AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

The Improving Mathematics Education in Schools (TIMES) Project

TRIGONOMETRIC FUNCTIONS

A guide for teachers - Year 10

MEASUREMENT AND GEOMETRY : Module 25

June 2011

YEAR 10

Trigonometric Functions

(Measurement and Geometry: Module 25)

For teachers of Primary and Secondary Mathematics

510

Cover design, Layout design and Typesetting by Claire Ho

The Improving Mathematics Education in Schools (TIMES) Project 2009-2011 was funded by the Australian Government Department of Education, Employment and Workplace Relations.

The Improving Mathematics Education in Schools (TIMES) Project

TRIGONOMETRIC FUNCTIONS

A guide for teachers - Year 10

MEASUREMENT AND GEOMETRY : Module 25

June 2011

Peter Brown
Michael Evans
David Hunt
Janine McIntosh
Bill Pender
Jacqui Ramagge

YEAR 10

THE TRIGONOMETRY FUNCTIONS

ASSUMED KNOWLEDGE

- Familiarity with the material in the modules, *Introduction to Trigonometry* and *Further Trigonometry*.
- Knowledge of basic coordinate geometry.
- Introductory graphs and functions.
- Facility with simple algebra, formulas and equations.

MOTIVATION

In the module, *Further Trigonometry*, we saw how to use points on the unit circle to extend the definition of the trigonometric ratios to include obtuse angles. That same construction can be extended to angles between 180° and 360° and beyond. The sine, cosine and tangent of negative angles can be defined as well.

Once we can find the sine, cosine and tangent of any angle, we can use a table of values to plot the graphs of the functions $y = \sin x$, $y = \cos x$ and $y = \tan x$. In this module, we will deal only with the graphs of the first two functions.

The graphs of the sine and cosine functions are used to model wave motion and form the basis for applications ranging from tidal movement to signal processing which is fundamental in modern telecommunications and radio-astronomy. This provides a breathtaking example of how a simple idea involving geometry and ratio was abstracted and developed into a remarkably powerful tool that has changed the world.

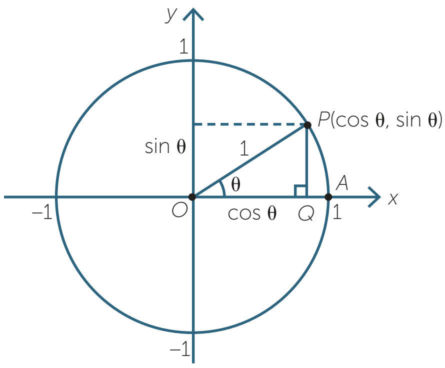
CONTENT

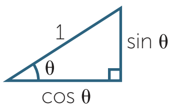
ANGLES IN THE FOUR QUADRANTS

Redefining the Trigonometric Ratios

We begin by taking the circle of radius 1, centre the origin, in the plane. From the point P on the circle in the first quadrant we can construct a right-angled triangle POQ with O at the origin and Q on the x-axis.

We mark the angle *POQ* as θ.





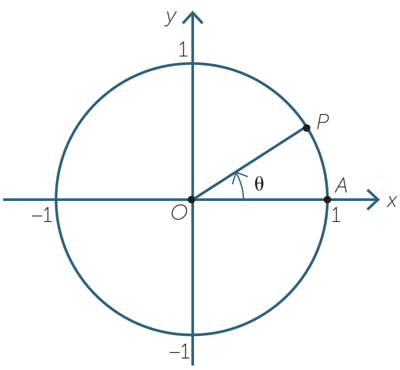
Since the length $OQ = \cos\theta$ is the *x*-coordinate of *P*, and $PQ = \sin\theta$ is the *y*-coordinate of *P*, we see that the point *P* has coordinates

$$(\cos\theta, \sin\theta).$$

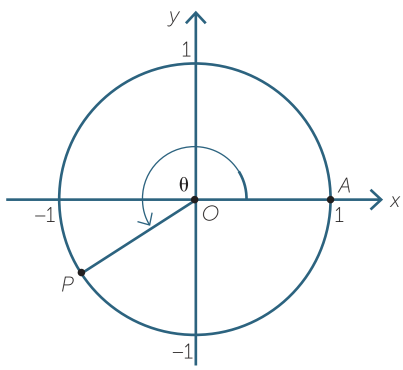
We measure angles anticlockwise from *OA* and call these **positive angles.** Angles measured clockwise from *OA* are called **negative angles**. For the time being we will concentrate on positive angles between 0° and 360°.

Since each angle θ determines a point *P* on the unit circle, we will define

- the cosine of θ to be the *x*-coordinate of the point *P*
- the sine of θ to be the *y*-coordinate of the point *P*.



$0^\circ < \theta < 90^\circ$



$180^\circ < \theta < 270^\circ$

For acute angles, we know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. For angles that are greater than 90° we define the tangent of θ by

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

unless cos In this case, we say that the tangent ratio is **undefined**. Between 0° and 360°, this will happen when θ = 90°, or θ = 270°. You will see in the following exercise why this is the case.

Note that this is the same as saying that $\tan\theta$ equals the $\frac{y \text{ coordinate of } P}{x \text{ coordinate of } P}$.

