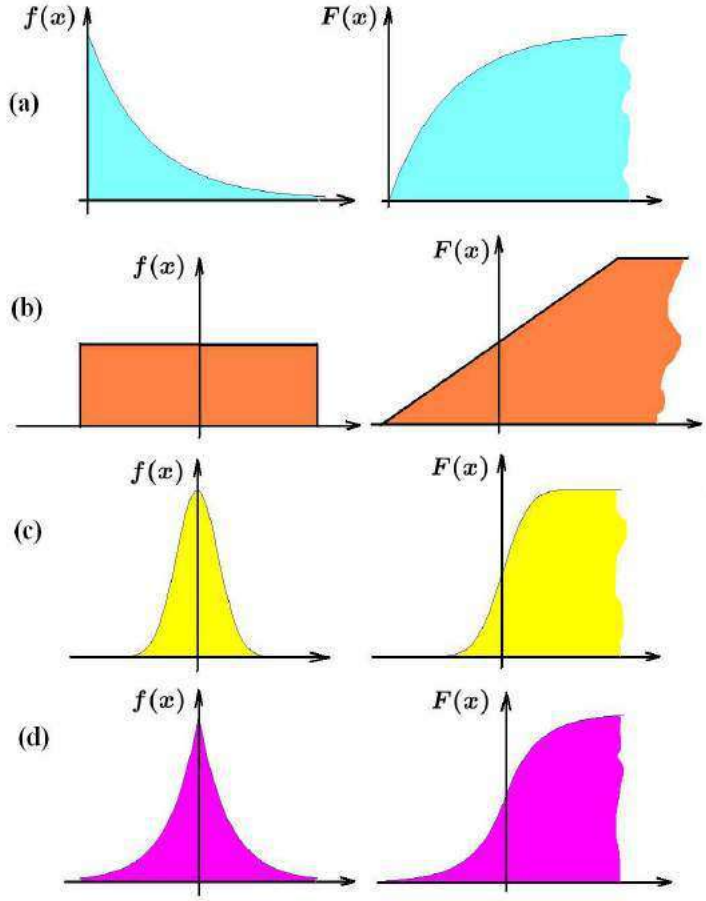
▶ **11-12.**

$$
\begin{aligned}
s^2 =& \frac{1}{N-1}\sum_{j=1}^{N}(x_j - \bar{x})^2 = \frac{1}{N-1}\sum_{j=1}^{N}(x_j^2 - 2\bar{x}x_j + \bar{x}^2) \\
=& \frac{1}{N-1}\left(\sum_{j=1}^{N} x_j^2 - 2\bar{x}\sum_{j=1}^{N} x_j + N\bar{x}^2\right) \\
=& \frac{1}{N-1}\left(\sum_{j=1}^{N} x_j^2 - 2\bar{x}N\bar{x} + N\bar{x}^2\right) = \frac{1}{N-1}\left(\sum_{j=1}^{N} x_j^2 - N\bar{x}^2\right)
\end{aligned}
$$

Use the fact that $\bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$ and write

$$
s^2 = \frac{1}{N(N-1)}\left\{N\sum_{j=1}^{N} x_j^2 - \left(\sum_{j=1}^{N} x_j\right)^2\right\}
$$

▶ **11-13.**



(a) $F(x) = \int_0^x \alpha e^{-\alpha x}\, dx = 1 - e^{-\alpha x}$

(c) $F(x) = \int_{-\infty}^{x} f(x)\, dx$

▶ **11-14.** (b) $\binom{n}{n+1} = \dfrac{n!}{(m+1)!(n-(m+1))!} = \dfrac{(n-m)n!}{(m+1)m!(n-m)!} = \dfrac{n-m}{m+1}\binom{n}{m}$

▶ **11-15.** (c) (i) Area from $-\infty$ to $z = 1$ is 0.8413 and area from $-\infty$ to $z = 0$ is 0.5. Therefore area between 0 and 1 is $0.8413 - 0.5 = 0.3413$ By symmetry, twice this is 0.6826 which represents $P(-1 < X \le 1)$

▶ **11-16.** (b) $P(Ace) = 4/52$ *and* $P(king) = 4/52$ *so that*
$P(Ace\ or\ King) = P(Ace) + P(King) = 8/52 = 2/13$

▶ **11-17.** (b) $P(E_1) = 4/52$ *and* $P(E_2) = 13/52$ *and*
$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 4/52 + 13/52 - 1/52 = 16/52 = 4/13$

▶ **11-19.** (i)

$$\begin{aligned}\sum_{x=0}^{n} x^2 f(x) =& \sum_{x=0}^{n}[x(x-1)+x]f(x)\\ =& \sum_{x=0}^{n} x(x-1)f(x) + \sum_{x=0}^{n} xf(x)\\ =& \sum_{x=0}^{n} x(x-1)f(x) + np\end{aligned}$$

Use $x(x-1)\binom{n}{x} = x(x-1)\dfrac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} = n(n-1)\binom{n-1}{x-2}$ and write

$$\sum_{x=0}^{n} x^2 f(x) = \sum_{x=2}^{n} n(n-1)\binom{n-2}{x-2}p^x q^{n-x} + np = n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}q^{n-x} + np$$

Note that by shifting the summation index

$$\sum_{X=0}^{n-2}\binom{n-2}{X}p^X q^{n-2-X} = \sum_{X=0}^{n-2}\binom{n-2}{n-2-X}p^X q^{n-2-X} = (p+q)^{n-2} = 1$$

so that $\sum_{x=0}^{n} x^2 f(x) = n(n-1)p^2 + np$

$$\begin{aligned}\sigma^2 =& \sum_{x=0}^{n}(x-\mu)^2 f(x) = \sum_{x=0}^{n}(x^2 - 2x\mu + \mu^2)f(x)\\ =& \sum_{x=0}^{n} x^2 f(x) - 2\mu\sum_{x=0}^{n} xf(x) + \mu^2\sum_{x=0}^{n} f(x)\\ =& \sum_{x=0}^{n} x^2 f(x) - \mu^2 = E[(x^2)] - (E[x])^2\\ =& n(n-1)p^2 + np - n^2p^2\\ =& np(1-p) = npq\end{aligned}$$

▶ **11-21.** (a) $f(x) = \dfrac{\binom{3}{x}\binom{9}{5-x}}{\binom{12}{5}}$ $\quad \sum_{x=1}^{5} f(x) = 1$

$$f(x) = 7/44, \quad f(1) = 21/44, \quad f(2) = 7/22, \quad f(3) = 1/12, \quad f(4) = f(5) = 0$$

(b) The probability that $n = 6$ items are selected with zero nondefectives is $f(x) = \dfrac{\binom{3}{x}\binom{9}{6-x}}{\binom{12}{6}}$ evaluated at $x = 0$ or $f(0) = 1/11$. The probability that 6 items are selected and there is 1 defective is $f(1) = 9/22$. We are not interested in $f(x)$ for x

larger than 1 because if 2 items or more are defective out of 6, we cannot obtain 5 nondefectives. $P(X \leq 1) = 1/11 + 9/22 = 1/2$

If $n = 7$ items are selected $f(x) = \dfrac{\binom{3}{x}\binom{9}{7-x}}{\binom{12}{7}}$. Here

$$f(0) = 1/22, \quad f(1) = 7/22, \quad f(2) = 21/44$$

then $P(X \leq 2) = \dfrac{1}{22} + \dfrac{7}{22} + \dfrac{21}{44} = \dfrac{37}{44} = 0.84 > 0.8$

▶ **11-24.** (b) $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

(c)

$$\frac{1}{32} = p^5 = (\frac{1}{2})^5 \text{ probability of 5 heads}$$
$$\frac{5}{32} = 5p^4q = 5(\frac{1}{2})^4(\frac{1}{2}) \text{ probability of 4 heads, 1 tail}$$
$$\frac{10}{32} = 10p^3q^2 = 10(\frac{1}{2})^3(\frac{1}{2})^2 \text{ probability of 3 heads, 2 tails}$$
$$\frac{10}{32} = 10p^2q^3 = 10(\frac{1}{2})^2(\frac{1}{2})^3 \text{ porobability of 2 heads, 1 tail}$$
$$\frac{13}{16} = p_5^5p^4q + 10p^3q^2 + 10p^2q^3 \text{ probability of getting at least 2 heads}$$