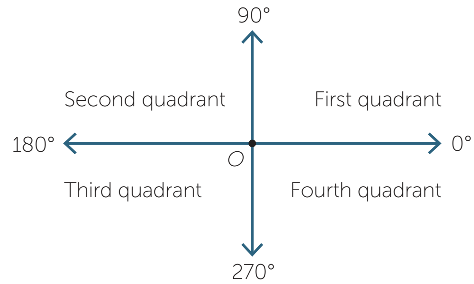
EXERCISE 1

Place the point *P* on the unit circle corresponding to each of the angles 0°, 90° , 180°, 270°. By considering the coordinates of each such point, complete the table below

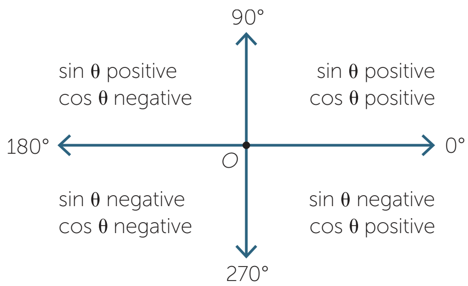
P	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
	0°			
	90°			undefined
	180°			
	270°			undefined
	360°			

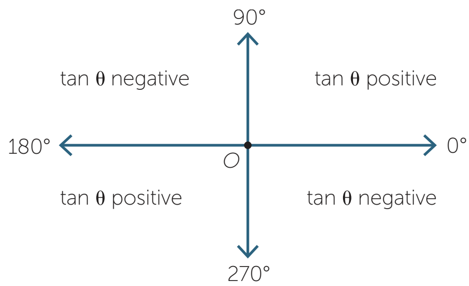
THE FOUR QUADRANTS

The coordinate axes divide the plane into four quadrants, labeled First, Second, Third and Fourth as shown. Angles in the third quadrant, for example, lie between 180° and 270°.



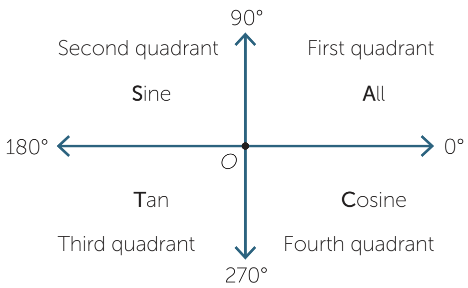
By considering the *x* and *y* coordinates of the point *P* as it lies in each of the four quadrants, we can identify the *sign* of each of the trigonometric ratios in a given quadrant. These are summarised in the following diagrams.





To obtain the second diagram, we used the definition $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

To assist in remembering the signs of the three trigonometric ratios in the various quadrants, we see that only one of the ratios is *positive* in each quadrant. Hence we can remember the signs by the picture:



The mnemonic *All Stations To Central* is sometimes used.

EXAMPLE

What is the sign of

a cos 150° **b** sin 300° **c** tan 235°?

SOLUTION

a 150° lies in the second quadrant so cos 150° is negative.

b 300° lies in the fourth quadrant so sin 300° is negative.

c 235° lies in the third quadrant so tan 235° is positive.

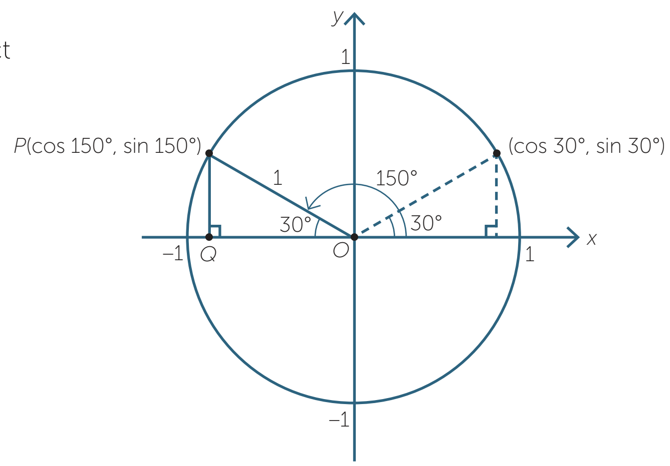
The related angle

To find the trigonometric ratio of angles beyond 90°, we introduce the notion of the *related angle*. We will examine this quadrant by quadrant.

The Second Quadrant

Suppose we wish to find the exact values of cos 150° and sin 150°.

The angle 150° corresponds to the point *P* in the second quadrant with coordinates (cos 150°, sin 150°) as shown.



The angle *POQ* is 30° and is called the **related angle** for 150°. From △*POQ* we can see that *OQ* = cos 30° and *PQ* = sin 30°, so the coordinates of *P* are –cos 30°, sin 30°).

Hence $\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$ and $\sin 150° = \sin 30° = \frac{1}{2}$.

Also, $\tan 150° = \frac{\sin 150°}{\cos 150°} = -\tan 30° = -\frac{1}{\sqrt{3}}$.

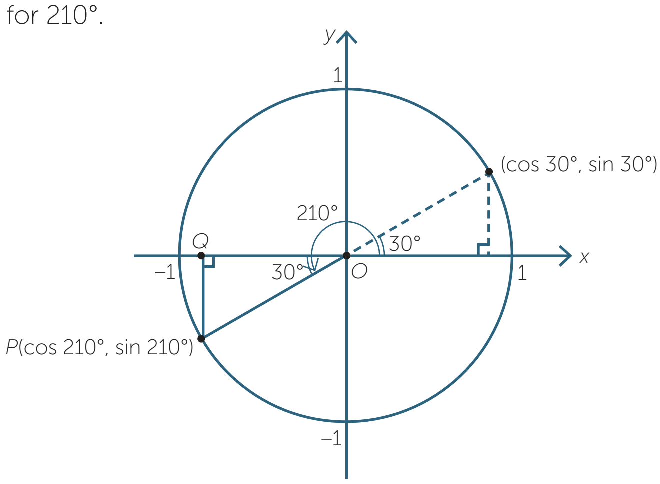
In general, if θ lies in the second quadrant, the acute angle $180° - \theta$ is called the **related angle** for θ.

We introduced this idea in the module, *Further Trigonometry.*

The Third Quadrant

An angle between 180° and 270°
will place the corresponding point *P* in the third quadrant. In this quadrant, we can see that the sine and cosine ratios are negative and the tangent ratio positive.

To find the sine and cosine of 210° we locate the corresponding point *P* in the third quadrant. The coordinates of *P* are (cos 210°, sin 210°). The angle *POQ* is 30° and is called the related angle for 210°.



So, $\cos 210° = -\cos 30° = -\frac{\sqrt{3}}{2}$ and $\sin 210° = -\sin 30° = -\frac{1}{2}$.

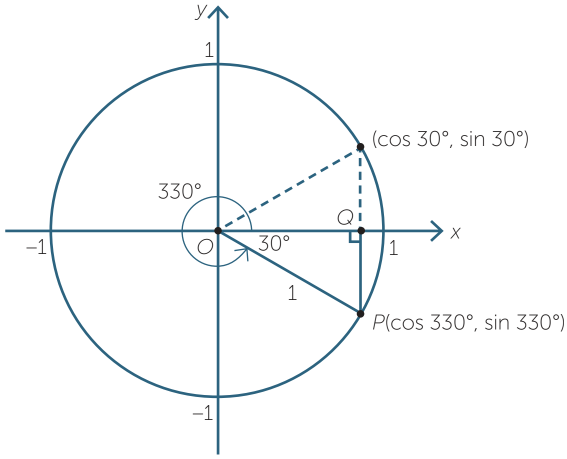
Hence $\tan 210° = \tan 30° = -\frac{1}{\sqrt{3}}$.

In general, if lies in the third quadrant, the acute angle $\theta - 180°$ is called the **related angle** for θ.

The Fourth Quadrant

Finally, if θ lies between 270° and 360° the corresponding point *P* is in the fourth quadrant. In this quadrant, we can see that the sine and tangent ratios are negative and the cosine ratio is positive.

To find the sine and cosine of 330° we locate the corresponding point *P* in the fourth quadrant. The coordinates of *P* are (cos 330°, sin 330°) .The angle *POQ* is 30° and is called the related angle for 330°.



So, $\cos 330° = \cos 30° = -\frac{\sqrt{3}}{2}$ and $\sin 330° = -\sin 30° = -\frac{1}{2}$.

Hence, $\tan 330° = -\tan 30° = -\frac{1}{\sqrt{3}}$.

In general, if lies in the fourth quadrant, the acute angle is called the **related angle** for θ.

In summary, to find the trigonometric ratio of an angle between 0° and 360° we

- find the related angle,
- obtain the *sign* of the ratio by noting the quadrant,
- evaluate the trigonometric ratio of the *related angle* and attach the appropriate sign.

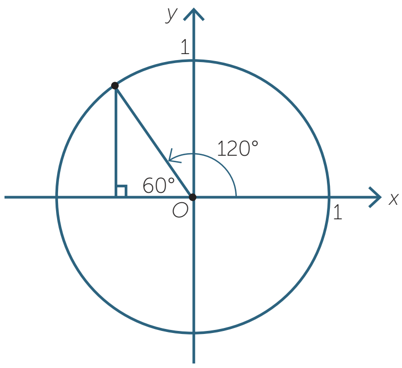
EXAMPLE

Use the related angle to find the axact value of:

a sin 120° **b** cos 150° **c** tan 300° **d** cos 240°

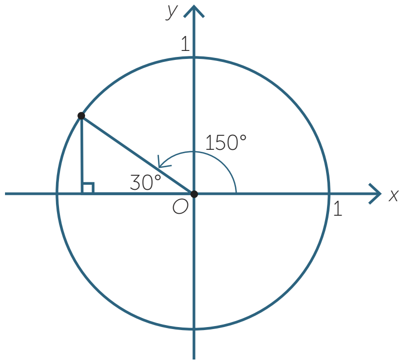
SOLUTION

a The related angle is 60°.
120° is in the second quadrant,
so $\sin 120° = \sin 60°$
$= \frac{\sqrt{3}}{2}$.



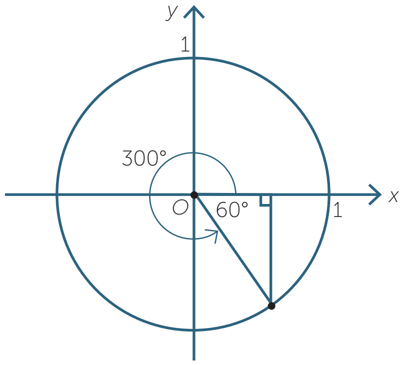
b $\cos 150° = -\cos 30°$

$= -\frac{\sqrt{3}}{2}$



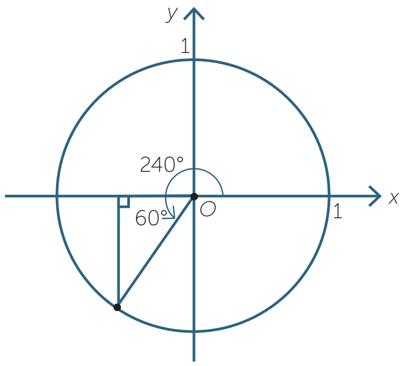
c $\tan 300° = -\tan 60°$

$= -\sqrt{3}$



d $\cos 240° = -\cos 60°$

$= -\frac{1}{2}$



Note that a calculator can be used to find, for example, including the correct sign. The calculator is, however, of limited use in finding angles and for solving trigonometric equations. Here a thorough understanding of the above material is required.

EXERCISE 2

a Find the exact value of

i $\sin 210°$ **ii** $\cos 330°$ **iii** $\tan 150°$

b Use your calculator to find $\cos 110° + \cos 70°$. Explain the answer.