▶ **11-25.**

x	$f(x) = \binom{40}{x}(0.8)^x(0.2)^{40-x}$
40	0.000132923
39	0.00132923
38	0.00647999
37	0.02052
36	0.0474524
35	0.0854143
34	0.124563
33	0.151255
32	0.155981
31	0.13865

(a) $f(33) = P(X = 33) = 0.151255$

(b) $P(X = 37) = f(37) = 0.02052$

(c) $\sum_{k=0}^{37} = 1 - f(40) - f(39) - f(38) = 0.992057963$

(d) $\sum_{k=32}^{40} f(k) = f(32) + \cdots + f(40) = 0.59312713$

▶ **11-26.**

x	$f(x)=\frac{9^x e^{-9}}{x!}$
0	0.00012341
1	0.00111069
2	0.0049981
3	0.0149943
4	0.0337372
5	0.0607269
6	0.0910903
7	0.117116
8	0.131756
9	0.131756
10	0.11858
11	0.0970201
12	0.072765

(a) $P(X>4)=1-\sum_{k=0}^{4} f(k)=0.94503636$
(b) $P(X\le 8)=\sum_{k=0}^{\infty} f(k)=0.45565260$
(c) $P(8<X\le 12)=\sum_{k=8}^{12} f(k)=0.5518747$

▶ **11-27.** $f(k)=\dfrac{2^k e^{-2}}{k!}$

$$(a)\quad f(0)=e^{-2}=0.1353$$
$$(b)\quad \sum_{k=6}^{\infty} f(k)=1-\sum_{k=0}^{5} f(k)=0.0166$$
$$(c)\quad 1-\sum_{k=0}^{1} f(k)=0.5940$$

▶ **11-28.** (a) Write $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-t^2/2}\,dt=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-t^2/2}\,dt+\dfrac{1}{\sqrt{2\pi}}\int_{0}^{x} e^{-t^2/2}\,dt$
The first integral equals $1/2$ and in the second integral make the substitution $\tau=t/\sqrt{2}$ to obtain result.

(b) Similar to part (a) but make the substitution $\tau=\frac{1}{\sqrt{2}}\left(\frac{t-\mu}{\sigma}\right)$ with $d\tau=\frac{1}{\sigma\sqrt{2}}\,dt$.

▶ **11-33.** $Area\approx 0.982923$

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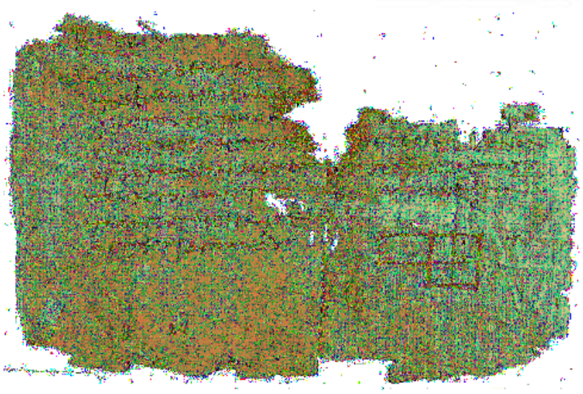
Isaac Newton (1642-1727)

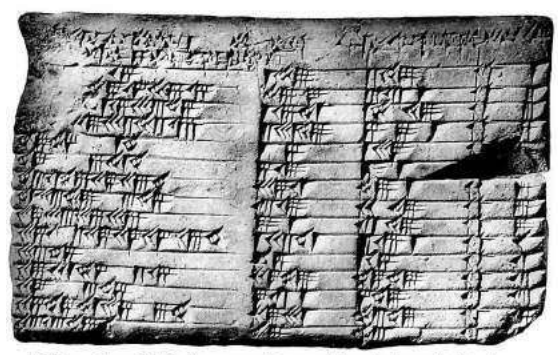
Gottfried Wilhelm Leibnitz (1646-1716)

Rhind mathematical papyrus written around 1650 BCE can be found at the British Museum.

Translation available on internet.



Papyrus containing part of Euclid's Elements is located at the University of Pennsylvania. Euclid's Elements consists of 13 books on geometry written around 300 BCE.



Plimpton 322 is a mathematical clay tablet from Babylon around 1800 BCE. For more details read, Eleanor Robson, "Words and pictures: new light on Plimpton 322", American Mathematical Monthly,109(2):105-120. Tablet can be found at Columbia University.