THE RECIPROCAL RATIOS

We have previously defined the reciprocal ratios for angles between 0° and 90°. We can extend these definitions to include any angles, so that

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \text{ for } \sin\theta \neq 0,$$

$$\sec\theta = \frac{1}{\cos\theta}, \text{ for } \cos\theta \neq 0,$$

$$\cot\theta = \frac{1}{\tan\theta}, \text{ for } \tan\theta \neq 0.$$

EXERCISE 3

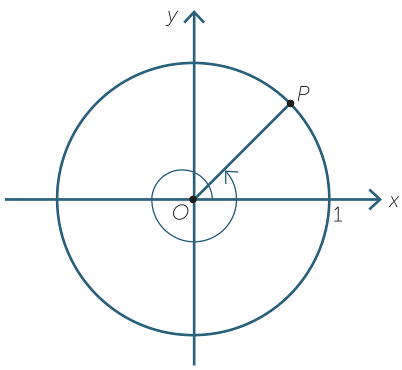
Find the exact value of

a cosec 45° **b** cot 210°

c $\sin^2 120°\operatorname{cosec} 270° - \cos^2 315°\sec 180° - \tan^2 225°\cot 315°$.

ANGLES OF ANY MAGNITUDE

As before, we take the point *P* on the unit circle. We have seen what happens as the point *P* moves through the four quadrants in an anti-clockwise direction. If the point continues a little further, it arrives back in the first quadrant.



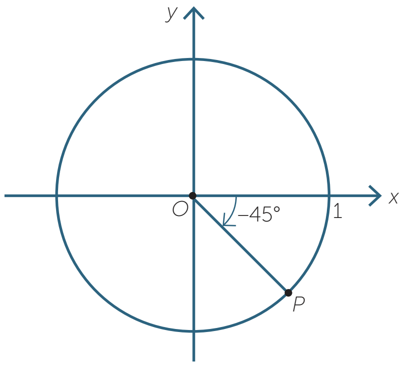
The angle shown is now greater than 360°. The point P, however, is back in the first quadrant. By considering the coordinates of *P*, we see that adding 360° to the angle does not change the x and y coordinates of *P*. Hence, for any θ,

$$\sin(360° + \theta) = \sin\theta \text{ and } \cos(360° + \theta) = \cos\theta$$

and so $\tan(360° + \theta) = \tan\theta$.

It is important to emphasise that although, for example, angles measuring 60° and 420° have the same sine, cosine and tangent ratios, they are *different* angles. They correspond to the same point on the unit circle, but are different angles.

We usually measure angles in an anti-clockwise direction and call them **positive angles**. Angles measured in a clockwise direction are called **negative angles**.



The diagram shows an angle of –45°.

EXAMPLE

Find sin 480° in surd form.

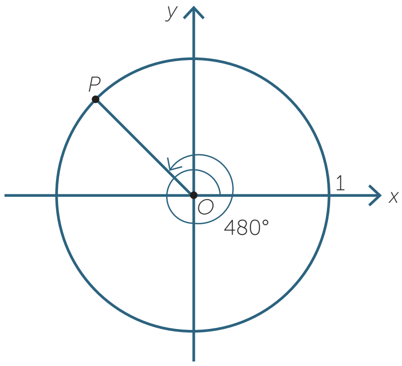
SOLUTION

sin 480° = sin (480° – 360°)

= sin 120° (120° lies in the second quadrant)

= sin 60° (the related angle is 60°)

$= \frac{\sqrt{3}}{2}$



EXAMPLE

Find cos (–120°) in surd form.

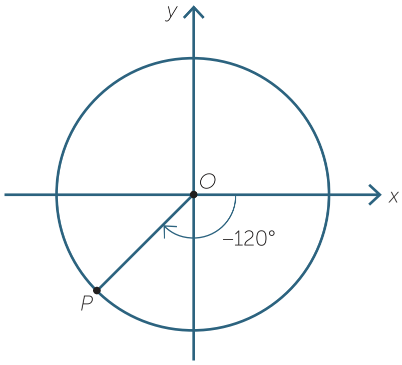
SOLUTION

cos (–120)° = cos (–120° + 360°)

= cos 240° (240° lies in the third quadrant)

= –cos 60° (the related angle is 60°)

$= \frac{1}{2}$



EXERCISE 4

Find the exact value of sin 1200°.

FINDING ANGLES

In this section we will, unless stated otherwise, restrict our angles to be between 0° and 360°.

We have seen, for example, that the angles 30° and 150° have the same sine ratio $\frac{1}{2}$. Hence, if we ask the question: *What angle has a sine of* $\frac{1}{2}$? There are two answers.

The calculator can only give at most one of the two correct values.

Since we know the trigonometric ratios of the special angles 30°, 45° and 60°, these make good examples to practice on.

EXAMPLE

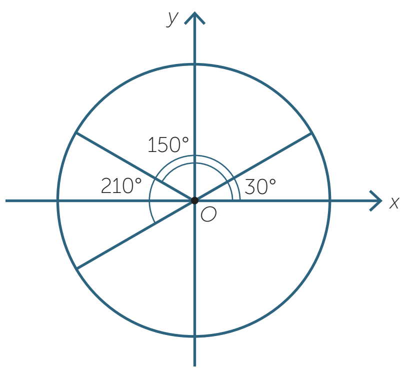
Find all angles in the range 0° to 360° such that

a $\cos\theta = -\frac{\sqrt{3}}{2}$

b $\tan\theta = -1.192$ (give the answer to the nearest degree).

SOLUTION

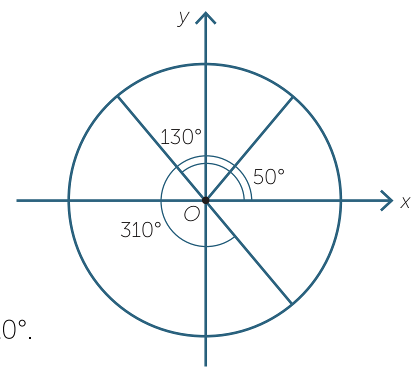
a We recognise the number $\frac{\sqrt{3}}{2}$ as the cosine of 30°. This is the related angle in this instance.



The negative sign tells us that the angle is in the second or third quadrant. Hence $\theta = 180° - 30° = 150°$ or $\theta = 180° + 30° = 210°$.

That is, $\theta = 150°$ or $210°$.

b We begin by finding the related angle, which we obtain from the calculator by entering $\tan^{-1} 1.192$. This produces the related angle 50° (to the nearest degree).



Now the negative sign tells us that the angle lies in the second or fourth quadrants. Hence, to the nearest degree,
$\theta = 180° - 50° = 130°$ or $\theta = 360° - 50° = 310°$.

That is, $\theta \approx 130°$ or $310°$.

Note then that we find the related angle by entering $\tan^{-1} 1.192$ into the calculator. If we enter $\tan^{-1} -1.192$ the calculator may produce 130°, or on most calculators −50°, neither of which is all that helpful in finding the two desired solutions.

It is best to tell students not to enter on their calculator the inverse trigonometric function of a negative number.

EXERCISE 5

Solve the equation $\sin\theta = -\frac{\sqrt{3}}{2}$ in the range, $0° \leq \theta \leq 720°$.

THE TRIGONOMETRIC FUNCTIONS AND THEIR SYMMETRIES

We will now think of the trigonometric ratios as functions. Thus, the function $y = \sin\theta$ has input values θ, consisting of angles, initially in the range 0° to 360°, and output values that are real numbers between –1 and 1.

In the module on *The Quadratic Function*, we saw how to draw the graph representing the points that lie on the parabola $y = x^2$. Initially this was done by drawing up a table of ordered pairs, plotting them and connecting them to produce a smooth curve.

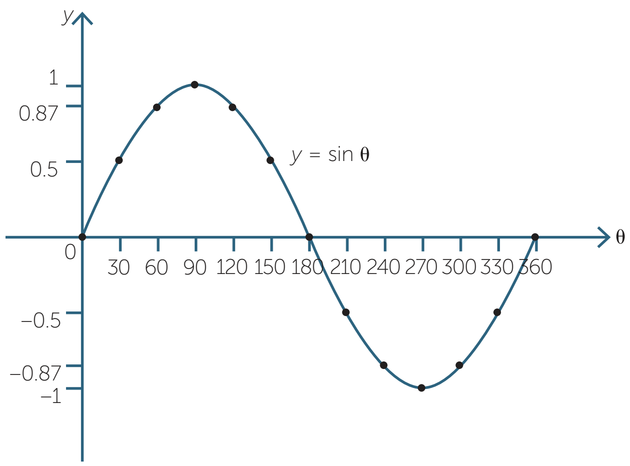
We can perform a similar task for the trigonometric functions $y = \cos\theta$ and $y = \cos\theta$.

The graph of $y = \sin\theta$

Using the values $\sin 30° = \frac{1}{2} = 0.5$ and $\sin 60° = \frac{\sqrt{3}}{2} \approx 0.87$, we can draw up the following table of values and then plot them.

θ°	0	30	60	90	120	150	180	210	240	270	300	330	360
sin θ	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0

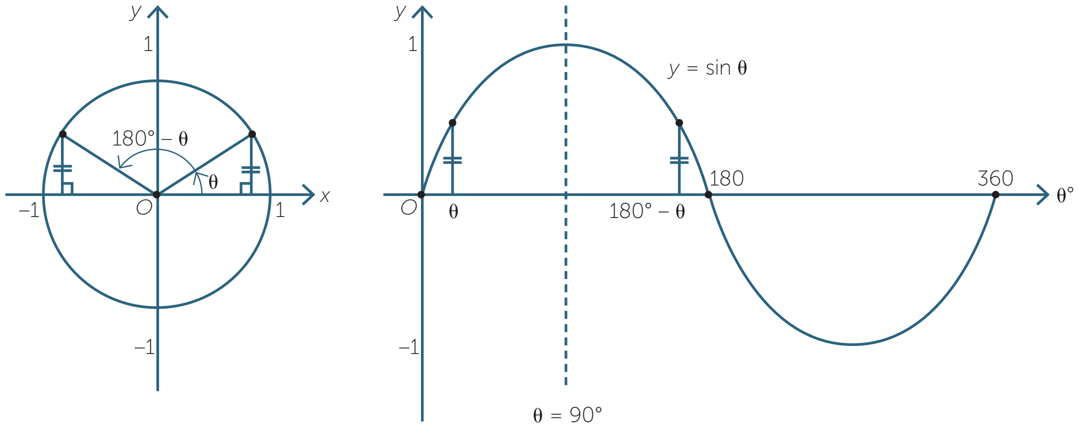
More points can be used to show that the shape of the graph is as shown below.



Electrical engineers and physicists call this a **sinewave**.

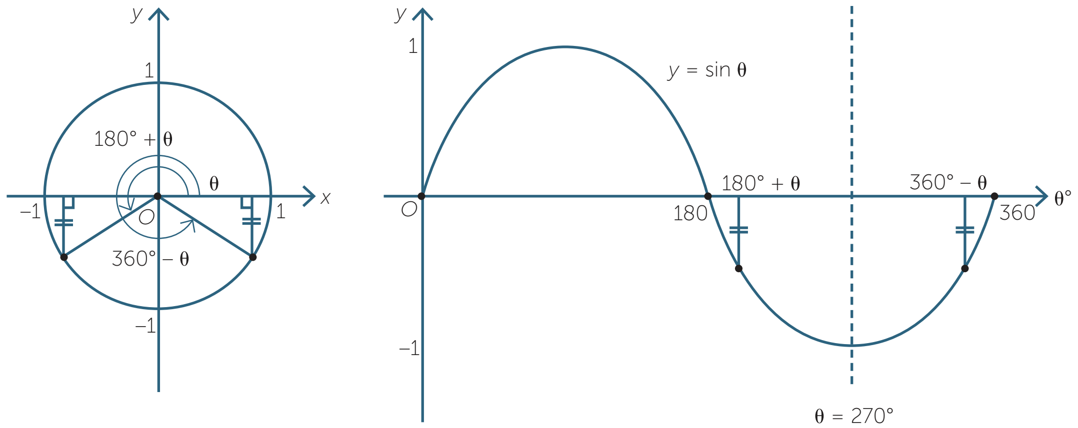
You can see many interesting symmetries in the graph of .

The model employing the unit circle helps to elucidate these. The sine of the angle θ is represented by the y-value of the point P on the unit circle. Thus, since $\sin\theta = \sin(180 - \theta)$, we mark the two equal intervals in the graph.

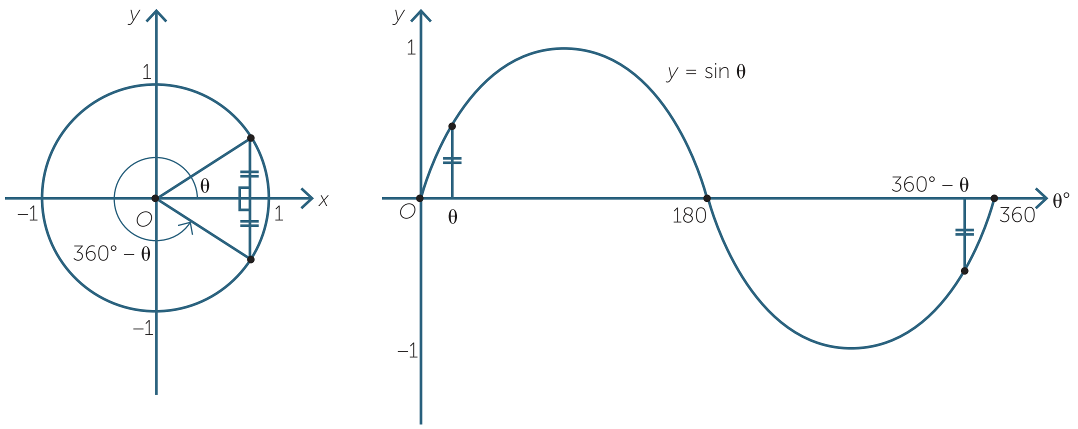


Hence, between 0° and 180°, the graph is symmetric about θ = 90°.

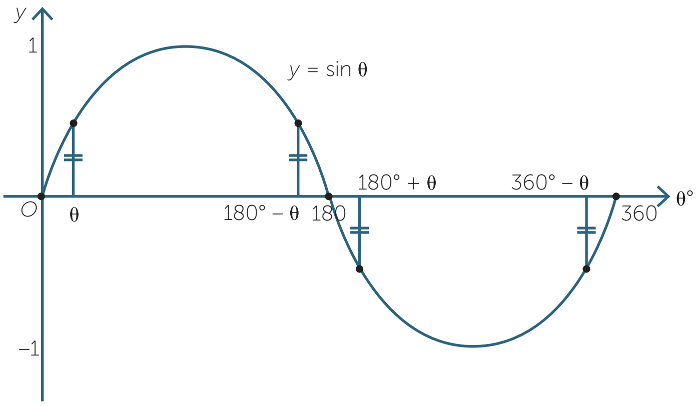
Similarly, between 180° and 360°, the graph is symmetric about θ = 270°.



Finally, the graph possesses a rotational symmetry about θ = 180° as the following diagrams demonstrate.



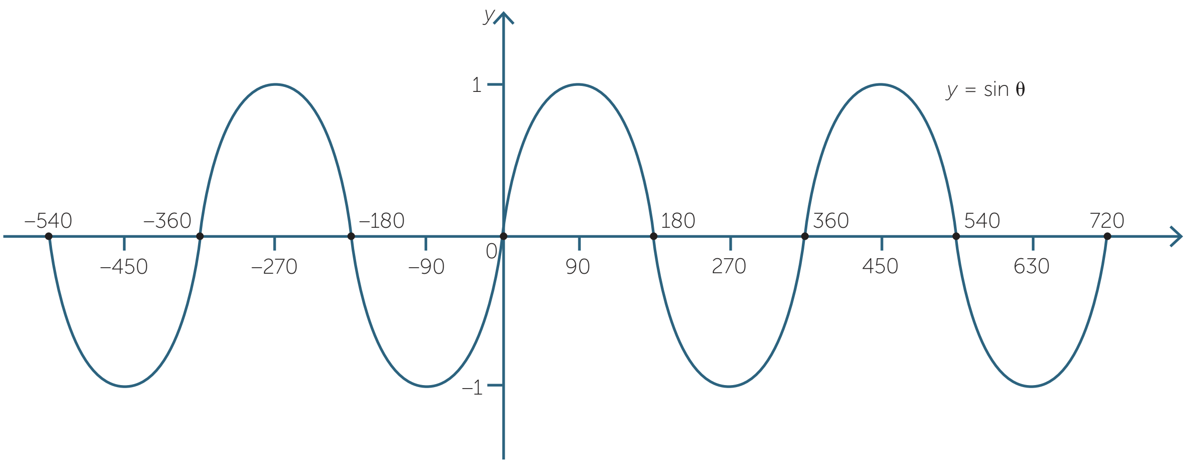
All these observations are summarised by the diagram:



This symmetry diagram illustrates the related angle, the quadrant sign rules and the symmetries discussed above.

Extending the graph

We noted above that the values of sine repeat as we move through an angle of 360°, that is, $\sin(360° + \theta) = \sin\theta$. We say that the function $y = \sin\theta$ is **periodic** with **period** 360°. Thus, the graph may be drawn for angles greater than 360° and less than 0°, to produce the full (or extended) graph of $y = \sin\theta$.



Note that the extended sine graph has even more symmetries. There is a translation (by 360°) symmetry, a reflection symmetry about any odd multiple of 90° and a rotational symmetry about vertical lines through any even multiple of 90°.

EXERCISE 6

a Make up a table of values (to 2 dec. places) for $y = \cos\theta$ similar to that shown for $y = \sin\theta$.

b Plot the points in the table and sketch the graph of $y = \cos\theta$ for θ in the range 0° to 360°.

c Mark in the symmetries for the graph in **b**

d Draw the extended graph of $y = \cos\theta$.

e In the extended graph of $y = \cos\theta$ what are the points of rotational symmetry and the lines of reflection symmetry?

Note that if we translate the extended graph of $y = \cos\theta$ by 90° to the right, we obtain the graph of $y = \sin\theta$.

LINKS FORWARD

TRIGONOMETRIC EQUATIONS

In various modules we have looked at how to solve linear equations, such as $3x - 7 = 8$ and quadratic equations such as $x^2 - 5x + 6 = 0$. We have also seen that such equations naturally arise in problem solving.

Similarly trigonometric equations naturally arise in problem solving. An equation such as $2\sin x - 1 = 0$ is an example of a linear trigonometric equation, since putting $a = \sin x$ produces the linear equation $2a - 1 = 0$. Equations such as these generally have infinitely many solutions, but in practice we often restrict the range of solutions to be between 0° and 360°, or for example between –180° and 180°.

We can solve this equation by firstly making $\sin x$ the subject and then, using the methods outlined above, find all solutions in the range $0° \le x \le 360°$.

$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

$$x = 30°, 150°$$

An equation such as $2\cos^2 x - 5\cos x - 3 = 0$ resembles a quadratic equation.
We can solve this for $0° \le x \le 360°$, by putting $a = \cos x$, thereby turning the equation into a quadratic.

$$2\cos^2 x - 5\cos x - 3 = 0$$

Let $a = \cos x.$

Hence, $2a^2 - 5a - 3 = 0$

$(2a - 1)(a + 3) = 0$

$a = \frac{1}{2}, \qquad a = -3$

Thus $\cos x = \frac{1}{2}$ or $\cos x = -3.$

The second equation has no solutions since $-1 \leq \cos x \leq -1$. The first equation has solutions $x = 60°$, $x = 300°$.

EXERCISE 7

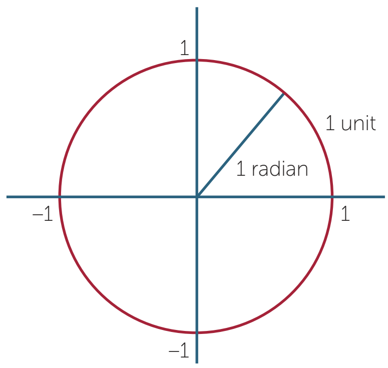
Use the identity $\cos^2 x + \sin^2 x = 1$ to find all solutions, in the range $0° \leq x \leq 360°$ for the equation $\sin^2 x - \cos^2 x = 1$.

RADIAN MEASURE

In all of our work on trigonometry and geometry, we have measured angles in degrees. This is the traditional unit of measurement in geometry and introductory trigonometry which was inherited from the Babylonians. It is, however, an arbitrary unit of angle measure – one can also measure angles in a decimal system in which a right-angle is 100 units, (known as *grads)*.

When we come to apply calculus to the trigonometric functions, it is more useful to provide a purely geometric procedure to measure an angle. This is done using the circumference of a unit circle.

We take a circle of radius 1 and define **1 radian** to be the angle subtended by an arc of 1 unit on the circumference of the circle.



Now since the entire circumference has length 2π, and this subtends an angle of 360°, we see that

2π radians equals to 360° or π radians are equal to 180°.

Now the number π is approximately, 3.14159 correct to 5 decimal places, so we can divide and say that 1 radian is approximately 57.29578 degrees, which is 57 degrees, correct to the nearest degree.

EXERCISE 8

Express each of the angles: 30°, 45°, 60°, 90°, 225° as rational multiples of π.

Measuring angles in this way, it can be shown, by means of ratios, that in a circle of radius r the length l of an arc subtending an angle θ measured in radians, is given by

$$l = r\theta.$$

(Note that if θ is in degrees, $l = \frac{\pi\theta}{180°}$.)

Furthermore, the area of the sector in this circle making an angle θ, again in radians, at the centre, is given by

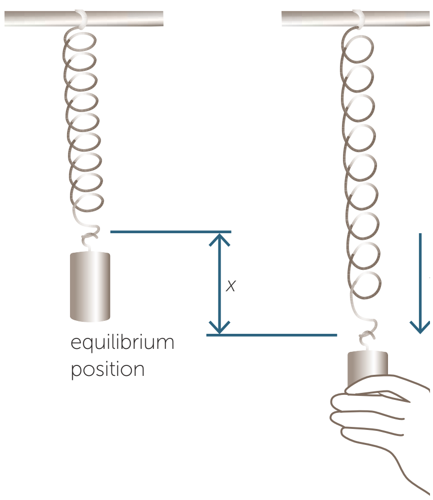
$$\text{Area} = \tfrac{1}{2}r^2\theta$$

CALCULUS

We saw above that the graphs of the functions $y = \sin\theta$ and $y = \cos\theta$ look like waves. In the same way that a parabola (upside down) can be used to model projectile motion, so the trigonometric functions with radians can be used to model wave motion. Thus the function $y = \cos t$ might be used to model the vertical displacement of a point travelling on a wave at time t. We can obtain the velocity of the particle by taking the derivative. It turns out that the derivative of $\sin t$ is $\cos t$. This is remarkably simple!

SIMPLE HARMONIC MOTION

Suppose that a mass is placed on the end of a spring, pulled down and released. The mass will oscillate about a central point. Such motion is an example of what physicists and mathematicians call **simple harmonic motion.**



If we plot the vertical displacement of the mass from the central point, at time t, we obtain a graph that is modeled by using the sine curve.

Many physical phenomena can be modelled using simple harmonic motion – for example sound waves and radio waves – and hence by the trigonometric functions.

HISTORY

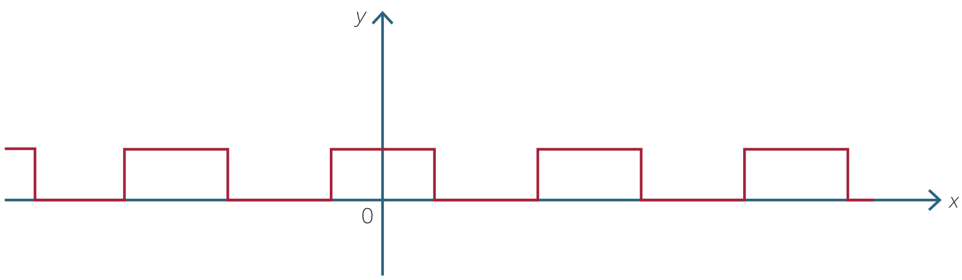
The early history of trigonometry was covered in the modules, *Introductory Trigonometry*, and *Further Trigonometry.*

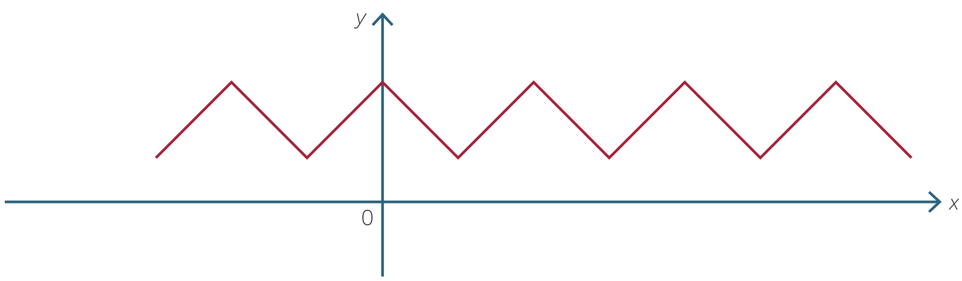
The history of the trigonometric functions really begins after the development of the calculus and was largely developed by Leonard Euler (1707–1783). The works of James Gregory in the 17th century and Colin Maclaurin in the 18th century led to the development of infinite series expansions for the trigonometric functions, such as

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

This enabled accurate tables of values to be tabulated for the trigonometric functions and also useful approximations to be made. For example, when a pendulum is swung, since the angle θ between the arm of the pendulum and the vertical is small, physicists approximate $\sin \theta$ by θ in the equation of motion and can thereby predict, approximately, the period of the pendulum.

The mathematician Joseph Fourier (1768–1830) used infinite series of sines and cosines to solve problems involving heat transfer and vibrations. Thus the modern theory of Fourier Series was born. Any periodic function (that is, a function whose graph repeats after a fixed interval, called the period of the function) can be approximated by taking a series of trigonometric functions. Thus, for example, a square wave, and the so-called *saw-tooth function,* both of which arise in signal processing, can also be modelled using Fourier Series.





ANSWERS TO EXERCISES

EXERCISE 1

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
90°	1	0	undefined
180°	0	–1	0
270°	–1	0	undefined
360°	0	1	0

EXERCISE 2

a $-\frac{1}{2}$ b $\frac{\sqrt{3}}{2}$ c $-\frac{\sqrt{3}}{2}$ d $\cos 110° = -\cos 70°$

EXERCISE 3

a $\sqrt{2}$ b $\sqrt{3}$ c $\frac{3}{4}$

EXERCISE 4

$\frac{\sqrt{3}}{2}$

EXERCISE 5

240°, 300°, 600°, 660°

EXERCISE 7

90°, 270°

EXERCISE 8

a $\frac{\pi}{6}$ b $\frac{\pi}{4}$ c $\frac{\pi}{3}$ d $\frac{\pi}{2}$ e $\frac{5\pi}{4}$

INTERNATIONAL CENTRE
OF EXCELLENCE FOR
EDUCATION IN
MATHEMATICS

The aim of the International Centre of Excellence for Education in Mathematics (ICE-EM) is to strengthen education in the mathematical sciences at all levels- from school to advanced research and contemporary applications in industry and commerce.

ICE-EM is the education division of the Australian Mathematical Sciences Institute, a consortium of 27 university mathematics departments, CSIRO Mathematical and Information Sciences, the Australian Bureau of Statistics, the Australian Mathematical Society and the Australian Mathematics Trust.

AUSTRALIAN MATHEMATICS TRUST

AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

The ICE-EM modules are part of *The Improving Mathematics Education in Schools* (TIMES) *Project*.

The modules are organised under the strand titles of the Australian Curriculum:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

The modules are written for teachers. Each module contains a discussion of a component of the mathematics curriculum up to the end of Year 10.

www.amsi.org.au